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RANDOM PROBABILITY MEASURES WITH FIXED MEAN DISTRIBUTIONS

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Linear functionals, or means, of discrete random probability measures are a natural probabilistic object and the investigation of their properties have a long and rich history. They appear in several areas of mathematics, including statistics, combinatorics, special functions, excursions of stochastic processes and financial mathematics, among others. Most contributions have aimed at determining their distribution starting from a fully specified random probability. This work addresses the inverse problem: the identification of the base measure of a discrete random probability measure yielding a specific mean distribution. Available results concern only the Dirichlet case for specific choices of the concentration parameter. Here we address the problem in much greater generality and our results cover generic Dirichlet processes, the normalized stable process and the Pitman–Yor process. In addition to their theoretical interest, the results are of practical relevance to Bayesian nonparametric inference, where the law of a random probability measure acts as a prior distribution: often pre-experimental information is available about a finite-dimensional projection of the data generating distribution, such as the mean, rather than about an infinite-dimensional parameter. We further extend our findings to mixture models, ubiquitous in Statistics and Machine Learning.

1. Introduction. We consider almost surely discrete random probability measures

$$\tilde P = \sum_{i\in\mathcal{I}} \omega_i\, \delta_{Z_i}, \tag{1}$$

on some space $\mathbb{X}$, where $\mathcal{I}$ is countable, the sequences $(\omega_i)_{i\in\mathcal{I}}$ and $(Z_i)_{i\in\mathcal{I}}$ are independent and $\sum_{i\in\mathcal{I}}\omega_i = 1$, almost surely. Moreover, the Z_i's are independent and identically distributed from some probability measure P_0 on $\mathbb{X}$, termed *parameter measure* or *base measure*. Clearly, $\mathbb{E}[\,\tilde P\,] = P_0$. When P_0 is non-atomic, $\tilde P$ is a *species sampling model*, a notion introduced in Pitman (1996). Various features of (1) have been thoroughly studied in probabilistic contexts, because of their intrinsic connection with the theory of random partitions and, in general, combinatorial stochastic processes. Among countless contributions we refer to the seminal ones of Kingman (1975, 1978, 1982), Kallenberg (1975, 2017), Pitman (1995, 2006) and Bertoin (2006). By virtue of de Finetti's representation theorem (de Finetti, 1937), the law of a random probability measure may also be the directing measure, according to the terminology of Aldous (1985), of exchangeable sequences of random elements. This provides a neat foundation of the Bayesian approach (Diaconis and Skyrms, 2018) and random

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probability measures represent key building blocks of Bayesian nonparametric procedures. See Ghosal and van der Vaart (2017) and Müller et al. (2015).

The focus of the paper is on linear functionals of discrete random probabilities (1), given by

$$\int_{\mathbb{X}} h(x)\tilde{P}(\mathrm{d}x) = \sum_{i\in\mathcal{I}} \omega_i\, h(Z_i), \tag{2}$$

with $h : \mathbb{X} \to \mathbb{R}$ some measurable function such that $\int |h|\mathrm{d}\tilde{P} < +\infty$ almost surely. These can be seen as means of $\tilde{P}$ or $\tilde{P}$-means as referred to by Pitman (2018). The study of distributional properties of (2) has a long and rich history dating back to von Neumann (1941) and Watson (1956), who characterized its distribution in the case of $\tilde{P}$ being a Dirichlet distribution with finite number of support points. In the infinite setup the problem has been pioneered in Cifarelli and Regazzini (1979, 1990) for $\tilde{P}$ being a Dirichlet process. Diaconis and Kemperman (1996) highlight several mathematical and statistical setups related to (2). An important such instance is represented by the Markov and Hausdorff moment problems, whose investigation is framed in terms of (2) in Kerov (1993, 1998), where a link with transition measures induced by continual Young diagrams is also established. Further developments along this line are given in Tsilevich (1999), Kerov and Tsilevich (2004) and Vershik, Yor and Tsilevich (2004). Another research topic, where (2) plays a prominent role, is represented by Lévy's arcsine laws (Lévy, 1939) and, in particular, its generalizations due to Lamperti (1958) and Barlow, Pitman and Yor (1989) as well as the study of excursions of Bessel processes, which are directly related to means of Pitman-Yor processes (Perman, Pitman and Yor, 1992; Pitman and Yor, 1997a). See also James (2010); James, Lijoi and Prünster (2008). Moreover, (2) appear also in the statistical physics literature, in relation to zero-range process models (Pulkkinen, 2007). Connections with the theory of multivariate hypergeometric functions have been studied in Lijoi and Regazzini (2004); Chamayou and Wesołowski (2009), whereas further analytic properties have been derived, among others, in Peccati (2004, 2008), El-Dakkak and Peccati (2008), Dello Schiavo (2019), Flint and Torrisi (2021). Extensive reviews on random means can be found in Lijoi and Prünster (2009) and Pitman (2018), whereas a historical perspective on the subject in Bayesian statistics can be found in Lijoi and Prünster (2011).

The standard approach undertaken in the existing literature, which we have broadly discussed above, first completely specifies $\tilde{P}$ in (1) and then studies the properties of the corresponding linear functional (2). For instance, the distribution of (2) has been derived for $\tilde{P}$ being a Dirichlet process (Cifarelli and Regazzini, 1990), a Pitman–Yor process (James, Lijoi and Prünster, 2008) or a normalized random measure with independent increments (Regazzini, Lijoi and Prünster, 2003).

The present work pursues a different, and in a sense opposite, task as it provides an answer to the following question: which $\tilde{P}$, within a specific class of discrete random probability measures, yields a specific distribution for the mean $\int h\,\mathrm{d}\tilde{P}$ in (2)? This amounts to identifying the parameter measure $\mathbb{E}[\tilde{P}] = P_0$, if there exists any, inducing the pre-specified law of the $\tilde{P}$-mean. This natural research question is still an open problem, since available results are limited to Dirichlet random means and, moreover, impose constraints on its concentration parameter. In particular, Romik (2004) investigates transition measures induced by a hook walk on continual Young diagrams and, by leveraging the relationship with Dirichlet means established in Kerov (1993), successfully tackles the case of a Dirichlet process mean with unit parameter. Furthermore, James, Roynette and Yor (2008) highlight the intriguing connection between Dirichlet random means and generalized gamma convolutions (see Bondesson, 1992). By relying on the Thorin measures associated with gamma integrals, they deduce a result for the Dirichlet case assuming the concentration parameter is less than 1. Here we address the problem without resorting to the above mentioned connections to combinatorics and generalized gamma convolutions, since these would prevent us to obtain results beyond the Dirichlet case, which represents our main goal. Nonetheless the relationship to continual Young diagrams established in Romik (2004) has been inspiring in the interpretation of our results and will be further developed in Gaffi, Lijoi and Prünster (2023).

A different approach to achieve a prescribed mean distribution is present in an elegant and stimulating paper by Hill and Monticino (1998), where the underlying random probability measure is characterized via random sequential barycenter arrays. An important point worth clarifying is that in Hill and Monticino (1998) the unknown is the random probability measure itself, whereas in our setup described above the unknown is the deterministic base measure of a given class of random probability measures. While this makes the two approaches not directly comparable, an advantage of our approach is that, in addition to investigating widely used random probability measures, we are able to explicitly determine the unknown base measure, whereas in their setup the unknown random probability measure is only implicitly characterized and to date neither results identifying their law nor concrete examples are available.

These limitations have motivated this endeavour to establish both general and explicit results. In particular, we obtain explicit expressions for the base measure P_0 inducing a broad class of distributions on the mean, when $\tilde{P}$ is either a Dirichlet process, a normalized stable process or a Pitman–Yor process. Interestingly, the techniques we introduce can be easily extended, on a case-by-case basis, to situations that are ruled out by the assumptions of our general results. We use them to discuss some noteworthy examples for which we are still able to identify the base measure: this is helpful to gain insight about the admissible sets of random mean distributions. From a technical perspective, taking the described inverse path poses several challenges. Even if we rely on integral identities, known as Markov–Krein correspondences or Cifarelli–Regazzini identities, as well as on generalized Cauchy–Stieltjes transform inversion formulas, these classical tools cannot be directly applied to our case. New proof strategies are required. Moreover, we need to assess existence and regularity for singular integrals, in order to identify hypotheses that allow a broad class of mean densities to be included. This allows to determine closed form expressions for the parameter measure of Dirichlet, normalized stable and Pitman–Yor processes inducing a broad class of mean distributions. Interestingly, our study unravels some relevant features of the underlying discrete random probability measures and of the corresponding space of mean distributions. For example, we show a surprising (at least to us) fact according to which not every absolutely continuous law with compact support is the mean distribution of a certain discrete random probability measure with given parameters.

Our findings are also of practical relevance for Bayesian nonparametric modeling. Indeed, as shown *e.g.* in Kessler, Hoff and Dunson (2015), in many applications one might have enough *a priori* information for eliciting the distribution of an interpretable (and finite-dimensional) parameter of a nonparametric prior, foremost its mean. Once the mean is specified, our results allow to specify the infinite-dimensional $\tilde{P}$ accordingly. Moreover, the results can be readily extended to cover mixture models, such as the Dirichlet process mixture model, which are hugely popular in Statistics and Machine Learning (Orbanz and Teh, 2010; Müller et al., 2015). These consist in random densities, $\tilde{f}(y) = \int k(y;x)\,\tilde{P}(\mathrm{d}x)$ with $\tilde{P}$ a discrete random probability as in (1) and $k(\,\cdot\,;\,\cdot\,)$ some transition kernel density such as, e.g. a Gaussian density with parameter $x = (\mu, \sigma^2)$. Then, the corresponding mean

$$\int_{\mathbb{R}} h(y)\tilde{f}(y)\,\mathrm{d}y = \int \bar{h}(x)\,\tilde{P}(\mathrm{d}x),$$

where $x \mapsto \bar{h}(x) = \int_{\mathbb{R}} h(y)\, k(y;x)\, \mathrm{d}y$, is still a linear functional of $\tilde{P}$. Thus, if pre-experimental information allows for the elicitation of the law of $\int \bar{h}\, \mathrm{d}\tilde{P}$ and $\tilde{P}$ is identified up to its parameter measure P_0, the latter can be specified so to enforce such prior knowledge on the mean. Regardless of the applied motivation, these results on mixtures have the merit of showcasing that our techniques encompass means of random probability measures that can be both discrete or absolutely continuous.

The structure of the paper is as follows. In Section 2 we recall some general concepts and fundamental results on random means and illustrate tools that will be crucial for achieving our goals. In Section 3 we provide our main results obtaining explicit expressions for the base measure P_0 inducing a broad class of distributions on the mean. In Section 4 we extend our results to cover nonparametric mixtures.

2. Transformations of completely random measures and random means. An effective strategy for defining (discrete) random probability measures is through transformations of completely random measures. Such constructions, combined together with powerful analytical tools that we are going to present, have been fundamental for the study of random means. Here we provide a brief overview of these aspects, as they play a key role also for the derivation of the main results in the manuscript.

2.1. *Completely random measures.* Let $\mathbb{X}$ be a complete and separable metric space equipped with the Borel σ-algebra $\mathscr{X}$ and $\mathscr{M}$ be the set of boundedly finite measures on $\mathbb{X}$ endowed with the corresponding Borel σ-algebra $\sigma(\mathscr{M})$. A completely random measure (CRM) $\tilde{\mu}$ on $(\mathbb{X}, \mathscr{X})$ is a measurable function on $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in $\mathscr{M}$ such that for finite collection of disjoint sets $A_1, \dots, A_n$ in $\mathscr{X}$, the random variables $\tilde{\mu}(A_1), \dots, \tilde{\mu}(A_n)$ are independent. See Kingman (1967, 1993) for a detailed treatment. In the following we will focus on CRMs without drift and fixed points of discontinuity. It is important to recall that such CRMs are almost surely discrete and that their Laplace functional admits *Lévy–Khintchine representation*

$$\mathbb{E}\left[\mathrm{e}^{-\int_{\mathbb{X}} f(x)\,\tilde{\mu}(\mathrm{d}x)}\right] = \exp\left\{-\int_{\mathbb{R}^+\times\mathbb{X}} \left[1-\mathrm{e}^{-vf(x)}\right]\, \nu(\mathrm{d}v, \mathrm{d}x)\right\} \tag{3}$$

where $f : \mathbb{X} \to \mathbb{R}^+$ is any measurable function and ν is a measure on $\mathbb{R}^+ \times \mathbb{X}$ such that

$$\int_{\mathbb{R}^+\times B} \min\{v, 1\}\, \nu(\mathrm{d}v, \mathrm{d}x) < \infty \tag{4}$$

for any bounded B in $\mathscr{X}$. The measure ν is known as the *Lévy intensity* of $\tilde{\mu}$ and regulates the intensity of the jumps of a CRM and their locations. By virtue of (3), it characterizes the CRM $\tilde{\mu}$.

