

# A Generalized Index of Fractionalization

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**Abstract.** This paper characterizes an index that is informationally richer than the commonly used ethno-linguistic fractionalization (*ELF*) index. Our measure of fractionalization takes as a primitive the individuals, as opposed to ethnic groups, and uses information on the similarity among them. Compared to existing indices, our measure does not require that individuals are pre-assigned to exogenously determined categories or groups. We provide an empirical illustration of how our index can be operationalized and what difference it makes as compared to the standard *ELF* index. This application pertains to the pattern of fractionalization in the United States. *Journal of Economic Literature* Classification Nos.: C43, D63.

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# 1 Introduction

The role of ethnic and cultural diversity has received increasing attention by economists in recent years. Numerous contributions have analyzed the relationship between ethnic heterogeneity and socioeconomic outcomes, including public good provision, growth, corruption and social capital. The transmission of cultural traits and the formation of heterogeneity have also been studied theoretically and empirically. Among the first group of studies, ethnic diversity has been shown to be associated with lower growth rates (Easterly and Levine, 1997), more corruption (Mauro, 1995), lower contributions to local public goods (Alesina, Baqir and Easterly, 1999), lower participation in groups and associations (Alesina and La Ferrara, 2000) and a higher propensity to form jurisdictions to sort into homogeneous groups (Alesina, Baqir and Hoxby, 2004). For a review of contributions on the relationship between ethnic diversity and economic performance, see Alesina and La Ferrara (2005). For the formation and transmission of cultural traits see, among others, Bisin and Verdier (2000), Fernandez, Fogli and Olivetti (2004), and Giuliano (2007). The growing interest in these topics is likely attributable to the upward trend in migration flows and the fact that many societies are becoming increasingly heterogeneous from a cultural point of view.

Yet the economics literature does not seem to have advanced very far in the measurement of ethnic and cultural diversity. This contrasts with the breadth of the literature on the measurement of income inequality, the traditional notion of heterogeneity employed by economists. While we can rely on a variety of indices of economic inequality, and these indices have been axiomatically characterized from a theoretical point of view, the economic literature on the measurement of categorical heterogeneity is much less developed. Virtually every empirical contribution on the topic uses the so-called index of ethnolinguistic fractionalization (*ELF*), which is a decreasing transformation of the Herfindahl

concentration index applied to population shares. The *ELF* index measures the probability that two randomly drawn individuals from the overall population belong to different (pre-defined) ethnic groups. While *ELF* has the advantage of being simple to compute and easy to interpret, its economic underpinnings seem inadequate. To our knowledge, the only paper that attempts to provide a theoretical background for the use of *ELF* is the one by Vigdor (2002). He proposes a behavioral interpretation of *ELF* in a model where individuals display differential altruism. He assumes that an individual's willingness to spend on local public goods depends partly on the benefits that other members of the community derive from the good, and that the weights of this altruistic component vary depending on how many members of the community share the same ethnicity of that individual. Notice that our goal here is to provide a characterization, rather than a behavioral interpretation, of a new index of fractionalization.

The implicit contention in economic models is often that different ethnic groups may have different preferences and this would generate conflicts of interest in economic decisions. It is hard to believe that population shares would be enough to capture the extent of divergence in preferences among society's members. Presumably, people of different culture or ethnicity will feel differently about each other depending on how similar they are in other dimensions. A second channel through which ethnic or cultural diversity may affect economic performance is the existence of possible skill complementarities among different types. But again, it is unlikely that simple population shares will capture the nature and extent of skill complementarities among groups.

If the rationale for including ethnic diversity effects in economic models lies in preferences or technological features, then measuring fractionalization purely as a function of population shares seems a severe limitation. Similarity between individuals should play a role. This similarity could depend, for example, on language spoken, age, educational

background, employment status, just to mention a few attributes. If preferences might be induced by these other characteristics, then considering similarities between individuals will give a better understanding of the potential conflict in economic decisions. Providing a measure of fractionalization that accounts for the degree of similarity among agents seems therefore an important task.

The goal of this paper is to characterize a *generalized fractionalization index (GELF)* that takes as primitive the individuals and uses information on their similarities to measure fractionalization. We propose to use as a building block a *similarity matrix* containing pairwise similarity values  $\{s_{ij}\}$  among any two individuals  $i$  and  $j$  in society. An entry equal to one in the matrix represents perfect similarity among individuals, an entry equal to zero complete dissimilarity. We then rely on four axioms to characterize *GELF*. The first axiom is normalization, requiring that in a society with maximal similarity our diversity index takes value zero and in a society with maximal dissimilarity it takes a positive value. The second axiom, anonymity, requires that individuals are treated impartially, that is, our diversity measure is invariant with respect to permutations. The third axiom, additivity, imposes a separability property on our index. The fourth and last axiom, replication invariance, requires the index to be invariant with respect to replications of the population. We prove that a diversity measure satisfies these four axioms if and only if it is a decreasing function of the sum of similarity values in the matrix, scaled by the square of the population size. We call this generalized index *GELF* and show that it is a natural extension of *ELF*. More generally, depending on the metric used to measure similarity among individuals and on the level of aggregation of the information (that is, similarity among individuals or among groups), our index nests a number of indices used in the literature. In the limit case where the information is purely categorical (that is, similarity is either zero or one), our measure reduces to *ELF*. In richer information

settings where measuring the distance among individuals is feasible and meaningful, our index conveys a broader measure of diversity. The flexibility of our formulation and its suitability to being applied in very different informational environments are an advantage of the measure we propose. Another advantage is that our index does not require that individuals are pre-assigned to exogenously determined categories or groups. Our theoretical framework can actually be used to determine an endogenous partition of society into groups. Relevant groups may be constituted by clusters of individuals who have perfect (or very high) similarity among themselves, and share the same (or very close) similarity values vis-a-vis the rest of society.

We also provide an empirical illustration of how *GELF* can be operationalized and what difference it makes as compared to the standard *ELF* index. This application pertains to the pattern of fractionalization in the United States. Using individual level data from the 1990 Census, we compute the two indices for all US states. We find that the ranking of several states changes significantly when we use *GELF* rather than *ELF*. For example, in 1990 Hawaii was the most diverse state in terms of ethnic diversity (*ELF*) and California was the fifth. When we compute *GELF* embedding information on similarity in income, education and employment, as well as ethnicity, Hawaii moves to the 42nd place and California to the 30th. This is because economic opportunities in these states are relatively more equal across races than they are in other states. The District of Columbia, on the other hand, is the second most fractionalized on the basis of *ELF* and becomes the most fractionalized—by a wide margin—when we use *GELF*. Finally, we compute *grouped* versions of the *GELF* index and show how each variable contributes to the pattern of similarity among races.

Our paper is related to several strands of the literature. First, it naturally relates to the economics literature on ethnic diversity and its economic effects. While the bulk of

this literature does not focus on the specific issue of measurement, a few contributions do. As the majority of applications have used language as a proxy for ethnicity, some authors have criticized the use of *ELF* on the grounds that linguistic diversity may not correspond to ethnic diversity. Among these, Alesina, Devleeschauwer, Easterly, Kurlat and Wacziarg (2003) have proposed a classification into groups that combines information on language with information on skin color. These authors propose three measures of fractionalization: one purely linguistic, one related to religion, and one that broadly defines ethnicity by combining language and skin color. Note that this approach differs from ours because it defines ethnic (or linguistic, or religious) categories on the basis of certain criteria and then applies the *ELF* formula to the resulting number of groups.

