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# Preference Models in Portfolio Construction and Evaluation

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## Introduction

What role should the preferences of an investor play in optimal portfolio decisions? If one adds a qualifier that the optimal portfolio decision concerns the very investor whose preferences are under investigation, the question seems trivial. Everyone would answer that preferences are crucial or at least very important ingredients, alongside other factors such as the asset menu, the dynamics of investment opportunities, and the relevant constraints. Modern portfolio theory affirms that such a question is far from trivial for two reasons: First, various asset allocation frameworks often disregard the role of preferences. This omission is often justified by results in asset pricing theory; for instance by the separation result of the celebrated capital asset pricing model (CAPM) implying that, independently of preferences, investors ought to simply demand a multiple of the market portfolio. Second, critical differences between ex-ante versus ex-post optimal portfolios exists, and preferences are often downplayed on an ex-ante basis. Strategies that seem to be optimal ex-ante may turn gravely disappointing ex-post. Further, strategies that in principle are suboptimal (e.g., ones disregarding the preferences of decision makers) may yield ex-post robust performance. This chapter investigates whether and how preference-based optimal asset allocation models may potentially contribute to producing appealing ex-ante and ex-post performances.

This chapter mixes the goals and methods of a review of the methodological literature with the objective of offering novel insights on whether and how the tools described herein may work in practice. The chapter is organized as follows: It begins by setting up the typical portfolio problem, providing relevant definitions and notations. Next, the chapter introduces the main types of preference frameworks used in the portfolio literature and, to a lesser extent, in the practice of applied wealth management, often borrowing from microeconomic theory. Because various researchers have proposed that Taylor approximations applied to the functional representation of standard preferences may replace more complex

mathematical constructs, the chapter includes an in-depth discussion of the advantages and disadvantages of using Taylor approximations, which emphasize the role played by statistical moments (mean, variance, skewness, and kurtosis) of either terminal wealth or portfolio returns. Some discussion is then devoted to new and exciting developments that have recently occurred at the intersection between decision theory and portfolio management in the form of frameworks that emphasize the concepts of robust decisions and ambiguity aversion. Providing an illustrative example that aims at investigating some aspects of the interaction between preferences and statistical models of investment opportunities is undertaken. In particular, optimal portfolio decisions are computed under regime-switching models that capture various features of time-varying investment opportunities.

## Preliminaries and Definitions

In the economic literature, a standard practice is to model the choices of economic agents among several goods using the concept of a utility function. In its cardinal form, a utility function,  $u(\cdot)$ , is used to assign a numeric value to all possible choices (e.g., bundles of goods) faced by an economic agent. These values, often referred to as the *utility index*, have the property that bundle  $r^1$  is (weakly) preferred to  $r^2$  if and only if the utility of  $r^1$  is higher than that of  $r^2$ , as in  $u(r^1) \geq u(r^2)$ . The higher the value of a particular choice, the greater the utility derived from that choice. Utility functions can represent a broad set of preference orderings.

The literature on utility functions, such as Elton, Gruber, Brown, and Goetzmann (2010), widely explores the precise conditions under which a preference ordering can be expressed through a utility function. At least at a superficial level, the properties of such conditions are usually held to imply several things. First, when a utility index is written as a function of either the wealth or the consumption of an agent, the condition implies that  $u(\cdot)$  should be monotonically increasing in its argument(s); this increase is known as the *nonsatiation property*, meaning that investors always prefer more to less. Second, the conditions necessary for a preference ordering imply that  $u(\cdot)$  should be concave, which can be proven to be equivalent to risk aversion. Risk aversion involves that investors prefer the expected value of a gamble (risky investment) to the risky gamble itself.

In portfolio theory, investors are faced with a set of choices under uncertainty. Different portfolios have different levels of risk,  $\kappa$ , and expected return,  $\mu$ , where risk may be measured in various ways. Besides variance, examples of alternative measures of risk are dispersion measures, such as mean absolute deviation of portfolio returns or wealth and downside risk measures. Investors are faced with the decision of choosing a portfolio from the set of all possible risk-return combinations, and obtain different levels of utility from different combinations. The utility obtained from a risk-return combination is expressed by the utility function, implicitly or explicitly capturing preferences in regard to perceived risk

and expected return. When such dependence is assumed to be explicit, the result is Equation 11.1,

$$V \equiv U(\mu, \kappa) \quad \partial U / \partial \mu > 0, \partial U / \partial \kappa < 0 \quad (11.1)$$

where  $\partial U / \partial \mu > 0$  derives from nonsatiation and  $\partial U / \partial \kappa < 0$  from risk aversion. These preference representations may be particularly simple and enlightening, giving rise to classical derivations, for instance of the CAPM in asset pricing theory. However, in such cases, how this risk-return preference is derived from an underlying preference ordering concerning bundles of goods and services under uncertainty is often unclear. When the agent has simple preference structures directly defined according to expected risk and returns, a utility function can be presented in graphical form by a set of indifference curves.

More often, such dependence is modeled in an indirect, implicit fashion, so that the links with the underlying, micro-founded preference ordering are easy to formalize, but the analysis tends to be more involved. In this case, the general idea is that a rational investor with utility  $u(\cdot)$  and initial wealth  $W_t$  chooses his portfolio  $\gamma_t$  at time  $t$  so as to maximize his expected utility of either of the terminal wealth  $T$  periods ahead, as shown in Equation 11.2,

$$\begin{aligned} \max_{\omega_t} E_t [u(W_{t+T})] \text{ s.t. (i) } W_{t+T} \\ = \exp \left[ \sum_{n=1}^N \omega_t^n \sum_{i=1}^T r_{t+i}^n + (1 - \sum_{n=1}^N \omega_t^n) T r^f \right] W_t; \text{ (ii) } \omega_t' 1_N = 1. \end{aligned} \quad (11.2)$$

or of the stream of consumption flows between  $t$  and  $t+T$ , as shown in Equation 11.3:

$$\begin{aligned} \max_{\{\omega_{t+i}, C_{t+i}\}_{i=0}^T} E_t [u(C_{t+i})] \text{ s.t.} \\ \text{(i) } W_{t+i} = \exp \left[ \sum_{n=1}^N \omega_t^n \sum_{i=1}^T r_{t+i}^n + (1 - \sum_{n=1}^N \omega_t^n) T r^f \right] (W_{t+i-1} - C_{t+i}); \\ \text{(ii) } \omega_{t+i}' 1_N = 1 \quad i = 0, 1, \dots, T-1; \\ \text{(iii) } C_{t+i} \geq 0 \quad i = 0, 1, \dots, T-1. \end{aligned} \quad (11.3)$$

In both Equations 11.2 and 11.3, the constraint in (i) is the dynamic law of motion of wealth (the net of consumption withdrawals in the case of Equation 11.3). Equation 11.2 illustrates the problem of an investor who commits her initial wealth  $W_t$  to a vector of weights  $\{\omega_t^n\}_{n=1}^N$  in order to maximize the expected utility of her final wealth,  $u(W_{t+T})$ , without the possibility of any interim withdrawals or consumption. This is also called a *buy-and-hold problem*. Of course, most investors are not concerned with the level of wealth for its own sake but with the standard of living that their wealth can support. In other words, they

consume out of wealth and derive utility from consumption rather than wealth. Therefore, Equation 11.3 illustrates the problem of an investor who selects a vector of weights,  $\{\omega_{t+i}^n\}_{n=1}^N$ , as well as of interim consumption levels  $C_{t+i}$  and who does this repeatedly over time to maximize her utility of the flow of consumption. However, in both equations 11.2 and 11.3, the utility indices play a key role.

The Arrow-Pratt coefficients of (local) absolute and relative risk aversion are two key measures describing the (local) properties of utility functions  $u(W_{t+T})$  and/or  $u(C_{t+i})$ , usually abbreviated as  $CARA(x)$  and  $CRRA(x)$ , respectively. They are illustrated in the equations shown in 11.4:

$$CARA(x) \equiv -\frac{u'(x)}{u''(x)} \quad CRRA(x) \equiv -\frac{u'(x)}{u''(x)}x = CARA(x)x. \quad (11.4)$$

where  $x$  is either terminal wealth  $W_{t+T}$  or consumption  $C_{t+i}$ . These properties give important insights into the nature and behavior of cardinal utility functions, as described in the following:  $CARA(x)$  is a scaled, normalized measure of an individual's risk aversion in a small neighborhood of her current (initial) wealth or consumption. Notice that if  $u'(x) > 0$  and  $u''(x) < 0$ , as normally required of utility functions used in portfolio theory, then  $CARA(x) > 0$ .  $CRRA(x)$ , on the other hand, is a normalized measure of an individual's risk aversion in a small neighborhood of her current (initial) wealth or consumption per unit of wealth or consumption. Because  $CRRA(x) = CARA(x) \cdot x$ , when  $CARA(x) > 0$ ,  $CRRA(x) > 0$ . Pratt (1964) shows that for a small degree of risk, the  $CARA$  coefficient determines the absolute dollar amount that an investor is willing to pay to avoid such a small risk. A common view is that  $CARA$  should decrease, or at least not increase, with wealth. Instead,  $CRRA$  determines the fraction of wealth than an investor will pay to avoid a small risk of a given size relative to wealth. Another common belief is that plausible preferences should imply that relative risk aversion should be independent of wealth (LeRoy and Werner, 2001). Moreover, Campbell and Viceira's (2002) discussion of the long-run behavior of most economies, characterized by substantial growth in real consumption but also by real interest rates and consumption-wealth ratios that fail to be trending, is consistent with relative risk aversion levels that are independent of wealth.

This chapter examines relatively simple portfolio selection problems under a variety of alternative assumptions concerning the preferences (objectives) of an atomistic investor who is not necessarily representative of the market. Therefore, the analysis is of a partial-equilibrium nature. In particular, unless otherwise stated, the chapter deals with an investor who has to choose a portfolio comprised of  $N$  risky assets, described by a vector  $\mathbf{r}_t$  of continuously compounded returns. If a riskless asset exists, its risk-free yield is denoted as  $r$  and the asset is indexed as 0. To keep things simple, the assumption is made not only that the riskless asset exists but also that the risk-free rate is constant over time. In fact, these assumptions are not far from describing how very-short-term interest rates behave in reality, over the investment horizons of interest in this chapter. The investor's choice is embodied in an  $N \times 1$  vector,  $\omega \equiv [\omega^0 \omega^1 \omega^2 \cdots \omega^N]$  of weights, where each weight  $\omega^n$  represents the percentage of the  $n$ th asset held

in the portfolio, and the sum of the weights, including the riskless asset weight  $\omega^0$  must be equal to 1 (i.e., no money should be left on the table, which derives from nonsatiation). Although, in general, short selling (the case in which weights can be negative or can exceed 1 is possible. In the illustrations given later, the restriction prohibiting short selling is imposed because this is realistic and often simplifies numerical optimization when short selling is undertaken.

## Subjective Expected Utility Preferences: Main Functional Classes

To get comfortable with the previously described frameworks and concepts, listing the most commonly used assumptions made in the literature about  $u(W_{t+T})$  and  $u(C_{t+i})$  and discussing their connections are useful. As a rule, examples are provided with explicit reference to the case of utility depending on terminal wealth, unless the presentation requires dealing with consumption.