Two special cases of CRM stand out for their analytical tractability: the σ-stable and gamma CRMs. Let α be a σ-finite measure on $(\mathbb{X}, \mathscr{X})$. Then, for $\sigma \in (0,1)$, the CRM with Lévy intensity

$$\nu(\mathrm{d}v, \mathrm{d}x) = \frac{\sigma}{\Gamma(1-\sigma)} v^{-1-\sigma}\, \mathrm{d}v\, \alpha(\mathrm{d}x). \tag{5}$$

is a σ*-stable CRM* $\tilde{\mu}_\sigma$ with parameter measure α on $\mathbb{X}$. Moreover, for any measurable function $f : \mathbb{X} \to \mathbb{R}^+$, the Laplace functional is of the form $\mathbb{E}\left[\mathrm{e}^{-\int f\, \mathrm{d}\tilde{\mu}_\sigma}\right] = \mathrm{e}^{-\int f^\sigma\, \mathrm{d}\alpha}$. Hence, for any $B \in \mathscr{X}$, the Laplace transform of $\tilde{\mu}_\sigma(B)$ is that of a positive stable random variable, namely $\mathbb{E}\left[\mathrm{e}^{-\lambda\tilde{\mu}_\sigma(B)}\right] = \mathrm{e}^{-\lambda^\sigma\, \alpha(B)}$, for any $\lambda > 0$. Instead, if we consider the Lévy intensity

$$\nu(\mathrm{d}v, \mathrm{d}x) = e^{-v} v^{-1}\, \mathrm{d}v\, \alpha(\mathrm{d}x) \tag{6}$$

a *gamma CRM* $\tilde{\mu}$ is obtained. In this case, for any measurable function $f : \mathbb{X} \to \mathbb{R}^+$ one has $\mathbb{E}\left[\mathrm{e}^{-\int f \, \mathrm{d}\tilde{\mu}}\right] = \mathrm{e}^{-\int \log(1+f) \, \mathrm{d}\alpha}$. Hence, for any $B \in \mathscr{X}$ the Laplace transform of $\tilde{\mu}(B)$, evaluated at $\lambda > 0$, equals $(1+\lambda)^{-\alpha(B)}$. This entails that $\tilde{\mu}(B)$ is gamma distributed with parameters $(1, \alpha(B))$.

2.2. *Discrete random probabilities derived from CRMs.* Most discrete random probability measures, popular in the Statistics and Machine Learning literature, can be obtained as transformations of CRMs. See Lijoi and Prünster (2010) for a review using CRMs as unifying concept.

The first of these transformations we consider is normalization. If $0 < \tilde{\mu}(\mathbb{X}) < \infty$ a.s., then

$$\tilde{P} = \frac{\tilde{\mu}}{\tilde{\mu}(\mathbb{X})} \tag{7}$$

is well defined and takes values in $\mathscr{P}$, the space of probability measures on $(\mathbb{X}, \mathscr{X})$ with $\sigma(\mathscr{P})$ the corresponding Borel σ-algebra. The resulting class of random probabilities is termed *normalized completely random measures*. It was introduced in Regazzini, Lijoi and Prünster (2003) on $\mathbb{R}$ as *normalized random measures with independent increments*, from which the acronym NRMI routinely used also for normalized CRMs. Clearly, any NRMI is characterized by its Lévy intensity ν.

The Dirichlet process (Ferguson, 1973), $\tilde{\mathscr{D}}_\alpha$, is readily obtained as NRMI by considering a gamma CRM (6) with finite α. Often it is convenient to write $\alpha := \theta \, P_0$ with $\theta = \alpha(\mathbb{X})$ the concentration parameter and $P_0 = \mathbb{E}[\tilde{P}]$. Moreover, starting from the σ-stable CRM (5), one obtains the *normalized stable process* (Kingman, 1975), which is henceforth referred to as *σ-stable NRMI* and denoted by $\tilde{P}_\sigma$.

A different transformation of the σ-stable CRM leads to Pitman–Yor process (Pitman and Yor, 1997b) $\tilde{P}_{\sigma,\theta}$, also known as two parameter Poisson–Dirichlet process. Denote by $\mathbb{P}_\sigma$ the law of the σ-stable CRM. For $\theta > -\sigma$, define a random measure $\tilde{\mu}_{\sigma,\theta}$ with distribution $\mathbb{P}_{\sigma,\theta}$ absolutely continuous with respect to $\mathbb{P}_\sigma$ and such that

$$\frac{\mathrm{d}\mathbb{P}_{\sigma,\theta}}{\mathrm{d}\mathbb{P}_\sigma}(\tilde{\mu}) = \frac{[\tilde{\mu}(\mathbb{X})]^{-\theta}}{\mathbb{E}[\tilde{\mu}_\sigma(\mathbb{X})^{-\theta}]} \tag{8}$$

Note that $\tilde{\mu}_{\sigma,\theta}$, obtained as polynomial titling of a σ-stable CRM, is not a CRM anymore. Nonetheless, one can still obtain a random probability measure via normalization

$$\tilde{P}_{\sigma,\theta} = \frac{\tilde{\mu}_{\sigma,\theta}}{\tilde{\mu}_{\sigma,\theta}(\mathbb{X})},$$

which is now a Pitman–Yor process.

2.3. *Means.* A key step for the determination of the distribution of linear functionals, or means, of a random probability measure $\tilde{P}$

$$M_h(\tilde{P}) := \int_{\mathbb{X}} h(x) \tilde{P}(\mathrm{d}x), \tag{9}$$

with $h : \mathbb{X} \to \mathbb{R}$ being some measurable function, typically consists in representing them via suitable integral transforms and, then, applying appropriate inversion formulae. We concisely recall the two most successful approaches to date, which we will partially exploit also in this paper.

A first convenient tool is the *generalized Cauchy–Stieltjes transform*: for a function $g:\ \mathbb{R}^+ \to \mathbb{R}$, it is defined as

$$\mathscr{S}_\lambda[z;g] := \int_{\mathbb{R}^+} \frac{g(x)}{(z+x)^\lambda}\,\mathrm{d}x \tag{10}$$

for any $\lambda > 0$ and $z \in \mathbb{C}$ such that $|\arg(z)| < \pi$. Inversion formulae for (3.1) are available and can be found in, *e.g.*, Sumner (1949) and Schwarz (2005). Under suitable conditions, for instance $|z^\beta\,\mathscr{S}_\lambda[z;g]|$ is bounded at infinity for some $\beta > 0$, from Schwarz (2005) one has

$$g(x) = -\frac{x^\lambda}{2\pi\mathrm{i}} \int_{\mathscr{W}} (1+w)^{\lambda-1}\,\mathscr{S}'_\lambda[xw;g]\,\mathrm{d}w \tag{11}$$

where $\mathscr{W}$ is a contour in the complex plane starting and ending at the point $w = -1$ and enclosing the origin in a counterclockwise sense, while $\mathscr{S}'_\lambda[xw;g] = \frac{\mathrm{d}}{\mathrm{d}z}\,\mathscr{S}_\lambda[z;g]\,|_{z=xw}$. If $\lambda > 1$, then one can integrate (11) by parts obtaining

$$g(x) = \frac{\lambda-1}{2\pi\mathrm{i}}\,x^{\lambda-1} \int_{\mathscr{W}} (1+w)^{\lambda-2}\,\mathscr{S}_\lambda[xw;g]\,\mathrm{d}w$$

For the case $\lambda = 1$, (11) reduces to Widder's inversion formula (Widder, 2015).

For the Dirichlet process case a closed form expression for the distribution of the mean has been derived in Cifarelli and Regazzini (1990) leveraging on the inversion formula by Sumner (1949). A similar strategy has been pursued in James, Lijoi and Prünster (2008) for means of a Pitman–Yor process.

A second fruitful approach relies on an inversion formula of the characteristic function due to Gurland (1948), which was first adopted by Regazzini, Guglielmi and Di Nunno (2002). If F is a cumulative distribution function (cdf) on $\mathbb{R}$ and ϕ the corresponding characteristic function, then

$$F(y) - F(y-) = 1 - \frac{2}{\pi} \lim_{\varepsilon\downarrow 0,\,T\uparrow\infty} \int_\varepsilon^T \frac{1}{t}\,\mathrm{Im}\left[\mathrm{e}^{-\mathrm{i}yt}\,\phi(t)\right]\,\mathrm{d}t, \tag{12}$$

where $F(x-)$ is the left limit of F at y and $\mathrm{Im}\,z$ stands for the imaginary part of $z \in \mathbb{C}$. Such an inversion formula is useful for determining the distribution of ratios of random variables and, thus, is suited for NRMIs. To see this, let h be such that $\int |h|\,\mathrm{d}\tilde{\mu} < \infty$ a.s. and denote the cdf of $\int h\,\mathrm{d}\tilde{P}$ by $y \mapsto Q(y) = \mathbb{P}[M_h(\tilde{p}) \le y]$ with $\tilde{P}$ a NRMI. A crucial step consists in noting that

$$Q(y) = \mathbb{P}\left[\int_{\mathbb{X}} [f(x) - y]\tilde{\mu}(\mathrm{d}x) \le 0\right],$$

which reduces the problem of studying a NRMI mean to the problem of studying a linear functional of a CRM. Importantly, the characteristic functions of linear functionals of CRMs, analogously to the Laplace functional transform (3), have Lévy–Khintchine representation in terms of the underlying Lévy intensity measure. Therefore, from (12) one obtains

$$\begin{aligned}
&\frac{1}{2}\,\{Q(y) + Q(y-)\}\\
&= \frac{1}{2} - \frac{1}{\pi} \lim_{\epsilon\downarrow 0, T\uparrow+\infty} \int_\epsilon^T \frac{1}{t}\,\mathrm{Im}\,\exp\left\{-\int_{\mathbb{X}\times\mathbb{R}^+} [1 - \mathrm{e}^{\mathrm{i}tv\,(h(x)-y)}]\nu(\mathrm{d}v,\mathrm{d}x)\right\}\mathrm{d}t.
\end{aligned}$$

See Regazzini, Lijoi and Prünster (2003) for details and James, Lijoi and Prünster (2010) for further developments.

3. Fixing the distribution of the mean. In this Section we provide the main results of the paper: given a random probability measure $\tilde{P}$ on $[0,1]$, our goal is to determine the parameter measure inducing a desired distribution on the random mean

$$M(\tilde{P}) := \int_0^1 x\, \tilde{P}(\mathrm{d}x), \tag{13}$$

provided that such a measure is unique.

REMARK 3.1. There is no loss in generality by considering simple means (13) instead of generic linear functionals (9) for measurable functions $h : \mathbb{X} \to \mathbb{R}$ such that $\int |h|\, \mathrm{d}\tilde{P} < \infty$ a.s.. This follows from the fact that

$$\int h\, \mathrm{d}\tilde{P} \stackrel{d}{=} \int x\, \tilde{P}_h(\mathrm{d}x) \tag{14}$$

where $\tilde{P}_h = \tilde{P} \circ h^{-1}$ is the pushforward random probability. Hence, if interest is in (2), it is enough to re-interpret P_0 as $P_0 \circ h^{-1}$. Moreover, if $\tilde{P}$ is a normalized CRM (7) with Lévy intensity $\nu(\mathrm{d}v, \mathrm{d}x)$, (14) is equivalent to saying that $\tilde{P}_h$ is obtained by normalizing a CRM $\tilde{\mu}_f$ whose Lévy intensity ν_h is such that

$$\int_B \int_A \nu_h(\mathrm{d}v,\, \mathrm{d}x) = \int_{h^{-1}(B)} \int_A \nu(\mathrm{d}v,\, \mathrm{d}x)$$

for any $A \in \mathscr{B}(\mathbb{R}^+)$ and $B \in \mathscr{B}(\mathbb{R})$.

REMARK 3.2. For clarity of the exposition, in the following we assume that $\tilde{P}$ in (1) or, equivalently, P_0 have $[0,1]$ support but all results can be easily extended to cover the case, where the support of $\tilde{P}$ is any bounded interval of $\mathbb{R}$. These assumptions are not restrictive neither from a theoretical nor from an applied perspective. On the one hand, the extension to the unbounded support case requires some strengthening of the hypotheses, without affecting the type of results we obtain, while at the same time not providing further insights. On the other hand, in applied contexts random probability measures with compact, or even finite, support are typically employed as models or as approximations when it comes to the computational implementation.

