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DISTRIBUTIONAL RESULTS FOR MEANS OF NORMALIZED RANDOM MEASURES WITH INDEPENDENT INCREMENTS[1]

BY EUGENIO REGAZZINI,[2] ANTONIO LIJOI AND IGOR PRÜNSTER

Università degli Studi di Pavia

Dedicated to the memory of Viatcheslav V. Sazonov

We consider the problem of determining the distribution of means of random probability measures which are obtained by normalizing increasing additive processes. A solution is found by resorting to a well-known inversion formula for characteristic functions due to Gurland. Moreover, expressions of the posterior distributions of those means, in the presence of exchangeable observations, are given. Finally, a section is devoted to the illustration of two examples of statistical relevance.

1. Introduction. This paper concerns a class of random probability measures that includes the Dirichlet process. The material that is presented here can be considered of some interest for two main reasons:

1. It represents a natural approach to the problem of defining a random probability measure.
2. It is possible to determine explicit forms for the distribution of a mean of any random probability measure yielded by this approach.

With reference to the former, one reasonable way to pass from deterministic to random real-valued functions defined on $\mathbb{R}$ is to consider their increments on disjoint intervals as independent random variables. Starting from these considerations, we have defined a random distribution function by resorting to what is called an increasing additive process. A suitable modification of the underlying Lévy measure yields an almost surely finite random measure with independent increments. A systematic account of these random measures is given, for example, in Kingman (1967, 1993), Morando (1969) and Skorohod (1991). Kingman designated this random measure with the term *completely random measure*. For our purposes we consider the normalized increments of an increasing

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[2]Also affiliated with CNR-IAMI, Milano, Italy.

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additive process somewhat in the spirit of the work of Kingman (1975). For an allied contribution, see, for example, Hjort (1990).

An analogous approach, which exploits a different transformation of an increasing additive process, is due to Doksum (1974). The resulting random probability measures are very popular in Bayesian literature, and their posterior distributions admit explicit forms. Among various contributions to this topic it is worth mentioning, for example, Ferguson (1974), Ferguson and Phadia (1979), Walker and Muliere (1997) and Walker and Damien (1998). Nonetheless, to our knowledge no exact distribution for their means, apart from the Dirichlet process, has been determined.

As for the operational aspect of our proposal, we would like to stress two points. The availability of an explicit expression for the distribution of means (of the random probability measure we introduce) makes it possible to provide its numerical evaluation with any prescribed error of approximation. On the other hand, if the analytic expression of the exact distribution is not easy to handle, we can adopt simulation techniques, based on those proposed, for example, in Damien, Laud and Smith (1995), as an alternative.

Following these guidelines, in Section 2 we present the basic elements of the theory of processes with positive and independent increments, introduce the notion of random measure with positive and independent increments and obtain the corresponding random probability measure by normalization. In Section 3 a result concerning (almost sure) existence of any mean of a random probability measure is given and an expression for its distribution is provided. The problem of finding an expression for the posterior distribution is tackled in Section 4 and two illustrative examples are developed in Section 5.

2. Preliminaries and basic definitions. Let A be a discrete distribution function (d.f.) on $\mathbb{R}$ with discontinuities at $s_1 < s_2 < \cdots < s_k$ of magnitude $\alpha_1, \dots, \alpha_k$. Suppose $\Delta_1, \dots, \Delta_k$ are independent random variables having distributions $G(\alpha_1), \dots, G(\alpha_k)$, respectively, $G(\eta)$ being the gamma distribution with shape parameter η and scale parameter 1, for any $\eta > 0$. Then, by defining $W_0 := 0$ and $W_j := \Delta_1 + \cdots + \Delta_j$ for each $j = 1, \dots, k$, the Laplace transform of the probability distribution of W_j coincides with

$$\lambda \mapsto (1+\lambda)^{-A(s)} = \exp\left\{-A(s)\int_0^{+\infty}(1-e^{-\lambda x})x^{-1}e^{-x}\,dx\right\}, \qquad s \in I_j,\ j = 0, \dots, k,$$

where $I_0, I_1, \dots, I_k$ denotes the partition of $\mathbb{R}$ determined by the discontinuities of A. Moreover, the random vector $(\Delta_1, \dots, \Delta_j)/W_k$ has the $(j-1)$-variate Dirichlet distribution $D(\alpha_1, \dots, \alpha_{j-1}, \sum_{i=j}^k \alpha_i)$. In other words, the Dirichlet distribution is definable as the joint distribution of a set of independent gamma variables divided by their sum. See, for example, Bilodeau and Brenner (1999).

Now, consider any nondegenerate d.f. A on $\mathbb{R}$ such that $\lim_{x\to+\infty} A(x) = a \in (0,+\infty)$. The above construction, which refers to the discrete case, can be extended to this more general framework as follows. Let $\{\xi_t : t \geq 0\}$ denote a gamma subordinator, that is, an increasing process with independent increments such that, for any $s < t$, $\xi_t - \xi_s$ is gamma distributed with shape parameter $t - s$ and scale parameter 1. The reader is referred to Tsilevich, Vershik and Yor (2001) for an exhaustive and stimulating treatment of gamma processes. The time change $t = A(x)$, with $x \in \mathbb{R}$, yields a "reparameterized gamma process" $\{\xi_{A(x)} : x \in \mathbb{R}\}$ whose increments $\xi_{A(x)} - \xi_{A(y)}$ are, for any $x > y$, gamma distributed with scale parameter still equal to 1 and shape parameter equal to $A(x) - A(y)$. Whence, combining the definition of the reparameterized gamma process with the above description of a Dirichlet distribution, we have that the normalized subordinator $x \mapsto \xi_{A(x)}/\xi_a$ has the same finite-dimensional distributions as those of a *Dirichlet process* with parameter A. At this stage we are in position to describe the main purpose of the present paper, that is, the generalization of the previous construction to cases in which gamma processes are replaced by increasing additive processes, yielding a wide class of priors for Bayesian inference in nonparametric form.

We begin with a short review of some basic facts of the theory of additive processes employed in this paper. Let $\{\nu_t : t \geq 0\}$ be a family of measures on $\mathscr{B}((0,+\infty))$ such that:

1. $\nu_0 \equiv 0$, $\int_{(0,+\infty)} (x \wedge 1)\nu_t(dx) < +\infty$ holds true for any $t > 0$.
2. $\nu_s(B) \leq \nu_t(B)$ for $s < t$ and $B \in \mathscr{B}((0,+\infty))$.
3. $\nu_s(B) \to \nu_t(B)$ as $s \to t$ in $[0,+\infty)$ with $B \in \mathscr{B}((0,+\infty))$ and $B \subset \{x : x > \varepsilon\}$, for some $\varepsilon > 0$.

According to Theorem 9.8 in Sato (1999), facts 1–3 assure the existence of a stochastic process $\{\xi_t : t \geq 0\}$ defined on some probability space $(\Omega, \mathscr{F}, P)$ satisfying:

4. For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables ξ_{t_0}, $\xi_{t_1} - \xi_{t_0}, \dots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent.
5. $P\{\xi_0 = 0\} = 1$.
6. $P\{|\xi_s - \xi_t| > \varepsilon\} \to 0$ as $s \to t$ in $[0,+\infty)$ for any $\varepsilon > 0$.
7. There exists $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$ such that $t \mapsto \xi_t(\omega)$ is increasing and right continuous for each $\omega \in \Omega_0$.
8. $E[e^{-\lambda\xi_t}] = \exp[-\int_{[0+\infty)} (1 - e^{-\lambda x})\nu_t(dx)]$ for any $\lambda \geq 0$, where $E[\,\cdot\,]$ denotes expectation with respect to P.
9. $\xi_t = \int_{(0,t]\times(0,+\infty)} x J(ds\,dx)$ a.s.-P, where J, defined by $J(B) := \#\{s : (s, \xi_s - \xi_{s-}) \in B\}$ for any $B \in \mathscr{B}((0,+\infty)^2)$, $\#D$ standing for the cardinality of set D, is a Poisson random measure with intensity measure $\tilde{\nu}$ such that $\tilde{\nu}((0,t] \times C) := \nu_t(C)$ for every $t \geq 0$ and $C \in \mathscr{B}((0,+\infty))$.

Any process that satisfies statements 4–9 is said to be an *increasing additive process* (IAP).

For our purposes, some slight modifications of this definition are useful. Suppose α is a nonnull finite measure on $\mathscr{B}(\mathbb{R})$ with d.f. A and assume that $\xi_{\alpha(\mathbb{R})}$ is strictly positive and finite a.s.-P. This condition can be restated in terms of the Lévy measure ν_t, since it is equivalent to

$$\nu_{\alpha(\mathbb{R})}\big((0,+\infty)\big)=+\infty. \tag{1}$$

Indeed, $\exp(-\int_{(0,+\infty)}(1-e^{-\lambda v})\nu_{\alpha(\mathbb{R})}(dv)) = E[e^{-\lambda\xi_{\alpha(\mathbb{R})}}] = P\{\xi_{\alpha(\mathbb{R})}=0\} + E[e^{-\lambda\xi_{\alpha(\mathbb{R})}}\mathbb{1}_{(0,+\infty)}(\xi_{\alpha(\mathbb{R})})]$, where, for any set B, $\mathbb{1}_B$ denotes the indicator function of B and, by the monotone convergence theorem, $P\{\xi_{\alpha(\mathbb{R})}=0\} = \lim_{\lambda\to+\infty}\exp((-\int_{(0,+\infty)}(1-e^{-\lambda v})\nu_{\alpha(\mathbb{R})}(dv)))$. This entails that $P\{\xi_{\alpha(\mathbb{R})}=0\}=0$ if and only if $\lim_{\lambda\to+\infty}\int_{(0,+\infty)}(1-e^{-\lambda v})\nu_{\alpha(\mathbb{R})}(dv)=+\infty$. Finally, we again apply monotone convergence so that $P\{\xi_{\alpha(\mathbb{R})}>0\}=1$ if and only if (1) holds true.