### AD-HOC MEAN-VARIANCE UTILITY FUNCTIONS

The development of the classical theory of finance has been characterized by using simple but ad-hoc mean-variance (MV) objective functions with structure, as shown in Equation 11.5:

$$MV \equiv E[W_{t+T}] - \frac{1}{2} \lambda \text{Var}[W_{t+T}]. \quad (11.5)$$

However, Equation 11.5 does not define a utility function in a technical sense. Instead of writing a mapping from either terminal wealth or consumption streams into investor's welfare, Equation 11.5 pins down a mapping between the final investor's objective—say, expected utility—and the first two moments of the distribution of wealth (i.e., mean and variance). For a long time, this representation has been just perceived as a convenient shortcut. Using Equation 11.5 generates problems that stem from its lack of microfoundations. For instance, Equation 11.5 does not allow computing either  $\text{CARA}(W_t)$  or  $\text{CRRA}(W_t)$  and therefore formally characterizing MV. However, finding interpretations of  $\lambda$  that assimilate this coefficient to a CARA measure is common. Although this interpretation is not formally correct, it is approximately the case under some special assumptions.

Interestingly, these remarks concerning Equation 11.5 do not apply to Markowitz's classical risk minimization framework that is sometimes referred to as being based on MV, although this labeling may be misleading. Markowitz (1952) argues that for any given level of expected return,  $\bar{\mu}_p$ , a rational investor would choose the portfolio with minimum variance from among the set of all possible portfolios, as illustrated in Equation 11.6:

$$\min_{\omega} \omega' \text{Var}[r_t] \omega \quad \text{s.t. (i) } \omega' E[r_t] = \bar{\mu}_p; \text{ (ii) } \omega' 1_N = 1. \quad (11.6)$$

The set of all possible portfolios that can be constructed is called the *feasible set*. MV portfolios are called *mean-variance efficient portfolios*. The set of all MV efficient portfolios, for different desired levels of expected return, is called the *efficient frontier*. Clearly, this algorithm does not lead to selecting a unique optimal portfolio but instead to locating the efficient frontier, represented by the set  $\hat{\omega}$  seen as a function of  $\bar{\mu}_p$ . However, a well-known alternative to Markowitz's risk minimization framework, shown in the optimization problem 11.6, is to explicitly model the trade-off between risk and return in the objective function using a fictitious risk-aversion coefficient,  $\lambda$ , which is commonly called the risk-aversion formulation of the efficient frontier problem, as shown in Equation 11.7:

$$\min_{\omega} \omega' E[r_t] - \lambda \omega' \text{Var}(r_t) \omega \quad \text{s.t. } \omega' 1_N = 1. \quad (11.7)$$

If  $\lambda$  is gradually increased from zero to infinity, and for each increase, the optimization problem is solved, the result is that all portfolios along the efficient frontier can be calculated. Equation 11.7 differs from Equation 11.5 in that the MV trade-off is explicitly formulated in terms of the expectation and variance of portfolio returns. Although useful in the development of the CAPM, this relationship cannot represent a benchmark in portfolio choice applications. In fact, the objective in Equation 11.7 is even more problematic than the one in Equation 11.5. When returns are discretely compounded, as in  $W_{t+T} = (1 + R_{t,T}^p(\omega_t))W_t$ , where  $R_{t,T}^p$  is the total portfolio return (from an investment strategy characterized by  $\omega_t$ ) between time  $t$  and time  $T$ , many researchers often plug this accounting into Equation 11.5 and obtain an equivalent objective, as shown in Equation 11.8:

$$\begin{aligned} MV^R &\equiv E[(1 + R_{t,T}^p(\omega_t))W_t] - \frac{1}{2} \lambda \text{Var}[(1 + R_{t,T}^p(\omega_t))W_t] \\ &= W_t + E[R_{t,T}^p(\omega_t)]W_t - \frac{1}{2} W_t^2 \lambda \text{Var}[R_{t,T}^p(\omega_t)]. \end{aligned} \quad (11.8)$$

If one further standardizes initial wealth to be 1 and observes that adding a constant to the objective of a maximization problem does not change the nature of the problem or affect the set of controls, then Equation 11.9 results:

$$MV^R \propto E[R_{t,T}^p(\omega_t)] - \frac{1}{2} \lambda \text{Var}[R_{t,T}^p(\omega_t)], \quad (11.9)$$

which is a new, ad-hoc objective functional in which the mean and variance are no longer defined with reference to terminal wealth but directly in terms of portfolio returns over the horizon  $[t, T]$ . However, the absence of a precise microfoundation is simply concealed by the deceptively intuitive nature of the objective

11.9, as defining  $CARA(R_{t,T}^p)$  or  $CRRA(R_{t,T}^p)$  remains impossible. An additional problem is caused by that fact that because of Equation 11.10,

$$R_{t,T}^p = \prod_{i=0}^{T-1} (1 + R_{t+i,t+i+1}^p) - 1 = \prod_{i=0}^{T-1} \left( 1 + \sum_{n=0}^N \omega_t^n r_{t+i,t+i+1}^n \right) - 1. \quad (11.10)$$

$Var[R_{t,T}^p(\omega_t)]$  has a complicated expression unless one assumes, usually in contrast with the data, that the returns on all risky assets are independent over time with zero cross-serial correlations. Probably as a result of this simple fact, Equation 11.9 has been most often employed only after setting  $T=1$ , when  $Var(R_{t+1}^p) = \omega_t' Var(\mathbf{r}_{t+1}) \omega_t$ .

Both equations 11.5 and 11.9 are frequently used in portfolio management, and, at least to some extent, in practice this is because they lead to a closed-form expression (Fabozzi, Focardi, and Kolm, 2006). For instance, in the case of Equation 11.5, Equation 11.11

$$\begin{aligned} \max_{\omega_t} E[R_{t+1}^p(\omega_t)] - \frac{1}{2} \lambda Var[R_{t+1}^p(\omega_t)] \\ \text{s.t. (i) } R_{t+1}^p = \omega_t' \mathbf{r}_{t+1} + (1 - \omega_t' \mathbf{1}_N) r^f W_t; \text{ (ii) } \omega_t' \mathbf{1}_N = 1, \end{aligned} \quad (11.11)$$

leads to the first-order conditions  $(\mathbf{r}_{t+1} - r^f \mathbf{1}_N) - \lambda Var(\mathbf{r}_{t+1}) \hat{\omega}_t = \mathbf{0}_N$  that yield the classical MV formula in Equation 11.12:

$$\hat{\omega}_t = \lambda^{-1} [Var(\mathbf{r}_{t+1})]^{-1} (\mathbf{r}_{t+1} - r^f \mathbf{1}_N). \quad (11.12)$$

## LINEAR UTILITY

Even though the case of linear utility, which is better known as maintaining risk-neutral preferences, can be easily derived from Equation 11.5 by setting  $\lambda=0$ , different than that done in both equations 11.5 and 11.9, linear utility has clean microfoundations, as shown in Equation 11.13:

$$u_{lin}(W_{t+T}) = W_{t+T}, \quad (11.13)$$

which implies that  $E[u_{lin}(W_{t+T})] = E[W_{t+T}]$ , implying that an investor ought to simply maximize her expected terminal wealth. Because the case of risk neutrality derives from an assumption of preferences for terminal wealth, then  $u'_{lin}(W_{t+T}) = 1 > 0$ ,  $u''_{lin}(W_{t+T}) = 0$ , which imply  $CARA(W_t) = CRRA(W_t) = 0$ . Even though this specification lacks an assumption of risk aversion, much commentary about market performance implicitly assumes that a simple, linear objective may characterize the behavior of important portions of investors, especially those with short-term goals.



## QUADRATIC UTILITY

One of the most traditional assumptions concerning  $u(W_{t+T})$  or  $u(C_{t+i})$  is that this relationship is a quadratic utility function (Hanoch and Levy, 1970). For instance, focusing on the simpler case of no interim consumption yields Equation 11.14:

$$u_{quad}(W_{t+T}) = W_{t+T} - \frac{1}{2} \lambda W_{t+T}^2. \quad (11.14)$$

In this case, risk aversion holds as  $u''_{quad}(W_{t+T}) = -\lambda < 0$  (but issues exist with nonsatiation because  $u'_{quad}(W_{t+T}) = 1 - \lambda W_{t+T}$ , which is positive if and only if  $W_{t+T} < 1/\lambda$ , putting an upper bound on the domain for wealth levels and therefore the portfolio choices.  $W_{t+T}^* = 1/\lambda$  is often also called the *bliss point* of quadratic utility. Notice that in the case of Equation 11.15, we have

$$\begin{aligned} E[u_{quad}(W_{t+T})] &= E[W_{t+T}] - \frac{1}{2} \lambda E[W_{t+T}^2] = E[W_{t+T}] - \frac{1}{2} \lambda \{ \text{Var}[W_{t+T}] + (E[W_{t+T}])^2 \} \\ &= \left\{ E[W_{t+T}] - \frac{1}{2} \lambda (E[W_{t+T}]^2) \right\} - \frac{1}{2} \lambda \text{Var}[W_{t+T}], \end{aligned} \quad (11.15)$$

which shows that quadratic utility is of a MV type. Even though under quadratic utility  $E[u_{quad}(W_{t+T})]$  declines in the variance of terminal wealth, some ambiguity remains over the behavior of  $E[u_{quad}(W_{t+T})]$  as a function of expected terminal wealth, as shown in Equation 11.16:

$$\frac{\partial E[u_{quad}(W_{t+T})]}{\partial E[W_{t+T}]} = 1 - \lambda E[W_{t+T}]. \quad (11.16)$$

The expression in Equation 11.16 is positive if and only if  $E[W_{t+T}] < 1/\lambda$ , which means that expected wealth is below the bliss point. However, this issue is only an apparent one, as  $W_{t+T} < 1/\lambda$  is sufficient for  $E[W_{t+T}] < 1/\lambda$  to hold, so that quadratic utility preferences are truly based on the MV framework. As a result, the decomposition in 11.15 indicates the way in which the portfolio objective in 11.5 may suffer from an ad-hoc nature. If Equation 11.5 derives from a quadratic utility function, then its functional form is misspecified, because Equation 11.5 differs from Equation 11.15 in the absence of the term  $-0.5\lambda(E[W_{t+T}])^2$ . Additionally, if the objective in 11.5 derives from a quadratic utility function, then one should emphasize that this representation is only valid for  $W_{t+T} < 1/\lambda$ , something that users of Equation 11.5 often forget.

As  $u_{quad}(W_{t+T})$  has a precise microfoundation, computing CARA and CRRA measures in Equation 11.17 is useful:

$$\text{CARA}_{quad}(W) \equiv -\frac{1 - \lambda W}{-\lambda} = \frac{1}{\lambda} - W \quad \text{CRRA}_{quad}(W) = W \left( \frac{1}{\lambda} - W \right). \quad (11.17)$$



Clearly, as long as wealth is below the bliss point,  $CARA_{quad}(W) > 0$ , but  $CARA$  is decreasing in wealth so that as wealth approaches the bliss point, then  $CARA_{quad}(W)$  converges to zero from the right,  $CARA_{quad}(W) \rightarrow 0^+$  and the investor stops being risk averse. Moreover, Equation 11.17 shows that in this case  $CRRA$  is decreasing in wealth. The finding that scaled, normalized risk aversion is declining as investors grow wealthier is an unrealistic feature of quadratic preferences, as is the property that utility may be defined only below the bliss point. In fact, while Equation 11.5 is often employed in practice because of the attractiveness of closed-form expressions such as Equation 11.12, its microfounded version in Equation 11.15 is hardly ever used. These relationships imply that the microfoundations of the portfolio objective in 11.5 have to be found elsewhere.