For the Dirichlet process $\tilde{\mathscr{D}}_\alpha$ the problem of determining the base measure $\alpha := \theta P_0$, where $\theta > 0$ and P_0 is a probability measure on $[0,1]$, leading to a specific probability distribution for the mean functional $M(\tilde{\mathscr{D}}_\alpha) = \int x\, \tilde{\mathscr{D}}_\alpha(\mathrm{d}x)$ was first hinted at in Cifarelli and Regazzini (1993). For a given α, we let $Q_\alpha = \mathbb{P} \circ (M(\tilde{\mathscr{D}}_\alpha))^{-1}$ stand for the probability distribution of $M(\tilde{\mathscr{D}}_\alpha)$. Moreover, with $\mathbb{F}$ denoting the set of finite and non-null measures on $([0,1], \mathscr{B}([0,1]))$, define $\mathbb{F}_\theta := \{\alpha \in \mathbb{F} :\ \alpha([0,1]) = \theta\}$ and

$$\mathbb{M}_\theta := \{Q_\alpha :\ \alpha \in \mathbb{F}_\theta\}. \tag{15}$$

The latter is the set of all probability distributions of the random Dirichlet mean $M(\tilde{\mathscr{D}}_\alpha)$ as α varies in $\mathbb{F}_\theta$. According to Theorem 2 in Lijoi and Regazzini (2004) any measure α in $\mathbb{F}_\theta$ is determined by the corresponding distribution Q_α in $\mathbb{M}_\theta$. This implies that, for random Dirichlet means, the total mass θ and Q_α in $\mathbb{M}_\theta$ uniquely identify the base measure $\alpha \in \mathbb{F}_\theta$. Furthermore, as a consequence of Theorem 10 in Lijoi and Regazzini (2004), Q_α is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ and its density function is indicated by q_α. The correspondence between q_α and α is expressed by the Cifarelli–Regazzini identity

$$\mathscr{S}_\theta[z; q_\alpha] = \exp\left\{ -\theta \int_0^1 \log(z + x)\, P_0(\mathrm{d}x) \right\} \qquad z \in \mathbb{C} \setminus [-1, 0] \tag{16}$$

where $\mathscr{S}_\theta$ denotes the generalized Cauchy–Stieltjes transform of order θ as defined in (3.1).

When $\theta = 1$, an explicit solution to the inverse problem, that is the determination of $\alpha = P_0$ inducing a suitably smooth q_α, can be extrapolated from Romik (2004). In this work, *continual Young diagrams* and the *transition measure* they induce on a compact interval via *hook walks* are considered. See also Kerov (1993) for definitions, early results, and links to the Markov moment problem. If the Young diagram is convex, then it can be seen as a primitive function of a cdf, which then corresponds to a probability distribution on the compact interval. In this case, the correspondence between the diagram and the induced transition measure is the same as the one between the base measure of a Dirichlet process with concentration parameter $\theta = 1$ and its mean distribution. Since in Romik (2004) an explicit expression of the derivative of the diagram as a function of the transition density is given, it is possible to leverage such result and obtain

$$P_0([0,x]) = \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi\, q(x)} \operatorname{PV} \int_0^1 \frac{q(t)}{t-x}\, \mathrm{d}t\right) \tag{17}$$

for P_0 being the base measure of a Dirichlet process with $\theta = 1$, q the density of the mean distribution and $\operatorname{PV}\int$ indicating the Cauchy principal value integral. See Estrada and Kanwal (2012) for an exhaustive account on these analytical tools. The identity (17) has been proved for q piecewice C^1 with bounded derivative. This implies that, however we choose a mean density q with such regularity, we can explicitly identify the parameter measure P_0 leading to $M(\tilde{P}) \sim q$. Hook walks on continual Young diagrams, transition measures and this surprising connection with the Dirichlet process are further investigated in Gaffi, Lijoi and Prünster (2023).

In the following we give an explicit expression for the cdf of the parameter measure of $\tilde{P}$ enforcing a broad class of distributions on the random mean for $\tilde{P}$ a Dirichlet, normalized stable or Pitman–Yor process.

3.1. *Base measure of a Dirichlet process.* First we solve the problem for any $\theta \in (0,1)$. To this end we derive a novel expression for the generalized Cauchy–Stieltjes transform of the cumulative distribution function of the base measure of a Dirichlet process, in terms of the transform of its mean density.

For a density function f such that

$$\int_0^1 \frac{f(x)}{|x-t|^\theta}\, \mathrm{d}x < \infty \qquad \forall t \in [0,1]$$

with $\theta \in (0,1)$, we define

$$\mathscr{I}_\theta[\, f;\, t\,] := \frac{\displaystyle\int_t^1 \frac{f(x)}{|x-t|^\theta}\, \mathrm{d}x}{\displaystyle\int_0^t \frac{f(x)}{|x-t|^\theta}\, \mathrm{d}x} \qquad t \in (0,1]. \tag{18}$$

Note that $\lim_{t\to 0} \mathscr{I}_\theta[\, f;\, t\,] = \infty$ and $\mathscr{I}_\theta[\, f;\, 1\,] = 0$. Moreover, $\mathscr{I}_\theta[f;\, \cdot\,]$ is monotonically decreasing and, hence, we can consider its right continuous version, suitably modifying it in its at most countable jump discontinuity points. With a slight abuse of notation, we denote also this version with $\mathscr{I}_\theta[\,\cdot\,;\, t\,]$. Our result is as follows.

THEOREM 3.3. *Let $\theta \in (0,1)$ and q_α be the density of $M(\tilde{\mathscr{D}}_\alpha)$ with* $\operatorname{supp}(q_\alpha) = [0,1]$. *If*

$$\int_0^1 \frac{q_\alpha(x)}{|x-t|^\theta}\, \mathrm{d}x < \infty \qquad \forall t \in [0,1] \tag{19}$$

then the cdf of the base measure P_0 is given by

$$F_0(t) = \left\{ \frac{1}{\theta\pi} \arctan\left(\frac{\sin(\theta\pi)}{\cos(\theta\pi) + \mathscr{I}_\theta[\,q_\alpha;\,t\,]} \right) + \frac{1}{\theta} \mathbb{1}_{(t_*,\infty)}(t) \right\} \mathbb{1}_{(0,1)}(t) + \mathbb{1}_{[1,\infty)}(t) \tag{20}$$

with

$$t_* = \inf\left\{ t \in [0,1] \,\middle|\, \mathscr{I}_\theta[\,q_\alpha;\,t\,] \le -\cos(\theta\pi) \right\} \tag{21}$$

REMARK 3.4. It is easy to verify that F_0 in (20) is indeed a cdf. Clearly, $F_0(0) = 0$, $F_0(1) = 1$ and F_0 is increasing since $\mathscr{I}_\theta[\,q_\alpha;\,\cdot\,]$ is decreasing. Moreover, $\mathscr{I}_\theta[\,q_\alpha;\,\cdot\,]$ is right continuous. Since

$$F_0(t_*^-) = \frac{1}{2\theta} = F_0(t_*^+),$$

when the set in (21) is non-empty, F_0 is also right continuous. Note that $t_* = \infty$ for $\theta < \frac{1}{2}$.

REMARK 3.5. For every $\theta < 1$, we have $F_0(0^+) = 0$ and $F_0(1^-) = 1$, that is the parameter measure cannot have positive mass on 0 or 1. The reason is that Dirichlet mean densities corresponding to such parameter measures are ruled out by the integrability assumption (19). Consider, for instance, $\alpha(\,\cdot\,) = \theta_0\delta_{\{0\}}(\,\cdot\,) + \theta_1\delta_{\{1\}}(\,\cdot\,)$, where $\theta = \theta_0 + \theta_1$. In this case $M(\tilde{\mathscr{D}}_\alpha) \overset{d}{=} \tilde{\mathscr{D}}_\alpha(\{1\})$, hence $M(\tilde{\mathscr{D}}_\alpha) \sim \text{beta}(\theta_1, \theta_0)$, and its density violates (19) for $t \in \{0,1\}$, since $\theta + 1 - \theta_i > 1$ for $i = 0,\,1$.

PROOF OF THEOREM 3.3. We start by rewriting the right-hand-side of the Cifarelli–Regazzini identity (16). Indeed, leveraging on the fact that

$$\int_0^x \frac{1}{t+z}\,\mathrm{d}t = \log(z+x) - \log(z) \qquad \text{for} \quad \operatorname{Im} z \neq 0$$

and obtaining

$$\exp\left\{ -\,\theta \int_0^1 \log(z+x)\, P_0(\mathrm{d}x) \right\} = \exp\left\{ -\,\theta \log(z) - \theta \int_0^1 \int_t^1 P_0(\mathrm{d}x) \frac{\mathrm{d}t}{t+z} \right\} =$$
$$= \frac{1}{(1+z)^\theta} \exp\left\{ \theta \int_0^1 \frac{F_0(t)}{t+z}\,\mathrm{d}t \right\},$$

where F_0 is the cdf of P_0. Hence, the right-hand-side of (16) becomes

$$\exp\left\{\theta \mathscr{S}_1[F_0; z]\right\} = (1+z)^\theta \int_0^1 \frac{q_\alpha(x)}{(z+x)^\theta}\,\mathrm{d}x$$

with $\mathscr{S}_1$ according to the definition in (3.1). Applying the principal value of the complex logarithm to both sides

$$\mathscr{S}_1[F_0; z] = \frac{1}{\theta} \log\left\{ (1+z)^\theta \mathscr{S}_\theta[q_\alpha; z] \right\} + \frac{2k(z)\pi \mathrm{i}}{\theta}$$

where

$$k(z) := -\theta \operatorname{Im} \mathscr{S}_1[F_0; z] \setminus \pi$$

for $\setminus$ denoting the integer division. Then, by the Cauchy–Stieltjes transform inversion formula in Widder (2015), we have for $t \in [0,1]$

$$F_0(t) = \lim_{\varepsilon \downarrow 0} \left\{ -\frac{1}{\theta\pi} \operatorname{Im} \left(\log\left\{ (1-t+\mathrm{i}\varepsilon)^\theta \mathscr{S}_\theta[q_\alpha; -t+\mathrm{i}\varepsilon] \right\} \right) + \frac{2k(-t+\mathrm{i}\varepsilon)}{\theta} \right\} =$$
$$= \lim_{\varepsilon \downarrow 0} \left\{ -\,\frac{1}{\theta\pi} \operatorname{Arg}\Big((1-t+\mathrm{i}\varepsilon)^\theta \mathscr{S}_\theta[q_\alpha; -t+\mathrm{i}\varepsilon] \Big) + \frac{2k(-t+\mathrm{i}\varepsilon)}{\theta} \right\}$$

where $\operatorname{Arg}(w)$ denotes the principal argument of $w \in \mathbb{C}$. If we write

$$\Im_\varepsilon[\,q_\alpha; t\,] := \operatorname{Im}\left((1-t+\mathrm{i}\varepsilon)^\theta \mathscr{S}_\theta[q_\alpha; -t+\mathrm{i}\varepsilon]\right) \tag{22}$$

and

$$\Re_\varepsilon[\,q_\alpha; t\,] := \operatorname{Re}\left((1-t+\mathrm{i}\varepsilon)^\theta \mathscr{S}_\theta[q_\alpha; -t+\mathrm{i}\varepsilon]\right) \tag{23}$$

then

$$\begin{aligned}\operatorname{Arg}\left((1-t+\mathrm{i}\varepsilon)^\theta \mathscr{S}_\theta[q_\alpha; -t+\mathrm{i}\varepsilon]\right) &= \arctan\left(\frac{\Im_\varepsilon[\,q_\alpha; t\,]}{\Re_\varepsilon[\,q_\alpha; t\,]}\right) + \\ &\quad + \pi \mathbb{1}_{\{\Re_\varepsilon[\,q_\alpha; t\,] < 0\}} \operatorname{sign}\left(\Im_\varepsilon[\,q_\alpha; t\,]\right)\end{aligned} \tag{24}$$