In this framework there exists a set $\Omega_1\in\mathscr{F}$ with $P(\Omega_1)=1$ such that $t\mapsto\xi_t(\omega)$ is continuous from the left at $\alpha(\mathbb{R})$ for any $\omega\in\Omega_1$. Hence, in accordance with the notation introduced in the first paragraph of this section, $x\mapsto\xi_{A(x)}(\omega)$ turns out to be a bounded d.f. on $\mathbb{R}$ for any ω in Ω_1 and

$$x\mapsto\tilde F(x)=\tilde F(x,\omega)=\begin{cases}\xi_{A(x)}(\omega)/\xi_{\alpha(\mathbb{R})}(\omega), & \text{if } \omega\in\Omega_1,\\ E[\tilde F(x)\mathbb{1}_{\Omega_1}], & \text{if } \omega\notin\Omega_1,\end{cases}$$

is a random probability d.f. on $\mathbb{R}$. By *random measure with independent increments* (RMI) we mean the random measure $\tilde\xi$ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ associated with $\xi_{A(\cdot)}$. Consistently, the random probability measure $\tilde\varphi$ associated with $\tilde F$, that is,

$$\tilde\varphi(B)=\int_{\mathbb{R}}\mathbb{1}_B\,d\tilde F,\qquad B\in\mathscr{B}(\mathbb{R}), \tag{2}$$

is said to be a *normalized* RMI. A natural representation of $\tilde F$ can be given using

$$\xi_{A(x)}=\int_{(-\infty,x]\times(0,+\infty)}v\,J_\alpha(ds\,dv), \tag{3}$$

which is a straightforward consequence of the right continuity of A, with $J_\alpha(ds\,dv):=J(G_\alpha^{-1}(ds)\,dv)$ on $\mathscr{B}(\mathbb{R}\times(0,+\infty))$ and $G_\alpha(x):=\inf\{z:A(z)\ge x\}$ for $x\in(0,\alpha(\mathbb{R}))$. Moreover, if $\tilde\nu_\alpha(ds\,dv):=\tilde\nu(G_\alpha^{-1}(ds)\,dv)$, we obtain

$$E[\xi_{A(x)}]=\int_{(-\infty,x]\times(0,+\infty)}v\tilde\nu_\alpha(ds\,dv).$$

Kingman (1975) was the first to undertake the approach we are describing. He addressed the problem of computing the expected value of a specific function of the random vector $(\tilde\varphi_1,\dots,\tilde\varphi_N)$ defined by $\tilde\varphi_j=\tilde\xi(I_j)/\tilde\xi(\mathbb{R})$, where $I_1,\dots,I_N$ is a partition of $\mathbb{R}$. For more recent papers that draw inspiration from Kingman's contribution and that are somehow connected to the present paper, refer to, for example, Perman (1993) and Pitman (1996). The former deals with ranked random discrete distributions derived from subordinators and provides the joint distribution of the n largest atoms, whereas the latter discusses the generalization of the usual Dirichlet updating rule which applies to a two-parameter family of random measures that includes both the normalized gamma and stable subordinators as special cases.

3. Distributional results for means of normalized RMI. If $\tilde{\xi}$ and $\tilde{\varphi}$ are the same as in the previous section, then the integral $\tilde{\varphi}|f| = \tilde{\varphi}(|f|, \omega) := \int_{\mathbb{R}} |f| \, d\tilde{\varphi}$ is well defined for each ω in Ω. In particular, using (3), we have

$$\tilde{\xi}(C) = \int_{C \times (0,+\infty)} v J_\alpha(dx\, dv), \qquad C \in \mathscr{B}(\mathbb{R}),$$

$$\tilde{\xi}|f| = \tilde{\xi}(|f|, \omega) := \int_{\mathbb{R}^+} x\bigl(\tilde{\xi} \circ |f|^{-1}\bigr)(dx) = \int_{\mathbb{R}^+ \times (0,+\infty)} x v J_\alpha\bigl(|f|^{-1}(dx)\, dv\bigr).$$

The last integral is finite a.s.-P when, for example, the support of α is finite. This plays a role in proving the following proposition, which specifies a criterion that is dependent only on the properties of the measure $\tilde{\nu}_\alpha$ to determine whether $\tilde{\xi}|f|$ is finite. Details about its proof are given in the Appendix.

PROPOSITION 1. *For any normalized RMI $\tilde{\varphi}$ and any measurable function $f : \mathbb{R} \to \mathbb{R}$, the following conditions are equivalent*:

(i) $P\{\tilde{\varphi}|f| < +\infty\} = P\{\tilde{\xi}|f| < +\infty\} = 1$.
(ii) $\int_{\mathbb{R} \times (0,+\infty)} [1 - \exp(-\lambda y |f(x)|)] \tilde{\nu}_\alpha(dx\, dy) < +\infty$ *holds for every* $\lambda > 0$.
(iii) $\int_{\mathbb{R} \times (0,+\infty)} [1 - \cos(yt|f(x)|)] \tilde{\nu}_\alpha(dx\, dy) < +\infty$ *and* $\int_{\mathbb{R} \times (0,+\infty)} |\sin(yt|f(x)|)| \tilde{\nu}_\alpha(dx\, dy) < +\infty$ *hold for every* $t \in \mathbb{R}$.

Proposition 1 represents a generalization of a well-known statement, concerning only Dirichlet random probability measures, studied by Feigin and Tweedie (1989) and Cifarelli and Regazzini (1990, 1996).

Approaching the problem of the determination of the probability distribution of $\tilde{\varphi}(f)$, we assume that the conditions of Proposition 1 hold. Since

$$P\{\tilde{\varphi}(f) \le \sigma\} = P\bigl\{\tilde{\xi}(f - \sigma) \le 0\bigr\}$$

holds true for any $\sigma \in \mathbb{R}$, then by the Gurland inversion formula for characteristic functions, we get

$$\begin{aligned} &\frac{1}{2}\bigl[P\{\tilde{\varphi}(f) \le \sigma\} + P\{\tilde{\varphi}(f) < \sigma\}\bigr] \\ &\qquad = \frac{1}{2} - \frac{1}{\pi} \lim_{\substack{\varepsilon \downarrow 0 \\ T \uparrow +\infty}} \int_\varepsilon^T \frac{1}{t} \operatorname{Im} E\bigl[\exp\bigl(it\tilde{\xi}(f - \sigma)\bigr)\bigr]\, dt, \end{aligned} \tag{4}$$

where $\operatorname{Im} z$ stands for the imaginary part of $z \in \mathbb{C}$ [cf. Gurland (1948)]. The final part of the proof of Proposition 1 can be used, with slight modifications, to prove the relationships

$$\begin{aligned}\gamma_f(t) &:= E\big[\exp\big(it\tilde{\xi}(f)\big)\big]\\ &= \exp\bigg[\int_{\mathbb{R}\times(0,+\infty)} (e^{itvf(x)} - 1)\tilde{\nu}_\alpha(dx\,dv)\bigg]\\ &= \exp\bigg[\int_{\mathbb{R}\times(0,+\infty)} (e^{itvx} - 1)\tilde{\nu}_\alpha\big(f^{-1}(dx)\,dv\big)\bigg].\end{aligned}$$

As a consequence, we can confine ourselves to studying the probability d.f. $\mathbb{F}$ of $\int x\tilde{\varphi}(dx)$.

PROPOSITION 2. *Letting $\mathbb{F}$ be the probability d.f. of $\int x\tilde{\varphi}(dx)$ with $\tilde{\varphi}$ defined as in* (2), *we have*

$$\begin{aligned}&\frac{1}{2}[\mathbb{F}(\sigma) + \mathbb{F}(\sigma - 0)]\\ &\quad= \frac{1}{2} - \frac{1}{\pi}\lim_{T\uparrow+\infty}\int_0^T \frac{1}{t}\exp\bigg\{\int_{\mathbb{R}\times(0,+\infty)}\big[\cos\big(tv(x-\sigma)\big) - 1\big]\tilde{\nu}_\alpha(dx\,dv)\bigg\}\\ &\qquad\qquad\times \sin\bigg(\int_{\mathbb{R}\times(0,+\infty)} \sin\big(tv(x-\sigma)\big)\tilde{\nu}_\alpha(dx\,dv)\bigg)\,dt\end{aligned}$$

for every $\sigma \in \mathbb{R}$.

The proof of Proposition 2 is provided in the Appendix.

4. Some results about the posterior distribution. This section provides some results concerning the posterior distribution of means of a normalized RMI in view of Bayesian applications. We restrict our attention to *exchangeable* observations. Assume that the probability space $(\Omega, \mathscr{F}, P)$, introduced in Section 2 to define the stochastic process $\{\xi_t : t \ge 0\}$, also supports a sequence $X = (X_n)_{n\ge1}$ of random variables that are conditionally i.i.d., given the random probability measure $\tilde{\varphi}$ defined as in (2). Hence, we have that $P(X_1 \in A_1, \dots, X_n \in A_n|\tilde{\varphi}) = \tilde{\varphi}(A_1)\cdots\tilde{\varphi}(A_n)$ a.s. for every $A_1, \dots, A_n$ and $n \ge 1$. Let $\mathbb{F}(\cdot\,; f)$ denote the probability d.f. of $\tilde{\varphi}(f)$, where f is any real-valued function satisfying the conditions in Proposition 1. Set $X^{(n)} = (X_1, \dots, X_n)$ for every $n \ge 1$ and define $\mathbb{F}_{X^{(n)}}(\cdot\,; f)$ to be a posterior d.f. for $\tilde{\varphi}(f)$ given $X^{(n)}$. Extend the definition of $\mathbb{F}_{X^{(n)}}$ to $n = 0$ by putting $\mathbb{F}_{X^{(0)}} = \mathbb{F}$.

We begin by discussing the case in which α has finite support, $\operatorname{supp}(\alpha) = \{s_1, \dots, s_N\}$, and (a, b) is an interval containing all the $f(s_j)$'s. The tool we use to determine $\mathbb{F}_{X^{(n)}}$ for $n \ge 1$ is the moment generating function of $\sum_{j=1}^N \tilde{\varphi}(s_j)t_j$. In particular, if $x^{(n)} = (x_1, \dots, x_n)$ is a sample including $n_{i_r} > 0$ terms equal to s_{i_r}, for $r = 1, \dots, k$, with $\sum_r n_{i_r} = n$, we have

$$g_{x^{(n)}}(\lambda; t_1, \dots, t_N) := \int_{(a,b)} e^{-\lambda\sigma}\, d\mathbb{F}_{x^{(n)}}(\sigma; t_1, \dots, t_N)$$
$$= C(x^{(n)}) \int \prod_{r=1}^{k} \varphi(s_{i_r})^{n_{i_r}} \exp\Biggl(-\lambda \sum_{j=1}^{N} t_j \varphi(s_j)\Biggr) Q(d\varphi),$$

where Q is the probability distribution of $\tilde{\varphi}$ and $C(x^{(n)})^{-1} = \int \prod_{r=1}^{k} \varphi(s_{i_r})^{n_{i_r}} \times Q(d\varphi)$. Observe that combination of Theorems 16.8 and 18.4 in Billingsley (1995) gives

$$g_{x^{(n)}}(\lambda; t_1, \dots, t_N) = (-1)^n C(x^{(n)})$$
$$\times \frac{\partial^n}{\partial t_{i_1}^{n_{i_1}} \cdots \partial t_{i_k}^{n_{i_k}}} \int_{(a,b)} e^{-\lambda\sigma} I_{a^+}^{n-1} \mathbb{F}(\sigma; t_1, \dots, t_N)\, d\sigma,$$

where $I_{a^+}^{n} h(\sigma) = \int_a^\sigma ((\sigma - u)^{n-1}/(n-1)!) h(u)\, du$ is the *Liouville–Weyl fractional integral* for $n \geq 1$ and $I_{a^+}^{0}$ represents the identity operator [see, e.g., Oldham and Spanier (1974)]. It is worth stressing that the evaluation of the posterior distribution of $\tilde{\varphi}(f)$, given $X^{(n)}$, can be based on the expression of its prior distribution. In particular, this is true when one assumes the validity of interchanging the derivative with the integral according to the following condition:

(H) There exists a Lebesgue null set $\mathscr{N} \subset \mathbb{R}$ such that for every $\sigma \in \mathscr{N}^c \cap (a,b)$, $k \geq 1$, $n \geq 1$, $i_1, \dots, i_k$ in $\{1, \dots, N\}$, $(t_1, \dots, t_N) \in (a,b)^N$ and for any sample $x^{(n)}$ as above $\partial^n I_{a^+}^{n-1} \mathbb{F}(\sigma; t_1, \dots, t_N)/\partial t_{i_1}^{n_{i_1}} \cdots \partial t_{i_k}^{n_{i_k}}$ exists and

$$\frac{\partial^n}{\partial t_{i_1}^{n_{i_1}} \cdots \partial t_{i_k}^{n_{i_k}}} \int_a^b e^{-\lambda\sigma} I_{a^+}^{n-1} \mathbb{F}(\sigma; t_1, \dots, t_N)\, d\sigma$$
$$= \int_a^b e^{-\lambda\sigma} \frac{\partial^n}{\partial t_{i_1}^{n_{i_1}} \cdots \partial t_{i_k}^{n_{i_k}}} I_{a^+}^{n-1} \mathbb{F}(\sigma; t_1, \dots, t_N)\, d\sigma$$

holds true for every $\lambda \in \mathbb{R}$.

In fact, if (H) is in force,

$$g_{x^{(n)}}(\lambda; t_1, \dots, t_N)$$
$$= (-1)^n C(x^{(n)}) \int_a^b e^{-\lambda\sigma} \frac{\partial^n}{\partial t_{i_1}^{n_{i_1}} \cdots \partial t_{i_k}^{n_{i_k}}} I_{a^+}^{n-1} \mathbb{F}(\sigma; t_1, \dots, t_N)\, d\sigma$$

and we obtain:

PROPOSITION 3. *Let the support of* α *be* $\{s_1, \dots, s_N\}$ *and suppose condition* (H) *is met. Then*

$$(-1)^n C(x^{(n)}) \frac{\partial^n}{\partial t_{i_1}^{n_{i_1}} \cdots \partial t_{i_k}^{n_{i_k}}} I_{a^+}^{n-1} \mathbb{F}(\sigma; t_1, \dots, t_N)\Big|_{(t_1,\dots,t_N)=(f(s_1),\dots,f(s_N))}$$

is a posterior probability density function (*with respect to the Lebesgue measure on* $\mathbb{R}$) *of* $\tilde{\varphi}(f)$, *given* $X^{(n)} = x^{(n)}$.

We move on to the case in which the support of α is arbitrary. Here we confine ourselves to writing the posterior probability d.f. of $\tilde{\varphi}(f)$, given $X^{(n)}$, as a limit, in the sense of weak convergence, of a suitable sequence of posterior probability d.f.'s determined according to Proposition 3, employing techniques similar to those introduced in Regazzini and Sazonov (2001). Suppose that f meets the conditions in Proposition 1. Let $\mu^{(n)}$ be the marginal distribution of $X^{(n)}$ and introduce a partition $\mathscr{P}_m := \{A_{m,i} : i = 1, \dots, k_m + 2\}$ of $\mathbb{R}$ that has the following properties:

1. $\mathscr{P}_{m+1}$ is a refinement of $\mathscr{P}_m$.
2. $\mathscr{B}(\mathbb{R})$ is generated by $\bigcup_{m\ge 1} \sigma(\mathscr{P}_m)$, where $\sigma(\mathscr{P}_m)$ denotes the σ-algebra generated by $\mathscr{P}_m$, $m \ge 1$.
3. $\varepsilon_m := 2 \max_{1\le i\le k_m} \operatorname{diam}(A_{m,i}) \downarrow 0$ (as $m \to +\infty$).
4. $\bigcup_{i=1}^{k_m} A_{m,i} = [-R_m, R_m]$, $A_{m,k_m+1} = (R_m, +\infty)$ and $A_{m,k_m+2} = (-\infty, -R_m)$, with $R_m > 0$ for any $m \ge 1$.

A simple example of such a sequence of partitions corresponds to fixing $A_{m,1}, \dots, A_{m,k_m}$ as the dyadic intervals of rank m that partition $[-R_m, R_m]$, with $R_m = m$. The sequence $(\mathscr{P}_m)_{m\ge 1}$ plays a key role in defining a discretization of $\tilde{\varphi}$ we need to approximate $\mathbb{F}_{x^{(n)}}$. This discretization requires selecting points $a_{m,i}$ in $A_{m,i}$ and, whenever the rth element, X_r, in the sample lies in $A_{m,i}$, it is as if we have observed $a_{m,i}$. In other words, the original sample $X^{(n)}$ is replaced by $\zeta_m^{(n)} := (\zeta_{m,1}, \dots, \zeta_{m,n})$, where $\zeta_{m,r} = \sum_{i=1}^{k_m+2} a_{m,i} \mathbb{1}_{A_{m,i}}(X_r)$, $r = 1, \dots, n$, with $a_{m,k_m+1} = R_m$ and $a_{m,k_m+2} = -R_m$. Define $\tilde{\varphi}_m := \sum_{j=1}^{k_m} \tilde{\varphi}(A_{m,j})\delta_{a_{m,j}} + \tilde{\varphi}(A_{m,k_m+1})\delta_{R_m} + \tilde{\varphi}(A_{m,k_m+2})\delta_{-R_m}$, where, as usual, δ_x denotes the point mass at x, and set $\mathbb{F}^*_{m,x^{(n)}}(\sigma; f) := P(\tilde{\varphi}_{1,m} \le \sigma | \zeta_m^{(n)})(x^{(n)})$, where $\tilde{\varphi}_{1,m} = \tilde{\varphi}_m(f)$. We are now in a position to state the following approximation result.

PROPOSITION 4. *There exists a* $\mu^{(n)}$*-null set* $\mathscr{N}$ *such that for each* $x^{(n)} \in \mathscr{N}^c$,

$$\lim_{m\to+\infty} \mathbb{F}^*_{m,x^{(n)}}(\sigma; f) = \mathbb{F}_{x^{(n)}}(\sigma; f)$$

is valid for every σ *belonging to the set of continuity points of* $\mathbb{F}_{x^{(n)}}$.

Proposition 4, the proof of which is provided in the Appendix, is interesting because of the concrete availability of $\mathbb{F}^*_{m,x^{(n)}}$, which can be determined by means of the procedures described above and related to Proposition 3.

5. Two illustrative examples. Here we focus our attention on two examples of Lévy measures and illustrate how the results given in the previous sections can be applied to determine distributions of means of random probability measures which are useful, for instance, in a Bayesian nonparametric setting. First we introduce and study a generalization of the Dirichlet process. The second example is related to the so-called γ-stable subordinator and was originally employed in this framework by Kingman (1975).

5.1. *A generalization of the Dirichlet process.* Let, for any $t \geq 0$,

$$\nu_t(dv) = t \frac{(1 - e^{-\gamma v})}{(1 - e^{-v})} \frac{e^{-v}}{v} \, dv, \qquad \gamma > 0.$$

Clearly, the family of measures $\{\nu_t : t \geq 0\}$ satisfies properties 1–3 in Section 2, and the corresponding IAP $\{\xi_t : t \geq 0\}$ is a gamma process when $\gamma = 1$. Moreover, given a nonnull finite measure α on $\mathbb{R}$ with d.f. A, it is easy to verify that

$$\begin{aligned} E\bigl[\exp(-\lambda \xi_{A(x)})\bigr] &= \exp\biggl(-A(x) \int_{(0,+\infty)} (1 - e^{-\lambda v}) \frac{(1 - e^{-\gamma v})}{(1 - e^{-v})} \frac{e^{-v}}{v} \, dv\biggr) \\ &= \biggl\{\frac{\Gamma(1+\lambda)\Gamma(1+\gamma)}{\Gamma(\lambda+\gamma+1)}\biggr\}^{A(x)} \end{aligned}$$

[cf. e.g., 3.413.1 in Gradshteyn and Ryzhik (2000)].

5.1.1. *Finiteness of* $\tilde{\varphi}(f)$. The first important issue we need to face, according to the program expounded in Section 3, concerns the (almost sure) existence of the mean $\tilde{\varphi}(f)$. A straightforward application of Proposition 1 leads us to write $\int_{\mathbb{R} \times (0,+\infty)} (1 - e^{-\lambda v |f(x)|})(1 - e^{-\gamma v}) e^{-v} \{(1 - e^{-v}) v\}^{-1} \alpha(dx) \, dv < +\infty$ as an equivalent condition for (almost sure) finiteness of $\tilde{\varphi}|f|$. This, by virtue of 3.413.1 in Gradshteyn and Ryzhik (2000), reduces to

$$\int_{\mathbb{R}} \log \frac{\Gamma(\gamma + 1 + \lambda |f(x)|)}{\Gamma(1 + \lambda |f(x)|)} \alpha(dx) < +\infty$$

and, if γ is any positive integer, to

$$\int_{\mathbb{R}} \log\bigl(\gamma + \lambda |f(x)|\bigr)_{\gamma} \alpha(dx) < +\infty$$

with $(a)_n := a(a-1) \cdots (a-n+1)$. When $\gamma = 1$, the latter inequality coincides with the Feigin and Tweedie condition. Moreover, we immediately see that conditions for finiteness of $\tilde{\varphi}(f)$ in the Dirichlet case and in the present generalization of the Dirichlet case are equivalent.