### NEGATIVE EXPONENTIAL UTILITY

This utility function is as popular as the quadratic utility function, both in its own right—for implying a rather realistic constant coefficient of absolute risk aversion for small risks—and because it provides an alternative to and more compelling microfoundation than Equation 11.5 under the specific assumptions shown in Equation 11.18:

$$u_{\exp}(W_{t+T}) = -\exp(-\lambda W_{t+T}) \quad (11.18)$$

Because  $u'_{\exp}(W_{t+T}) = \lambda \exp(-W_{t+T})$  and  $u''_{\exp}(W_{t+T}) = -\lambda^2 \exp(-W_{t+T})$ , Equation 11.19 results in:

$$CARA_{\exp}(W) \equiv -\frac{-\lambda^2 \exp(-W_{t+T})}{\lambda \exp(-W_{t+T})} = \lambda \quad CRRA_{quad}(W) = \lambda W, \quad (11.19)$$

which means that  $CARA$  is constant and independent of wealth, with  $\lambda$  being the  $CARA$  coefficient, while  $CRRA$  is monotone, increasing in wealth. Of course, this latter property is unrealistic, and concerns about the usefulness of this utility function have stemmed from this  $CRRA$  behavior.

When using expected utility as an objective in portfolio optimization,  $u_{\exp}(W_{t+T})$  fails to yield particularly enlightening insights, such as  $E[u_{\exp}(W_{t+T})] = -E[\exp(-\lambda W_{t+T})]$ , because the convexity of the exponential function prevents the claim that  $E[\exp(-\lambda W_{t+T})] = \exp(-\lambda E[W_{t+T}])$ . However, when  $W_{t+T}$  has a lognormal distribution, the function in Equation 11.20 results from the properties of the moment-generating function of terminal wealth:

$$E[\exp(-\lambda W_{t+T})] = \exp(-\lambda E[W_{t+T}]) \exp\left(\frac{1}{2} \lambda^2 \text{Var}[W_{t+T}]\right). \quad (11.20)$$

Because the standard features of convex optimization ensure that when  $\lambda > 0$ , then choosing  $\omega_t$  to maximize  $-E[u_{\exp}(W_{t+T}(\omega_t))]$  is identical to maximizing

$-\ln E[-\lambda^{-1}u_{\exp}(W_{t+T}(\omega_t))]$ . Thus, maximizing the function in Equation 11.21,

$$-\ln E[-\lambda^{-1}u_{\exp}(W_{t+T}(\omega_t))]=E[W_{t+T}(\omega_t)]-\frac{1}{2}\lambda\text{Var}[W_{t+T}(\omega_t)], \quad (11.21)$$

delivers the same optimal weights as the maximization of the original expected utility functional  $E[u_{\exp}(W_{t+T})]$ . In turn, for any given choice of weights  $\mathbf{w}_t$ , it follows that the expression in Equation 11.22,

$$W_{t+T}(\omega_t)=\exp(\omega_t'\mathbf{r}_{t,T})=\exp\left(\omega_t'\sum_{i=1}^T\mathbf{r}_{t+i}\right), \quad (11.22)$$

has a lognormal distribution (i.e.,  $\ln W_{t+T}(\omega_t)=\omega_t'\sum_{i=0}^N\mathbf{r}_{t+i}$  has a normal distribution) if and only if  $\mathbf{r}_{t+i}$  has a multivariate normal distribution for all cases of  $i\geq 1$ . Therefore, under negative exponential utility, the fact that  $\mathbf{r}_{t+i}$  has a multivariate, joint normal distribution is sufficient to lead to an expected utility functional with a structure identical to Equation 11.5. In that case, the parameter  $\lambda$  in Equation 11.5 can be interpreted as a constant CARA coefficient.

## LOGARITHMIC UTILITY

As discussed later in this chapter, the case of logarithmic utility corresponds to a special limit parameterization of power, known as isoelastic preferences. The structure of this utility function is shown in Equation 11.23:

$$u_{\log}(W_{t+T})=\ln W_{t+T}, \quad (11.23)$$

which implies  $u'_{\log}(W_{t+T})=1/W_{t+T}$  and  $u''_{\log}(W_{t+T})=-1/W_{t+T}^2$ , so that the expressions in Equation 11.24 are obtained:

$$\text{CARA}_{\log}(W)\equiv-\frac{1/W^2}{-1/W}=\frac{1}{W}\text{CRRA}_{\log}(W)=1, \quad (11.24)$$

which reveals that a logarithmic utility function implies a monotone decreasing CARA coefficient (i.e., the investor becomes decreasingly risk-averse as she gets wealthier) and a constant unit CRRA coefficient.

## POWER UTILITY

This functional generalizes the logarithmic case in the utility function 11.25,

$$u_{\text{power}}(W_{t+T})=\frac{W_{t+T}^{1-\gamma}}{1-\gamma}\gamma\neq 1, \quad (11.25)$$

which implies  $u'_{power}(W_{t+T}) = W_{t+T}^{-\gamma}$  and  $u''_{power}(W_{t+T}) = -\gamma W_{t+T}^{-\gamma-1}$ , so that expressions in Equation 11.26,

$$CARA_{power}(W) \equiv -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} = \frac{\gamma}{W} CRR A_{power}(W) = \gamma, \quad (11.26)$$

confirm the same properties obtained under logarithmic utility. Yet, in this case, all the results concerning CARA and CRRA appear to have been scaled by a factor of  $\gamma \neq 1$ . However, one notable limit result applied to Equation 11.26 leads to.

$$\lim_{\gamma \rightarrow 1} u_{power}(W_{t+T}) = \ln W_{t+T} = u_{\log}(W_{t+T}). \quad (11.27)$$

Unless one is ready to resort to some form of approximations, power utility preferences, of which the logarithmic case may be simply interpreted as a special case for  $\gamma \rightarrow 1$ , generally do not lead to closed-form expressions for optimal portfolio weights. Hence, one has to resort to numerical methods to compute optimal allocations.

## EPSTEIN-ZIN PREFERENCES

Despite the many attractive features of the power utility model, it has one highly restrictive feature, which is that power utility implies that the consumer's elasticity of intertemporal substitution,  $\Psi$ , is the reciprocal of the coefficient of relative risk aversion,  $\gamma$ . Yet, whether these two concepts should be linked so tightly is unclear. Risk aversion describes the consumer's reluctance to substitute consumption across states of the world and is meaningful even in a temporal setting. By contrast, the elasticity of intertemporal substitution describes the consumer's willingness to substitute consumption over time and is meaningful even in a deterministic setting. Epstein and Zin (1989) offer a more flexible version of the basic power utility model. The Epstein-Zin model retains the desirable scale independence of power utility (i.e., the fact that CRRA does not depend on wealth) but breaks the link between the parameters  $\Psi$  and  $\gamma$ . In this section, the assumption is made that utility is defined over a stream of consumption. This is the only case for which the recursive structure of Epstein-Zin preferences is sensible, because defining utility over consumption streams drives a wedge between  $\Psi$  and  $\gamma$ . The Epstein-Zin objective function is defined recursively, as shown in Equation 11.28:

$$V_t = \left\{ (1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta \left( E_t[V_{t+1}^{1-\gamma}] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\lambda}}, \quad (11.28)$$

where  $\delta \in (0,1)$  is the subjective discount rate that reflects the impatience of investors and  $\theta \equiv (1-\gamma)/(1-1/\Psi)$ . When  $\gamma=1/\Psi$ ,  $\theta=1$  and the recursion in Equation 11.28 becomes linear, so that it can be solved forward to yield the familiar (time-separable) power utility model.

The nonlinear recursion in Equation 11.28 is generally difficult to work with in consumption/portfolio problems. However, when risky returns are IID (identically and independently distributed, i.e., investment opportunities are constant), then consumption is a constant fraction of wealth and covariance with consumption growth equals covariance with portfolio return. In this case, showing that if  $\gamma=1$ , then  $\theta=0$  is straightforward, so that the standard myopic MV portfolio rule in Equation 11.12 results.

## MOMENT-BASED APPROXIMATIONS

Both power and, at least under general dynamics for portfolio returns, negative exponential utility fail to lead to a closed-form solution for optimal portfolio weights, while MV preferences do not account for skewness and kurtosis in either the (portfolio) return distribution or in interim or terminal wealth under models of time-varying predictive densities. To compensate for these weaknesses, a growing body of literature that goes back to seminal papers by Arditti and Levy (1975) and Kraus and Litzenberger (1976) has adopted a different approach. The key idea of this strand of papers is that expanding one of the utility functions (e.g., power utility or negative exponential) previously specified is useful to obtain a tractable expression that depends only on the first  $M$  moments of the wealth or portfolio return distribution. In fact, although in the classical development of financial economics, MV-based portfolio selection and performance evaluation have been dominant, some papers (Arditti, 1967; Samuelson, 1970) stress that, unless either asset returns are multivariate and normally distributed or utility functions are quadratic, higher moments cannot be neglected. Indeed in the 1960s, the literature shows that security returns were hardly Gaussian (Fama, 1965). More recently, Harvey and Siddique (2000) show that skewness in stock returns is relevant to portfolio selection based on asset pricing fundamentals. If asset returns exhibit nondiversifiable coskewness (the covariance between portfolio returns and the variance of market returns), investors must be rewarded for coskewness, resulting in increased expected returns. In fact, in the presence of positive coskewness, investors may be willing to accept a negative return. Guidolin and Timmermann (2008) extend these intuitions to international asset allocation applications and derive results that explain why US investors may hesitate before aggressively diversifying their equity portfolios internationally.

In particular, Samuelson (1970) shows that the possibility of using MV preferences to approximate any properly defined utility function (as discussed by Tsiang, 1972, and Levy and Markowitz, 1979) extends to all finite  $M$ th moment approximations (obtained by taking a Taylor expansion) and to the generic utility of final wealth functions,  $u(W_{t+T})$ . The approximation will work and will generate a sensible representation of preferences (for instance, in terms of global nonsatiation and risk aversion) only when riskiness is limited in a very precise sense. Assume that the  $t+T$  period returns on the  $N$  risky assets are drawn from a family of compact (small-risk) distributions; for instance, a multivariate distribution illustrated in Equation 11.29

$$F(r_{t \rightarrow t+T}) = P \left( \frac{r_{t,T}^1 - r^f - \sigma_1 T}{\sigma_1 \sqrt{T}}, \frac{r_{t,T}^2 - r^f - \sigma_2 T}{\sigma_2 \sqrt{T}}, \dots, \frac{r_{t,T}^N - r^f - \sigma_N T}{\sigma_N \sqrt{T}} \right), \quad (11.29)$$

such that the condition will do (in other words, this condition will only be sufficient). Intuitively, compactness implies that as the time horizon vanishes asset returns all converge to the riskless rate. Samuelson shows that an  $M$ th moment approximation of a utility function  $u(W_{t+T})$  has a precision that increases as the horizons gets small. Furthermore, given the order of  $M$  of the approximation in Equation 11.30,

$$\lim_{T \rightarrow 0^+} \frac{\partial \hat{w}_n^M(T)}{(\partial T)^m} = \frac{\partial \hat{w}_n(T)}{(\partial T)^m} \quad m = 0, 1, \dots, M \text{ and } n = 1, \dots, N, \quad (11.30)$$

where  $\partial \hat{w}_n^M(T)/(\partial T)^0 = \hat{w}_n^M(T)$ , the apex indicates that an optimal portfolio weight has been computed from an  $M$ th order approximation and  $\hat{w}_n^M$  is the optimizing weight under the utility function  $u(W_{t+T})$ . The implication is that the gain in taking expansions that go beyond the simple MV model is that not only portfolio weights but also their overall behavior as a function of the time horizon can be better approximated the higher that the expansion order  $M$  is. These local approximations involving derivatives of the control variable are referred to as being high contact. Conversely, notice that the result holds only asymptotically and irrespective of the order  $m$ : if  $T$  is too large even under a high  $M$ , the resulting  $\hat{w}_n^M$  may have nothing to do with the correct  $\hat{w}_n$ . Samuelson's paper stresses that two components are needed for approximations to work in asset allocation problems: (1) the asset returns must be drawn from well-behaved distribution families (such as normals), and/or (2) the investment horizon must be very short, in principle infinitesimal.