Since

$$\lim_{\varepsilon \downarrow 0} \operatorname{Im}\left((1-t+\mathrm{i}\varepsilon)^\theta\right) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \downarrow 0} \operatorname{Re}\left((1-t+\mathrm{i}\varepsilon)^\theta\right) = (1-t)^\theta$$

we shall neglect a summand and obtain

$$\begin{aligned}\lim_{\varepsilon \downarrow 0} \Im_\varepsilon[\,q_\alpha; t\,] &= -(1-t)^\theta \lim_{\varepsilon \downarrow 0} \int_0^1 \frac{\sin\left(\theta \arctan\left(\varepsilon/(x-t)\right) + \theta\pi \mathbb{1}_{(0,t)}(x)\right)}{\left((x-t)^2+\varepsilon^2\right)^{\theta/2}} q_\alpha(x)\,\mathrm{d}x = \\ &= -(1-t)^\theta \sin(\theta\pi) \int_0^t \frac{q_\alpha(x)}{|x-t|^\theta}\,\mathrm{d}x\end{aligned} \tag{25}$$

and

$$\begin{aligned}\lim_{\varepsilon \downarrow 0} \Re_\varepsilon[\,q_\alpha; t\,] &= (1-t)^\theta \lim_{\varepsilon \downarrow 0} \int_0^1 \frac{\cos\left(\theta \arctan\left(\varepsilon/(x-t)\right) + \theta\pi \mathbb{1}_{(0,t)}(x)\right)}{\left((x-t)^2+\varepsilon^2\right)^{\theta/2}} q_\alpha(x)\,\mathrm{d}x = \\ &= (1-t)^\theta \left\{\cos(\theta\pi) \int_0^t \frac{q_\alpha(x)}{|x-t|^\theta}\,\mathrm{d}x + \int_t^1 \frac{q_\alpha(x)}{|x-t|^\theta}\,\mathrm{d}x\right\}\end{aligned} \tag{26}$$

applying Lebesgue's dominated convergence theorem, which holds because of (19). To deal with the indicator and sign functions in (24), it suffices to check their discontinuity points. We have

$$-(1-t)^\theta \sin(\theta\pi) \int_0^t \frac{q_\alpha(x)}{|x-t|^\theta}\,\mathrm{d}x \le 0$$

with equality holding for $t \in \{0, 1\}$, while

$$(1-t)^\theta \left\{\cos(\theta\pi) \int_0^t \frac{q_\alpha(x)}{|x-t|^\theta}\,\mathrm{d}x + \int_t^1 \frac{q_\alpha(x)}{|x-t|^\theta}\,\mathrm{d}x\right\} \lesseqgtr 0 \iff \mathscr{I}_\theta[q_\alpha; t] \gtreqless -\cos(\theta\pi).$$

for $t \neq 0$, with equality holding at most in one point, by monotonicity. Finally we just need to prove that $k(-t+\mathrm{i}\varepsilon) \to 0$ for $\varepsilon \downarrow 0$. Since $\varepsilon > 0$, by properties of the Cauchy–Stieltjes transform, reported for instance in Karp and Prilepkina (2012), $\operatorname{Im} \mathscr{S}_1[F_0; -t+\mathrm{i}\varepsilon] \le 0$. Hence, it suffices to note that, as $\varepsilon \to 0$, one has

$$-\operatorname{Im} \mathscr{S}_1[F_0; -t+\mathrm{i}\varepsilon] \le \int_0^1 \frac{\varepsilon}{(s-t)^2 - \varepsilon^2}\,\mathrm{d}s = \arctan\left(\frac{1-t}{\varepsilon}\right) - \arctan\left(-\frac{t}{\varepsilon}\right) \longrightarrow \pi$$

□

As an application of the previous general result we consider two interesting cases: the determination of the base measure of a Dirichlet process such that the corresponding mean $M(\tilde{\mathscr{D}}_\alpha)$ has a uniform and a triangular distribution on $[0,1]$.

EXAMPLE (Uniform case). Let $q_\alpha(x) = \mathbb{1}_{[0,1]}(x)$. Since

$$\mathscr{I}_\theta[\,\mathbb{1}_{[0,1]};\,t\,] = \left(\frac{1-t}{t}\right)^{1-\theta}$$

the cdf of P_0 is

$$F_0(t) = \frac{1}{\theta\pi}\arctan\left(\frac{\sin(\theta\pi)}{\cos(\theta\pi) + \left(\frac{1-t}{t}\right)^{1-\theta}}\right) + \frac{1}{\theta}\mathbb{1}_{(t_*,1)}(t)\mathbb{1}_{(\frac{1}{2},1)}(\theta) \tag{27}$$

for $t \in (0,1)$, where for $\theta \in \left(\frac{1}{2},1\right)$

$$t_* := \frac{1}{1+(-\cos(\theta\pi))^{\theta}}$$

In particular, for $\theta = \frac{1}{2}$

$$F_0(t) = \frac{2}{\pi}\arctan\sqrt{\frac{t}{1-t}} \qquad t \in (0,1)$$

EXAMPLE (Triangular case). Let $q_\alpha(x) = 4x\mathbb{1}_{[0,\frac{1}{2})}(x) + 4(1-x)\mathbb{1}_{[\frac{1}{2},1]}(x)$. We have

$$\mathscr{I}_\theta[\,q_\alpha;\,t\,] = \left\{\frac{(1-t)^{2-\theta} - 2\left(\frac{1}{2}-t\right)^{2-\theta}}{t^{2-\theta}}\right\}\mathbb{1}_{(0,\frac{1}{2}]}(t) + \left\{\frac{(1-t)^{2-\theta}}{t^{2-\theta} - 2\left(t-\frac{1}{2}\right)^{2-\theta}}\right\}\mathbb{1}_{(\frac{1}{2},1]}(t).$$

Since $\mathscr{I}_\theta[\,q_\alpha;\,t\,]$ is decreasing and $\mathscr{I}_\theta\left[\,q_\alpha;\,\frac{1}{2}\,\right] = 1$, the cdf of P_0, for $t \in (0,1)$, is

$$\begin{aligned}F_0(t) = &\left\{\frac{1}{\theta\pi}\arctan\left(\frac{t^{2-\theta}\sin(\theta\pi)}{t^{2-\theta}\cos(\theta\pi) + (1-t)^{2-\theta} - 2\left(\frac{1}{2}-t\right)^{2-\theta}}\right)\right\}\mathbb{1}_{(0,\frac{1}{2}]}(t) + \\ &+ \left\{\frac{1}{\theta\pi}\arctan\left(\frac{\left(t^{2-\theta} - 2(t-\frac{1}{2})^{2-\theta}\right)\sin(\theta\pi)}{\left(t^{2-\theta} - 2(t-\frac{1}{2})^{2-\theta}\right)\cos(\theta\pi) + (1-t)^{2-\theta}}\right)\right\}\mathbb{1}_{(\frac{1}{2},1)}(t) + \\ &+ \frac{1}{\theta}\mathbb{1}_{(t_*,1)}(t)\mathbb{1}_{(\frac{1}{2},1)}(\theta)\end{aligned}$$

where t_* is such that

$$\frac{(1-t_*)^{2-\theta}}{t_*^{2-\theta} - 2\left(t_*-\frac{1}{2}\right)^{2-\theta}} = -\cos(\theta\pi).$$

Now, we deal with the case of $\theta > 1$. Given the constructive nature of the proof of Theorem 3.3, it is possible to leverage its rationale and retrieve consistent results for a large class of densities. For $\theta > 1$, the integrability condition (19) rules out every probability density. However, (19) is only needed to perform a limit/integral switch in (25) and (26). Hence, for any choice of mean density q leading to explicit expressions of the generalized Cauchy–Stieltjes transform in (22) and (23), *i.e.* such that the integral

$$\int_0^1 \frac{q(x)}{(-t+\mathrm{i}\varepsilon+x)^{\theta}}\,\mathrm{d}x$$

has a tractable form, results analogous to those of Theorem 3.3 can be obtained. Yet the set $\mathbb{M}_\theta$ varies with θ and there is no guarantee that any absolutely continuous distribution can be a Dirichlet(θ) mean distribution for any θ; therefore, with this procedure, one may obtain a function which is not a cdf. This phenomenon is in line with the $\theta = 1$ case: in Romik (2004) a homeomorphism is built between diagrams and probability distributions on $[0,1]$; however, since convex diagrams form a proper subset of all diagrams, some transition measures are bound to correspond to non-convex diagrams, which do not represent base probability measures. Such possibilities are treated in the following proposition and example.

PROPOSITION 3.6. *Let $\theta \in (1,2)$ and $M(\tilde{\mathscr{D}}_\alpha) \sim q_\alpha$ with $q_\alpha(x) = \mathbb{1}_{[0,1]}(x)$. Then the cdf of the base measure P_0 is*

$$F_0(t) = \left\{ \frac{1}{\theta\pi} \arctan\left(\frac{\sin(\theta\pi)}{\cos(\theta\pi) + \left(\frac{t}{1-t}\right)^{\theta-1}} \right) + \frac{1}{\theta}\mathbb{1}_{(t_*,1)}(t) \right\} \mathbb{1}_{(0,1)}(t) + \mathbb{1}_{(1,\infty)}(t) \tag{28}$$

with

$$t_* = \frac{(-\cos(\theta\pi))^\theta}{1 + (-\cos(\theta\pi))^\theta} \mathbb{1}_{(1,\frac{3}{2})}(\theta) \tag{29}$$

REMARK 3.7. In contrast to the $\theta \in (0,1)$ case, F_0 in (28) defines a distribution with $\frac{\theta-1}{\theta}$ masses in 0 and 1, while the rest of the mass is (symmetrically) diffuse in $(0,1)$.

PROOF OF PROPOSITION 3.6. Since

$$\mathscr{S}_\theta[\mathbb{1}_{[0,1]}, -t + \mathrm{i}\varepsilon] = \frac{1}{\theta - 1} \left\{ \frac{1}{(-t + \mathrm{i}\varepsilon)^{\theta-1}} - \frac{1}{(1 - t + \mathrm{i}\varepsilon)^{\theta-1}} \right\}$$

we have

$$\begin{aligned}\mathrm{Im}\left(\mathscr{S}_\theta[\mathbb{1}_{[0,1]}, -t + \mathrm{i}\varepsilon]\right) = \frac{\left(t^2+\varepsilon^2\right)^{\frac{1-\theta}{2}}}{\theta-1} \sin\left((\theta-1)\arctan\left(\frac{\varepsilon}{t}\right) - (\theta-1)\pi\right) + \\ + \frac{\left((1-t)^2+\varepsilon^2\right)^{\frac{1-\theta}{2}}}{\theta-1} \sin\left((\theta-1)\arctan\left(\frac{\varepsilon}{1-t}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\left(\mathscr{S}_\theta[\mathbb{1}_{[0,1]}, -t + \mathrm{i}\varepsilon]\right) = \frac{\left(t^2+\varepsilon^2\right)^{\frac{1-\theta}{2}}}{\theta-1} \cos\left((\theta-1)\arctan\left(\frac{\varepsilon}{t}\right) + (\theta-1)\pi\right) - \\ - \frac{\left((1-t)^2+\varepsilon^2\right)^{\frac{1-\theta}{2}}}{\theta-1} \cos\left((\theta-1)\arctan\left(\frac{\varepsilon}{1-t}\right)\right).\end{aligned}$$

Therefore, using the notation set in (25) and (26), we obtain

$$\lim_{\varepsilon\downarrow 0} \Im_\varepsilon[\mathbb{1}_{[0,1]}; -t + \mathrm{i}\varepsilon] = \frac{(1-t)^\theta}{\theta - 1} t^{\theta-1} \sin(\theta\pi)$$

and

$$\lim_{\varepsilon\downarrow 0} \Re_\varepsilon[\mathbb{1}_{[0,1]}; -t + \mathrm{i}\varepsilon] = \frac{(1-t)^\theta}{1 - \theta} t^{\theta-1} \left\{ \cos(\theta\pi) + \left(\frac{t}{1-t}\right)^{\theta-1} \right\}.$$

Hence, proceeding as in the proof of Theorem 3.3, we obtain the desired expression. □

EXAMPLE. Let $\theta \in (2,3)$ and $M(\tilde{\mathscr{D}}_\alpha) \sim q_\alpha$ with $q_\alpha(x) = \mathbb{1}_{[0,1]}(x)$. Reasoning as in Proposition 3.6, one again obtains

$$\frac{1}{\theta\pi} \arctan\left(\frac{\sin(\theta\pi)}{\cos(\theta\pi) + \left(\frac{t}{1-t}\right)^{\theta-1}}\right)$$

having disregarded additive constants. However, for $\theta \in (2,3)$ this is a decreasing function. Hence, it cannot be the cdf of a probability measure. This implies that the uniform distribution cannot be the mean distribution of a Dirichlet process with concentration parameter $\theta \in (2,3)$.