Other authors, in particular Fearon (2003), have criticized standard applications of *ELF* on the grounds that they would fail to account for the salience of ethnic distinctions in different contexts. For example, the same two ethnic groups may be allies in one country and opponents in another, and using simply their shares in the population would fail to capture this. We share Fearon's concerns on this point, and indeed we hope that our index can be a first step towards incorporating issues of salience in the measurement of fractionalization, albeit in a simplistic way. In particular, if one thinks that differences in income, or education, or any other measurable characteristic may be the reason why ethnicity matters only in certain contexts, our *GELF* index already weighs ethnic categories by their salience.

Turning to the notion of distance among ethnic groups, relatively little has been done. Using a heuristic approach, Laitin (2000) and Fearon (2003) rely on measures of distance between languages to assess how different linguistic groups are across countries. In particular, in his 2003 contribution Fearon proposes a measure of 'cultural fractionalization' that adapts Greenberg's (1956) formula by weighing population shares with a

resemblance factor that depends on the number of shared classifications between any two languages. This measure intuitively captures the expected cultural distance between two people drawn at random from the population. As we show below, this measure can be derived as a special case of our *GELF* index. Caselli and Coleman (2006) stress the importance of ethnic distance in a theoretical model and propose to measure it using surveys of anthropologists. Finally, a few recent contributions underline the correlation between genetic distance and pairwise income differences, trust and trade flows (Giuliano, Spilimbergo and Tonon, 2006, Spolaore and Wacziarg, 2009, and Guiso, Sapienza and Zingales, forthcoming).

Second, the paper relates to the literature on ethnic polarization. In her original contribution, Reynal-Querol (2002) adapts the measure of polarization developed by Esteban and Ray (1994) to the case of categorical variables, such as ethnicity or religion, and proposes an index of ethnic polarization,  $RQ$ , which captures how far the distribution of ethnic groups is from the bipolar case. Montalvo and Reynal-Querol (2005) show that the  $RQ$  index is a more powerful predictor of the incidence of civil wars than  $ELF$ . The authors also show that  $RQ$  is highly correlated with  $ELF$  at low levels, uncorrelated at intermediate levels and negatively correlated at high levels. In a recent contribution (Montalvo and Reynal-Querol, 2008), the same authors analyze the theoretical properties of  $RQ$  and show that the explanatory power of  $RQ$  for the incidence of wars is greater the higher the intensity of the conflict. Desmet, Ortuño-Ortín and Weber (2005) focus on ethno-linguistic conflict that arises between a dominant central group and peripheral minority groups. To this end, the authors propose an index of peripheral ethno-linguistic diversity,  $PD$ , which can capture both the notion of diversity and of polarization. The relationship between  $RQ$ ,  $PD$  and  $GELF$  is discussed in section 4.

Third, the paper is related to the theoretical economics literature on the measurement

of diversity. For example, Weitzman (1992) suggests an index that is primarily intended to measure biodiversity. Moreover, the measurement of diversity has become an increasingly important issue in the recent literature on the ranking of opportunity sets in terms of freedom of choice, where opportunity sets are interpreted as sets of options available to a decision maker. Examples for such studies include Weitzman (1998), Pattanaik and Xu (2000), Nehring and Puppe (2002) and Bossert, Pattanaik and Xu (2003). A fundamental difference between the above-mentioned contributions and the approach followed in this paper is the informational basis employed which results in a very different set of axioms that are suitable for a measure of diversity. Both Weitzman's (1992) seminal paper and the literature on incorporating notions of diversity in the context of measuring freedom of choice proceed by constructing a ranking of *sets* of objects, interpreted as sets of species in the case of biodiversity and as sets of available options in the context of freedom of choice. On the other hand, we operate in an informationally richer environment: not only whether a group is present may influence the measure of fractionalization, but also the relative population shares of these groups along with the pairwise similarities among them. We are interested in capturing a different aspect of diversity than Nehring and Puppe (2002), namely the instrumental—as opposed to intrinsic—value of diversity, where the number of individuals plays a key role.

Finally, *ELF* is also used in the literature on network formation as a measure of heterogeneity in the underlying population, where distances in characteristics translate into distances in connections in the network (see, for example, Moody, 2001).

The remainder of the paper is organized as follows. In section 2 we introduce the notion of a similarity matrix, we present the formula of our diversity index, and we provide some examples to show how it compares with the *ELF* index and how our framework can be used to derive an endogenous partition of society into groups. Section 3 contains our main



theoretical result, namely, the axiomatic characterization of *GELF*. The relationship between *GELF* and alternative measures that appear in the literature is discussed in section 4. Section 5 provides an empirical illustration and section 6 concludes with a summary of our results and possible extensions.

## 2 Similarity and fractionalization: notation and examples

In this section we introduce the notion of a similarity matrix, which is the building block of our index. We then present our proposed diversity measure, *GELF*, and show that the commonly employed *ELF* is a special case of our index. Finally, we briefly illustrate how our framework can be used to partition the population into groups.

### 2.1 Similarity

While the existing literature on the measurement of fractionalization relies on exogenous partitions of the population into groups, our starting point is a society composed of individuals. We believe that a measure of fractionalization of a society should take as primitive the individual and consider attributes such as ethnicity like any other personal characteristic in determining the similarity among individuals. In our informal discussion, we shall occasionally refer to ethnic groups in order to be in line with the literature to which we aim at contributing. Similarly, the empirical application will also make use of ethnic categories for comparison purposes with standard indices. However, the characterization result we provide in this paper is very general and we do not need to impose any predefined partition of the population into groups.

Our reasoning proceeds as follows. Imagine a society composed of individuals with

personal characteristics, whatever they might be. Any two individuals may be perfectly identical according to the characteristics under consideration, completely dissimilar or similar to different degrees. For simplicity, we normalize the similarity values to be in the interval  $[0, 1]$ , assign the value one to perfect similarity and a value of zero to maximum dissimilarity. If the society is composed of  $n$  individuals, the comparison process will generate  $n^2$  similarity values. These values are collected in a matrix that we call the *similarity matrix*. Each row  $i$  of this matrix contains the similarity values of individual  $i$  with respect to all members of society. Naturally, all entries on the main diagonal of such a matrix—the entries representing the similarity of each individual to itself—are equal to one. Furthermore, a similarity matrix is symmetric: the similarity between individuals  $i$  and  $j$  is equal to that between  $j$  and  $i$ . The possibility of including non-symmetric similarity matrices is discussed later.

Let  $\mathbb{N}$  denote the set of positive integers and  $\mathbb{R}$  the set of all real numbers. The set of all non-negative real numbers is  $\mathbb{R}_+$  and the set of positive real numbers is  $\mathbb{R}_{++}$ . For  $n \in \mathbb{N}$ ,  $\mathbb{R}^n$  is Euclidean  $n$ -space and  $\Delta^n$  is the  $n$ -dimensional unit simplex. Furthermore,  $\mathbf{0}^n$  is the vector consisting of  $n$  zeroes. A *similarity matrix of dimension*  $n \in \mathbb{N} \setminus \{1\}$  is an  $n \times n$  matrix  $S = (s_{ij})_{i,j \in \{1, \dots, n\}}$  such that:

- (a)  $s_{ij} \in [0, 1]$  for all  $i, j \in \{1, \dots, n\}$ ;
- (b)  $s_{ii} = 1$  for all  $i \in \{1, \dots, n\}$ ;
- (c)  $[s_{ij} = 1 \Rightarrow s_{ik} = s_{kj}]$  for all  $i, j, k \in \{1, \dots, n\}$ .