Tsiang (1972) seems to offer the deepest theoretical background to finite-order Taylor expansions of generally accepted utility functions. Tsiang notes that although rigorous, Samuelson's (1970) asymptotic results could be improved, so that risk would be nonnegligible and small enough in a relative sense for Taylor approximations (possibly MV analysis) to be sufficiently accurate. Finite-order Taylor expansions may be applied to utility functions that display these properties, provided that the power series converges (equivalently, provided the remainder term can be ignored). Tsiang carefully considers this aspect for two classes of utility functions. First, in the case of negative exponentials shown in Equation 11.31,

$$\begin{aligned} u_{app}^M(W_{t+T}; E_t[W_{t+T}]) &= -\exp(-\lambda E_t[W_{t+T}]) + \lambda(W_{t+T} - E_t[W_{t+T}])\exp(-\lambda E_t[W_{t+T}]) + \\ &\quad -\frac{1}{2}\lambda^2(W_{t+T} - E_t[W_{t+T}])^2\exp(-\lambda E_t[W_{t+T}]) + \\ &\quad +\frac{1}{6}\lambda^3(W_{t+T} - E_t[W_{t+T}])^3\exp(-\lambda E_t[W_{t+T}]) + \dots + \\ &\quad -(-1)^M \frac{1}{M!}\lambda^M(W_{t+T} - E_t[W_{t+T}])^M\exp(-\lambda E_t[W_{t+T}]) \\ &= -\exp(-\lambda E_t[W_{t+T}]) \left[ 1 + \lambda(W_{t+T} - E_t[W_{t+T}]) - \right. \\ &\quad \left. \frac{1}{2}\lambda^2(W_{t+T} - E_t[W_{t+T}])^2 + \frac{1}{6}\lambda^3(W_{t+T} - E_t[W_{t+T}])^3 + \dots \right. \\ &\quad \left. -(-1)^M \frac{1}{M!}\lambda^M(W_{t+T} - E_t[W_{t+T}])^M \right]. \end{aligned} \quad (11.31)$$

which gives an approximation to the expected utility in Equation 11.32 of

$$E_t[u_{app}^M(W_{t+T}; E_t[W_{t+T}])] = -\exp(-\lambda E_t[W_{t+T}]) \left[ 1 + \frac{1}{2} \lambda^2 \text{Var}_t[W_{t+T}] + \frac{1}{6} \lambda^3 (\text{Var}_t[W_{t+T}])^{3/2} \right. \\ \left. \times \text{Skew}_t[W_{t+T}] + \cdots - (-1)^M \frac{1}{M!} \lambda^M E_t[(W_{t+T} - E_t[W_{t+T}])^M] \right] \quad (11.32)$$

From well-known mathematical results, the series emerging in Equation 11.33

$$1 + \lambda h - \frac{1}{2} \lambda^2 h^2 + \frac{1}{6} \lambda^3 h^3 + \cdots - (-1)^M \frac{1}{M!} \lambda^M h^M, \quad (11.33)$$

converges for all  $h$ s (provided they are finite). This relationship also guarantees that the approximation can be accurate provided  $M$  is high enough ( $M \rightarrow \infty$ ). In particular, Tsiang argues that if  $\lambda$  is bounded by  $1/E_t[W_{t+T}]$ , then setting  $M = 2$  or 3 may be enough.

With reference to power utility, and when the approximation is taken around  $v \equiv E_t[W_{t+T}]$ ,  $v = E_t[W_{t+T}]$  as shown in Equation 11.34,

$$E[u_{app}^M(W_{t+T}; E_t[W_{t+T}])] = \frac{(E_t[W_{t+T}])^{1-\gamma}}{1-\gamma} - \frac{1}{2} \gamma (E_t[W_{t+T}])^{-\gamma-1} \text{Var}_t[W_{t+T}] + \\ + \frac{1}{6} \gamma(\gamma+1) (E_t[W_{t+T}])^{-\gamma-2} (\text{Var}_t[W_{t+T}])^{3/2} \text{Skew}_t[W_{t+T}] + \\ + \cdots - (-1)^M \frac{1}{M!} \prod_{j=0}^{M-2} (\gamma+j) (E_t[W_{t+T}])^{-\gamma-M+1} E[(W_{t+T} - E_t[W_{t+T}])^M]. \quad (11.34)$$

Tsiang (1972) shows that the condition  $|h| = |W_{t+T} - E_t[W_{t+T}]| \leq E_t[W_{t+T}]$  is required for the series in Equation 11.35,

$$\frac{(E_t[W_{t+T}])^{1-\gamma}}{1-\gamma} + h (E_t[W_{t+T}])^{-\gamma} - \frac{1}{2} (E_t[W_{t+T}])^{-\gamma-1} h^2 \\ + \frac{1}{6} (E_t[W_{t+T}])^{-\gamma-2} h^3 + \cdots - (-1)^M \frac{1}{M!} (E_t[W_{t+T}])^{-\gamma-M+1} h^M, \quad (11.35)$$

to converge. In general, convergence is much slower than in the exponential utility case, and it turns out to depend on  $T$ . For large values of  $T$ ,  $\Pr\{|W_{t+T} - E_t[W_{t+T}]| \leq E_t[W_{t+T}]\} = 1$  is unlikely to hold (depending on the distribution of asset returns), and as such approximations may not be viable. Moreover,

from an asset allocation perspective, Equation 11.34 has the disadvantage of being taken around expected time  $t+T$  wealth, which depends on a portfolio choice that is supposed to be derived endogenously from the maximization of Equation 11.34, which is a circular reasoning (Kane, 1982).

Of course, moment-based expansions developed around points that differ from conditional expected wealth can also be considered. In the case of power utility, suppose  $W_t = 1$  and consider a fourth-order Taylor series expansion, such as a polynomial approximation arrested to the fourth-term of a standard power function  $W_{t+T}^{1-\gamma}/(1-\gamma)$  ( $\gamma > 0$ ) around  $v \equiv \exp(r^f T)$  (i.e., a 100 percent investment in the riskless asset), as shown in Equation 11.36:

$$\begin{aligned} u(W_{t+T}) \cong & \frac{v^{1-\gamma}}{1-\gamma} + v^{-\gamma}(W_{t+T} - v) - \frac{1}{2} \gamma v^{-(\gamma+1)}(W_{t+T} + v)^2 + \\ & + \frac{1}{6} \gamma(\gamma+1)v^{-(\gamma+2)}(W_{t+T} - v)^3 - \frac{1}{24} \gamma(\gamma+1)(\gamma+2)v^{-(\gamma+3)}(W_{t+T} - v)^4, \end{aligned} \quad (11.36)$$

where  $u'(v) = v^{-\gamma}$ ,  $u''(v) = -\gamma v^{-(\gamma+1)}$ ,  $u'''(v) = \gamma(\gamma+1)v^{-(\gamma+2)}$ , and  $u''''(v) = -\gamma(\gamma+1)(\gamma+2)v^{-(\gamma+3)}$ . Expanding the powers of  $(W_{t+T} - v)$  and taking the expectation conditional on information up to time  $t$ , one obtains the expression for a fourth-order approximation in Equation 11.37:

$$\begin{aligned} E_t[u_{app}^4(W_{t+T})] \cong & \kappa_0(\gamma) + \kappa_1(\gamma)E_t[W_{t+T}] + \kappa_2(\gamma)E_t[W_{t+T}^2] \\ & + \kappa_3(\gamma)E_t[W_{t+T}^3] + \kappa_4(\gamma)E_t[W_{t+T}^4]. \end{aligned} \quad (11.37)$$

Here, variable definitions are shown in the following expressions:

$$\begin{aligned} \kappa_0(\gamma) \equiv & v^{1-\gamma} \left[ (1-\gamma)^{-1} - 1 - \frac{1}{2} \gamma - \frac{1}{6} \gamma(\gamma+1) - \frac{1}{24} \gamma(\gamma+1)(\gamma+2) \right] \\ \kappa_1(\gamma) \equiv & \frac{1}{6} v^{-\gamma} [6 + 6\gamma + 3\gamma(\gamma+1) + \gamma(\gamma+1)(\gamma+2)] > 0 \\ \kappa_2(\gamma) \equiv & -\frac{1}{4} \gamma v^{-(1+\gamma)} [2 + 2(\gamma+1) + (\gamma+1)(\gamma+2)] < 0 \\ \kappa_3(\gamma) \equiv & \frac{1}{6} \gamma(\gamma+1)(\gamma+3)v^{-(2+\gamma)} > 0 \\ \kappa_4(\gamma) \equiv & -\frac{1}{24} \gamma(\gamma+1)(\gamma+3)v^{-(3+\gamma)} < 0. \end{aligned} \quad (11.38)$$

Equation 11.37 has highly intuitive implications: the (conditional) expected utility from final wealth increases in  $E_t[W_{t+T}]$  and  $E_t[W_{t+T}^3]$ , i.e., the higher the expected portfolio returns are and the more skewed to the right the induced distribution of final wealth is. These are all signed statistics measuring the location



of the distribution of final wealth. By contrast, expected utility is a decreasing function of even noncentral moments, such as  $E_t[W_{t+T}^2]$  and  $E_t[W_{t+T}^4]$ , which are statistics related to the thickness of the tails of the distribution of time  $t+T$  wealth.

As for the economic interpretation of the coefficients  $\kappa_3(\gamma)$  and  $\kappa_4(\gamma)$  in the expressions in Equation 11.39, in the former case (with reference to a third-order Taylor expansion of power utility around expected future wealth  $W_{t+T}$ ), Kraus and Litzenberger (1976) observe that as long as the bound is imposed, a three-moment Taylor expansion has three desirable properties besides the existence of expected utility: (1) positive marginal utility of wealth, (2) decreasing marginal utility (risk aversion), and (3) nonincreasing absolute risk aversion, which implies  $\kappa_3(\gamma) > 0$ . Scott and Horvath (1980) show that a strictly risk-averse individual who always prefers more to less and who consistently (i.e., for all wealth levels) likes skewness will necessarily dislike kurtosis,  $\kappa_4(\gamma) < 0$ . Since global risk aversion and nonsatiation seem plausible and preference for skewness may be obtained under very weak assumptions, assuming kurtosis aversion may be justified.