3.2. *Base measure of a σ-stable NRMI.* Our goal is now to determine the base measure P_0 yielding a specific probability distribution for $M(\tilde{P}_\sigma)$ where, as before, $\tilde{P}_\sigma$ is a σ-stable NRMI. For the case of Dirichlet means, a preliminary step consisted in establishing a correspondence between $\mathbb{F}_\theta$ and $\mathbb{M}_\theta$. An analogous preliminary step to verify whether a similar correspondence holds in the σ-stable NRMI case is carried out in the following result.

PROPOSITION 3.8. *The probability distribution of the mean $M(\tilde{P}_\sigma)$ of a σ-stable NRMI $\tilde{P}_\sigma$ is determined by σ and $\mathbb{E}[\tilde{P}_\sigma] = P_0$.*

PROOF. We give a proof that relies on the evaluation of the moments of $M(\tilde{P}_\sigma)$ and noting they do not depend on θ. It is worth noting that an alternative proof to the direct one we provide can be given by exploiting the properties of σ-stable CRMs. Set

$$Z(n,k) := \{\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n : \sum_i i m_i = n, \sum_i m_i = k\}$$

where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and $r_j = \int_0^1 x^j \, P_0(\mathrm{d}x)$. Relying on Theorem 3.3 in Lijoi and Prünster (2009) we obtain

$$\begin{aligned}\mathbb{E}\left[M^n(\tilde{P}_\sigma)\right] &= \frac{1}{\Gamma(n)} \sum_{k=1}^n \sum_{\boldsymbol{m} \in Z(n,k)} \frac{n!}{\prod_{j=1}^n (j!)^{m_j} \, m_j!} \prod_{j=1}^n (r_j \, (1-\sigma)_{j-1})^{m_j} \\ &\qquad \times \, \sigma^k \, \theta^k \int_0^\infty u^{k\sigma-1} \, \mathrm{e}^{-\theta u^\sigma} \, \mathrm{d}u \\ &= \sum_{k=1}^n \frac{\sigma^{k-1} \, \Gamma(k)}{\Gamma(n)} \sum_{\boldsymbol{m} \in Z(n,k)} \frac{n!}{\prod_{j=1}^n (j!)^{m_j} \, m_j!} \prod_{j=1}^n (r_j \, (1-\sigma)_{j-1})^{m_j},\end{aligned}$$

which depends on the base measure $\alpha = \theta P_0$ only through the moments r_j of P_0. Hence, for fixed P_0, any θ yields the same probability distribution for $M(\tilde{P}_\sigma)$. □

Unlike the Dirichlet process case, the previous result implies that any α in (5) such that $\alpha = \theta P_0$ leads to the same probability distribution for $M(\tilde{P}_\sigma)$, regardless of the value of θ. For this reason we henceforth set $\theta = 1$ and focus solely on P_0.

Before stating the main result, it is important to point out that the key idea of the proof makes use of an analogue of the Cifarelli–Regazzini identity (16), for the mean of a σ-stable NRMI. If q_σ is the density function of the random mean $M(\tilde{P}_\sigma)$, where $\tilde{P}_\sigma$ is obtained by normalizing a σ-stable CRM with Lévy intensity as in (5) with $\alpha = P_0$, then, as shown in Tsilevich (1999), one has

$$\exp\left\{\int \log(z+x)^\sigma\, q_\sigma(x)\,\mathrm{d}x\right\} = \int (z+x)^\sigma\, P_0(\mathrm{d}x). \tag{30}$$

This identity is leveraged to obtain a novel representation of the generalized Cauchy–Stieltjes transform of the cumulative distribution function of the base measure of a σ-stable NRMI in terms of a suitable integral transform of its mean density. Moreover, we resort to existence results for singular integrals, in order to formulate explicit and reasonable assumptions on the mean density. Here, we again consider q_σ and P_0 supported on $[0,1]$.

THEOREM 3.9. *Let the density q_σ of $M(\tilde{P}_\sigma)$ be piecewise Hölder continuous and such that*

$$\int_0^1 |\log|x-t||\; q_\sigma(x)\,\mathrm{d}x < \infty \tag{31}$$

Lebesgue-almost everywhere. Then the base measure P_0 has cdf given by

$$\begin{aligned} F_0(y) = \frac{1}{\pi}\int_0^y\; (y-t)^{-\sigma}\,\mathrm{e}^{\sigma\int_0^1 \log|x-t|\, q_\sigma(x)\,\mathrm{d}x} \\ \left\{\pi\, q_\sigma(t)\,\cos(\sigma\pi\, Q_\sigma(t)) + \sin(\sigma\pi Q_\sigma(t))\,\mathrm{PV}\int_0^1 \frac{q_\sigma(x)}{t-x}\,\mathrm{d}x\right\}\,\mathrm{d}t \end{aligned} \tag{32}$$

for any $y \in (0,1)$, where Q_σ is the cdf of q_σ.

PROOF. First note that $(z+x)^\sigma = z^\sigma + \sigma\int_0^x (z+s)^{\sigma-1}\,\mathrm{d}s$ for any x in $[0,1]$ and $\mathrm{Im}(z) \neq 0$. This implies that

$$\begin{aligned} \int_0^1 (z+x)^\sigma\, P_0(\mathrm{d}x) &= z^\sigma + \sigma\int_0^1 (z+s)^{\sigma-1}\int_s^1 P_0(\mathrm{d}x)\,\mathrm{d}s \\ &= z^\sigma + \sigma\int_0^1 \frac{1-F_0(s)}{(z+s)^{1-\sigma}}\,\mathrm{d}s \end{aligned}$$

which leads to

$$\int_0^1 \frac{F_0(s)}{(z+s)^{1-\sigma}}\,\mathrm{d}s = \frac{(z+1)^\sigma}{\sigma} - \frac{1}{\sigma}\int_0^1 (z+x)^\sigma\, P_0(\mathrm{d}x)$$

By virtue of the identity (30), one can rewrite the right hand side of the previous equation to get

$$\mathscr{S}_{1-\sigma}[z;F_0] = \frac{(z+1)^\sigma}{\sigma} - \frac{1}{\sigma}\exp\left\{\sigma\int_0^1 \log(z+x)\, q_\sigma(x)\,\mathrm{d}x\right\}. \tag{33}$$

At this stage, F_0 is obtained by applying an inversion formula for the generalized Cauchy–Stieltjes transform $\mathscr{S}_{1-\sigma}$. To this end, we apply the following alternative version of the inversion formula in (11), displayed in Schwarz (2005)

$$F_0(y) = \int_0^y (y-t)^{-\sigma}\Delta'(t)\,\mathrm{d}t, \tag{34}$$

where

$$\Delta(t) := \frac{1}{2\pi\mathrm{i}}\lim_{\varepsilon\downarrow 0}\left\{\mathscr{S}_{1-\sigma}[-t-\mathrm{i}\varepsilon;F_0] - \mathscr{S}_{1-\sigma}[-t+\mathrm{i}\varepsilon;F_0]\right\}. \tag{35}$$

This holds true whenever the involved integral does exist. Since the generalized Cauchy–Stieltjes transform is a holomorphic function on $\mathbb{C}\setminus\mathbb{R}^-$, we have

$$\Delta(t)=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\,(\mathscr{S}_{1-\sigma}[-t-\mathrm{i}\varepsilon;\,F_0])=\frac{-1}{\pi\sigma}\lim_{\varepsilon\downarrow 0}\,\mathrm{Im}\,(\mathscr{L}_\sigma(-t-\mathrm{i}\varepsilon))$$

because $\lim_{\varepsilon\downarrow 0}\mathrm{Im}(1-t-\mathrm{i}\varepsilon)^\sigma=0$, where

$$\mathscr{L}_\sigma(z):=\exp\left\{\sigma\int_0^1\log(x+z)\,q_\sigma(x)\,\mathrm{d}x\right\}\qquad z\in\mathbb{C}.$$

Now, since

$$\log(x-t-\mathrm{i}\varepsilon)=\frac{1}{2}\log((x-t)^2+\varepsilon^2)+\mathrm{i}\left\{\arctan\left(\frac{-\varepsilon}{x-t}\right)-\mathbb{1}_{\{(0,t)\}}(x)\pi\right\},$$

we have

$$\mathrm{Im}\,(\mathscr{L}_\sigma(-t-\mathrm{i}\varepsilon))=-\exp\left\{\frac{\sigma}{2}\int_0^1\log((x-t)^2+\varepsilon^2)q_\sigma(x)\,\mathrm{d}x\right\}\times$$
$$\times\sin\left(\sigma\int_0^1\arctan\left(\frac{\varepsilon}{x-t}\right)q_\sigma(x)\,\mathrm{d}x+\sigma\pi Q_\sigma(t)\right).$$

Hence, by monotone and Lebesgue's dominated convergence theorems (the latter of which applies because of (31)) we obtain

$$\Delta(t)=\frac{1}{\sigma\pi}\,\mathrm{e}^{\sigma\int_0^1\log|x-t|q_\sigma(x)\,\mathrm{d}x}\,\sin\left(\sigma\pi Q_\sigma(t)\right).\tag{36}$$

Finally, as can be found in Estrada and Kanwal (2012), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1\log|x-t|q_\sigma(x)\,\mathrm{d}x\right)=\mathrm{PV}\int_0^1\frac{q_\sigma(x)}{t-x}\,\mathrm{d}x\tag{37}$$

whenever the Cauchy principal value integral in the right hand side exists. It is easy to show that if q_α is Hölder continuous in the singularity point t, then the principal value in (37) exists and it is finite. See *e.g.* Estrada and Kanwal (2012). For arguments which weaken this condition, involving even and odd part of the density function, see Martin and Rizzo (1996). Hence, since q_σ is piecewice Hölder continuous, (37) holds for Lebesgue-almost every $t\in[0,1]$. Therefore, differentiating (36) and substituting in (34), we get the expression in (32). □

Also in this σ-stable NRMI setup, we consider the two noteworthy special cases of $M(\tilde{P}_\sigma)$ having a uniform and a triangular distribution on $[0,1]$ and determine the associated P_0.

EXAMPLE (Uniform case). Let $q_\sigma(x)=\mathbb{1}_{[0,1]}(x)$. Since

$$\mathrm{PV}\int_0^1\frac{\mathrm{d}x}{t-x}=\lim_{\varepsilon\downarrow 0}\left(\int_0^{t-\varepsilon}+\int_{t+\varepsilon}^1\right)\frac{\mathrm{d}x}{t-x}$$
$$=\lim_{\varepsilon\downarrow 0}\{\log t-\log\varepsilon-\log(1-t)+\log\varepsilon\}=\log\frac{t}{1-t},$$

by Theorem 3.9 the cdf of P_0 is

$$F_0(y)=\mathbb{1}_{[1,\infty)}(y)+\frac{1}{\mathrm{e}^\sigma\pi}\int_0^y\left(\frac{1-t}{y-t}\right)^\sigma\left(\frac{t}{1-t}\right)^{\sigma t}\times\left\{\pi\cos(\sigma\pi t)-\sin(\sigma\pi t)\,\log\frac{t}{1-t}\right\}\mathrm{d}t\,\mathbb{1}_{[0,1)}(y)\tag{38}$$

EXAMPLE (Triangular case). Let $q_\sigma(x) = 4x\mathbb{1}_{[0,\frac{1}{2})}(x) + 4(1-x)\mathbb{1}_{[\frac{1}{2},1]}(x)$. If $t \in \left[0, \frac{1}{2}\right)$ then

$$
\begin{aligned}
\mathrm{PV}\int_0^1 \frac{q_\sigma(x)}{t-x}\,\mathrm{d}x &= 4\lim_{\varepsilon\downarrow 0}\left\{\int_0^{t-\varepsilon} + \int_{t+\varepsilon}^{\frac{1}{2}}\right\}\frac{x}{t-x}\,\mathrm{d}x + 4\int_{\frac{1}{2}}^1 \frac{1-x}{t-x}\,\mathrm{d}x \\
&= 4\lim_{\varepsilon\downarrow 0}\left\{-t+\varepsilon - t\log\varepsilon + t\log t - \frac{1}{2}\log\left(\frac{1}{2}-t\right) + t\varepsilon + t\log\varepsilon\right\} + \\
&\quad + 2 - 4(1-t)\log(1-t) + 4(1-t)\log\left(\frac{1}{2}-t\right) = \\
&= 2t\log t - 2(1-t)\log(1-t) + 2(1-2t)\log\left|\frac{1}{2}-t\right|
\end{aligned}
$$