5.1.2. *Exact distribution of* $\int x\tilde{\varphi}(dx)$. As far as the problem of determining the distribution of $\tilde{\varphi}(f)$ is concerned, we confine ourselves to considering the case $f(x) \equiv x$. By resorting to Proposition 2, we can easily show that the probability d.f., $\mathbb{F}$, of $\int x\tilde{\varphi}(dx)$ is characterized by

$$
\begin{aligned}
&\frac{1}{2}[\mathbb{F}(\sigma) + \mathbb{F}(\sigma - 0)] \\
&\quad = \frac{1}{2} - \frac{1}{\pi} \lim_{T\uparrow+\infty} \int_0^T \frac{1}{t} \\
&\qquad \times \exp\biggl\{-\frac{1}{2}\int_{\mathbb{R}}\biggl(\log\frac{\sinh(\pi t(x-\sigma))}{\pi t(x-\sigma)} \\
&\qquad\qquad + \log\frac{\Gamma(1+\gamma - it(x-\sigma))\Gamma(1+\gamma+it(x-\sigma))}{(\Gamma(1+\gamma))^2}\biggr)\alpha(dx)\biggr\} \\
&\qquad \times \sin\biggl(\frac{1}{2i}\int_{\mathbb{R}} \log\frac{B(\gamma; 1 - it(x-\sigma))}{B(\gamma; 1+it(x-\sigma))}\alpha(dx)\biggr)\,dt, \\
&\qquad\qquad \sigma \in \mathbb{R},
\end{aligned}
\tag{5}
$$

where B denotes the *beta* function. For computational details, refer to the Appendix. If it is further supposed that $\gamma \in \mathbb{N}$, then

$$
\begin{aligned}
&\frac{1}{2}[\mathbb{F}(\sigma) + \mathbb{F}(\sigma - 0)] \\
&\quad = \frac{1}{2} - \frac{(\gamma!)^{\alpha(\mathbb{R})}}{\pi} \\
&\qquad \times \lim_{T\uparrow+\infty}\int_0^T \frac{1}{t}\exp\biggl\{-\frac{1}{2}\sum_{k=1}^{\gamma}\int_{\mathbb{R}}\log(k^2 + t^2(x-\sigma)^2)\alpha(dx)\biggr\} \\
&\qquad\qquad \times \sin\biggl(\sum_{k=1}^{\gamma}\int_{\mathbb{R}}\arctan\frac{t(x-\sigma)}{k}\alpha(dx)\biggr)dt.
\end{aligned}
\tag{6}
$$

In this case the integrand is absolutely integrable in $(M, +\infty)$ for any $M > 0$ and, thus, the previous integral can be thought of as a Lebesgue integral on $\mathbb{R}$. In particular, if we set $\gamma = 1$, we obtain the distribution of the mean of a Dirichlet random probability measure; that is, $1/2 - (1/\pi)\int_0^{+\infty}(1/t)\exp\{-(1/2) \times \int_{\mathbb{R}}\log(1 + t^2(x-\sigma)^2)\alpha(dx)\}\sin(\int_{\mathbb{R}}\arctan(t(x-\sigma))\alpha(dx))\,dt$ [cf., e.g., (3) in Regazzini, Guglielmi and Di Nunno (2002)]. Figures 1 and 2 display graphs of the probability density function and of the probability d.f., respectively, of $\int x\tilde{\varphi}(dx)$. Thicker lines refer to the generalized Dirichlet process with $\gamma = 2$, whereas the other situation refers to the usual Dirichlet process (i.e., $\gamma = 1$). The measure α

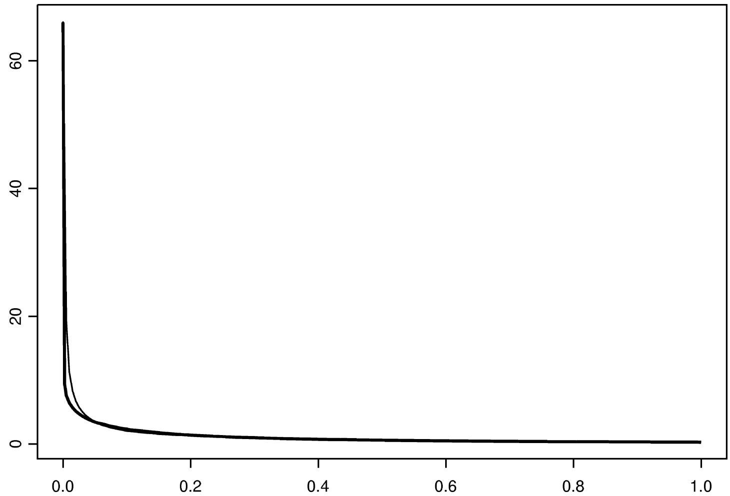

FIG. 1. *Prior densities of* $\int x\tilde{\varphi}(dx)$, *where* $\tilde{\varphi}$ *corresponds to a Dirichlet process* $(\gamma = 1)$ *and to a generalized Dirichlet process* $(\gamma = 2)$. *The measure* α *is the beta distribution with parameters* $1/9$ *and* 1 *and total mass equal to* $1/2$.

corresponds to a beta distribution, with parameters $1/9$ and 1, and total mass, $\alpha(\mathbb{R})$, equal to $1/2$. We do not give details about numerical aspects to be faced at this stage, but the procedure adopted is similar to that in Regazzini, Guglielmi and Di Nunno (2002).

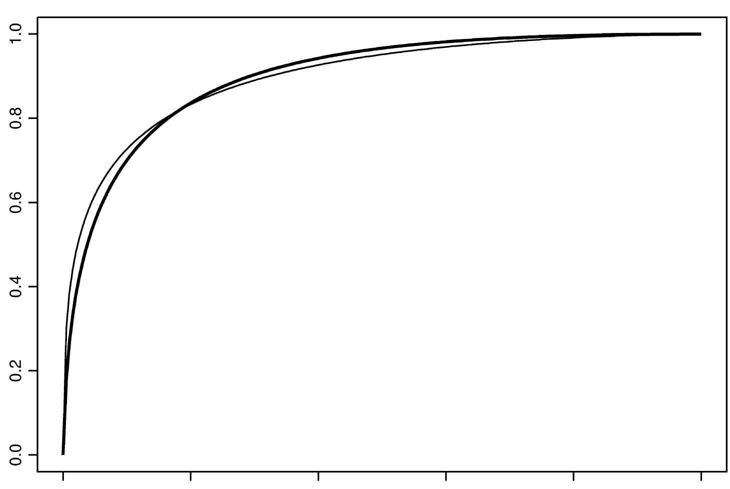

FIG. 2. *Prior distribution functions of* $\int x\tilde{\varphi}(dx)$, *where* $\tilde{\varphi}$ *coincides with a Dirichlet process* $(\gamma = 1)$ *and with a generalized Dirichlet process* $(\gamma = 2)$. *The measure* α *is the beta distribution with parameters* $1/9$ *and* 1 *and total mass equal to* $1/2$.

5.1.3. *Posterior distribution in the discrete case with $\gamma \in \mathbb{N}$.* In moving on to the computation of the posterior distribution of $\int x\tilde{\varphi}(dx)$ given $x^{(n)}$, we undertake the procedure sketched in Section 4, and, from now on, γ is taken to be a positive integer. We start by considering the case in which the support of α is discrete, consisting of the points $s_1, \dots, s_N$. In such a case, we can show that $\mathbb{F}$, that is, the prior d.f. of $\int x\tilde{\varphi}(dx)$, is continuous and coincides with

$$\mathbb{F}(\sigma) = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} \frac{1}{t}\operatorname{Im}\left(\exp\left\{-\sum_{j=1}^N \alpha_j \log\frac{\Gamma(\gamma+1+it(s_j-\sigma))}{\Gamma(\gamma+1)\Gamma(1+it(s_j-\sigma))}\right\}\right)dt, \tag{7}$$

where $\alpha_j = \alpha(s_j)$, $j = 1, \dots, N$. According to Proposition 3, (7) is the starting point for the determination of the posterior. To this end, it is useful to introduce some new notation. For any $(r_1, \dots, r_n)$ in $\{1, \dots, \gamma\}^n$, let $\bar{\alpha}_j(r_1, r_2) = \alpha_j + \delta_{s_{j_1}}(\{s_j\})$ if $r_1 = r_2$, $\bar{\alpha}_j(r_1, r_2) = \alpha_j$ if $r_1 \neq r_2$, $\bar{\alpha}_j(r_1, \dots, r_k) = \bar{\alpha}_j(r_1, \dots, r_{k-2}, r_k) + \delta_{s_{j_{k-1}}}(\{s_j\})$ if $r_k = r_{k-1}$ and $\bar{\alpha}_j(r_1, \dots, r_k) = \bar{\alpha}_j(r_1, \dots, r_{k-2}, r_k)$ if $r_k \neq r_{k-1}$, for any $k \geq 3$. At this stage, following Proposition 3 with $\mathbb{F}$ as in (7), we have

$$\begin{aligned}
C(x^{(n)}) &\frac{\partial^n}{\partial s_{i_1}^{n_{i_1}} \cdots \partial s_{i_k}^{n_{i_k}}} \int_0^{+\infty} \frac{1}{t}\exp\left\{-\sum_{p=1}^N \sum_{q=1}^{\gamma} \alpha_j \log(q + it(s_p - \sigma))\right\} dt \\
&= C(x^{(n)})\alpha_{j_1} \sum_{\{(r_1,\dots,r_n)\in\{1,\dots,\gamma\}^n\}} \bar{\alpha}_{j_1}(r_1, r_2)\cdots\bar{\alpha}_{j_n}(r_1, \dots, r_n) \\
&\qquad \times \int_0^{+\infty} t^{n-1}\exp\left(-\sum_{q=1}^{\gamma}\sum_{p=1}^{N} \bar{\alpha}_k(r_1, \dots, r_n, q) \right. \\
&\qquad\qquad \left. \times \log(q + it(s_p - \sigma))\right) dt \\
&=: \psi(\sigma),
\end{aligned}$$

where $(j_1, \dots, j_n)$ is a vector whose components are in $\{i_1, \dots, i_k\}$ and $k \leq n$. Resort again to Proposition 3 to obtain: If $n = 2p$, a posterior probability density function of $\int x\tilde{\varphi}(dx)$ given $x^{(n)}$ is

$$\rho_{x^{(n)}}(\sigma) = \frac{(-1)^{p+1}(\gamma!)^{\alpha(\mathbb{R})}}{\pi} I_{a^+}^{n-1} \operatorname{Im}\psi(\sigma); \tag{8}$$

if $n = 2p + 1$, a posterior probability density function of $\int x\tilde{\varphi}(dx)$ given $x^{(n)}$ is

$$\rho_{x^{(n)}}(\sigma) = \frac{(-1)^{p+1}(\gamma!)^{\alpha(\mathbb{R})}}{\pi} I_{a^+}^{n-1} \operatorname{Re}\psi(\sigma). \tag{9}$$