### MV PREFERENCES AS A SPECIAL CASE

As a special case of Equation 11.35, one can obtain a MV objective function that can be interpreted as a two-moment approximation to a power utility objective, the argument of which is time  $t+T$  wealth, similarly to that done by Tsiang (1972) and Levy and Markowitz (1979) in Equation 11.39:

$$E_t[u_{app}^2(W_{t+T})] \cong \kappa_0(\gamma) + \kappa_1(\gamma)E_t[W_{t+T}] + \kappa_2(\gamma)E_t[W_{t+T}^2]. \quad (11.39)$$

This result derives from the fact that  $E_t[W_{t+T}^2] = \text{Var}_t[W_{t+T}] + \{E_t[W_{t+T}]\}^2$ , implying the following expression:

$$E_t[u_{app}^2(W_{t+T})] \cong \kappa_0(\gamma) + \kappa'_1(\gamma)E_t[W_{t+T}] + \kappa_2(\gamma)\text{Var}_t[W_{t+T}] \propto E_t[W_{t+T}] - \lambda\text{Var}_t[W_{t+T}]. \quad (11.40)$$

The expressions in the former equation are defined in Equation 11.41,

$$\begin{aligned} \kappa'_1(\gamma) &\equiv \kappa_1(\gamma) + \kappa_2(\gamma)E_t[W_{t+T}] = \frac{1}{6}v^{-\gamma}[6 + 3\gamma + 3\gamma(\gamma+1) + \gamma(\gamma+1)(\gamma+2)] + \\ &\quad - \frac{1}{4}\gamma v^{-(1+\gamma)}[2 + 2(\gamma+1) + (\gamma+1)(\gamma+2)]E_t[W_{t+T}] \\ &\equiv \frac{1}{12}v^{-(1+\gamma)}\{2v[6 + 3\gamma + 3\gamma(\gamma+1) + \gamma(\gamma+1)(\gamma+2)] \\ &\quad - 3\gamma[2 + 2(\gamma+1) + (\gamma+1)(\gamma+2)]E_t[W_{t+T}]\}, \end{aligned} \quad (11.41)$$

while  $\kappa_2(\gamma)$  has a definition identical to Equation 11.38.  $\kappa'_1(\gamma)$  can be shown to be positive provided  $\gamma$  is not too high. However, the sign of  $\kappa'_1(\gamma)$  is hard to assess, as it depends on  $E_t[W_{t+T}]$  and hence on the portfolio strategy implemented by the investor.

## Exotic, Nonstandard Preferences

A new strand of research that straddles the empirical finance, theoretical microeconomics, and portfolio management literatures develops techniques of robust portfolio management. This work contributes to establishing important connections between the role played by preferences in the practice of asset allocation and applications of optimal decisions under ambiguity. Traditional models assume the following: (1) that investors maximize (subjective) expected utility ([S]EU), (2) that agents are perfectly aware of their own preferences, and (3) that investors' expectations are not systematically biased and are made up of rational expectations. However, a growing body of empirical evidence suggests that this traditional paradigm does not well describe investors' behavior in that actual choices are incompatible with (S)EU predictions. As a result, a new line of research entertains agents whose choices are consistent with models that are less restrictive than the standard (S)EU framework in the sense that the underlying axioms are less demanding. In this area, particular attention has recently been dedicated to ambiguity. Under (S)EU, if preferences satisfy certain axioms, numerical utilities and probabilities are used to represent decisions under uncertainty by a standard weighted sum of the utilities, where the weights are (subjective) probabilities for each of the states. As innocuous as this basic principle may seem, a long, rich tradition questions whether it adequately describes behavior. Knight (1921) distinguishes risk, or known probability, from uncertainty. He suggests that economic returns could be earned for bearing uncertainty but not for bearing risk. However, Ellsberg's (1961) paradox most directly provides the modern attack to (S)EU as a descriptive theory. Ellsberg's thought-provoking article led researchers to assemble massive experimental evidence indicating that people generally prefer the least ambiguous acts. Such a pattern is inconsistent with Savage's (1954) *sure-thing principle*, the axiom by which a state with a consequence common to a pair of acts is irrelevant in determining preference between the acts. The implication for portfolio management would be that investors would select optimal portfolios not only by taking the risk of portfolios into account but also by considering their overall uncertainty, which cannot be simply measured as risk.

Although a brief survey of this literature follows, a more complete discussion is available in textbooks devoting chapters to the topic of robust portfolio decisions, such as Fabozzi et al. (2006), or in papers reviewing applications of ambiguity to finance, such as Guidolin and Rinaldi (2010).

### MV ANALYSIS UNDER AMBIGUITY

As previously shown, under the assumption that  $\mathbf{r}_t$  follows a joint multivariate normal distribution with known variance-covariance matrix  $\Sigma \equiv \text{Var}[\mathbf{r}_t]$  and

known mean  $\mu \equiv E[\mathbf{r}_t]$ , a MV investor will invest using the well-known formula in Equation 11.12. Kogan and Wang (2003) extend this result to the case in which  $\mathbf{r}_t$  the investor does not have perfect knowledge of the distribution of the risky returns; specifically, the case in which follows a joint multivariate normal distribution with a known variance-covariance matrix and an unknown vector of mean returns,  $\mu$ . Here, the agent displays a special kind of preferences, which Gilboa and Schmeidler (1989) call the multiple priors type (MPP). In this classification, a rational decision maker evaluates expected utility using a multivalued set of priors to capture the existence of ambiguity on the distribution of the set of random outcomes that may affect either the wealth or consumption of the investor. In this case, Schmeidler (1989) proves that standard SEU optimization may be replaced by max-min problems, in which an investor minimizes her maximum expected utility with reference to a set of candidate probability measures, as defined by the multiple priors. Assuming a unique source of information, so that the agent is able to derive only a reference joint normal distribution of asset returns,  $\hat{\mathbf{f}} \sim N(\mu, \Sigma)$ , the set of effective priors  $\wp(\hat{\mathbf{f}})$  is shown in Equation 11.42:

$$\wp(\hat{\mathbf{f}}) = \left\{ q : E[z \ln z] \leq \eta z \equiv \frac{dq}{df} \right\}, \quad (11.42)$$

where  $\eta$  captures ambiguity aversion (a larger  $\eta$  means higher aversion). The investment problem can then be reformulated as a typical max-min problem illustrated in Equation 11.43:

$$\max_{\omega} V(W, \wp(\hat{\mathbf{f}})) = \max_{\omega} \min_{q \in \wp(p)} \{E_q[u(W)]\} \quad \text{s.t. } W = [\omega'(r - r_N^f) + (1 + r^f)], \quad (11.43)$$

(under the standard constraint that portfolio weights sum to one,  $\omega' \mathbf{1}_N = 1$ ) where  $W$  is final wealth, and the set  $\wp(\hat{\mathbf{f}})$  constrains the statistical models for the vector process  $\mathbf{r}$  to be not too distant from the benchmark  $\hat{\mathbf{f}}$ , with maximum distance given by  $\eta$ . Letting  $\theta \equiv \mu - \hat{\mu}$  be the divergence between one of the possible mean vectors under MPP and the vector of expected returns under the benchmark model, the problem in Equation 11.43 can be rewritten in Equation 11.44 as

$$\max_{\omega} \min_{\theta \in \left\{ \theta : \frac{1}{2} \theta' \Sigma^{-1} \theta \leq \eta \right\}} E(z(r)u(W)) \quad z(r) \equiv \exp \left\{ \frac{1}{2} \theta' \Sigma^{-1} \theta - \theta' \Sigma^{-1} \theta(r - \mu + \theta) \right\}, \quad (11.44)$$

subject to a budget constraint, which is a transformation of the constraint on the set of admissible models under the parameter  $\eta$  into a (multiplicative) factor that appears in the objective function.

Garlappi, Uppal, and Wang (2007) extend these early results and show how to use a confidence interval framework that appears to be natural in the portfolio literature. Their starting point is that the parameters of the joint normal density

characterizing the ambiguous asset returns  $\mathbf{r}$  have to be estimated. Assuming that a time series of length  $T$  of past asset returns  $\mathbf{h}_t$  is available, the conditional density distribution of returns  $g(\mathbf{r}|\mathbf{h}_t)$  must be derived. Assuming that the returns depend on some unknown parameters  $\theta$ , whose prior is  $\pi(\theta)$ , the predictive density  $g(\mathbf{R}|\mathbf{h}_t)$  is  $g(\mathbf{r}|\mathbf{h}_t) = \int g(\mathbf{r}|\theta)p(\theta|\mathbf{h}_t)d\theta$ , where  $p(\theta|\mathbf{h}_t)$  is the data posterior. If an investor solves the classical MV problem using the predictive density of the data, Equation 11.45 emerges:

$$\max_{\omega} \omega' \int \mathbf{r} g(\mathbf{r}|\mathbf{h}_t) d\mathbf{r} - \frac{1}{2} \gamma \omega' \Sigma \omega \quad (11.45)$$

The resulting portfolio is of a Bayesian type, taking into full account the existence of parameter uncertainty (Barberis, 2000). The portfolios in which only parameter uncertainty is considered often perform poorly out of sample, even in comparison to portfolios selected according to some simple ad-hoc rules (Miguel, Garlappi, and Uppal, 2009). One reason for this result is that the vector of expected asset returns  $\mu$  is hard to estimate with any precision. This induces Garlappi et al. (2007) to introduce ambiguity on the appropriate statistical model, as identified here by the vector of expected returns  $\mu$ . When using MPP, the optimization takes the form shown in Equation 11.46:

$$\max_{\omega} \min_{\mu} \omega' \mu - \frac{1}{2} \gamma \omega' \Sigma \omega \quad \text{s.t. } \mathbf{f}(\mu, \hat{\mu}, \Sigma) \leq \eta \quad \omega' \mathbf{1} = 1 \quad (11.46)$$

where  $\mathbf{f}$  is a vector-valued function and  $\hat{\mu}$  is the estimate of  $\mu$  derived from the predictive density  $g(\mathbf{r}|\mathbf{h}_t)$ . One can prove that the max-min problem in Equation 11.46 is equivalent to the simpler maximization problem in Equation 11.47:

$$\max_{\omega} \omega' (\hat{\mu} - \mu^{adj}) - \frac{1}{2} \gamma \omega' \Sigma \omega, \quad (11.47)$$

where  $\hat{\mu} - \mu^{adj}$  is the adjusted estimated expected return (the adjustment has the role of incorporating ambiguity), and define a vector  $\mu^{adj}$  to satisfy Equation 11.48,

$$\mu^{adj} = \left[ \text{sgn}(\omega_1) \frac{\sigma_1}{\sqrt{T}} \sqrt{\eta_1} \text{sgn}(\omega_2) \frac{\sigma_2}{\sqrt{T}} \sqrt{\eta_2} \cdots \text{sgn}(\omega_N) \frac{\sigma_N}{\sqrt{T}} \sqrt{\eta_N} \right]. \quad (11.48)$$

The adjustment depends on the precision with which parameters are estimated, the length of the data series, and the investor's aversion to ambiguity ( $\eta$ ).