A similar expression holds true when $t \in \left[\frac{1}{2}, 1\right]$. Moreover, for any $t \in (0,1)$ one has

$$
\int_0^1 \log|t-x|\, q_\sigma(x)\,\mathrm{d}x = 2t^2\log t + 2(1-t)^2\log(1-t) - 4\left(t-\frac{1}{2}\right)^2\log\left|t-\frac{1}{2}\right|
$$

Hence by Theorem 3.9

$$
\begin{aligned}
F_0(y) = \frac{1}{\pi}\int_0^y (y-t)^{-\sigma}\,&\frac{(1-t)^{2\sigma(1-t)^2}\, t^{2\sigma t^2}}{\left|\frac{1}{2}-t\right|^{4\sigma(t-\frac{1}{2})^2}} \times \\
&\times \left\{\pi\, q_\sigma(t)\,\cos(\sigma\pi Q_\sigma(t)) - \sin(\sigma\pi Q_\sigma(t))\,\log\frac{t^{2t}\,\left|t-\frac{1}{2}\right|^{2(1-2t)}}{(1-t)^{2(1-t)}}\right\}\mathrm{d}t
\end{aligned}
$$

for $y \in (0,1)$, where $Q_\sigma(t) = \mathbb{1}_{[1,\infty)}(t) + 2t^2\,\mathbb{1}_{[0,\frac{1}{2})}(t) + \left\{-2t^2+4t-1\right\}\mathbb{1}_{[\frac{1}{2},1)}(t)$

3.3. *Base measure of a Pitman–Yor process.* In order to determine the base measure P_0 leading to a specific distribution of a Pitman–Yor mean, we need three ingredients. The first one is our general result in Theorem 3.9. The second one is a useful representation of Pitman–Yor means as a combination of Dirichlet and σ-stable NRMI means given in Theorem 2.1 of James, Lijoi and Prünster (2008). Specifically, let q_σ be the density of $M(\tilde{P}_\sigma)$ with $\tilde{P}_\sigma$ a σ-stable NRMI with base measure P_0 and consider a Dirichlet process with base measure $\alpha(B) = \theta\int_B q_\sigma(x)\,\mathrm{d}x$ for any Borel set B. Then one has

$$
\int x\,\tilde{P}_{\sigma,\theta}(\mathrm{d}x) \stackrel{d}{=} \int x\,\tilde{\mathscr{D}}_\alpha(\mathrm{d}x) \tag{39}
$$

with $\mathbb{E}[\tilde{P}_{\sigma,\theta}] = P_0$. Hence, a Pitman–Yor($\sigma$, θ) mean has the same distribution as a Dirichlet mean with concentration parameter θ and base measure Q given by a normalized σ-stable mean. The third ingredient is a novel identity that can be regarded as a real version of the original Cifarelli–Regazzini identity. This result is crucial to obtain sufficient conditions on the density of a Pitman–Yor mean, which allow to recover an expression of the cdf of the base measure by combining results on Dirichlet and σ-stable means via the distributional identity (39).

PROPOSITION 3.10. *Let q_α be the density of the mean $M(\tilde{\mathscr{D}}_\alpha)$, with $\alpha = P_0$. If q_α is piecewise Hölder continuous, then*

$$
\cos\left(\pi P_0([0,t)) + \frac{\pi}{2}P_0(\{t\})\right)\mathrm{e}^{-\int_0^1 \log|x-t|\,P_0(\mathrm{d}x)} = \mathrm{PV}\int_0^1 \frac{q_\alpha(x)}{x-t}\,\mathrm{d}x \tag{40}
$$

for Lebesgue-almost every $t \in [0,1]$.

PROOF. Consider the Cifarelli–Regazzini identity (16) for $\theta = 1$, which becomes

$$\exp\left\{-\int_0^1 \log(z+x)\, P_0(\mathrm{d}x)\right\} = \int_0^1 \frac{q_\alpha(x)}{z+x}\,\mathrm{d}x \qquad z \in \mathbb{C}\setminus[-1,0] \tag{41}$$

Substituting $z = -t + \mathrm{i}\varepsilon$, with $t \in [0,1]$ and $\varepsilon > 0$, and taking the real part we have on the left hand side

$$\cos\left(-\int_0^1 \left\{\arctan\left(\frac{\varepsilon}{x-t}\right) + \pi \mathbb{1}_{[0,t)}(x) + \frac{\pi}{2}\mathbb{1}_{\{t\}}(x)\right\} P_0(\mathrm{d}x)\right) \times$$
$$\times \exp\left\{-\int_0^1 \log\sqrt{(x-t)^2+\varepsilon^2}\, P_0(\mathrm{d}x)\right\}$$

and on the right hand side

$$\int_0^1 \frac{x-t}{(x-t)^2+\varepsilon^2}\, q_\alpha(x)\,\mathrm{d}x.$$

Since q_α is piecewice Hölder continuous, we get

$$\lim_{\varepsilon\downarrow 0}\int_0^1 \frac{x-t}{(x-t)^2+\varepsilon^2}\, q_\alpha(x)\,\mathrm{d}x = \mathrm{PV}\int_0^1 \frac{q_\alpha(x)}{x-t}\,\mathrm{d}x$$

and the limit is finite for Lebesgue-almost every $t \in [0,1]$. Hence, taking the limit for $\varepsilon \downarrow 0$ also in the left hand side, by virtue of the monotone convergence theorem, we obtain (40). Note that, *a fortiori*,

$$\int_0^1 |\log|x-t||\, P_0(\mathrm{d}x) < \infty$$

for Lebesgue-almost every $t \in [0,1]$.

□

Now we are in a position to state and prove the general result for Pitman–Yor means. Together with the identity in Proposition 3.10, regularity results for singular integrals are employed to determine suitable assumptions on the mean density.

THEOREM 3.11. *Consider a Pitman–Yor process with parameters $(\sigma, 1)$, $\tilde{P}_{\sigma,1}$. Assume the density $q_{\sigma,1}$ of its mean $M(\tilde{P}_{\sigma,1})$ is piecewise C^1 with piecewise Hölder continuous derivative. Then the base measure P_0 of $\tilde{P}_{\sigma,1}$ has cdf given by*

$$F_0(y) = \frac{1}{\pi}\int_0^y (y-t)^{-\sigma}\, e^{\sigma\int_0^1 \log|x-t|\, q_\sigma(x)\,\mathrm{d}x}$$
$$\left\{\pi\, q_\sigma(t)\cos(\sigma\pi\, Q_\sigma(t)) + \sin(\sigma\pi Q_\sigma(t))\,\mathrm{PV}\int_0^1 \frac{q_\sigma(x)}{t-x}\,\mathrm{d}x\right\}\mathrm{d}t \tag{42}$$

with q_σ having cdf given by

$$Q_\sigma(t) = \frac{1}{\pi}\,\mathrm{arccot}\left(\frac{1}{\pi\, q_{\sigma,1}(t)}\,\mathrm{PV}\int_0^1 \frac{q_{\sigma,1}(x)}{x-t}\,\mathrm{d}x\right) \tag{43}$$

PROOF. The result follows in a straightforward way by resorting to the distributional identity in (39), which allows to apply iteratively the representation in (17) and Theorem 3.9. We only need to check that the conditions on $q_{\sigma,1}$ are sufficient to apply the results for Dirichlet

and σ-stable NRMI means. First, since $q_{\sigma,1}$ is piecewice C^1 with bounded derivative, by (17), Q_σ as defined in (43) is the cdf of the base measure of a Dirichlet process whose mean has density $q_{\sigma,1}$.
Now, since $q_{\sigma,1}$ is piecewise Hölder continuous, we can apply Proposition 3.10 and obtain

$$\cos\left(\pi Q_\sigma(t)\right)\exp\left\{-\int_0^1 \log|x-t|\, q_\sigma(x)\,\mathrm{d}x\right\} = \mathrm{PV}\int_0^1 \frac{q_{\sigma,1}(x)}{x-t}\,\mathrm{d}x \tag{44}$$

for Lebesgue-almost every $t\in[0,1]$. Therefore we immediately recover the integrability condition (31) on q_σ. Hence in order to apply Theorem 3.9, we only need the Hölder continuity of q_σ, which is used in the proof to establish the derivative in (37). But, as recalled in Martin and Rizzo (1996), the derivative of the singular integral

$$\mathrm{PV}\int_0^1 \frac{f(x)}{x-t}\,\mathrm{d}x$$

exists and it is equal to the hypersingular integral

$$\mathrm{H}\int_0^1 \frac{f(x)}{(x-t)^2}\,\mathrm{d}x$$

named *Hadamard finite part integral*, whenever the density f is Hölder continuous with Hölder continuous derivative. Therefore, since this is the case for $q_{\sigma,1}$, both sides of (44) are differentiable, (37) holds and we can apply Theorem 3.9. □

REMARK 3.12. An extension of Theorem 3.11 to cover the case of a generic linear functional (2) rather than a simple mean is readily achieved by noting that (39) can be rewritten as

$$\int f(x)\,\tilde{P}_{\sigma,\theta}(\mathrm{d}x) \stackrel{d}{=} \int x\,\tilde{\mathscr{D}}_{q_\sigma}(\mathrm{d}x) \tag{45}$$

where now q_σ is the density function of $\int f\,\mathrm{d}\tilde{P}_\sigma$ and $\mathbb{E}[\tilde{P}_{\sigma,\theta}] = \mathbb{E}[\tilde{P}_\sigma] = P_0$.

The following Corollary of Proposition 3.10 highlights the connection between a Pitman–Yor mean density and the mean density of the σ-stable NRMI obtained by normalizing the σ-stable CRM underling the Pitman–Yor, according to the construction displayed in (8).

COROLLARY 3.13. *Let $q_{\sigma,1}$ be the density of the mean $M(\tilde{P}_{\sigma,1})$ of a Pitman–Yor process with parameters $(\sigma,1)$. If $q_{\sigma,1}$ is piecewise Hölder continuous, then*

$$\frac{\cos\left(\pi Q_\sigma(t)\right)}{1-t}\exp\left\{\mathrm{PV}\int_0^1 \frac{Q_\sigma(x)}{x-t}\,\mathrm{d}x\right\} = \mathrm{PV}\int_0^1 \frac{q_{\sigma,1}(x)}{x-t}\,\mathrm{d}x \tag{46}$$

for Lebesgue-almost every $t\in(0,1)$, where Q_σ is the mean distribution function of the normalized σ-stable $\tilde{P}_\sigma$ in (8).

PROOF. In view of representation (39), it suffices to apply Proposition 3.10, as done in the proof of Theorem 3.11, and consider that

$$\int_0^1 \log|x-t| q_\sigma(x)\,\mathrm{d}x = \int_0^1 \frac{Q_\sigma(t)-Q_\sigma(x)}{x-t}\,\mathrm{d}x + Q_\sigma(t)\log t + (1-Q_\sigma(t))\log(1-t)$$

and

$$\mathrm{PV}\int_0^1 \frac{Q_\sigma(x)}{x-t}\,\mathrm{d}x = Q_\sigma(t)\,\mathrm{PV}\int_0^1 \frac{1}{x-t}\,\mathrm{d}x + \int_0^1 \frac{Q_\sigma(x)-Q_\sigma(t)}{x-y}\,\mathrm{d}x = \tag{47}$$

$$= Q_\sigma(t)\log\left(\frac{1-t}{t}\right) - \int_0^1 \frac{Q_\sigma(t)-Q_\sigma(x)}{x-y}\,\mathrm{d}x$$

Note that, by Proposition 3.10, the Hölder continuity of $q_{\sigma,1}$ entails that q_σ integrates logarithmic singularities, which in turns implies the existence (and finiteness) of the principal value in (47). □

We close this section with an application of Theorem 3.11 to the uniform case, that is determine the parameter measure that makes the distribution of $M(\tilde{P}_{\sigma,1})$ uniform on $[0,1]$. Set $q_{\sigma,1}(x) = \mathbb{1}_{(0,1)}(x)$ and from (43) we get

$$Q_\sigma(x) = \mathbb{1}_{[1,\infty)}(x) + \frac{1}{\pi}\,\mathrm{arccot}\left(\frac{1}{\pi}\,\log\frac{1-x}{x}\right)\,\mathbb{1}_{(0,1)}(x). \tag{48}$$

Denote by q_σ the density function corresponding to Q_σ. Finally, set

$$\begin{aligned}\xi(t) &= \frac{1}{t(1-t)}\int_0^1 (1-x-t)\,q_\sigma(x)\,\mathrm{d}x\\ &\quad + \frac{2}{t(1-t)}\int_0^1 \frac{(\log|t-x|)\,\left(\log\frac{1-x}{x}\right)}{\pi^2+\log^2\frac{1-x}{x}}\,q_\sigma(x)\,\mathrm{d}x\end{aligned} \tag{49}$$

for any $t \in (0,1)$. By virtue of Theorem 3.11 one can state the following

PROPOSITION 3.14. *The distribution of the mean $M(\tilde{P}_{\sigma,1})$ of a Pitman–Yor process with parameters $(\sigma,1)$ is uniform on $(0,1)$ if and only if its base measure P_0 has cdf*

$$F_0(y) = \frac{1}{\pi}\mathrm{e}^{\sigma\int_0^1 \log|y-x|\,q_\sigma(x)\,\mathrm{d}x}\,\{\pi\,q_\sigma(t)\,\cos(\sigma\pi\,Q_\sigma(t)) + \xi(y)\,\sin(\sigma\pi Q_\sigma(t))\} \tag{50}$$

for any $y \in (0,1)$, where Q_σ and ξ are as in (48) and (49), respectively.