The three restrictions on the elements of a similarity matrix have very intuitive interpretations. (a) is consistent with a normalization requiring that complete dissimilarity is assigned a value of zero and full similarity is represented by one. Clearly, this requires that each individual has a similarity value of one when assessing the similarity to itself, as stipulated in (b). Condition (c) requires that if two individuals are fully similar, it is

not possible to distinguish between them as far as their similarity to others is concerned. Because  $i = j$  is possible in (c), the conjunction of (b) and (c) implies that a similarity matrix is symmetric. Finally, (c) implies that full similarity is transitive in the sense that, if  $s_{ij} = s_{ji} = s_{jk} = s_{kj} = 1$ , then  $s_{ik} = s_{ki} = 1$  for all  $i, j, k \in \{1, \dots, n\}$ . Our characterization result remains valid if restriction (c) is dropped—that is, our index can be characterized on a larger domain where the notion of similarity is not necessarily symmetric, as may be the case if the similarity values are obtained from people’s subjective views on the degree to which they differ from others. We state our main result with restriction (c) to emphasize that we do not need non-symmetric similarity matrices and, thus, our characterization is not dependent on an artificially large domain. See the appendix for details.

## 2.2 Measuring diversity: *GELF* and *ELF*

Let  $\mathcal{S}^n$  be the set of all  $n$ -dimensional similarity matrices, where  $n \in \mathbb{N} \setminus \{1\}$  and  $\mathcal{S} = \cup_{n \in \mathbb{N} \setminus \{1\}} \mathcal{S}^n$ . A *diversity measure* is a function  $D: \mathcal{S} \rightarrow \mathbb{R}_+$ . The specific measure we propose in this paper is what we call the *generalized fractionalization (GELF)* index  $G$ . It is defined as

$$(1) \quad G(S) = 1 - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n s_{ij}$$

for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$  (or any positive multiple; clearly, multiplying the index value by  $\gamma \in \mathbb{R}_{++}$  leaves all diversity comparisons unchanged). *GELF* is the expected dissimilarity between two individuals drawn at random. If the dissimilarity between two individuals  $i$  and  $j$  is given by the absolute difference between their incomes,  $G$  may be interpreted as a variant of the well-known Gini coefficient adapted to our environment. There are axiomatizations of the Gini index and some of its generalizations in the context of income inequality measurement (see, for instance, Donaldson and Wey-

mark, 1980, Weymark, 1981, and Bossert, 1990). However, these characterizations are fundamentally different from ours because we operate on a different domain—a domain of similarity matrices rather than a domain of income vectors. In accordance with this difference, we formulate our axioms in terms of the primitives of the framework—the similarity matrices—rather than in terms of income vectors. This difference distinguishes our axiomatic approach from those that appear in income inequality measurement. Furthermore, because we are interested in measuring fractionalization based on similarities rather than in measuring the inequality of income distributions, our set of axioms does not include a ‘transfer’ property.

As an example, suppose a three-dimensional similarity matrix is given by

$$S = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 0 \\ 1/4 & 0 & 1 \end{pmatrix}.$$

The corresponding value of  $G$  is given by

$$G(S) = 1 - \frac{1}{9} \left[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + 1 + 0 + \frac{1}{4} + 0 + 1 \right] = \frac{1}{2}.$$

It is easy to show that  $G(S)$  is indeed a generalization of the commonly-employed ethno-linguistic fractionalization (*ELF*) index. The application of *ELF* is restricted to an environment where the only information available is the vector  $p = (p_1, \dots, p_K) \in \Delta^K$  of population shares for  $K \in \mathbb{N}$  predefined groups. No partial similarity values are taken into consideration—individuals are either fully similar or completely dissimilar, that is,  $s_{ij}$  can assume the values one and zero only. Letting  $\Delta = \cup_{K \in \mathbb{N}} \Delta^K$ , the *ELF* index  $E: \Delta \rightarrow \mathbb{R}_+$  is defined by letting

$$E(p) = 1 - \sum_{k=1}^K p_k^2$$

for all  $K \in \mathbb{N}$  and for all  $p \in \Delta^K$ . Thus,  $ELF$  is one minus the well-known Herfindahl index of concentration.

In our setting, the  $ELF$  environment can be described by a subset  $\mathcal{S}_{01} = \cup_{n \in \mathbb{N} \setminus \{1\}} \mathcal{S}_{01}^n$  of our class of similarity matrices where, for all  $n \in \mathbb{N} \setminus \{1\}$ , for all  $S \in \mathcal{S}_{01}^n$  and for all  $i, j \in \{1, \dots, n\}$ ,  $s_{ij} \in \{0, 1\}$ . By properties (b) and (c), it follows that, within this subclass of matrices, the population  $\{1, \dots, n\}$  can be partitioned into  $K \in \mathbb{N}$  non-empty and disjoint subgroups  $N_1, \dots, N_K$  with the property that, for all  $i, j \in \{1, \dots, n\}$ ,

$$s_{ij} = \begin{cases} 1 & \text{if there exists } k \in \{1, \dots, K\} \text{ such that } i, j \in N_k; \\ 0 & \text{otherwise.} \end{cases}$$

Letting  $n_k \in \mathbb{N}$  denote the cardinality of  $N_k$  for all  $k \in \{1, \dots, K\}$ , it follows that  $\sum_{k=1}^K n_k = n$  and  $p_k = n_k/n$  for all  $k \in \{1, \dots, K\}$ . For  $n \in \mathbb{N} \setminus \{1\}$  and  $S \in \mathcal{S}_{01}^n$ , we obtain

$$G(S) = 1 - \frac{1}{n^2} \sum_{k=1}^K n_k^2 = 1 - \sum_{k=1}^K p_k^2 = E(p).$$

For example, suppose that

$$S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

that is, we are analyzing a society composed of three individuals. Two of them (individuals 1 and 2) are fully similar: the similarity values  $s_{12}$  and  $s_{21}$  are equal to one and, furthermore, they have the same degree of similarity—zero—with respect to the remaining member of society (individual 3). Because individual 3 is not completely similar to anyone else, it forms a group on its own. The corresponding value of  $G$  is given by

$$G(S) = 1 - \frac{1}{9} [1 + 1 + 0 + 1 + 1 + 0 + 0 + 0 + 1] = \frac{4}{9}.$$

Because  $S \in \mathcal{S}_{01}^3$ , we can alternatively calculate this diversity value using  $ELF$ . We have

$K = 2$ ,  $N_1 = \{1, 2\}$ ,  $N_2 = \{3\}$ ,  $p_1 = 2/3$  and  $p_2 = 1/3$ . Thus,

$$E(p) = 1 - \left[ \left( \frac{2}{3} \right)^2 + \left( \frac{1}{3} \right)^2 \right] = \frac{4}{9} = G(S).$$

### 2.3 Partitioning society into groups

Our framework allows us to obtain population subgroups endogenously from similarity matrices even if similarity values can assume values other than zero and one. A plausible method of doing so is the following. Any two individuals  $i$  and  $j$  belong to the same group if the similarity between  $i$  and  $j$  is equal to one and, moreover, the similarities of  $i$  with respect to all other individuals  $k$  are the same as those of  $j$ . Using this process, a group partition emerges naturally from the similarity matrix without having to impose it in advance. This method has several advantages: (i) it releases the researcher of the choice of the one characteristic that determines fractionalization in the society of interest; (ii) it makes it possible to consider simultaneously multiple characteristics; (iii) it allows group formation across characteristics; (iv) it considers the intensity of similarities between groups.