Additional papers have recently shown that using ambiguity aversion to solve realistic, large-scale portfolio problems is possible. Boyle et al. (2009), who study the role of ambiguity in determining portfolio underdiversification and the flight to familiarity episodes, offer the simplest of such papers. Consider a MV portfolio

problem in which the asset menu is composed of  $N$  identical risky assets and one riskless asset. Each asset has (unknown) expected excess return  $\mu_i$  and common volatility  $\sigma$ . Using the framework developed by Boyle et al., the authors write the optimization problem as shown in Equation 11.49:

$$\max_{\omega} \min_{\mu} \omega' \mu - \frac{\gamma}{2} \omega' \Sigma \omega \quad \text{s.t.:} \quad \frac{(\mu_i - \hat{\mu}_i)}{\sigma_{\hat{\mu}_i}^2} \leq \eta_i \quad \omega' \mathbf{1}_N = 1. \quad (11.49)$$

Under this specification, the ambiguity problem can be interpreted in terms of classical statistical analysis, because by letting  $\hat{\mu}_i$  represent the estimated value of the mean return of asset  $i = 1, \dots, N$  and  $\sigma_{\hat{\mu}_i}^2$  represent the variance of  $\hat{\mu}_i$ , defining the confidence interval  $\{T(\mu_i - \hat{\mu}_i) / \sigma_{\hat{\mu}_i}^2 \leq \eta_i \text{ for expected returns is possible. Hence } \sqrt{\eta_i}, \text{ is the critical value determining the size of the confidence interval, which can be interpreted as a measure of the amount of ambiguity of the estimate of expected returns. The authors find that an investor holds familiar assets but balances this investment by also holding a portfolio of all the other assets (as advocated by Markowitz 1952), which remains biased toward more familiar assets.}$

## SMOOTH RECURSIVE PREFERENCES

One novel approach to modeling ambiguity in asset allocation decisions exploits a generalization of Klibanoff, Marinacci, and Mukerji's (2005; hereafter, KMM) class of smooth preferences. These authors propose that the ambiguity of a risky act or decision can be characterized by a set  $\wp = \{P_1, \dots, P_n\}$  of subjectively plausible cumulative probability distributions. They give the hypothetical example that letting  $W_j$  denote the random variable distributed as  $P_j$ ,  $j = 1, \dots, n$  the decision maker, based on her subjective information, associates a distribution  $q_1, \dots, q_n$  over  $\wp$ , where  $q_j$  is the subjective probability of  $P_j$  being the true distribution of  $W$ . They that show that resulting preferences have the following representation:

$$\sum_{j=1}^n q_j \zeta \left( \int u(W) dP_j \right), \quad (11.50)$$

where  $\zeta(\cdot)$  is an increasing real-valued function whose shape describes the investor's attitude toward ambiguity. Using Equation 11.50, the decision maker first evaluates the expected utility of  $W$  with respect to all the priors in  $\wp$ : each prior  $P_j$  is indexed by  $j$ , so in the end, a set of expected utilities results, each being indexed by  $j$ . Then, instead of taking the minimum of these expected utilities, as MPP would, the investor takes an expectation of distorted expected utilities. The role  $\zeta$  of is crucial here: If  $\zeta$  were linear, the criterion would simply reduce to (S) EU maximization with respect to the combination of the  $q$ s representing the probabilities, and the  $P$ s, representing the possible distributions. When  $\zeta$  is not linear, one cannot combine  $q$ s and  $P$ s to construct a reduced probability distribution. In

this event, the decision maker takes the expected  $\zeta$ -utility (with respect to  $q$ ) of the expected  $u$ -utility (with respect to the  $P_s$ ). A concave  $\zeta$  will reflect ambiguity aversion, in the sense that it places a larger weight on poor expected  $u$ -utility realizations. One important implication of the two-stage approach is that the decision maker is not forced to be so pessimistic as to select the act that maximizes the minimum expected utility as a consequence of the separation between ambiguity and her attitude toward ambiguity. In this sense, KMM preferences may be interpreted as a smooth extension of Gilboa and Schmeidler's (1989) classical MPP. MPP is a limiting case of Equation 11.50. Up to ordinal equivalence, MPP is obtained in the limit as the degree of concavity of  $\zeta$  increases without bound. Although the type of portfolio problems that have been analyzed so far, assuming simple KMM's preferences, remain limited, this smooth class will acquire increasing weight in the asset allocation literature.

## An Illustrative Application

To illustrate the effects of preferences on optimal portfolio decisions, providing an empirical example is useful. Assume the following two experiments: The first experiment involves calculating optimal weights under a range of alternative utility functions (preferences). Such optimal weights are computed over a range of alternative scenarios describing the state of the economy, where the state is defined through the lenses of a simple but powerful two-state Markov switching model (henceforth, MSM). The second experiment illustrates the power of alternative preference frameworks as tools to evaluate performance. To keep things simple, a recursive evaluation of three portfolio strategies is undertaken. The first strategy is the optimal recursive strategy computed under a given preference framework. The second strategy is an equally weighted strategy (also called  $1/N$  after Miguel et al., 2009) that has been shown to be highly performing in spite of its complete disregard for any kind of utility optimization. The third strategy involves deriving the value-weighted market portfolio implied by the CAPM. Notice that  $1/N$  basically results in a simple 50–50 allocation between stocks and cash. This result avoids both estimation and model specification errors that are implicit when relying on the CAPM. The following are the preference frameworks employed in the examples:

- The linear utility framework characterizes the behavior of a risk-neutral investor.
- Ad-hoc MV preferences are defined over portfolio returns.
- Ad-hoc MV preferences are defined over terminal wealth.
- Three- and four-moment ( $M = 3$  and  $M = 4$ ) Taylor expansions approximate the expected power utility of terminal wealth as a function of the first three and four moments of terminal wealth. In this case, calculations are performed around two approximation points,  $v = r^f T$  and  $v = E_{t-1}[W_t]$ .
- Additional frameworks include:

- The power utility function characterizes the behavior of a risk-averse investor with constant relative risk-aversion.
- The negative exponential utility function characterizes the behavior of a risk-averse investor with constant absolute risk-aversion; entertaining this utility function as a separate framework from the standard MV one is advisable when stock returns do not follow an IID Gaussian distribution.
- A KMM smooth ambiguity functional characterized by  $u(W_{t+T})$  set to be a power utility function and  $\zeta(E_t[u(W_{t+T})]) = -\exp(-E_t[u(W_{t+T})])$  captures aversion to ambiguity; the set  $\mathcal{P} = \{P_1, \dots, P_n\}$  of plausible cumulative probability distributions is specified to correspond to the set of possible states/scenarios according to the estimated MSM along a grid  $\{0, 0.1, 0.2, \dots, 1\}$  with each of the regimes being equally weighted.

This list delivers a total of 10 alternative preference frameworks, in which alternative choices of the approximation points  $\nu$  are taken into account. In fact, each such framework is implemented with three alternative values for the unique parameter characterizing the degree of risk aversion:  $\zeta = 0.5, 2$ , and  $4$  in the case of MV and negative exponential utility and  $\gamma = 2, 5$ , and  $10$  in the case of power utility and KMM. Finally, calculations are performed for four horizons of 1, 3, 6, and 12 months. Short sales are ruled out throughout. However, calculations that are done without imposing the short sales restriction have been performed, obtaining results that are qualitatively similar.

## ECONOMETRIC ESTIMATES

Monthly data from the Center for Research in Security Prices (CRSP) at the University of Chicago from July 1926 through December 2010 on value-weighted stock returns on the New York Stock Exchange (NYSE), American Stock Exchange (AMEX), and NASDAQ markets are employed to estimate two alternative statistical models that describe the dynamics of stock returns. The first model is a simple Gaussian IID model that is consistent with the hypothesis of geometric random walk in (cum dividend) stock index prices and with the absence of predictability (robust standard errors are in parenthesis) shown in the estimated model 11.51:

$$r_t = 0.920 + 5.465\varepsilon_t \quad \varepsilon_t \sim \text{IID } N(0,1). \quad (11.51)$$

[0.172]

Moreover, a simple, two-state MSM is estimated in model 11.52:

$$r_t = -1.223S_t + 1.293(1-S_t) + [10.607S_t + 3.801(1-S_t)]\varepsilon_t \quad \varepsilon_t \sim \text{IID } N(0,1). \quad (11.52)$$

[0.920]      [0.152]

Regime 1 ( $S_t = 1$ ) is a bear state with negative expected returns and high volatility, while regime 0 ( $S_t = 0$ ) is a bull state with positive expected return and moderate



equity volatility. The estimated persistence on the main diagonal of the transition probability matrix is  $p_{11} = 0.984$  and  $p_{22} = 0.907$ . These imply an average duration of eleven months for the bear regime and sixty-three months for the bull regime. As a result, the ergodic, long-run probabilities (that one would obtain from an infinite sample from the process) of the two regimes are 0.147 and 0.853, respectively. Figure 11.1 plots the full-sample, smoothed-state probabilities of the two regimes. In the bear state plot, various episodes of declining and turbulent aggregate stock prices can be singled out: two spikes corresponding to the Great Depression in the 1930s, two spikes for the oil shocks of 1974–1975 and 1980, and one spike each for the crash of late 1987, the Asian crisis of the summer 1998, the dot-com bubble crash of 2000–2001, and more recently the great financial crisis of 2008–2009.

The reason the MSM is entertained as an alternative statistical framework is that dynamic econometric frameworks from this family are well known to capture,

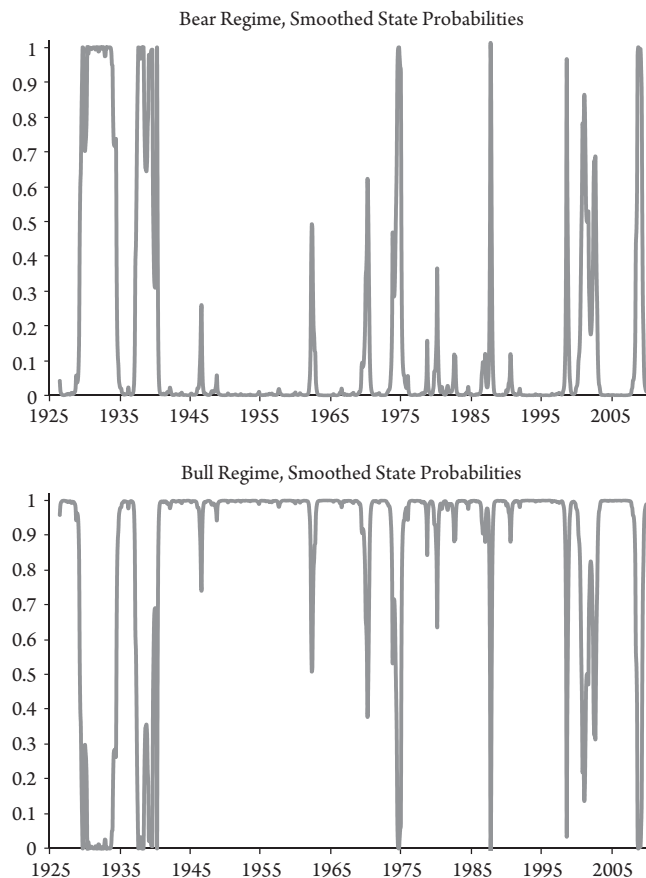


Figure 11.1 Smoothed (full-sample) probabilities from two-state Markov switching model. These two plots show the full-sample smooth probabilities derived from a two-state Markov switching model in which means, variances, and covariances are assumed to be a function of the Markov state.

in an intuitive way, the most salient features of the dynamics of investment opportunities. MSM also has rich implications for the time variation of means, variances, and especially skewness and kurtosis (Timmermann, 2000). Guidolin and Timmermann (2008) and Guidolin (2011) provide additional details on the role of MSMs in dynamic portfolio selection and estimation. These authors also show how to derive closed-form expressions for the first four noncentral moments of the time  $t+T$  wealth. Guidolin and Timmermann prove that when the risky asset return follows a Markov switching process, conditional expectations of the type  $E_t[\exp(m \sum_{i=1}^T r_{t+i})]$  can be calculated in a recursive fashion.

### OPTIMAL PORTFOLIO WEIGHTS UNDER ALTERNATIVE SCENARIOS

Table 11.1 reports optimal weights computed under different preferences, risk aversion parameters, and investment horizons. The optimal allocation to US stocks in five representative scenarios has been computed: (1) when at time  $t$  the regime is bull, (2) when it is bear, (3) when the investor is in a state of ignorance on the nature of the regime and guesses that each state carries a current probability equal to its ergodic frequency, (4) when the investor is in such a state of ignorance to understand only the presence of two regimes but is unable to compute the ergodic frequencies and she attributes equal probabilities to both, and (5) when the investor ignores regimes altogether. Notice that in scenarios (1)-(4), even when knowledge of the starting regime is assumed in the scenario simulation, this never implies that the regime is known in advance or observable at times  $t+1$ ,  $t+2$ , ...,  $t+T$ . Under scenario (5), optimal weights are computed assuming (counterfactually) that stock returns are generated by a simpler single state, Gaussian IID model, when only one regime is possible at all times.