5.1.4. *Posterior distribution in a general setting.* Finally, we apply results in Section 4 to achieve a representation for the posterior probability density function $\rho_{x^{(n)}}$ of $\int x\tilde{\varphi}(dx)$ given $X^{(n)} = x^{(n)}$ when the support of α is arbitrary. Referring to the same partition defined by properties 1–4 in Section 4, we take $\rho^*_{m,x^{(n)}}$ to be the probability density function corresponding to $\mathbb{F}^*_{m,x^{(n)}}$, with $x^{(n)} = (x_1, \ldots, x_n) \in A_{m,i_1} \times \cdots \times A_{m,i_n}$. First, observe that (3) can be used to show that $\mu^{(n)}$ is absolutely continuous with respect to the product measure $\bar{\alpha}^{(n)}$ of the n factor measures $\alpha, \bar{\alpha}(\cdot\,; r_1, r_2), \ldots, \bar{\alpha}(\cdot\,; r_1, \ldots, r_n)$, where, for any $(r_1, \ldots, r_n)$ in $\{1, \ldots, \gamma\}^n$, we set $\bar{\alpha}(B; r_1, r_2) := \alpha(B) + \delta_{x_1}(B)$ if $r_1 = r_2$, $\bar{\alpha}(B; r_1, r_2) := \alpha(B)$ if $r_1 \neq r_2$ and, for $k \geq 3$, $\bar{\alpha}(B; r_1, \ldots, r_{k-1}, r_k) = \bar{\alpha}(B; r_1, \ldots, r_{k-2}, r_k) + \delta_{x_{k-1}}(B)$ if $r_k = r_{k-1}$ and $\bar{\alpha}(B; r_1, \ldots, r_{k-1}, r_k) = \bar{\alpha}(B; r_1, \ldots, r_{k-2}, r_k)$ otherwise, for any $B \in \mathscr{B}(\mathbb{R})$. Thus, from a well-known result, we have, as m goes to $+\infty$, convergence (almost everywhere with respect to $\bar{\alpha}^{(n)}$) of $C(a_{m,i_1}, \ldots, a_{m,i_n})\alpha_{m,i_1}\bar{\alpha}_{m,i_2}(r_1, r_2) \cdots \bar{\alpha}_{m,i_n}(r_1, \ldots, r_n)$ to the Radon–Nikodým derivative, $g(x^{(n)}; r_1, \ldots, r_n)$, of $\mu^{(n)}$ with respect to $\bar{\alpha}^{(n)}$ on $\mathscr{B}(\mathbb{R}^n)$. See, for example, Theorem 35.7 in Billingsley (1995). Moreover, the dominated convergence theorem yields, for any bounded interval (a, b), $\lim_{m\to+\infty} \int_a^b \rho^*_{m,x^{(n)}}(\sigma)\,d\sigma = \int_a^b \rho_{x^{(n)}}(\sigma)\,d\sigma$, where $\rho_{x^{(n)}}$ is expressed as in (8) or (9) with

$$\psi(\sigma) = \mathbb{1}_{\mathcal{C}(\bar{A})}(\sigma) \sum_{\{(r_1,\ldots,r_n)\in\{1,\ldots,\gamma\}^n\}} \frac{1}{g(x^{(n)}; r_1, \ldots, r_n)}$$
$$\times \int_0^{+\infty} t^{n-1} \exp\Biggl(-\sum_{q=1}^{\gamma} \int_{\mathbb{R}} \log(q + it(s - \sigma))$$
$$\times\, \bar{\alpha}(ds; r_1, \ldots, r_n, q)\Biggr)\,dt,$$

$\bar{A}$ being the d.f. corresponding to $\bar{\alpha}$ and $\mathcal{C}(\bar{A})$ denoting the set of continuity points of $\bar{A}$. Therefore, in view of Proposition 4, $\rho_{x^{(n)}}$ is a probability density function for $\mathbb{F}_{x^{(n)}}$. Moreover, by Scheffé's theorem,

$$\lim_{m\to+\infty} \sup_{A\in\mathscr{B}(\mathbb{R})} \left| \int_A \rho^*_{m,x^{(n)}}(\sigma)\,d\sigma - \int_A \rho_{x^{(n)}}(\sigma)\,d\sigma \right| = 0$$

[see, e.g., Theorem 16.12 in Billingsley (1995)].

In particular, in the Dirichlet case ($\gamma = 1$) and for $n = 1$ (i.e., $\bar{\alpha} = \alpha + \delta_x$), we obtain

$$\rho_{x^{(1)}}(\sigma) = \mathbb{1}_{\mathcal{C}(\bar{A})}(\sigma)\frac{a}{\pi}\int_0^{+\infty} \mathrm{Re}\Big(\exp\Big(-\int_{\mathbb{R}} \log(1+it(s-\sigma))\bar{\alpha}(ds)\Big)\Big)\,dt,$$

which is a useful alternative representation of the posterior probability density function of the mean of the Dirichlet random probability measure, given $X^{(1)} = x$. See Proposition 5 in Regazzini, Guglielmi and Di Nunno (2002).

It is clear that to obtain the results we have just illustrated, existence of an R_m satisfying (A.5) is crucial. One might wonder whether it is possible to determine such an R_m concretely. An answer for the Dirichlet case can be given, for example, by applying the Markov inequality to bound the conditional probability appearing in (A.5) from above and, then, by resorting to the expression of the predictive probability d.f. for censored observations given, for example, in Regazzini (1978).

As far as the numerical evaluation of the posterior density of $\int x\tilde{\varphi}(dx)$ is concerned, we can use (8) and (9), after discretizing α (if necessary). For illustrative purposes, we consider an example already investigated, just for the Dirichlet case, by Regazzini, Guglielmi and Di Nunno (2002), in which $n = 2$, $x^{(2)} = (0.05, 0.1)$ and α is the beta distribution, with parameters $(1/9, 1)$ and total mass equal to $1/2$. Here we set $N = 20$, $s_j = j/N$ and $\alpha_j = \alpha((s_{j-1}, s_j])$ $(j = 1, \dots, N)$. Plots of posterior densities and d.f.'s are sketched in Figures 3 and 4. Again, thicker lines correspond to the generalized Dirichlet process with $\gamma = 2$. Notice that one can also consider more observations, although with a greater computational effort.

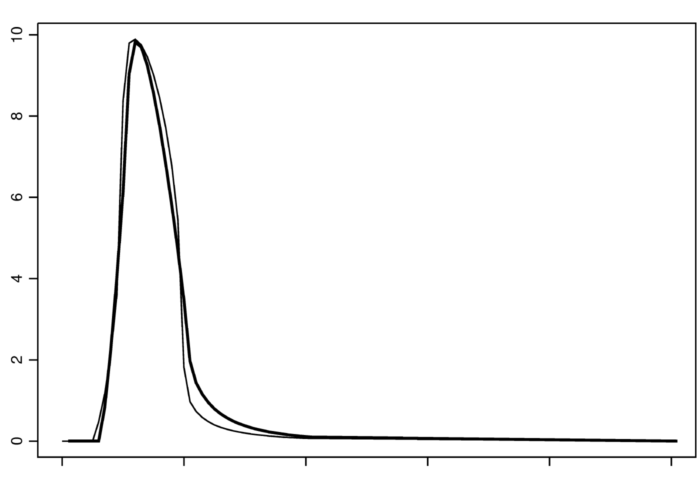

FIG. 3. *Posterior densities of* $\int x\tilde{\varphi}(dx)$, *where* $\tilde{\varphi}$ *coincides with a Dirichlet process* ($\gamma = 1$) *and with a generalized Dirichlet process* ($\gamma = 2$). *The measure* α *is the beta distribution with parameters* $1/9$ *and* 1 *and total mass* $1/2$.

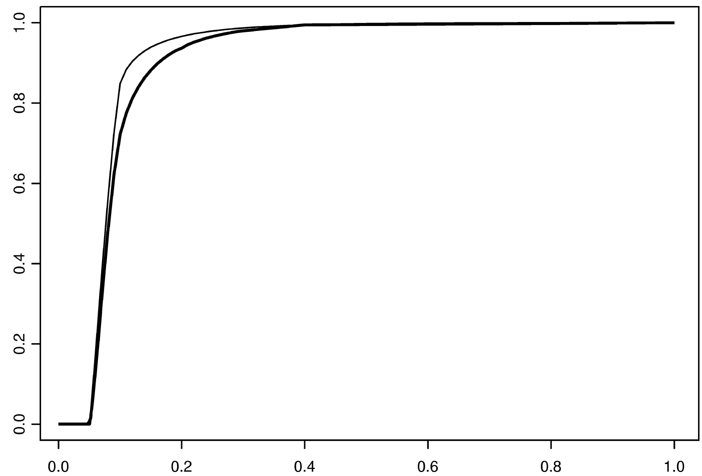

FIG. 4. *Posterior distribution functions of* $\int x\tilde{\varphi}(dx)$, *where* $\tilde{\varphi}$ *coincides with a Dirichlet process* $(\gamma = 1)$ *and with a generalized Dirichlet process* $(\gamma = 2)$. *The measure* α *coincides with the beta distribution with parameters* $1/9$ *and* 1 *and total mass* $1/2$.

5.2. *A normalized γ-stable subordinator.* Let

$$\nu_t(dv) = ctv^{-\gamma-1}\,dv, \qquad \gamma \in (0,1), c > 0,$$

and notice that the family of measures $\{\nu_t : t \geq 0\}$ meets conditions 1–3 in Section 2. The IAP $\{\xi_t : t \geq 0\}$ associated with $\{\nu_t : t \geq 0\}$ is the so-called γ-*stable* subordinator. Its use is relevant in certain problems in applied probability as pointed out, for example, in Kingman (1975). An analysis similar to the one we are going to develop below was presented in Barlow, Pitman and Yor (1989), where the authors discussed the generalization of the multivariate Dirichlet obtained by replacing the gamma subordinator by a stable one. See also Pitman and Yor (1997). The expression of the Laplace transform of $\xi_{A(x)}$ is given by

$$E\big[\exp(-\lambda\xi_{A(x)})\big] = \exp\biggl(-A(x)c\int_{(0,+\infty)}(1-e^{-\lambda v})v^{-\gamma-1}\,dv\biggr)$$

$$= \exp\biggl(-\frac{A(x)c\Gamma(1-\gamma)\lambda^{\gamma}}{\gamma}\biggr).$$

5.2.1. *Finiteness of* $\tilde{\varphi}(f)$. Before approaching the problem of determining the distribution of the mean $\tilde{\varphi}(f)$, we are going to verify the equivalent conditions for existence given in Proposition 1. In this case we observe that the second of those conditions is equivalent to requiring

$$\int_{\mathbb{R}} |f(x)|^{\gamma}\alpha(dx) < +\infty.$$

5.2.2. *Exact distribution of* $\tilde{\varphi}(f)$. When dealing with an f satisfying the above inequality, Proposition 2 leads to the following expression for the probability d.f. of $\tilde{\varphi}(f)$:

$$\begin{aligned}
&\frac{1}{2}[\mathbb{F}(\sigma)+\mathbb{F}(\sigma-0)] \\
&\quad = \frac{1}{2} - \frac{1}{\pi}\lim_{T\uparrow+\infty}\int_0^T \frac{1}{t}\exp\biggl\{-ct^{\gamma}\frac{\Gamma(1-\gamma)}{\gamma}\cos\frac{\pi\gamma}{2}\int_{\mathbb{R}}|f(x)-\sigma|^{\gamma}\alpha(dx)\biggr\} \\
&\qquad\qquad \times \sin\biggl(ct^{\gamma}\frac{\Gamma(1-\gamma)}{\gamma}\sin\frac{\pi\gamma}{2} \\
&\qquad\qquad\qquad \times \int_{\mathbb{R}}\operatorname{sgn}\bigl(f(x)-\sigma\bigr)|f(x)-\sigma|^{\gamma}\alpha(dx)\biggr)\,dt.
\end{aligned}$$