Formally, we define a partition of  $\{1, \dots, n\}$  into  $K \in \mathbb{N}$  non-empty and disjoint subgroups  $N_1, \dots, N_K$ . By properties (b) and (c), these subgroups are such that, for all  $k \in \{1, \dots, K\}$ , for all  $i, j \in N_k$  and for all  $h \in \{1, \dots, n\}$ ,  $s_{ij} = s_{ji} = 1$  and  $s_{ih} = s_{hi} = s_{hj} = s_{jh}$ . Thus, for all  $k, \ell \in \{1, \dots, K\}$ , we can unambiguously define  $\bar{s}_{k\ell} = s_{ij}$  for some  $i \in N_k$  and some  $j \in N_\ell$ . Again using  $n_k \in \mathbb{N}$  to denote the cardinality of  $N_k$  for all  $k \in \{1, \dots, K\}$ , it follows that  $\sum_{k=1}^K n_k = n$  and  $p_k = n_k/n$  for all  $k \in \{1, \dots, K\}$ . For  $n \in \mathbb{N} \setminus \{1\}$  and  $S \in \mathcal{S}^n$ , we obtain

$$(2) \quad G(S) = 1 - \frac{1}{n^2} \sum_{k=1}^K \sum_{\ell=1}^K n_k n_\ell \bar{s}_{k\ell} = 1 - \sum_{k=1}^K \sum_{\ell=1}^K p_k p_\ell \bar{s}_{k\ell}.$$

Clearly, the *ELF* index  $E$  is obtained for the case where all off-diagonal entries of  $S$  are

equal to zero.

To provide a numerical illustration of this case, let

$$S = \begin{pmatrix} 1 & 1 & 1/2 \\ 1 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix},$$

that is, we consider another society of three individuals. Again, two of them (individuals 1 and 2) are fully similar: the similarity values  $s_{12}$  and  $s_{21}$  are equal to one and, furthermore, they have the same degree of similarity with respect to the remaining member of society (individual 3). This time, however, the similarity between the members of the first group and the remaining individual is equal to  $1/2$  rather than zero. Individual 3 is not completely similar to anyone, thus is in a group by itself. The corresponding index value is

$$G(S) = 1 - \frac{1}{9} \left[ 1 + 1 + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 \right] = \frac{2}{9}.$$

According to the method outlined above, we can alternatively partition the population  $\{1, 2, 3\}$  into two groups  $N_1 = \{1, 2\}$  and  $N_2 = \{3\}$ . The population shares of these groups are  $p_1 = 2/3$  and  $p_2 = 1/3$ . We obtain the intergroup similarity values  $\bar{s}_{11} = \bar{s}_{22} = s_{11} = s_{22} = s_{12} = s_{21} = 1$  and  $\bar{s}_{12} = \bar{s}_{21} = s_{i3} = s_{3i} = 1/2$  for  $i \in \{1, 2\}$  which, using (2), leads to the index value

$$G(S) = 1 - \left[ \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \right] = \frac{2}{9}.$$

### 3 A characterization of *GELF*

We now turn to a characterization of *GELF*. Our characterization relies on four axioms, which we proceed to illustrate in order. We then state and prove the main theorem containing the formula of our diversity index.

### 3.1 Axiom 1: Normalization

Let  $I^n$  denote the  $n \times n$  identity matrix and  $\mathbf{1}^n$  denote the  $n \times n$  matrix all of whose entries are equal to one. Clearly, both of these matrices are in  $S^n$  and they represent extreme cases within this class.  $I^n$  can be thought of as having maximal diversity: any two distinct individuals are completely dissimilar and, therefore, each individual is in a group by itself.  $\mathbf{1}^n$ , on the other hand, represents maximal concentration (and, thus, minimal diversity) because there is but a single group in the population all members of which are fully similar. Our first axiom is a straightforward normalization property. It requires that the value of  $D$  at  $\mathbf{1}^n$  is equal to zero and the value of  $D$  at  $I^n$  is positive for all  $n \in \mathbb{N} \setminus \{1\}$ .

Given that the matrix  $\mathbf{1}^n$  is associated with minimal diversity, it is a very plausible restriction to require that  $D$  assumes its minimal value for these matrices. Note that this minimal value is the same across population sizes. This is plausible because, no matter what the population size  $n$  might be, there is but a single group of perfectly similar individuals and, thus, there is no diversity at all.

In contrast, it would be much less natural to require that the value of  $D$  at  $I^n$  be identical for all population sizes  $n$ . It is quite plausible to argue that having more distinct groups each of which consists of a single individual leads to more diversity than a situation where there are fewer groups containing one individual each. Our first axiom can thus be formalized as follows.

NORMALIZATION. For all  $n \in \mathbb{N} \setminus \{1\}$ ,

$$D(\mathbf{1}^n) = 0 \quad \text{and} \quad D(I^n) > 0.$$



### 3.2 Axiom 2: Anonymity

Our second axiom is very uncontroversial as well. It requires that individuals are treated impartially, paying no attention to their identities. For  $n \in \mathbb{N} \setminus \{1\}$ , let  $\Pi^n$  be the set of permutations of  $\{1, \dots, n\}$ , that is, the set of bijections  $\pi: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ . For  $n \in \mathbb{N} \setminus \{1\}$ ,  $S \in \mathcal{S}^n$  and  $\pi \in \Pi^n$ ,  $S_\pi$  is obtained from  $S$  by permuting the rows and columns of  $S$  according to  $\pi$ . Anonymity requires that  $D$  be invariant with respect to permutations.

ANONYMITY. For all  $n \in \mathbb{N} \setminus \{1\}$ , for all  $S \in \mathcal{S}^n$  and for all  $\pi \in \Pi^n$ ,

$$D(S_\pi) = D(S).$$

### 3.3 Axiom 3: Additivity

Our third axiom is additivity. As the role of this axiom may be less obvious compared to the two previous ones, it is worth providing a brief motivation for its use. Notions of additivity are among the most commonly-used properties employed in axiomatic approaches not only in the context of social index numbers but also in numerous other areas within economics. For instance, additivity plays a major role in fields such as game theory (Shapley, 1953), the analysis and design of cost sharing methods (Moulin, 1995, 2002; Moulin and Vohra, 2003; Moulin and Sprumont, 2005) and bargaining theory (Peters, 1992), to name but a few. While the exact formulation of additivity depends, of course, on the objects to be studied (here: measures of fractionalization), the intuition behind the principle itself is shared by the variations that can be found in the literature. Roughly speaking (we will be more precise below), additivity means that the value of a function (here: the value of a fractionalization index) calculated at the sum of two values of its variables (in our case, the sum of two similarity matrices) can be obtained on the basis

of the sum of the values of the function at the two values of its variables.