Independent of the preference framework for low risk aversion (i.e.,  $\zeta = 0, 0.5$ , and  $2$ ; moreover  $\gamma = 2$ ), in the IID case, a common finding is that an investor ought to invest 100 percent of her wealth in stocks. Equivalently, the market portfolio advocated by the CAPM is ex-ante optimal. This result is easy to understand because over the sample period, the US market portfolio yielded a handsome average monthly return of 0.9 percent, which exceeds the 0.3 percent average monthly return on one-month Treasury bills. The difference of about 7 percent per year is called the *equity premium*, which appears to be sufficiently high to lead any moderately risk-averse investor to bet all of her wealth on stocks. Under some preference assumptions such as power utility and KMM, higher risk aversion levels would induce an investor to a much more balanced approach, in which the optimal share invested in US stocks might be as low as 30 percent, independent of the horizon.

For zero- or low-risk-aversion coefficients and especially for short investment horizons of one and three months, all asset allocations simulated under alternative MSM scenarios reveal the following: Investors ought to aggressively time the market in a simple way. They should invest 100 percent in stocks in bull markets when limited uncertainty leads them to think that the current state does not depart much from ergodic probabilities. During bear markets, they should invest 0 percent in stocks (100 percent in cash). The case of 50–50 uncertainty is of some interest because this

Table 11.1 Optimal portfolio weights under alternative preferences, risk aversion coefficients, and investment horizons

Preferences	State/model	Risk aversion coefficient/investment horizon											
		T = 1			T = 3			T = 6			T = 12		
Linear (risk neutral)	Bull regime	1.000			1.000			1.000			1.000		
	Ergodic probabilities	1.000			1.000			1.000			1.000		
	Equal probabilities												
	Bear regime	0.000			0.000			0.000			1.000		
	Gaussian IID	1.000			0.000			0.000			0.000		
(Ad-hoc) mean-		$\lambda = 0.5$			$\lambda = 2$			$\lambda = 4$					
		T = 1	T = 3	T = 6	T = 1	T = 3	T = 6	T = 1	T = 3	T = 6	T = 1	T = 3	T = 6
	Bull regime	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.695
	Ergodic probabilities	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.340
	Equal probabilities	0.000	0.000	0.000	0.107	0.000	0.000	0.000	0.000	0.027	0.000	0.000	0.013
variance in portfolio returns	Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

(Continued)

Table 11.1 (Continued)

Preferences	State/model	Risk aversion coefficient/investment horizon											
		$\gamma = 2$				$\gamma = 4$				$\gamma = 10$			
		T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12
(Ad-hoc mean- variance in ter- minal wealth)	Bull regime	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.955	1.000	0.895	0.760	0.590
	Ergodic probabilities	1.000	1.000	1.000	1.000	0.700	0.680	0.660	0.610	0.420	0.410	0.395	0.365
Three-moment preferences (approximate past exponential wealth)	Equal probabilities	0.000	0.085	0.200	0.380	0.000	0.040	0.100	0.190	0.000	0.025	0.060	0.115
	Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.985
		$\gamma = 2$				$\gamma = 4$				$\gamma = 10$			
		T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12
	Bull regime	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.575	0.545	0.520	1.000
	Ergodic probabilities	1.000	1.000	1.000	1.000	0.535	0.535	0.540	0.595	0.215	0.210	0.215	1.000
	Equal probabilities	0.000	0.065	0.155	0.305	0.000	0.030	0.075	0.150	0.000	0.015	0.030	0.060
	Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

		$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
		T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12		
Three-moment preferences (approximately $r^2T$ )	Bull regime	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.575	0.560	1.000	1.000	0.560	1.000	1.000	1.000		
	Ergodic probabilities	1.000	1.000	1.000	1.000	0.535	0.535	0.545	0.670	0.215	0.210	0.215	1.000	0.215	0.210	0.215	1.000		
	Equal probabilities	0.000	0.065	0.155	0.305	0.000	0.030	0.075	0.150	0.000	0.015	0.030	0.060	0.000	0.015	0.030	0.060		
	Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000		
	Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
Four-moment		$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
		T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12		
preferences (approximately bilities)	Bull regime	1.000	1.000	1.000	1.000	1.000	1.000	0.910	0.695	0.540	0.460	0.375	0.285	0.540	0.460	0.375	0.285		
	Ergodic probabilities	1.000	0.965	0.910	0.840	0.520	0.495	0.470	0.440	0.210	0.200	0.190	0.180	0.210	0.200	0.190	0.180		
	Equal probabilities	0.000	0.065	0.150	0.290	0.000	0.030	0.075	0.145	0.000	0.015	0.030	0.060	0.000	0.015	0.030	0.060		
past exponential wealth)	Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000		
	Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
		(Continued)																	

(Continued)

Table 11.1 (Continued)

	$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
	T = 1		T = 3		T = 6		T = 1		T = 3		T = 6		T = 1		T = 3		T = 6	
	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3
Four-moment	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.880	0.640	0.540	0.455	0.365	0.270		
Bull regime	1.000	0.960	0.900	0.810	0.520	0.495	0.470	0.430	0.210	0.200	0.190	0.175						
Ergodic proba- bilities																		
Equal proba- bilities	0.000	0.065	0.150	0.290	0.000	0.030	0.075	0.145	0.000	0.015	0.030	0.060						
Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000						
Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000						
Power utility (CRRA)	$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
	T = 1		T = 3		T = 6		T = 1		T = 3		T = 6		T = 1		T = 3		T = 6	
	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3	T = 1	T = 3
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990	0.930	0.710	0.550	0.430			
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.940	0.870	0.420	0.390	0.360			
	Ergodic proba- bilities																	
	Equal proba- bilities	0.250	0.420	0.520	0.660	0.130	0.210	0.260	0.330	0.050	0.080	0.100	0.130					
Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	0.890	0.890	0.890	0.890	0.890	0.890	0.350	0.350	0.350	0.350	0.350	0.350

		$\lambda = 0.5$						$\lambda = 2$						$\lambda = 4$					
		T = 1	T = 3	T = 6	T = 12	T = 1	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 12
Negative exponential utility	Bull regime	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990
	Ergodic probabilities	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.940	0.870	0.870
	Equal probabilities	1.000	1.000	1.000	1.000	1.000	1.000	0.250	0.420	0.520	0.660	0.130	0.660	0.130	0.210	0.260	0.330	0.330	0.330
	Bear regime	0.000	0.000	0.000	0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(CARA)	Gaussian IID	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.890	0.890	0.890	0.890	0.890
		$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
Klibanoff, Marinacci, and Mukerji's smooth ambiguity preferences (power/negative exponential utility)	Bull regime	1.000	1.000	0.980	0.910	0.990	0.990	0.990	0.920	0.860	0.780	0.930	0.780	0.930	0.710	0.550	0.430	0.430	0.430
	Ergodic probabilities	1.000	0.970	0.860	0.780	0.960	0.960	0.960	0.900	0.830	0.750	0.420	0.750	0.420	0.400	0.390	0.360	0.360	0.360
	Equal probabilities	0.250	0.420	0.520	0.660	0.110	0.660	0.110	0.195	0.245	0.305	0.050	0.305	0.050	0.080	0.100	0.130	0.130	0.130
	Bear regime	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	Gaussian IID	1.000	1.000	1.000	1.000	0.810	1.000	0.810	0.810	0.810	0.810	0.295	0.810	0.295	0.295	0.295	0.295	0.295	0.295

Note: The table shows optimal portfolio weights under alternative assumptions concerning the investment horizon, coefficient that captures risk-aversion, regime scenario under which the calculation has been performed, and preference framework.



is when different preferences lead to heterogeneous indications. For example, linear utility, MV, and all moment-based preference models suggest exiting the stock market. Power utility and KMM favor mixed portfolios, and negative exponential utility oddly indicates a 100 percent optimal investment in stocks. Instead, for higher risk aversion and especially longer investment horizons, most preference frameworks give balanced indications in which the optimal share of stocks is not zero or one, although it remains the case that starting from a bear regime, all preferences and horizons indicate the optimality of exiting the stock market. This strategy is not only sensible, because in a bear regime, US stocks yield negative expected returns, but also plausible, in the light of the finding that the average duration of a bear regime is 11 months. Thus, even an investor with  $T=12$  may expect to remain, on average, in the bear regime over her entire horizon. However, in the remaining three switching scenarios, the heterogeneity in optimal portfolio weights across preferences is remarkable. Unlike the case of constant investment opportunities, an investor should pin down a preference framework that accurately describes her risk attitudes in order to be able to employ quantitative frameworks of portfolio optimization.

### BACK-TESTING THE REALIZED PERFORMANCE OF ALTERNATIVE PREFERENCES

Table 11.1 cannot be used to show that modeling portfolio decisions in a utility-maximizing setup give investors any advantage. A simple recursive back-testing exercise was implemented for each month between January 1980 and December 2010, both econometric models (IID and MSM) were recursively reestimated, and the resulting parameter estimates were used to compute optimal portfolio weights for the same range of preferences, the same risk aversion coefficients, and the same horizons as those in Table 11.1. After computing the weights, realized wealth (or portfolio returns) over the assumed horizons were computed from the data, such as the actual value-weighted CRSP stock returns and one-month Treasury bill yields. This yields a total of  $372 - T$  measures of realized wealth for each combination of preferences, risk aversion parameter, and horizon. Such time series of realized wealth were converted in time series of realized utility using the assumed structure for preferences and for instances in the case of power utility, so that Equation 11.53 emerges:

$$u_{power}(\hat{w}_{t,T}) = \frac{[W_{t+T}(\hat{w}_{t,T})]^{1-\gamma}}{1-\gamma} \quad (11.53)$$

For the time index  $t$  that ranges between January 1980 and December 2010. Finally, such time series of realized utility were averaged and converted into certainty equivalent returns (CERs), as illustrated in Equation 11.54:

$$\begin{aligned} \frac{[(1 + CER_{power}^{\gamma,T})W_t]^{1-\gamma}}{1-\gamma} &= \frac{1}{372-T} \sum_{t=1980:01}^{2010:12-T} u_{power}(\hat{w}_{t,T}) \Rightarrow \\ CER_{power}^{\gamma,T} &= [(1-\gamma)\bar{u}_{power}(\hat{w}_{t,T})]^{\frac{1}{1-\gamma}} - 1, \end{aligned} \quad (11.54)$$

where  $\bar{u}_{power}(\hat{w}_{t,T}) \equiv (372 - T)^{-1} \sum_{t=1980:01}^{2010:12-T} u_{power}(\hat{w}_{t,T})$ . The CER represents the certain return that one investor would be ready to accept in replacement of the risky portfolio strategy defined by  $\{\hat{w}_{t,T}\}_{t=1980:01}^{2010:12-T}$  because it gives her the same average realized utility. Obviously, the better a strategy is, the higher its CER. Moreover, the CER has the additional advantage of converting the realized performance of potentially heterogeneous strategies into a comparable measure. Obviously, in the case of simple MV frameworks, the ranking provided by the CER is identical to rankings based on Sharpe ratios. In the case of linear utility, the CER is identical to average realized portfolio returns. However, in other cases, including moment-based preferences, power utility, and KMM, the CER is likely to give indications that significantly differ from simple Sharpe ratios (Guidolin and Ria, 2011).