PROOF. The density function corresponding to (48) is

$$q_\sigma(x) = \frac{1}{x(1-x)}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathbb{1}_{(0,1)}(x).$$

As for the evaluation of the principal value integral appearing in (42), note that

$$\begin{aligned}&\left(\int_0^{t-\varepsilon}+\int_{t+\varepsilon}^1\right)\frac{1}{x(t-x)}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathrm{d}x\\ &= \frac{1}{t}\left(\int_0^{t-\varepsilon}+\int_{t+\varepsilon}^1\right)\frac{1}{x}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathrm{d}x\\ &\quad+\frac{1}{t}\left(\int_0^{t-\varepsilon}+\int_{t+\varepsilon}^1\right)\frac{1}{t-x}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathrm{d}x =: I_{1,\varepsilon}+I_{2,\varepsilon}.\end{aligned}$$

To shorten the notation below, set $\zeta_{1,\varepsilon} = (\log\varepsilon)/\{\pi^2+\log^2[(1-t+\varepsilon)/(t-\varepsilon)]\}$ and $\zeta_{2,\varepsilon} = (\log\varepsilon)/\{\pi^2+\log^2[(1-t-\varepsilon)/(t+\varepsilon)]\}$. A simple change of variable leads to

$$\begin{aligned}I_{2,\varepsilon} &= \frac{1}{t}\int_\varepsilon^t \frac{1}{y}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-t+y}{t-y}\right\}}\,\mathrm{d}y - \frac{1}{t}\int_\varepsilon^{1-t}\frac{1}{y}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-t-y}{t+y}\right\}}\,\mathrm{d}y\\ &= \frac{1}{t}\left\{-\zeta_{1,\varepsilon}+2\int_\varepsilon^t \frac{(\log y)\,\left(\log\frac{1-t+y}{t-y}\right)}{(t-y)(1-t+y)\left\{\pi^2+\log^2\frac{1-t+y}{t-y}\right\}^2}\,\mathrm{d}y\right.\end{aligned}$$

$$+ \zeta_{2,\varepsilon} + 2\int_{\varepsilon}^{1-t} \frac{(\log y)\left(\log\frac{1-t-y}{t+y}\right)}{(t+y)(1-t-y)\left\{\pi^2+\log^2\frac{1-t-y}{t+y}\right\}^2}\,\mathrm{d}y\Bigg\}.$$

Note also that

$$\left(\int_0^{t-\varepsilon} + \int_{t+\varepsilon}^1\right)\frac{1}{(1-x)(t-x)}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathrm{d}x$$

$$= -\frac{1}{1-t}\left(\int_0^{t-\varepsilon} + \int_{t+\varepsilon}^1\right)\frac{1}{1-x}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathrm{d}x$$

$$+\frac{1}{1-t}\left(\int_0^{t-\varepsilon} + \int_{t+\varepsilon}^1\right)\frac{1}{t-x}\,\frac{1}{\left\{\pi^2+\log^2\frac{1-x}{x}\right\}}\,\mathrm{d}x =: J_{1,\varepsilon} + J_{2,\varepsilon}.$$

Moreover, $J_{2,\varepsilon} = tI_{2,\varepsilon}/(1-t)$, for $i = 1, 2$. □

4. Application to mixture models. Our findings are also of practical relevance for Bayesian nonparametric modeling, where random probability measures play a fundamental role, since their law acts as nonparameteric prior distribution. When one has enough *a priori* information for eliciting the distribution of the mean (or some linear functional), as is often the case, our results allow to specify the corresponding infinite-dimensional $\tilde{P}$ accordingly. This represents an important step forward in nonparametric prior elicitation. However, random probability measures are typically not used to model directly the data but rather as main ingredient of more complex models, most notably mixture models, which are ubiquitous in the Statistics and Machine Learning literature (Orbanz and Teh, 2010; Müller et al., 2015).

Letting $\mathbb{Y}$ be a complete and separable metric space equipped with the Borel σ-algebra $\mathscr{Y}$, a random mixture density (absolutely continuous with respect to some σ-finite measure ν on $\mathbb{Y}$) is defined as

$$\tilde{f}(y) = \int_{\mathbb{X}} k(y;x)\,\tilde{P}(\mathrm{d}x) \tag{51}$$

where $\{k(\,\cdot\,;x) : x \in \mathbb{X}\}$ is a collection of density functions on $\mathbb{Y}$ indexed by a parameter taking values in $\mathbb{X}$. When $\tilde{P}$ is a Dirichlet process one obtains the popular Dirichlet process mixture introduced by Lo (1984). Mixtures based on σ-stable NRMIs or Pitman–Yor processes represent valid alternatives with appealing features especially in terms of clustering and robustness. See, e.g., Ishwaran and James (2001), Lijoi, Mena and Prünster (2007) and Barrios et al. (2013).

Also for mixture models, the procedure of assigning a prescribed distribution of the mean, such as in, *e.g.*, Kessler, Hoff and Dunson (2015), and then determining the underlying base measure P_0, is of great interest. In this scenario the data are assumed exchangeable from $\tilde{f}$ in (51), namely $Y_1,\ldots,Y_n \,|\, \tilde{f} \overset{\text{iid}}{\sim} \tilde{f}$ for any $n \geq 1$. Hence, one could aim at specifying the base measure P_0 of $\tilde{P}$ in (51) such that a prescribed distribution for the random mean $\int_{\mathbb{Y}} y\,\tilde{f}(y)\,\nu(\mathrm{d}y)$ is attained. As for cases discussed in Section 3, the distribution of the population mean $\mathbb{E}[Y_i \,|\, \tilde{f}]$ is easier to elicit from experts' opinions since it is a univariate random element.

Here we thus extend the results of the previous section to cover mixture models. This is achieved by combining Remark 3.1 with the observation that studying a linear functional of the mixture (51) reduces to studying a (different) linear functional of the underlying $\tilde{P}$, since

$$\int_{\mathbb{Y}} g(y)\, \tilde{f}(y)\nu(\mathrm{d}y) = \int_{\mathbb{X}} h(x)\, \tilde{P}(\mathrm{d}x) \tag{52}$$

where $h(x) = \int_{\mathbb{Y}} g(y)k(y;x)\nu(\mathrm{d}y)$. This strategy was applied in Nieto-Barajas, Prünster and Walker (2004) and James, Lijoi and Prünster (2010) for deriving the distribution of means of Dirichlet process and NRMI mixtures. From (52), it follows immediately that Theorems 3.3, 3.9 and 3.11 hold also for mixture models by suitably adapting the specification of f. Furthermore, note that the mixture procedure allows to extend our results on discrete random probability measures to absolutely continuous ones.

THEOREM 4.1. *Let $\tilde{f}$ be a mixture density as in* (51) *and let q be the density of the corresponding mean* (52) *for $g : \mathbb{Y} \to \mathbb{R}^+$ a measurable function.*

(a) *Assume $\tilde{P} = \tilde{\mathscr{D}}_\alpha$ in* (51)*, $\alpha = P_0$, q is pointwise C^1 with bounded derivative and* $\mathrm{supp}(q) = [0,1]$. *Then*

$$P_0 \circ h^{-1}([0,x]) = \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi\, q_\alpha(x)} \operatorname{PV} \int_0^1 \frac{q_\alpha(t)}{t-x}\, \mathrm{d}t \right) \tag{53}$$

for any $x \in (0,1)$, where $h(x) = \int_{\mathbb{Y}} g(y)\, k(y;x)\, \nu(\mathrm{d}x)$.
Instead, for $\alpha = \theta P_0$ with $\theta \in (0,1)$ and q also satisfying condition (19)*, we have*

$$P_0 \circ h^{-1}([0,x]) = \frac{1}{\theta\pi} \arctan\left(\frac{\sin(\theta\pi)}{\cos(\theta\pi) + \mathscr{I}_\theta[\, q;\, x\,]} \right) + \frac{1}{\theta} \mathbb{1}_{(x_*,\infty)}(x) \tag{54}$$

for any $x \in (0,1)$, where $\mathscr{I}_\theta$ is defined in (18) *and*

$$x_* = \inf\left\{ x \in [0,1] \,\middle|\, \mathscr{I}_\theta[\, q_\alpha;\, x\,] \le -\cos(\theta\pi) \right\}$$

(b) *Assume $\tilde{P} = \tilde{P}_\sigma$ in* (51) *with $P_0 = \mathbb{E}[\tilde{P}_\sigma]$ and q also satisfies condition* (31) *Lebesgue-almost everywhere. Then*

$$\begin{aligned} P_0 \circ h^{-1}([0,x]) = \frac{1}{\pi} \int_0^x\ & (x-t)^{-\sigma}\, \mathrm{e}^{\sigma \int_0^1 \log|s-t|\, q(s)\, \mathrm{d}s} \\ & \left\{ \pi\, q(t)\, \cos(\sigma\pi\, Q(t)) + \sin(\sigma\pi Q(t)) \operatorname{PV} \int_0^1 \frac{q(s)}{t-s}\, \mathrm{d}s \right\} \mathrm{d}t \end{aligned} \tag{55}$$

for any $x \in (0,1)$, where Q is the distribution function of the mean (52).

The expressions in (53), (54) and (55) can be used to determine the parameter measure yielding a specified probability distribution for a mean of a mixture model governed by either a Dirichlet or a σ-stable NRMI.

As for the practical implications of Theorem 4.1, we recall that in Bayesian density estimation problems the availability of P_0 is crucial for the implementation of computational algorithms that evaluate functionals of the posterior distribution, given exchangeable observations $Y_1, \ldots, Y_n$ from $\tilde{f}$. For example, in the Pólya urn Gibbs sampler one has to generate a sample from a distribution that is proportional to $\int_{\mathbb{X}} k(y;x)P_0(\mathrm{d}x)$. This is straightforward if P_0 is conjugate to $k(\,\cdot\,;x)$, but also for the non-conjugate scenario well-established algorithms are available (see e.g. MacEachern and Müller, 1998; Neal, 2000). The latter represents the most likely if P_0 is fixed according to Theorem 4.1 so to ensure that a certain density function q_α of the mean $\int_Y g(y)\, \tilde{f}(y)\, \nu(\mathrm{d}y)$ is attained.

We conclude describing an example involving the σ-stable case.

EXAMPLE. Consider $\tilde{f}$ as in (51) with $\tilde{P} = \tilde{P}_\sigma$. Moreover, set $k(y;x) = x\,\mathrm{e}^{-xy}\mathbb{1}_{(0,\infty)}(y)$ and $g(y) = y$, which leads to $h(x) = x^{-1}$ in (52). Thus, the goal is to determine the base measure P_0 that induces a specified distribution for the mean of a σ-stable NRMI mixture of exponential densities: if we set $q_\sigma(x) = \mathbb{1}_{[0,1]}(x)$, it can be easily seen that

$$P_0 \circ g^{-1}((0,x]) = \mathbb{1}_{[1,\infty)}(x) + \frac{\mathrm{e}^\sigma}{\pi} \int_0^x (x-t)^{-\sigma}\,(1-t)^{\sigma(1-t)}\,t^{\sigma t} \times \left\{ \pi\cos(\sigma\pi t) - \sin(\sigma\pi t)\,\log\frac{1-t}{t} \right\} \mathrm{d}t\,\mathbb{1}_{(0,1)}(x).$$

This implies that $\mathrm{supp}(P_0) = [1,\infty)$ and

$$P_0((0,x]) = \mathbb{1}_{[1,\infty)}(x)\,\left\{ 1 - \frac{\mathrm{e}^\sigma\,x^\sigma}{\pi} \int_0^{1/x} (1-xt)^{-\sigma}\,(1-t)^{\sigma(1-t)}\,t^{\sigma t} \times \left[\pi\cos(\sigma\pi t) - \sin(\sigma\pi t)\,\log\frac{1-t}{t} \right] \mathrm{d}t \right\}.$$

One can proceed in a similar fashion for different linear functionals of interest such as $g(y) = \mathbb{1}_{[T,\infty)}(y)$ for some $T > 0$, which yields $h(x) = \exp\{-xT\}$. The distribution of the mean of the mixture is the distribution of an average survival probability $\int \exp\{-xT\}\,\tilde{P}_\sigma(\mathrm{d}x)$ at T. Assuming again a uniform distribution is the desired mean distribution, P_0 has to be of the form

$$P_0((0,x]) = \mathbb{1}_{[0,\infty)}(x) \left\{ 1 - \frac{\mathrm{e}^\sigma}{\pi} \int_0^{\mathrm{e}^{-xT}} \left(\mathrm{e}^{-xT} - t\right)^{-\sigma}\,(1-t)^{\sigma(1-t)}\,t^{\sigma t} \times \left[\pi\cos(\sigma\pi t) - \sin(\sigma\pi t)\,\log\frac{1-t}{t} \right] \mathrm{d}t \right\}.$$

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REFERENCES

ALDOUS, D. J. (1985). *Exchangeability and related topics*. *Lecture Notes in Math., 1117*. Springer.