A crucial feature of additivity is that it entails a *separability* property: the contribution of any variable to the overall index value can be assessed in isolation, without having to know the values of the other variables. Thus, additivity properties are often linked to *independence* conditions of various forms. To take a simple example, consider a function  $f$  of three real-valued variables.  $f$  can be interpreted as an index number assigning welfare or inequality values to income distributions involving three individuals, for instance. Suppose we want to compare the value of  $f$  at  $(x, y, z)$  to the value of  $f$  at  $(x', y', z)$ , that is, we want to examine which of the values  $f(x, y, z)$  and  $f(x', y', z)$  exceeds the other or if the two values are the same. Note that the value of the third variable is the same in the two vectors to be compared—namely,  $z$ . Independence demands that the comparison of the values of  $f$  at  $(x, y, z)$  and at  $(x', y', z)$  be independent of the value of  $z$ —that is, if  $z$  is replaced with  $z'$  *in both vectors to be compared*, the relative ranking of the vectors according to  $f$  should not change. In consequence, the influence of the first two variables on  $f$  can be assessed without reference to the value of the third variable. For instance, independence requires that  $f(2, 3, 1)$  is greater than or equal to  $f(6, 1, 1)$  if and only if  $f(2, 3, 13)$  is greater than or equal to  $f(6, 1, 13)$ : as long as the third variable has the same value in the vectors to be compared (in our example, 1 or 13 but it could be any value whatsoever), it does not matter what this value is. This type of independence property is a consequence of additivity, suitably formulated for the environment under consideration. See, for instance, Blackorby, Primont and Russell (1978) for a detailed discussion of separability properties and their link to notions of additivity.

With a minor qualification to be discussed shortly, additivity means in our context that fractionalization is additive in similarity matrices in the sense that the fractionalization of the normalized sum of two matrices is the sum of the fractionalization values for the

original matrices, corrected to account for the normalization in the definition of the sum. Note that an additive structure is occasionally imposed without formulating it explicitly as an axiom on the primitives of the problem. See, for instance, Esteban and Ray (1994) who consider polarization measures that can be expressed as additive functions of entities that do not represent the primitives of the setup, and then impose further axioms on these derived entities rather than the primitives. In contrast, our formulation imposes all axioms on the primitives of the problem. Our additivity property is standard, except that we have to respect the restrictions imposed by the definition of  $\mathcal{S}^n$ . In particular, we cannot simply add two similarity matrices  $S$  and  $T$  of dimension  $n$  because, according to ordinary matrix addition, all entries on the diagonal of the sum  $S + T$  will be equal to two rather than one and, therefore,  $S + T$  is not an element of  $\mathcal{S}^n$ . For that reason, we define the following operation  $\oplus$  on the sets  $\mathcal{S}^n$  by letting, for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S, T \in \mathcal{S}^n$ ,  $S \oplus T = (s_{ij} \oplus t_{ij})_{i,j \in \{1, \dots, n\}}$  with

$$s_{ij} \oplus t_{ij} = \begin{cases} 1 & \text{if } i = j; \\ s_{ij} + t_{ij} & \text{if } i \neq j. \end{cases}$$

The standard additivity axiom has to be modified in another respect in our setting. Because the diagonal is unchanged when moving from  $S$  and  $T$  to  $S \oplus T$ , it would be questionable to require the value of  $D$  at  $S \oplus T$  to be given by the sum of  $D(S)$  and  $D(T)$  because, in doing so, we would double-count the diagonal elements in  $S$  and in  $T$ . Therefore, this sum has to be corrected by the value of  $D$  at  $I^n$ , and we obtain the following axiom.

**ADDITIVITY.** For all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S, T \in \mathcal{S}^n$  such that  $(S \oplus T) \in \mathcal{S}^n$ ,

$$D(S \oplus T) = D(S) + D(T) - D(I^n).$$

### 3.4 Axiom 4: Replication invariance

With the partial exception of the normalization condition (which implies that our diversity measure assumes the same value for the matrix  $\mathbf{1}^n$  for all population sizes  $n$ ), the first three axioms apply to diversity comparisons involving fixed population sizes only. Our last axiom imposes restrictions on comparisons across population sizes. We consider specific replications and require the index to be invariant with respect to these replications. The scope of the axiom is limited to what we consider clear-cut cases and, therefore, represents a rather mild variable-population requirement. In particular, consider the  $n$ -dimensional identity matrix  $I^n$ . As argued before, this matrix represents an extreme degree of diversity: each individual is in a group by itself and shares no similarities with anyone else. Now consider a population of size  $nm$  where there are  $m$  copies of each individual  $i \in \{1, \dots, n\}$  such that, within any group of  $m$  copies, all similarity values are equal to one and all other similarity values are equal to zero. Thus, this particular replication has the effect that, instead of  $n$  groups of size one that do not have any similarity to other groups, now we have  $n$  groups each of which consists of  $m$  identical individuals and, again, all other similarity values are equal to zero. As before, the population is divided into  $n$  homogeneous groups of equal size. Adopting a relative notion of diversity, it would seem natural to require that diversity has not changed as a consequence of this replication. To provide a precise formulation of the resulting axiom, we use the following notation. For  $n, m \in \mathbb{N} \setminus \{1\}$ , we define the matrix  $R_m^n = (r_{ij})_{i,j \in \{1, \dots, nm\}} \in \mathcal{S}^{nm}$  by

$$r_{ij} = \begin{cases} 1 & \text{if } \exists h \in \{1, \dots, n\} \text{ such that } i, j \in \{(h-1)m + 1, \dots, hm\}; \\ 0 & \text{otherwise.} \end{cases}$$

Now we can define our replication invariance axiom.

REPLICATION INVARIANCE. For all  $n, m \in \mathbb{N} \setminus \{1\}$ ,

$$D(R_m^n) = D(I^n).$$

### 3.5 Characterization

The four axioms introduced earlier in this section characterize *GELF*, as we state in the following theorem.

**Theorem 1** *A diversity measure  $D: \mathcal{S} \rightarrow \mathbb{R}_+$  satisfies normalization, anonymity, additivity and replication invariance if and only if  $D$  is a positive multiple of*

$$G(S) = 1 - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n s_{ij}$$

for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ .

**Proof.** That any positive multiple of  $G$  satisfies the axioms is straightforward to verify. Conversely, suppose  $D$  is a diversity measure satisfying normalization, anonymity, additivity and replication invariance. Let  $n \in \mathbb{N} \setminus \{1\}$ , and define the set  $\mathcal{X}^n \subseteq \mathbb{R}^{n(n-1)/2}$  by

$$\begin{aligned} \mathcal{X}^n = \{x = (x_{ij})_{\substack{i \in \{1, \dots, n-1\} \\ j \in \{i+1, \dots, n\}}} \mid \exists S \in \mathcal{S}^n \text{ such that } s_{ij} = x_{ij} \text{ for all } i \in \{1, \dots, n-1\} \\ \text{and for all } j \in \{i+1, \dots, n\}\}. \end{aligned}$$

Define the function  $F^n: \mathcal{X}^n \rightarrow \mathbb{R}$  by letting, for all  $x \in \mathcal{X}^n$ ,

$$(3) \quad F^n(x) = D(S) - D(I^n)$$

where  $S \in \mathcal{S}^n$  is such that  $s_{ij} = x_{ij}$  for all  $i \in \{1, \dots, n-1\}$  and for all  $j \in \{i+1, \dots, n\}$ . This function is well-defined because  $\mathcal{S}^n$  contains symmetric matrices with ones on the main diagonal only. Because  $D$  is bounded below by zero, it follows that  $F^n$  is bounded

below by  $-D(I^n)$ . Furthermore, the additivity of  $D$  implies that  $F^n$  satisfies Cauchy's basic functional equation

$$(4) \quad F^n(x + y) = F^n(x) + F^n(y)$$

for all  $x, y \in \mathcal{X}^n$  such that  $(x + y) \in \mathcal{X}^n$ ; see Aczél (1966, section 2.1). We have to address a slight complexity in solving this equation because the domain  $\mathcal{X}^n$  of  $F^n$  is not a Cartesian product, which is why we provide a few further details rather than invoking the corresponding standard result immediately.