Table 11.2 presents the realized certainty equivalent returns (CERs) for the three alternative strategies. Corresponding to each combination of preferences and risk aversion coefficients/horizon, the best performing CER is shown in boldface. The table shows that preference-based, optimizing asset allocation models may help investors to maximize their realized performance but may also occasionally provide disappointing results. Although a detailed analysis of the point estimates of the CERs in Table 11.2 is beyond the scope of this illustration, the usefulness of preference-based asset allocation seems to be “U-shaped” with respect to both risk aversion and horizon. The utility-based CER outperforms the benchmarks for risk-neutral investors and for highly risk-averse investors, while it underperforms the benchmarks for investors who display intermediate levels of risk aversion. The utility-based CER outperforms the benchmarks for shorter (one-month) and especially longer (6- and 12-month) horizons. However, when preferences are MV or MV skewness, utility-based CERs are never superior to those of the benchmarks. The equally weighted portfolio does not perform as well as expected in light of the recent academic literature. However, its realized performance tends to be strong for long-horizon investors whose risk aversion is relatively high. Yet, even when the utility-optimizing framework fails to turn out the best realized performance, the distance to the benchmarks remains modest. For instance, consider the case of a three-month investor favoring MV skewness whose approximation is taken around a power utility function with  $\gamma = 4$ . Independent of the details of the Taylor approximation, her CER is 6.3 percent per year from the utility-maximizing framework, 7.1 percent from 1/N, and 7.2 percent from the market portfolio that always invests 100 percent in stocks. One final finding is intriguing. The ex-ante preference-optimizing strategy is ex-post the most successful strategy under KMM’s ambiguity-averse preferences. This finding confirms the importance of performing quantitative portfolio optimization in the case where investors are sensitive to parameter and model uncertainty.

## Summary and Conclusions

Surveying the portfolio theory literature that uses preference-based frameworks to compute optimal portfolios, this chapter’s key empirical result is that basing

Table 11.2 Realized, recursive back-tested certainty equivalent return performance of alternative portfolio strategies

Preferences	State/model	Risk aversion coefficient/investment horizon																																																																																																																															
		T = 1				T = 3				T = 6				T = 12																																																																																																																			
Linear risk neutral	Optimal strategy	10.903				11.682				12.096				12.760																																																																																																																			
	Market portfolio	-0.359				7.813				9.718				10.468																																																																																																																			
	1/N	8.273				8.504				8.589				8.703																																																																																																																			
		$\lambda = 0.5$																																																																																																																															
(Ad-hoc) mean-variance in portfolio returns	Optimal strategy	T = 1				T = 3				T = 6				T = 12				T = 1				T = 3				T = 6				T = 12																																																																																																			
	Market portfolio	10.323				10.819				11.174				11.736				8.644				8.989				8.912				9.011				6.708				6.270				5.864				5.741																																																																																			
	1/N	12.338				12.705				12.808				12.907				10.404				10.489				10.420				10.216				7.732				7.353				6.975				6.399																																																																																			
		8.115				8.325				8.397				8.487				7.637				7.781				7.814				7.830				6.989				7.037				7.010				6.925																																																																																			
		$\lambda = 0.5$																$\lambda = 2$																$\lambda = 4$																																																																																															
			T = 1																T = 3																T = 6																T = 12																T = 1																T = 3																T = 6																T = 12														
(Ad-hoc) mean-variance in terminal wealth	Optimal strategy	9.311				9.719				9.876				10.339				7.657				7.702				7.422				7.506				4.630				3.359				3.158				5.125																																																																																			
	Market portfolio	11.049				11.248				11.274				11.288				9.055				8.908				8.688				8.449				4.796				3.547				2.133				1.565																																																																																			
	1/N	7.797				7.969				8.025				8.091				7.315				7.421				7.441				7.451				6.326				6.272				6.193				6.080																																																																																			
		$\gamma = 2$																$\gamma = 4$																$\gamma = 10$																																																																																															
Three-moment preferences (ap-proximate past exponential wealth)	Optimal strategy	T = 1				T = 3				T = 6				T = 12				T = 1				T = 3				T = 6				T = 12				T = 1				T = 3				T = 6				T = 12																																																																																			
	Market portfolio	8.783				9.085				9.110				9.529				6.611				6.293				5.823				5.647				2.493				-3.445				-10.220				-8.635																																																																																			
	past exponential	10.393				10.488				10.449				10.385				7.678				7.235				6.744				6.328				-1.644				-5.667				-10.715				-8.595																																																																																			
	1/N	7.637				7.788				7.833				7.881				6.988				7.045				7.037				7.007				4.947				4.609				4.303				4.063																																																																																			
		$\gamma = 2$																$\gamma = 4$																$\gamma = 10$																																																																																															
			T = 1																T = 3																T = 6																T = 12																T = 1																T = 3																T = 6																T = 12														

	$\gamma = 2$			$\gamma = 4$			$\gamma = 10$					
	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12
Three-moment preferences (ap-proximate 1/N i around $r^2_t$ )	8.783	9.086	9.105	9.534	6.611	6.296	5.839	5.647	2.493	-3.746	-10.201	-8.635
	10.393	10.488	10.449	10.385	7.678	7.235	6.744	6.328	-1.644	-5.667	-10.715	-8.595
	7.637	7.788	7.833	7.881	6.988	7.045	7.037	7.007	4.947	4.609	4.303	4.063
Four-moment preferences (ap-proximate 1/N past exponential wealth)	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12
	8.770	9.076	9.088	9.373	<b>6.681</b>	6.240	5.778	<b>6.568</b>	3.562	3.793	<b>4.329</b>	<b>4.814</b>
	<b>9.728</b>	<b>9.709</b>	<b>9.588</b>	<b>9.439</b>	6.261	5.454	4.575	4.025	-2.527	-7.024	-12.595	-9.757
	7.477	7.605	7.639	7.668	6.659	<b>6.663</b>	<b>6.622</b>	6.551	<b>4.770</b>	<b>4.389</b>	4.045	3.798
Four-moment preferences (ap-proximate 1/N past exponential wealth)	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12
	8.770	9.075	9.085	9.369	<b>6.681</b>	6.245	5.915	<b>6.717</b>	3.562	3.884	<b>4.492</b>	<b>4.873</b>
	<b>9.728</b>	<b>9.709</b>	<b>9.588</b>	<b>9.439</b>	6.261	5.454	4.575	4.025	-2.527	-7.024	-12.595	-9.757
	7.477	7.605	7.639	7.668	6.659	<b>6.663</b>	<b>6.622</b>	6.551	<b>4.770</b>	<b>4.389</b>	4.045	3.798
Four-moment preferences (ap-proximate 1/N (i around $r^2_t$ ))	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12
	8.742	9.030	9.059	9.324	6.554	6.305	5.731	5.654	2.311	2.857	3.549	4.307
	<b>9.924</b>	<b>10.160</b>	<b>10.410</b>	<b>10.001</b>	<b>7.293</b>	<b>6.993</b>	6.281	6.008	-1.631	-5.286	-10.325	-7.846
	7.365	7.524	7.831	7.710	6.774	6.352	<b>6.730</b>	<b>6.615</b>	<b>4.491</b>	<b>4.185</b>	<b>4.115</b>	3.885
(Continued)												

(Continued)

Table 11.2 (Continued)

	$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3
	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy
Power utility (CRRA)	10.326	10.939	11.228	11.692	8.704	9.046	9.002	9.100	6.738	6.346	6.382	6.593	6.346	6.382	6.593	6.346	6.382	6.593
Market portfolio 1/N	<b>12.338</b>	<b>12.705</b>	<b>12.808</b>	<b>12.907</b>	<b>10.404</b>	<b>10.489</b>	<b>10.420</b>	<b>10.216</b>	<b>7.732</b>	<b>7.353</b>	<b>6.975</b>	<b>6.399</b>	<b>7.010</b>	<b>6.925</b>	<b>6.399</b>	<b>7.010</b>	<b>6.925</b>	<b>6.399</b>
	8.115	8.325	8.397	8.487	7.637	7.781	7.814	7.830	6.989	7.037	7.010	6.925	7.037	7.010	6.925	7.037	7.010	6.925
Negative exponential utility (CARA)	$\gamma = 2$						$\gamma = 4$						$\gamma = 10$					
	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3	T = 6	T = 12	T = 1	T = 3
	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy	Optimal strategy
	8.437	8.274	<b>7.662</b>	<b>8.690</b>	<b>6.046</b>	5.172	4.744	4.139	2.523	2.980	<b>4.064</b>	<b>4.450</b>	2.980	2.980	<b>4.064</b>	<b>4.450</b>	2.980	2.980
Market portfolio 1/N	<b>8.575</b>	<b>8.793</b>	7.587	6.931	5.814	4.973	4.561	3.980	-3.416	-8.051	-13.901	-10.271	-3.416	-8.051	-13.901	-10.271	-3.416	-8.051
	6.207	7.420	6.561	7.183	5.843	<b>5.726</b>	<b>5.869</b>	<b>5.930</b>	<b>4.338</b>	<b>3.934</b>	3.574	3.209	<b>4.338</b>	<b>3.934</b>	3.574	3.209	<b>4.338</b>	<b>3.934</b>

**Note:** The table reports the annualized certainty equivalent return (CER) corresponding to average realized utility under alternative assumptions concerning the investment horizon, the coefficient that captures risk-aversion, the regime scenario under which the calculation has been performed, and the preference framework. Average realized utilities are computed with reference to the back-testing period 1980:01–2010:12. The best-performing CERs are shown in bold-face.

asset allocation decisions on preferences may pay off not only in ex-ante but also ex-post terms under two important conditions: First, asset returns must be generated from non-Gaussian frameworks characterized by nonlinear predictability dynamics, which translates into rich time variation in skewness and kurtosis. Second, the preferences must overweight the importance of higher-order moments and of the (conditional) tails of the distribution of portfolio returns, such as power utility or ambiguity-averse preferences.

Although this finding fits the key results in the literature, its characterization under Markov switching dynamics when ambiguity aversion is called into play appears novel. For instance, Kallberg and Ziemba (1983) already report that in many practical applications, one can choose the utility function that allows for the most efficient numerical solution. As the utility function that is most easily tractable in terms of computation, finding that quadratic utility is by far the most commonly used in practice is not surprising. These authors note, however, that they performed most of their calculations using assets exhibiting return distributions not too far away from normality; for instance, in the case of the so-called elliptical distributions (such as the normal, Student  $t$ , and Levy distributions). This chapter further emphasizes that when such elliptical properties are absent, results for MV preferences may differ from those derived under more complex and arguably realistic preferences.

## Discussion Questions

1. What are the possible combinations of assumptions about individual's preferences and about the statistical distribution of asset (portfolio) returns that may justify a simple MV approach to portfolio optimization, such as that in Equation 11.5?
2. Why is computing the standard (small) risk measures  $CARA(W)$  and  $CRRA(W)$  impossible in the case of MV preferences? Explain the source from which deficiencies stem.
3. Describe the intuition underlying Klibanoff, Marinacci, and Mukerji's (2005) smooth ambiguity-averse preferences. Explain how these smooth preferences can nest both Gilboa and Schmeidler's (1989) max-min type, multiple priors preferences, and the standard subjective expected utility case.
4. Why is a dynamic model of risky asset returns such as a Markov switching model likely to bring out the power of smooth ambiguity preferences to improve realized performance?

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