BARLOW, M., PITMAN, J. and YOR, M. (1989). A multidimensional extension of the arc-sine law. In *Séminaire de Probabilités, XXIII*. *Lecture Notes in Math.* **1372** 294–314. Springer.

BARRIOS, E., LIJOI, A., NIETO-BARAJAS, L. E. and PRÜNSTER, I. (2013). Modeling with normalized random measure mixture models. *Stat. Sci.* **28** 313 – 334.

BERTOIN, J. (2006). *Random Fragmentation and Coagulation Processes*. *Cambridge Studies in Advanced Mathematics*. Cambridge University Press.

BONDESSON, L. (1992). *Generalized gamma convolutions and related classes of distributions and densities*. *Lecture Notes in Statistics* **76**. Springer-Verlag, New York.

CHAMAYOU, J.-F. and WESOŁOWSKI, J. (2009). Lauricella and Humbert functions through probabilistic tools. *Integral Transforms Spec. Funct.* **20** 529–538.

CIFARELLI, D. M. and REGAZZINI, E. (1979). A general approach to Bayesian analysis of nonparametric problems. *Decis. Econ. Finance* **2** 39-52.

CIFARELLI, D. M. and REGAZZINI, E. (1990). Distribution functions of means of a Dirichlet process. *Ann. Stat.* **18** 429–442.

CIFARELLI, D. M. and REGAZZINI, E. (1993). Some remarks on the distribution functions of means of a Dirichlet process. Technical Report No. 93.4, IMATI–CNR, Milano.

DE FINETTI, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Ann. Henri Poincaré* **1** 1-68.

DELLO SCHIAVO, L. (2019). Characteristic functionals of Dirichlet measures. *Electron. J. Probab.* **24** 1 – 38.

DIACONIS, P. and KEMPERMAN, J. (1996). Some new tools for Dirichlet priors. In *Bayesian Statistics*, (J. M. Bernardo, J. O. Berger, A. P. Dawid and A. F. M. Smith, eds.) **5** 97–106. Oxford University Press.

DIACONIS, P. and SKYRMS, B. (2018). *Ten Great Ideas about Chance*. Princeton University Press.

EL-DAKKAK, O. and PECCATI, G. (2008). Hoeffding decompositions and urn sequences. *Ann. Probab.* **36** 2280 – 2310.

ESTRADA, R. and KANWAL, R. P. (2012). *Singular integral equations*. Birkhäuser Boston.

FERGUSON, T. S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Stat.* **1** 209 – 230.

FLINT, I. and TORRISI, G. L. (2021). An Integration by Parts Formula for Functionals of the Dirichlet-Ferguson Measure, and Applications. *Potential Anal.*

GAFFI, F., LIJOI, A. and PRÜNSTER, I. (2023). Transition probabilities of continual Young diagrams and Dirichlet random means. *Work in progress*.

GHOSAL, S. and VAN DER VAART, A. (2017). *Fundamentals of nonparametric Bayesian inference*. *Cambridge Series in Statistical and Probabilistic Mathematics* **44**. Cambridge University Press, Cambridge.

GURLAND, J. (1948). Inversion formulae for the distribution of ratios. *Ann. Math. Stat.* **19** 228 – 237.

HILL, T. and MONTICINO, M. (1998). Constructions of random distributions via sequential barycenters. *Ann. Stat.* **26** 1242 – 1253.

ISHWARAN, H. and JAMES, L. F. (2001). Gibbs sampling methods for stick-breaking priors. *J. Am. Stat. Assoc.* **96** 161–173.

JAMES, L. F. (2010). Lamperti-type laws. *Ann. Appl. Probab.* **20** 1303 – 1340.

JAMES, L. F., LIJOI, A. and PRÜNSTER, I. (2008). Distributions of linear functionals of two parameter Poisson–Dirichlet random measures. *Ann. Appl. Probab.* **18** 521–551.

JAMES, L. F., LIJOI, A. and PRÜNSTER, I. (2010). On the posterior distribution of classes of random means. *Bernoulli* **16** 155–180.

JAMES, L. F., ROYNETTE, B. and YOR, M. (2008). Generalized Gamma Convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surv.* **5** 346 – 415.

KALLENBERG, O. (1975). *Random measures*. Akademie-Verlag.

KALLENBERG, O. (2017). *Random measures, theory and applications*. Springer.

KARP, D. and PRILEPKINA, E. (2012). Generalized Stieltjes functions and their exact order. *J. Class. Anal.* **1** 53–74.

KEROV, S. V. (1993). Transition probabilities for continual Young diagrams and Markov moment problem. *Funct. Anal. its Appl.* **27** 104-117.

KEROV, S. V. (1998). Interlacing measures. In *Kirillov's Seminar on Representation Theory*. *AMS Translation Series 2* **181** 35-83. American Mathematical Society, Providence.

KEROV, S. V. and TSILEVICH, N. (2004). The Markov–Krein correspondence in several dimensions. *J. Math. Sci.* **121** 2345–2359.

KESSLER, D. C., HOFF, P. D. and DUNSON, D. B. (2015). Marginally specified priors for nonparametric Bayesian estimation. *J. R. Stat. Soc. Series B Stat. Methodol.* **77** 35-58.

KINGMAN, J. F. C. (1967). Completely random measures. *Pac. J. Math.* **21** 59 – 78.

KINGMAN, J. F. C. (1975). Random discrete distributions. *J. R. Stat. Soc. Series B Stat. Methodol.* **37** 1–22.

KINGMAN, J. F. C. (1978). The Representation of Partition Structures. *J. Lond. Math. Soc.* **s2-18** 374-380.

KINGMAN, J. F. C. (1982). The coalescent. *Stoch. Process. Their Appl.* **13** 235-248.

KINGMAN, J. F. C. (1993). *Poisson processes*. *Oxford Studies in Probability*. Oxford University Press.

LAMPERTI, J. W. (1958). An occupation time theorem for a class of stochastic processes. *Trans. Am. Math. Soc.* **88** 380-387.

LÉVY, P. (1939). Sur certains processus stochastiques homogènes. *Compos. Math.* **7** 283-339.

LIJOI, A., MENA, R. H. and PRÜNSTER, I. (2007). Controlling the reinforcement in Bayesian non-parametric mixture models. *J. R. Stat. Soc. Series B Stat. Methodol.* **69** 715–740.

LIJOI, A. and PRÜNSTER, I. (2009). Distributional properties of means of random probability measures. *Stat. Surv.* **3** 47–95.

LIJOI, A. and PRÜNSTER, I. (2010). Models beyond the Dirichlet process. In *Bayesian nonparametrics* (N. L. Hjort, C. Holmes, P. Müller and S. G. Walker, eds.) 80-136. Cambridge University Press, Cambridge.

LIJOI, A. and PRÜNSTER, I. (2011). A conversation with Eugenio Regazzini. *Stat. Sci.* **26** 647 – 672.

LIJOI, A. and REGAZZINI, E. (2004). Means of a Dirichlet process and multiple hypergeometric functions. *Ann. Probab.* **32** 1469–1495.

LO, A. Y. (1984). On a class of Bayesian nonparametric estimates: I. Density estimates. *Ann. Stat.* **12** 351 – 357.

MACEACHERN, S. N. and MÜLLER, P. (1998). Estimating Mixture of Dirichlet Process Models. *J. Comput. Graph. Stat.* **7** 223–238.

MARTIN, P. A. and RIZZO, F. J. (1996). Hypersingular integrals: how smooth must the density be? *Int. J. Numer. Methods Eng.* **39** 687-704.

MÜLLER, P., QUINTANA, F. A., JARA, A. and HANSON, T. (2015). *Bayesian nonparametric data analysis*. *Springer Series in Statistics*. Springer, Cham.

NEAL, R. M. (2000). Markov Chain Sampling Methods for Dirichlet Process Mixture Models. *J. Comput. Graph. Stat.* **9** 249–265.

NIETO-BARAJAS, L. E., PRÜNSTER, I. and WALKER, S. G. (2004). Normalized random measures driven by increasing additive processes. *Ann. Stat.* **32** 2343–2360.

ORBANZ, P. and TEH, Y. (2010). Bayesian Nonparametric Models. *Encyclopedia of Machine Learning*.

PECCATI, G. (2004). Hoeffding-ANOVA decompositions for symmetric statistics of exchangeable observations. *Ann. Probab.* **32** 1796 – 1829.

PECCATI, G. (2008). Multiple integral representation for functionals of Dirichlet processes. *Bernoulli* **14** 91 – 124.

PERMAN, M., PITMAN, J. and YOR, M. (1992). Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Relat. Fields* **92** 21–39.

PITMAN, J. (1995). Exchangeable and partially exchangeable random partitions. *Probab. Theory Relat. Fields* **102** 145–158.

PITMAN, J. (1996). Some developments of the Blackwell–MacQueen urn scheme. In *Lecture Notes - Monograph Series*, (T. S. Ferguson, J. B. Shapley and J. B. MacQueen, eds.). *Statistics, probability and game theory: papers in honor of David Blackwell* 245–267. Institute of Mathematical Statistics.

PITMAN, J. (2006). *Combinatorial stochastic processes: Ecole d'eté de probabilités de Saint-Flour XXXII-2002 (J. Picard ed.)*. *Lecture Notes in Mathematics* **1875**. Springer.

PITMAN, J. (2018). Random weighted averages, partition structures and generalized arcsine laws. *Probab. Surveys*, to appear.

PITMAN, J. and YOR, M. (1997a). On the relative lengths of excursions derived from a stable subordinator. In *Séminaire de probabilités XXXI* (J. Azéma, M. Yor and M. Émery, eds.) 287–305. Springer, Berlin.

PITMAN, J. and YOR, M. (1997b). The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator. *Ann. Probab.* 855–900.

PULKKINEN, O. (2007). Boundary driven zero-range processes in random media. *J. Stat. Phys.* **128** 1289-1305.

REGAZZINI, E., GUGLIELMI, A. and DI NUNNO, G. (2002). Theory and numerical analysis for exact distributions of functionals of a Dirichlet process. *Ann. Stat.* **30** 1376 – 1411.

REGAZZINI, E., LIJOI, A. and PRÜNSTER, I. (2003). Distributional results for means of normalized random measures with independent increments. *Ann. Stat.* **31** 560–585.

ROMIK, D. (2004). Explicit formulas for hook walks on continual Young diagrams. *Adv. Appl. Math.* **32** 625–654.

SCHWARZ, J. H. (2005). The generalized Stieltjes transform and its inverse. *J. Math. Phys.* **46**.

SUMNER, D. B. (1949). An inversion formula for the generalized Stieltjes transform. *Bull. Am. Math. Soc.* **55** 174–183.

TSILEVICH, N. (1999). Distribution of the mean value for certain random measures. *J. Math. Sci.* **96** 3616–3623.

VERSHIK, A. M., YOR, M. and TSILEVICH, N. (2004). On the Markov–Krein identity and quasi-invariance of the gamma process. *J. Math. Sci.* **121** 2303–2310.

VON NEUMANN, J. (1941). Distribution of the Ratio of the Mean Square Successive Difference to the Variance. *Ann. Math. Stat.* **12** 367 – 395.

WATSON, G. S. (1956). On the joint distribution of the circular serial correlation coefficients. *Biometrika* **43** 161-168.

WIDDER, D. V. (2015). *Laplace transform*. Princeton University Press.