Fix  $i \in \{1, \dots, n-1\}$  and  $j \in \{i+1, \dots, n\}$ , and define the function  $f_{ij}^n: [0, 1] \rightarrow \mathbb{R}$  by

$$f_{ij}^n(x_{ij}) = F^n(x_{ij}; \mathbf{0}^{n(n-1)/2-1})$$

for all  $x_{ij} \in [0, 1]$ , where the vector  $(x_{ij}; \mathbf{0}^{n(n-1)/2-1})$  is such that the component corresponding to  $ij$  is given by  $x_{ij}$  and all other entries (if any) are equal to zero. Note that this vector is indeed an element of  $\mathcal{X}^n$  and, therefore,  $f_{ij}^n$  is well-defined. The function  $f_{ij}^n$  is bounded below because  $F^n$  is and, as an immediate consequence of (4), it satisfies the Cauchy equation

$$(5) \quad f_{ij}^n(x_{ij} + y_{ij}) = f_{ij}^n(x_{ij}) + f_{ij}^n(y_{ij})$$

for all  $x_{ij}, y_{ij} \in [0, 1]$  such that  $(x_{ij} + y_{ij}) \in [0, 1]$ . Because the domain of  $f_{ij}^n$  is an interval containing the origin and  $f_{ij}^n$  is bounded below, the only solutions to (5) are linear functions; see Aczél (1966, section 2.1). Thus, there exists  $c_{ij}^n \in \mathbb{R}$  such that

$$(6) \quad F^n(x_{ij}; \mathbf{0}^{n(n-1)/2-1}) = f_{ij}^n(x_{ij}) = c_{ij}^n x_{ij}$$

for all  $x_{ij} \in [0, 1]$ .

Let  $S \in \mathcal{S}^n$ . By additivity, the definition of  $F^n$  and (6),

$$F^n \left( (s_{ij})_{\substack{i \in \{1, \dots, n-1\} \\ j \in \{i+1, \dots, n\}}} \right) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n F^n(s_{ij}; \mathbf{0}^{n(n-1)/2-1}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n f_{ij}^n(s_{ij}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_{ij}^n s_{ij}$$

and, defining  $d^n = D(I^n)$  and substituting into (3), we obtain

$$(7) \quad D(S) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n c_{ij}^n s_{ij} + d^n.$$

Now fix  $i, k \in \{1, \dots, n-1\}$ ,  $j \in \{i+1, \dots, n\}$  and  $\ell \in \{k+1, \dots, n\}$ , and let  $S \in \mathcal{S}^n$  be such that  $s_{ij} = s_{ji} = 1$  and all other off-diagonal entries of  $S$  are equal to zero. Let the bijection  $\pi \in \Pi^n$  be such that  $\pi(i) = k$ ,  $\pi(j) = \ell$ ,  $\pi(k) = i$ ,  $\pi(\ell) = j$  and  $\pi(h) = h$  for all  $h \in \{1, \dots, n\} \setminus \{i, j, k, \ell\}$ . By (7), we obtain

$$D(S) = c_{ij}^n + d^n \quad \text{and} \quad D(S_\pi) = c_{k\ell}^n + d^n,$$

and anonymity implies  $c_{ij}^n = c_{k\ell}^n$ . Therefore, there exists  $c^n \in \mathbb{R}$  such that  $c_{ij}^n = c^n$  for all  $i \in \{1, \dots, n-1\}$  and for all  $j \in \{i+1, \dots, n\}$ , and substituting into (7) yields

$$D(S) = c^n \sum_{i=1}^{n-1} \sum_{j=i+1}^n s_{ij} + d^n$$

for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ .

Normalization requires

$$D(\mathbf{1}^n) = c^n \frac{n(n-1)}{2} + d^n = 0$$

and, therefore,  $d^n = -c^n n(n-1)/2$  for all  $n \in \mathbb{N} \setminus \{1\}$ . Using normalization again, we obtain

$$D(I^n) = -c^n \frac{n(n-1)}{2} > 0$$

which implies  $c^n < 0$  for all  $n \in \mathbb{N} \setminus \{1\}$ . Thus,

$$(8) \quad D(S) = c^n \sum_{i=1}^{n-1} \sum_{j=i+1}^n s_{ij} - c^n \frac{n(n-1)}{2}$$

for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ .

Let  $n$  be an even integer greater than or equal to four. By replication invariance and (8),

$$D(R_{n/2}^2) = c^n \frac{n}{2} \left( \frac{n}{2} - 1 \right) - c^n \frac{n(n-1)}{2} = -c^2 = D(I^2).$$

Solving, we obtain

$$(9) \quad c^n = 4 \frac{c^2}{n^2}.$$

Now let  $n$  be an odd integer greater than or equal to three. Thus,  $q = 2n$  is even, and the above argument implies

$$(10) \quad c^q = 4 \frac{c^2}{q^2} = \frac{c^2}{n^2}.$$

Furthermore, replication invariance requires

$$D(R_2^n) = D(R_2^{q/2}) = c^q \frac{q}{2} - c^q \frac{q(q-1)}{2} = -c^n \frac{n(n-1)}{2} = D(I^n).$$

Solving for  $c^n$  and using the equality  $q = 2n$ , it follows that  $c^n = 4c^q$  and, combined with (10), we obtain (9) for all odd  $n \in \mathbb{N} \setminus \{1\}$  as well.

Substituting into (8), simplifying and defining  $\gamma = -2c^2 > 0$ , it follows that, for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ ,

$$\begin{aligned} D(S) &= 4 \frac{c^2}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n s_{ij} - 2 \frac{c^2}{n^2} n(n-1) \\ &= 2 \frac{c^2}{n^2} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n s_{ij} - 2c^2 + 2 \frac{c^2}{n} \\ &= -2c^2 \left[ 1 - \frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n s_{ij} - \frac{1}{n} \right] \\ &= -2c^2 \left[ 1 - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n s_{ij} \right] \\ &= \gamma G(S). \quad \blacksquare \end{aligned}$$

## 4 Alternative and related approaches

In this section we discuss the differences between *GELF* and related indices proposed in various literatures. We start briefly with the linguistics and statistical literature and



compare *GELF* with Greenberg’s (1956) index and with the quadratic entropy index (*QE*). We then proceed with the economics literature, focusing on the indices of ethnic polarization (*RQ*) and peripheral diversity (*PD*).

What is known in the economics literature as *ELF* is, in the statistical literature, the Gini-Simpson index, introduced first by Gini (1912) and then by Simpson (1949) as a measure of diversity of the multinomial distribution. The same index has been proposed by the linguist Greenberg (1956) who termed it the ‘*A* index.’ In his 1956 article, Greenberg suggested a way to measure the degree of resemblance among  $K$  languages. Indicating by  $r_{k\ell} \geq 0$  the resemblance between language  $k$  and  $\ell$ , the proposed  $B$  index is:

$$B = 1 - \sum_{k=1}^K \sum_{\ell=1}^K p_k p_\ell r_{k\ell}.$$

This is the index used by Fearon (2003) in his empirical contribution on cultural fractionalization.