The integrand is absolutely integrable in $(M,+\infty)$ for any $M>0$ and, therefore, we have

$$\begin{aligned}
&\frac{1}{2}[\mathbb{F}(\sigma)+\mathbb{F}(\sigma-0)] \\
&\quad = \frac{1}{2} - \frac{1}{\pi\gamma}\arctan\biggl(\frac{\int_{\mathbb{R}}\operatorname{sgn}(f(x)-\sigma)|f(x)-\sigma|^{\gamma}\alpha(dx)}{\int_{\mathbb{R}}|f(x)-\sigma|^{\gamma}\alpha(dx)}\tan\frac{\pi\gamma}{2}\biggr).
\end{aligned} \tag{10}$$

For technical details, refer to the Appendix.

5.2.3. *Posterior distribution.* Finally we face the problem of finding an expression for the posterior distribution of $\tilde{\varphi}(f)$ given $x^{(1)}$ by resorting to Proposition 3. Let $y_1<y_2<\cdots<y_h$ be the h distinct values among $f(s_1),\dots,f(s_N)$ arranged in increasing order ($h=1,\dots,N$). Thus, if $\sigma\in(y_{i-1},y_i)$ is a point at which $\mathbb{F}$ is continuous, we get

$$\begin{aligned}
&\mathbb{F}\bigl(\sigma; f(s_1),\dots,f(s_N)\bigr) \\
&\quad = \frac{1}{2} - \frac{1}{\pi\gamma}\arctan\biggl(\frac{-\sum_{j=1}^{i-1}(\sigma-y_j)^{\gamma}\alpha_j+\sum_{j=i}^{h}(y_j-\sigma)^{\gamma}\alpha_j}{\sum_{j=1}^{i-1}(\sigma-y_j)^{\gamma}\alpha_j+\sum_{j=i}^{h}(y_j-\sigma)^{\gamma}\alpha_j}\tan\frac{\pi\gamma}{2}\biggr),
\end{aligned}$$

where $\alpha_j=\sum_{\{i\,:\,f(s_i)=y_j\}}\alpha(s_i)$, $j=1,\dots,h$. Suppose x_k is any point such that $f(x_k)=y_r$. Application of Proposition 3 yields the following expression, up to a constant, for the posterior density of a mean of the normalized γ-stable subordinator given $X_1=x_k$:

$$\frac{\alpha_r(\sigma-y_r)^{\gamma-1}\sum_{j=i}^{h}(y_j-\sigma)^{\gamma}\alpha_j}{(\sum_{j=1}^{h}|y_j-\sigma|^{\gamma}\alpha_j)^2+(\sum_{j=1}^{h}\operatorname{sgn}(y_j-\sigma)|y_j-\sigma|^{\gamma}\alpha_j)^2\tan^2(\pi\gamma/2)} \tag{11}$$

if $y_r \le y_{i-1} < \sigma$ and

$$\frac{\alpha_r(y_r-\sigma)^{\gamma-1}\sum_{j=1}^{i-1}(\sigma-y_j)^\gamma\alpha_j}{(\sum_{j=1}^h|y_j-\sigma|^\gamma\alpha_j)^2+(\sum_{j=1}^h \operatorname{sgn}(y_j-\sigma)|y_j-\sigma|^\gamma\alpha_j)^2\tan^2(\pi\gamma/2)} \tag{12}$$

if $y_r \ge y_i > \sigma$.

General results given in Section 4 can be used, by taking $f(x) \equiv \mathbb{1}_{\{s_j\}}(x)$ in (11) and (12), to provide an expression for the distribution of the random probability $\tilde{\varphi}(s_j)$, as well. If we observe $X_1 = x_k$, it is enough to consider two cases: $x_k \ne s_j$ and $x_k = s_j$. Hence, for any $\sigma \in (0,1)$, the posterior density function is, up to a constant, equal to

$$\frac{\sigma^{\gamma-1}(1-\sigma)^\gamma\alpha_1\alpha_2}{(\sigma^\gamma\alpha_1+(1-\sigma)^\gamma\alpha_2)^2+((1-\sigma)^\gamma\alpha_2-\sigma^\gamma\alpha_1)^2\tan^2(\pi\gamma/2)} \tag{13}$$

in the first case and equal to

$$\frac{\sigma^{\gamma}(1-\sigma)^{\gamma-1}\alpha_1\alpha_2}{(\sigma^\gamma\alpha_1+(1-\sigma)^\gamma\alpha_2)^2+((1-\sigma)^\gamma\alpha_2-\sigma^\gamma\alpha_1)^2\tan^2(\pi\gamma/2)} \tag{14}$$

in the second case. It is worth noting an interesting, although not surprising, feature of (13) and (14): If $x_k = s_j$, then, from (14), we observe that the posterior density function of $\tilde{\varphi}(s_j)$, given $X_1 = x_k$, tends to $+\infty$ (0, respectively) as $\sigma \to 1$ ($\sigma \to 0$, respectively). On the other hand, if $x_k \ne s_j$, then, from (13), it is possible to conclude that the posterior density function of $\tilde{\varphi}(s_j)$, given $X_1 = x_k$, tends to $+\infty$ (0, respectively) as $\sigma \to 0$ ($\sigma \to 1$, respectively).

APPENDIX

PROOF OF PROPOSITION 1. To prove this proposition, define

$$g_{|f|}(\lambda) := E\big[\exp(-\lambda\tilde{\xi}|f|)\big], \qquad \lambda \ge 0,$$

and

$$\gamma_{|f|}(t) := E\big[\exp(it\tilde{\xi}|f|)\big], \qquad t \in \mathbb{R},$$

whenever (i) is valid. Moreover, let $(f_n)_{n\ge1}$ be a sequence of measurable simple functions such that $0 \le |f_1| \le \cdots \le |f|$, $f_n \to f$ pointwise and $f_n \to f$ uniformly on any set on which f is bounded. By virtue of fact 4 in Section 2 and (3), we obtain the equalities

$$g_{|f_n|}(\lambda) = \exp\bigg[\int_{\mathbb{R}\times(0,+\infty)}\big[\exp\big(-\lambda v|f_n(x)|\big)-1\big]\tilde{\nu}_\alpha(dx\,dv)\bigg], \qquad \lambda \ge 0, \tag{A.1}$$

$$\gamma_{|f_n|}(t) = \exp\bigg[\int_{\mathbb{R}\times(0,+\infty)}\big[\exp\big(itv|f_n(x)|\big)-1\big]\tilde{\nu}_\alpha(dx\,dv)\bigg], \qquad t \in \mathbb{R}. \tag{A.2}$$

Finally, if (iii) is in force, by the Fubini theorem,

$$\begin{aligned}&\int_{\mathbb{R}\times(0,+\infty)}[1-\cos(yt|f(x)|)]\tilde{\nu}_\alpha(dx\,dy)\\ &\qquad=\int_{(0,+\infty)}\sin u\int_{A_{u/|t|}}\tilde{\nu}_\alpha(dx\,dy)\,du,\qquad t\neq 0,\end{aligned}\tag{A.3}$$

with $A_{u/|t|}=\{(x,y)\in\mathbb{R}\times(0,+\infty): y|tf(x)|>u\}$. Note that the (generalized) monotone convergence theorem, together with (A.1), entails $g_{|f_n|}(\lambda)\to\exp[\int_{\mathbb{R}\times(0,+\infty)}(e^{-\lambda v|f(x)|}-1)\tilde{\nu}_\alpha(dx\,dv)]$ as $n\to+\infty$, for any $\lambda\geq 0$. Moreover, pointwise convergence of $(f_n)_{n\geq 1}$ to f implies that $\tilde{\xi}(f_n)\xrightarrow{d}\tilde{\xi}(f)$ in $\bar{\mathbb{R}}$ as $n\to+\infty$, so that $g_{|f_n|}(\lambda)\to g_{|f|}(\lambda)$ and, under (i), $\gamma_{|f_n|}(t)\to\gamma_{|f|}(t)$ pointwise. Hence,

$$g_{|f|}(\lambda)=\exp\left[\int_{\mathbb{R}\times(0,+\infty)}(e^{-\lambda v|f(x)|}-1)\ \tilde{\nu}_\alpha(dx\,dv)\right],\qquad \lambda\geq 0,$$

and, since $P(\tilde{\xi}|f|<+\infty)=\lim_{\lambda\downarrow 0}g_{|f|}(\lambda)$, we obtain the equivalence between (i) and (ii).

Next, given $t\neq 0$ and $\varepsilon\in(0,\frac{\pi}{2})$, combine the monotone convergence theorem with the properties of $(f_n)_{n\geq 1}$ to prove that

$$\int_{A^c_{\varepsilon/|t|}}[1-\cos(yt|f_n(x)|)]\tilde{\nu}_\alpha(dx\,dy)\to\int_{A^c_{\varepsilon/|t|}}[1-\cos(yt|f(x)|)]\tilde{\nu}_\alpha(dx\,dy)$$

and

$$\int_{A^c_{\varepsilon/|t|}}\sin(yt|f_n(x)|)\tilde{\nu}_\alpha(dx\,dy)\to\int_{A^c_{\varepsilon/|t|}}\sin(yt|f(x)|)\tilde{\nu}_\alpha(dx\,dy)$$

as $n\to\infty$. Moreover, if (i) and (ii) hold, then we have

$$(1-e^{-\varepsilon})\int_{A_{\varepsilon/|t|}}\tilde{\nu}_\alpha(dx\,dy)\leq\int_{A_{\varepsilon/|t|}}[1-e^{-|tyf(x)|}]\tilde{\nu}_\alpha(dx\,dy)<+\infty,\tag{A.4}$$

which allows us to apply the dominated convergence theorem to obtain

$$\int_{A_{\varepsilon/|t|}}[1-\cos(yt|f_n(x)|)]\tilde{\nu}_\alpha(dx\,dy)\to\int_{A_{\varepsilon/|t|}}[1-\cos(yt|f(x)|)]\tilde{\nu}_\alpha(dx\,dy)$$

and

$$\int_{A_{\varepsilon/|t|}}\sin(yt|f_n(x)|)\tilde{\nu}_\alpha(dx\,dy)\to\int_{A_{\varepsilon/|t|}}\sin(yt|f(x)|)\tilde{\nu}_\alpha(dx\,dy)$$

as $n \to +\infty$. Thus, under (i) and (ii),

$$\gamma_{|f|}(t) = \exp\biggl[\int_{\mathbb{R}\times(0,+\infty)} \bigl[\cos\bigl(tv|f(x)|\bigr) - 1 + i\sin\bigl(tv|f(x)|\bigr)\bigr]\tilde{\nu}_\alpha(dx\,dv)\biggr]$$

and this is enough to assert that (i) entails (iii).