In an independent contribution, Rao (1982) suggested exactly the same generalization of *ELF*, the *quadratic entropy* index (*QE*), in order to take into account different distance values,  $d_{k\ell} \geq 0$ , of different pairs of categories,  $k$  and  $\ell$ . As opposed to Greenberg (1956), Rao (1984) and Rao and Nayak (1985) provide various axiomatizations of the measure. *QE* is an index that, rewritten in the settings of our paper, considers distances other than zero–one between individuals belonging to different groups, that is

$$QE = \sum_{k=1}^K \sum_{\ell=1}^K p_k p_\ell d_{k\ell}.$$

Recall the definition of  $\bar{s}_{k\ell}$  in section 2 and the formula for *GELF* (2). Letting  $d_{k\ell} = 1 - \bar{s}_{k\ell}$ , we immediately see that *GELF* is *QE*, and hence  $B$ , when the population is partitioned ex-ante into groups on the basis of a characteristic.

The inspection of the indices  $B$  and *QE* gives further insights into the relationship between *GELF* and *ELF*. As we said above, *GELF* is the expected dissimilarity between

two individuals drawn at random from the population.  $ELF$  is the likelihood that two randomly drawn individuals belong to different (exogenous) categories. Rewriting  $ELF$  as  $1 - \sum_{k=1}^K p_k p_k$ , we see that  $ELF$  can be interpreted as one minus a weighted sum of population shares  $p_k$ , where the weights are these shares themselves.  $GELF$ , on the other hand, is its natural generalization: it can be written as one minus a weighted sum of the population shares. However, the weight assigned to  $p_k$  is now not merely  $p_k$  itself but a considerably more refined expression that takes account of the similarities of the group members to the individuals in other groups. In calculating  $GELF$ , each individual counts in two capacities. Through its membership in its *own* group, an individual contributes to the population share of the group. In addition, there is a secondary contribution via the similarities to individuals of *other* groups.

It should be noted that, when the distance values are differences in income,  $QE$  is twice the well-known absolute Gini coefficient. The latter, when normalized by mean income, is one among the most popular indices of income inequality.

In economics, the index of ethnic polarization  $RQ$  (see Reynal-Querol, 2002, and Montalvo and Reynal-Querol, 2005) shares a structure similar to that of  $ELF$  and of  $GELF$ . It is defined by

$$RQ = 1 - \sum_{k=1}^K \left( \frac{1/2 - p_k}{1/2} \right)^2 p_k.$$

As is the case for  $ELF$ ,  $RQ$  employs a weighted sum of population shares. The weights employed in  $RQ$  capture the deviation of each group from the maximum polarization share  $1/2$  as a proportion of  $1/2$ . Analogously to  $ELF$ , underlying the formula of  $RQ$  is the implicit assumption that any two groups are either completely similar or completely dissimilar and, thus, the weights depend on population shares only.

The index of peripheral diversity  $PD$  (see Desmet, Ortuño-Ortín and Weber, 2005) is a specification of the original Esteban and Ray (1994) polarization index. It is derived from

the alienation-identification framework proposed by Esteban and Ray (1994), applied to distances between languages spoken rather than to income distances as in Esteban and Ray (1994). Desmet, Ortuno-Ortín and Weber (2005) distinguish between the effective alienation felt by the dominant group and that of the minorities. In particular, expressed in the setting of our paper, the index is defined by

$$PD = \sum_{k=1}^K [p_k^{1+\alpha} (1 - \bar{s}_{0k}) + p_k p_0^{1+\alpha} (1 - \bar{s}_{0k})],$$

where  $\alpha \in \mathbb{R}$  is a parameter indicating the importance given to the identification component, 0 is the dominant group and the other  $K$  are minority groups. When  $\alpha < 0$ ,  $PD$  is an index of peripheral diversity; when  $\alpha > 0$ ,  $PD$  is an index of peripheral polarization. The structure of this index is different from that of those previously discussed. As is the case for *GELF*, it does incorporate a notion of dissimilarity between groups, given by the complement to one of the similarity value. On the other hand, as opposed to the previous indices, the identification component plays a crucial role enhancing (when  $\alpha > 0$ ) or diminishing (when  $\alpha < 0$ ) the alienation produced by distances between groups. An additional difference to the other indices discussed in this section is the distinction between the dominant groups and the minorities.

## 5 An empirical illustration

In this section we provide an application of *GELF* to the pattern of diversity in the United States. Our goal is to compare the extent of diversity across states taking into account different dimensions of similarity among individuals, in particular: racial identity, household income, education and employment status.

## 5.1 Method and results

The data set used is the 5 percent IPUMS from the 1990 Census. We use individual level information on all household heads in the sample and record the following characteristics:

(a) **Race.** Each individual is attributed to one of five racial groups, that is, (i) White; (ii) Black; (iii) American Indian, Eskimo or Aleutian; (iv) Asian or Pacific Islander; and (v) Other. The last category includes any other race except the four mentioned. The 1990 Census does not identify Hispanic as a separate racial category. However, Alesina, Baqir and Easterly (1999), who construct *ELF* from the same five categories, report that the category Hispanic (obtained from a different source) has a correlation of more than 0.9 with the category Other in the Census data.

(b) **Income.** Total household income.

(c) **Education.** The years of education of the individual.

(d) **Employment.** Each individual is attributed to one of four categories, namely, (i) Civilian employed or armed forces, at work; (ii) Civilian employed or armed forces, with a job but not at work; (iii) Unemployed; and (iv) Not in labor force.

Drawing on the above information, we construct *GELF* in several ways. The first, and most general, is an implementation of formula (1) that takes into account all four dimensions at the same time without imposing an exogenous partition into groups. The second and third approaches rely on an ex-ante partition of the population and implement the grouped version of *GELF*, expression (2). Specifically, the five racial groups described under (a) are used in our illustration to generate the partition. We then measure the similarity among these groups along the remaining dimensions. The choice of race as the exogenously given category is purely instrumental to comparing our results to the widely used *ELF* index that relies exclusively on racial shares. Obviously, depending on the specific application, the grouping could be done on the cleavage that is most relevant for

the phenomenon under study. The idea underlying this second set of results is to propose a way to compute *GELF* that is less data intensive and to see whether the qualitative pattern of results differs from that obtained using the full similarity matrix. This second set of results, in turn, is obtained under two alternative methods. The first requires the availability of the entire distribution of individual characteristics, and can be used when individual survey data is available. The second relies only on aggregate data on *mean* characteristics by group.

## 5.2 Similarity of individuals

To implement our index (1), we start from the variables (a) to (d) and apply principal component analysis. To extract principal components, we employ a polychoric correlation matrix to take into account the fact that some of our variables are categorical (see for example, Kolenikov and Angeles, 2009). Specifically, we extract for each individual  $i$  a synthetic measure  $x_i$ , the first principal component, that we employ to compute pairwise distances among all individuals living in the same state, that is,  $|x_i - x_j|$ . To generate similarity values  $s_{ij}$  that are bounded between zero and one, we normalize this distance by the difference between the maximum and the minimum value of the  $x_i$  in the entire US sample, and we subtract the resulting value from one. Once we have the full set of similarity values  $\{s_{ij}\}_{i,j \in \{1, \dots, n\}}$  computation of (1) is straightforward. We refer to this index as *GELF* with no further specifications.