Finally, assume (iii). Then $\int_{A_{u/|t|}} d\tilde{\nu}_\alpha < +\infty$ holds true for every $u > 0$. Moreover, letting $A^{(n)}_{u/|t|} = \{(x, y) : y|tf_n(x)| > u\}$, we get $A^{(n)}_{u/|t|} \uparrow A_{u/|t|}$ and

$$\begin{aligned}|\gamma_{|f_n|}(t)| &= \exp\biggl[-\int_{(0,+\infty)} \sin u \int_{A^{(n)}_{u/|t|}} \tilde{\nu}_\alpha(dx\,dy)\,du\biggr]\\ &\to \exp\biggl[-\int_{(0,+\infty)} \sin u \int_{A_{u/|t|}} \tilde{\nu}_\alpha(dx\,dy)\,du\biggr]\end{aligned}$$

by the dominated convergence theorem. Analogously, under (iii) and with ε in $(0, \pi/2)$,

$$\int_{A^c_{\varepsilon/|t|}} \sin\bigl(tv|f_n(x)|\bigr)\tilde{\nu}_\alpha(dx\,dv) \to \int_{A^c_{\varepsilon/|t|}} \sin\bigl(tv|f(x)|\bigr)\tilde{\nu}_\alpha(dx\,dv)$$

(by monotone convergence),

$$\int_{A_{\varepsilon/|t|}} \sin\bigl(tv|f_n(x)|\bigr)\tilde{\nu}_\alpha(dx\,dv) \to \int_{A_{\varepsilon/|t|}} \sin\bigl(tv|f(x)|\bigr)\tilde{\nu}_\alpha(dx\,dv)$$

(by dominated convergence)

and both limits are finite. Thus, we can state that

$$\gamma_{|f_n|}(t) \to \exp\biggl[\int_{\mathbb{R}\times(0,+\infty)} \bigl[\cos\bigl(tv|f(x)|\bigr) - 1 + i\sin\bigl(tv|f(x)|\bigr)\bigr]\tilde{\nu}_\alpha(dx\,dv)\biggr]$$

as $n \to +\infty$ for every $t \in \mathbb{R}$, where the limiting function is continuous at $t = 0$. Then (i) holds true by the continuity theorem for characteristic functions. □

PROOF OF PROPOSITION 2. In view of (4) it is enough to prove that the integrand is absolutely integrable on $(0, \varepsilon)$ for any $\varepsilon > 0$. Fix a constant $a > 0$ and observe that the absolute integral on $(0, \varepsilon)$ is bounded from above by

$$\begin{aligned}&\int_0^\varepsilon \frac{1}{t}\int_{v|x-\sigma|\le a} \left|\frac{\sin(tv(x-\sigma))}{tv(x-\sigma)}\right| tv|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\,dt\\ &\quad + \int_0^\varepsilon \frac{1}{t}\int_{v|x-\sigma|>a} \bigl|\sin\bigl(tv(x-\sigma)\bigr)\bigr|\tilde{\nu}_\alpha(dx\,dv)\,dt\\ &\quad =: I_1 + I_2.\end{aligned}$$

Moreover,

$$\begin{aligned}
I_1 &\le \varepsilon \int_{v|x-\sigma|\le a} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&= \varepsilon \int_{\{v|x-\sigma|\le a\}\cap\{0<v\le 1\}} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&\quad + \varepsilon \int_{\{v|x-\sigma|\le a\}\cap\{v>1\}} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&\le \varepsilon \int_{0<v|x-\sigma|\le a\wedge|x-\sigma|} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&\quad + \varepsilon \int_{\{|x-\sigma|<a\}\cap\{|x-\sigma|<v|x-\sigma|<a\}} v\,|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&= \varepsilon \int_{\{|x-\sigma|\le a\}\cap\{0<v\le 1\}} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&\quad + \varepsilon \int_{\{|x-\sigma|>a\}\cap\{0<v|x-\sigma|\le a\}} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&\quad + \varepsilon \int_{\{|x-\sigma|<a\}\cap\{|x-\sigma|<v|x-\sigma|<a\}} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv).
\end{aligned}$$

Indicate by I_{11} and I_{12} the second and third integrals, respectively. First, we immediately note that $\varepsilon\int_{\{|x-\sigma|\le a\}\cap\{0<v\le 1\}} v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv) \le a \times \int_{\{|x-\sigma|\le a\}\cap\{v\le 1\}} v\tilde{\nu}_\alpha(dx\,dv) < +\infty$. Next, (ii) of Proposition 1 guarantees that $\int_{\{|x-\sigma|>a\}\cap\{v|x-\sigma|\le a\}}[1-e^{-\lambda|x-\sigma|v}]\tilde{\nu}_\alpha(dx\,dv)$ is finite and, for some $\delta>0$, independent of (x,v), we have

$$\begin{aligned}
&\int_{\{|x-\sigma|>a\}\cap\{v|x-\sigma|\le a\}} \bigl[1-e^{-\lambda|x-\sigma|}\bigr]\tilde{\nu}_\alpha(dx\,dv)\\
&\qquad \ge \int_{\{|x-\sigma|>a\}\cap\{v|x-\sigma|\le a\}} \delta\lambda v|x-\sigma|\tilde{\nu}_\alpha(dx\,dv)\\
&\qquad = \delta\lambda I_{11}.
\end{aligned}$$

An analogous argument shows that $I_{12} < +\infty$.

Next, rewrite I_2 as

$$\begin{aligned}
I_2 &= \int_{|x-\sigma|>a}\left(\int_0^{\varepsilon} t^{-1}\bigl|\sin(tv|x-\sigma|)\bigr|\,dt\right)\tilde{\nu}_\alpha(dx\,dv)\\
&= \int_0^{+\infty} |\sin z|z^{-1}\int_{v|x-\sigma|>(a\vee z/\varepsilon)} \tilde{\nu}_\alpha(dx\,dv)\,dz
\end{aligned}$$

$$= \int_0^{a\varepsilon} |\sin z|\, z^{-1} \int_{v|x-\sigma|>a} \tilde{\nu}_\alpha(dx\, dv)\, dz$$
$$+ \int_{a\varepsilon}^{+\infty} |\sin z|\, z^{-1} \int_{v|x-\sigma|>z/\varepsilon} \tilde{\nu}_\alpha(dx\, dv)\, dz.$$

This yields

$$I_2 \le a\varepsilon \int_{v|x-\sigma|>a} \tilde{\nu}_\alpha(dx\, dv) + (a\varepsilon)^{-1} \int_0^{+\infty} |\sin z| \int_{v|x-\sigma|>z/\varepsilon} \tilde{\nu}_\alpha(dx\, dv)\, dz,$$

the right-hand side of which is finite because of (ii) in Proposition 1, (A.3) and (A.4). □

PROOF OF PROPOSITION 4. Without loss of generality, assume that $f(x) \equiv x$. Denote the value of $P(\cdot|\zeta_m^{(n)})$ on the set $\{X^{(n)} = x^{(n)}\}$ by $P(\cdot|\zeta_m^{(n)})(x^{(n)})$. For notational convenience, set $\tilde{\phi} = \int x \tilde{\varphi}(dx)$ and abbreviate $P(\tilde{\phi} \le \sigma|\zeta_m^{(n)})(x^{(n)})$ to $\mathbb{F}_{m,x^{(n)}}(\sigma)$. Since (i) in Proposition 1 holds true, then, for some $\mu^{(n)}$-null set $\mathcal{N}_1 \in \mathscr{B}(\mathbb{R}^n)$, $P(\int_{\mathbb{R}} |x|\tilde{\varphi}(dx) < +\infty|\zeta_m^{(n)})(x^{(n)}) = 1$ for each $x^{(n)} \in \mathcal{N}_1^c$ and it is possible to choose $(R_m)_{m\ge 1}$ in such a way that

$$P\left(\int_{[-R_m, R_m]^c} |x|\tilde{\varphi}(dx) > \frac{\varepsilon_m}{4}\middle|\zeta_m^{(n)}\right)(x^{(n)}) < \varepsilon_m, \qquad \forall\, x^{(n)} \in \mathcal{N}_1^c. \tag{A.5}$$

We now show that $\mathbb{F}_{m,x^{(n)}}$ may be used to approximate $\mathbb{F}_{x^{(n)}}$.

PROPOSITION A1. *There exists a set $\mathcal{N}_2 \in \mathscr{B}(\mathbb{R}^n)$ such that $\mu^{(n)}(\mathcal{N}_2) = 0$ and, for every $x^{(n)} \in \mathcal{N}_2^c$,*

$$\lim_{m\to+\infty} \mathbb{F}_{m,x^{(n)}}(\sigma) = \mathbb{F}_{x^{(n)}}(\sigma) \tag{A.6}$$

holds true for each $\sigma \in \mathcal{C}(\mathbb{F}_{x^{(n)}})$.

Indeed, by virtue of a martingale convergence theorem [cf. Theorem 35.6 in Billingsley (1995)], for every $\sigma \in \mathbb{R}$ there is a $\mu^{(n)}$-null set $\mathcal{N}_\sigma \in \mathscr{B}(\mathbb{R}^n)$ such that $\mathbb{F}_{m,x^{(n)}}(\sigma) \to \mathbb{F}_{x^{(n)}}(\sigma)$ for any $x^{(n)} \in \mathcal{N}_\sigma^c$. Let $\mathcal{N}_2 = \bigcup_{\sigma\in\mathbb{Q}} \mathcal{N}_\sigma$. If $x^{(n)} \in \mathcal{N}_2^c$, then

$$\lim_{m\to+\infty} \mathbb{F}_{m,x^{(n)}}(\sigma) = \mathbb{F}_{x^{(n)}}(\sigma), \qquad \forall\, \sigma \in \mathbb{Q} \cap \mathcal{C}(\mathbb{F}_{x^{(n)}}). \tag{A.7}$$

Taking $\mathcal{S} := \{(\sigma_1, \sigma_2] : \sigma_1 < \sigma_2, \sigma_1, \sigma_2 \in \mathbb{Q}\}$, (A.7) entails $P(\tilde{\phi} \in A|\zeta_m^{(n)})(x^{(n)}) \to \int_A d\mathbb{F}_{x^{(n)}}(\sigma)$ as $m \to +\infty$ for all $x^{(n)} \in \mathcal{N}_2^c$ and for every $A \in \mathcal{S}$. Hence, (A.6) follows by resorting to Theorem 2.3 in Billingsley (1999).

To complete our proof, we need an intermediate approximation result.

PROPOSITION A2. *For any $x^{(n)} \in \mathcal{N}_1^c$, we have*

$$d_L(\mathbb{F}_{m,x^{(n)}}, \mathbb{F}^*_{m,x^{(n)}}) < \varepsilon_m, \qquad m = 1, 2, \dots, \tag{A.8}$$

where d_L is the Lévy metric for probability d.f.'s.

In fact, inequalities

$$\begin{aligned}|\tilde{\varphi}_{1,m} - \tilde{\phi}| &\le \sum_{i=1}^{k_m} \int_{A_{m,i}} |a_{m,i} - x| \tilde{\varphi}(dx) + 2\int_{[-R_m, R_m]^c} |x| \tilde{\varphi}(dx) \\ &\le \frac{\varepsilon_m}{2} + 2\int_{[-R_m, R_m]^c} |x| \tilde{\varphi}(dx)\end{aligned}$$

combined with (A.5) give

$$P\big(|\tilde{\varphi}_{1,m} - \tilde{\phi}| > \varepsilon_m \big| \zeta_m^{(n)}\big)(x^{(n)}) \le P\bigg(\int_{A_{m,k_m+1}} |x| \tilde{\varphi}(dx) > \varepsilon_m/4 \bigg| \zeta_m^{(n)}\bigg)(x^{(n)}) < \varepsilon_m$$

for every $x^{(n)} \in \mathcal{N}_1^c$. These inequalities, in turn, entail

$$\begin{aligned}\mathbb{F}_{m,x^{(n)}}(\sigma) &= P\big(\tilde{\phi} \le \sigma, |\tilde{\varphi}_{1,m} - \tilde{\phi}| \le \varepsilon_m \big| \zeta_m^{(n)}\big)(x^{(n)}) \\ &\qquad + P\big(\tilde{\phi} \le \sigma, |\tilde{\varphi}_{1,m} - \tilde{\phi}| > \varepsilon_m \big| \zeta_m^{(n)}\big)(x^{(n)}) \\ &\le \mathbb{F}^*_{m,x^{(n)}}(\sigma + \varepsilon_m) + \varepsilon_m,\end{aligned}$$

and, analogously, $\mathbb{F}^*_{m,x^{(n)}}(\sigma) \le \mathbb{F}_{m,x^{(n)}}(\sigma + \varepsilon_m) + \varepsilon_m$. The proof of (A.8) is now complete, and Proposition 4 follows from combination of (A.6) and (A.8). $\square$

DETAILS FOR THE DETERMINATION OF (5) AND (6). Straightforward application of (4) yields

$$\begin{aligned}&\frac{1}{2}[\mathbb{F}(\sigma) + \mathbb{F}(\sigma - 0)] \\ &\quad = \frac{1}{2} - \frac{1}{\pi} \lim_{T \uparrow +\infty} \int_0^T \frac{1}{t} \exp\bigg\{\int_{\mathbb{R} \times (0,+\infty)} \big[\cos\big(tv(x - \sigma)\big) - 1\big] \\ &\qquad\qquad \times \frac{(1 - e^{-\gamma v})}{(1 - e^{-v})} \frac{e^{-v}}{v} \alpha(dx)\, dv\bigg\} \\ &\qquad\qquad \times \sin\bigg(\int_{\mathbb{R} \times (0,+\infty)} \sin\big(tv(x - \sigma)\big) \\ &\qquad\qquad\qquad \times \frac{(1 - e^{-\gamma v})}{(1 - e^{-v})} \frac{e^{-v}}{v} \alpha(dx)\, dv\bigg)\, dt.\end{aligned}$$

To simplify the previous expression, consider

$$\int_{\mathbb{R}\times(0,+\infty)} \bigl[\cos\bigl(tv(x-\sigma)\bigr)-1\bigr]\alpha(dx)\frac{(1-e^{-\gamma v})}{(1-e^{-v})}\frac{e^{-v}}{v}\,dv$$
$$= \int_{\mathbb{R}} \alpha(dx)\int_0^{+\infty}\left\{\frac{1-\cos(tv(x-\sigma))}{v(e^v-1)} - \frac{e^{-(\gamma+1)v}(1-\cos(tv(x-\sigma)))}{v(1-e^{-v})}\right\}dv.$$

From 3.951.21 in Gradshteyn and Ryzhik (2000),

$$\int_0^{+\infty}\frac{1-\cos(tv(x-\sigma))}{v(e^v-1)}\,dv = 2\int_0^{+\infty}\frac{\sin^2(tv(x-\sigma)/2)}{v(e^v-1)}\,dv = \frac{1}{2}\log\frac{\sinh(\pi t(x-\sigma))}{\pi t(x-\sigma)}.$$

On the other hand,

$$\int_0^{+\infty}\frac{e^{-(\gamma+1)v}(1-\cos(tv(x-\sigma)))}{v(1-e^{-v})}\,dv = \frac{1}{2}\log\frac{\Gamma(1+\gamma-it(x-\sigma))\Gamma(1+\gamma+it(x-\sigma))}{(\Gamma(1+\gamma))^2}$$

from 2.2.4.26 in Prudnikov, Brychkov and Marichev (1992). Moreover, observe that

$$\int_{\mathbb{R}\times(0,+\infty)} \sin\bigl(tv(x-\sigma)\bigr)\frac{(1-e^{-\gamma v})}{(1-e^{-v})}\frac{e^{-v}}{v}\alpha(dx)\,dv$$
$$= \frac{1}{2i}\int_{\mathbb{R}}\alpha(dx)\int_0^{+\infty}\frac{(1-e^{-2it(x-\sigma)v})(1-e^{-\gamma v})e^{-(1-it(x-\sigma))v}}{v(1-e^{-v})}\,dv$$
$$= \frac{1}{2i}\int_{\mathbb{R}}\log\frac{B(\gamma;1-it(x-\sigma))}{B(\gamma;1+it(x-\sigma))}\alpha(dx),$$

where the last equality follows from 3.413.1 in Gradshteyn and Ryzhik (2000). In light of these considerations, we get (5).

Further suppose that $\gamma\in\mathbb{N}$. Using well-known properties of the gamma function and 8.332.3 in Gradshteyn and Ryzhik (2000),

$$\log\Gamma\bigl(1+\gamma-it(x-\sigma)\bigr)\Gamma\bigl(1+\gamma+it(x-\sigma)\bigr) = \log\frac{\pi t(x-\sigma)}{\sinh(\pi t(x-\sigma))}+\sum_{k=1}^{\gamma}\log\bigl(k^2+t^2(x-\sigma)^2\bigr).$$

By virtue of 1.622.3 in Gradshteyn and Ryzhik (2000) we also have

$$\frac{1}{2i}\log\frac{B(\gamma;1-it(x-\sigma))}{B(\gamma;1+it(x-\sigma))}=\sum_{k=1}^{\gamma}\arctan\frac{t(x-\sigma)}{k}$$

and (6) clearly holds.

DETAILS FOR THE DETERMINATION OF (10). Application of Proposition 2 in this case leads to

$$\begin{aligned}
&\frac{1}{2}[\mathbb{F}(\sigma)+\mathbb{F}(\sigma-0)]\\
&\quad=\frac{1}{2}-\frac{1}{\pi}\lim_{T\uparrow+\infty}\int_0^T\frac{1}{t}\exp\biggl\{-c\int_{\mathbb{R}\times(0,+\infty)}\bigl(1-\cos\bigl(tv(x-\sigma)\bigr)\bigr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times v^{-\gamma-1}\alpha(dx)\,dv\biggr\}\\
&\qquad\qquad\times\sin\biggl(c\int_{\mathbb{R}\times(0,+\infty)}\sin\bigl(tv(x-\sigma)\bigr)v^{-\gamma-1}\alpha(dx)\,dv\biggr)\,dt.
\end{aligned}$$

To obtain an explicit form for the distribution, take the integral that appears in the exponential and use Lemma 14.11 in Sato (1999) to prove that, for any $t>0$,

$$\begin{aligned}
&\int_0^{+\infty}\bigl(1-\cos\bigl(tv(x-\sigma)\bigr)\bigr)v^{-\gamma-1}\,dv\\
&\quad=\frac{1}{2}\biggl\{-|t|^{\gamma}|x-\sigma|^{\gamma}\Gamma(-\gamma)\biggl(\exp\biggl(i\frac{\pi\gamma}{2}|x-\sigma|\,\mathrm{sgn}(t)\biggr)\\
&\qquad\qquad+\exp\biggl(-i\frac{\pi\gamma}{2}|x-\sigma|\,\mathrm{sgn}(t)\biggr)\biggr)\biggr\}\\
&\quad=t^{\gamma}|x-\sigma|^{\gamma}\frac{\Gamma(1-\gamma)}{\gamma}\cos\frac{\pi\gamma}{2},
\end{aligned}$$

where $\Gamma(-\gamma)=-\Gamma(1-\gamma)/\gamma$. Moreover, using 3.761.4 in Gradshteyn and Ryzhik (2000), we find that

$$\int_0^{+\infty}\sin\bigl(tv(x-\sigma)\bigr)v^{-\gamma-1}\,dv=\mathrm{sgn}(x-\sigma)\,t^{\gamma}|x-\sigma|^{\gamma}\,\frac{\Gamma(1-\gamma)}{\gamma}\sin\frac{\pi\gamma}{2}.$$

These computations lead to the expression for the distribution function $\mathbb{F}$ provided in (10).

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E. REGAZZINI
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI PAVIA
VIA FERRATA, 1
27100 PAVIA
ITALY
E-MAIL: eugenio@dimat.unipv.it

A. LIJOI
I. PRÜNSTER
DIPARTIMENTO DI ECONOMIA POLITICA
E METODI QUANTITATIVI
UNIVERSITÀ DEGLI STUDI DI PAVIA
VIA SAN FELICE, 5
27100 PAVIA
ITALY
E-MAIL: lijoi@unipv.it
igor.pruenster@unipv.it