[Insert Figure 1]

The main result of our empirical analysis is summarized in figure 1. On the horizontal axis we plot values of ethno-linguistic fractionalization (*ELF*) for all states in the US in 1990. The vertical axis reports the corresponding value of *GELF*. While the two

are positively correlated, their relationship is far from linear: the correlation coefficient is only 0.59. In particular, states like Hawaii, California and Nevada are much more heterogeneous if one only looks at racial shares than if all dimensions are considered jointly. This is because in these states the distribution of income, education and employment is relatively more similar among races than in other states. At the opposite end we have states like Alaska, Kentucky, Rhode Island, Massachusetts and in general New England, where diversity measured in terms of racial shares is relatively low, but different races differ in the distribution of the remaining characteristics to such an extent that they are actually more diverse when the full similarity *GELF* is employed.

**[Insert Table 1]**

Table 1 provides the counterpart to the graphical analysis, as it reports the full set of states listed in decreasing order of ethno-linguistic fractionalization, the corresponding values of *ELF*, *GELF* and the difference in ranks between *ELF* and *GELF* for each state. We prefer to rely on a comparison of ranks because the absolute values of the two indices are not comparable. In particular, in the last column of table 1 we report the difference  $ELF_{rank} - GELF_{rank}$ , so that negative values indicate that a given state is less fractionalized according to *GELF* than according to *ELF*, while positive values indicate the opposite. The magnitude of the difference gives a rough approximation of how big a difference it makes for a particular state to use one index over the other, in terms of relative rankings.

### 5.3 Similarity of distributions

Once the population is exogenously partitioned into racial groups, we can assess the distance among these groups by comparing the distributions of individual characteristics

such as income, education, employment. We refer to the indices as *GroupedGELF<sub>income</sub>* etc. Consider for example income. We first estimate non-parametrically the distributions of household income by race of the head of the household,  $\widehat{f}^k(y)$ . The estimation method applied in the paper is the *adaptive* or *variable* kernel modified to take into account sample weights attached to each observation in each group. After estimating the densities of household income by race, we measure the overlap among them, implying that two racial groups whose income distributions perfectly overlap are considered perfectly similar. The measure of overlap of distributions applied is the Kolmogorov measure of variation distance:

$$Kov_{k\ell} = \frac{1}{2} \int \left| \widehat{f}^k(y) - \widehat{f}^\ell(y) \right| dy.$$

$Kov_{k\ell}$  is a measure of the lack of overlap between groups  $k$  and  $\ell$ . It ranges between zero and one, taking value zero if  $\widehat{f}^k(y) = \widehat{f}^\ell(y)$  for all  $y \in \mathbb{R}$  and one if  $\widehat{f}^k(y)$  and  $\widehat{f}^\ell(y)$  do not overlap at all. The distance is sensitive to changes in the distributions only when both take positive values, being insensitive to changes whenever one of them is zero. It will not change if the distributions move apart, provided that there is no overlap between them or that the overlapping part remains unchanged. The resulting measure of similarity between any two groups  $k$  and  $\ell$ , that we employ to implement formula (2) for grouped *GELF*, is

$$\bar{s}_{k\ell} = 1 - Kov_{k\ell}.$$

This method is also applied on the distribution of the synthetic measure  $x_i$  obtained for each individual in each group by principal component analysis. In this case we estimate  $\widehat{f}^k(x)$ , the distribution of the synthetic measure by race, compute the Kolmogorov measure of variation distance and the measure of similarity as described above. The results are displayed in table 2.

[Insert Table 2]

States in table 2 are listed in decreasing order of *GELF* and two additional indices (with the corresponding ranks) are reported. The first index, which we denote as *GroupedGELF<sub>a</sub>*, employs the Kolmogorov distance among distributions of the synthetic index to compute similarity values that are the used in formula (2). The second index, denoted as *GroupedGELF*, is simpler in that only the *average* value of the synthetic index for each racial group is used when computing distances (differences). While the use of means or of the entire distribution yield very similar results, the comparison with *GELF* suggests that for some states the exogenous definition of racial categories does make a difference: these are the same states for which the difference between *ELF* and *GELF* in figure 1 was more pronounced. In this sense, and not surprisingly, the *GroupedGELF* index calculated according to (2) is more similar to *ELF* than the *GELF* index (1) calculated on the full similarity matrix.

## 5.4 Similarity of means

As an alternative to the distance among distributions, we compute a crude measure of similarity based on the *expected value* of the distribution of the characteristic analyzed. This is to illustrate the performance of *GELF* in case of grouped data or poor availability of information in the data set.

We can measure similarity with respect to continuous or to categorical variables. For continuous variables, such as household income or education, we indicate by  $\lambda^k$  the sample mean of the distribution for group  $k$ , by  $\lambda_{Max}$  the maximum mean value among all groups in all states, and by  $\lambda_{Min}$  the minimum. Then we can compute  $\bar{s}_{k\ell}$  for each state as

$$(11) \quad \bar{s}_{k\ell} = 1 - \left| \frac{\lambda^k - \lambda^\ell}{\lambda_{Max} - \lambda_{Min}} \right|.$$

Note that expression (11) is bounded between zero and one by construction.



For categorical variables like employment, we create a dummy variable that assumes the value one if the household head is employed, and zero if it is unemployed or not in the labor force. We have also experimented with a different definition where one corresponds to households whose head is employed or not in the labor force, and zero to unemployed. The results were not significantly affected and are available from the authors. Indicating by  $\delta^k$  the sample means of this variable for group  $k$  (that is, the share of the population assuming value one), similarity between any two groups  $k$  and  $\ell$  is

$$\bar{s}_{k\ell} = 1 - |\delta^k - \delta^\ell|.$$

Again, sample weights are used in the computations for these variables.

**[Insert Table 3 and Figure 2]**

Results are contained in table 3. We denote as  $GroupedGELF_{income}$  the index obtained when distance among racial groups is measured solely in terms of differences in average income, as  $GroupedGELF_{edu}$  when differences is in average years of education, and as  $GroupedGELF_{empl}$  when the difference is in the share of people employed. For each index, we report the value and the rank, and states are still listed in decreasing order of the full similarity  $GELF$ . The results are quite informative and are more easily visualized through figure 2. Panel A of the figure plots the original values of  $ELF$  on the horizontal axis against  $GroupedGELF_{income}$  on the vertical. The two measures are closely correlated with two extreme outliers: Hawaii is much less fractionalized when we use  $GroupedGELF_{income}$  than when we use  $ELF$ , while the opposite occurs for the District of Columbia. The intuition is similar to that provided when commenting on figure 1: in states like Hawaii or California average income levels are relatively more similar among races than they are in DC or in Connecticut, for example. A similar picture is offered in panel B with respect to years of education. Interestingly, however, when we look at employment levels (panel

C) the relationship between the two indices becomes hump-shaped. The maximum value of diversity according to  $GroupedGELF_{empl}$  corresponds to *intermediate* levels of ethnic fractionalization; on the other hand, very low or very high levels of  $ELF$  translate into middle range values of diversity when both race and similarity in employment status are taken into account. A possible interpretation of this result is that sizeable differences in employment status (such as high unemployment levels for minorities) may be politically difficult to sustain in states where a relatively high fraction of the population is non-white. On the other hand, the same does not hold for income, as if income differences were more easily acceptable compared to the universal right of access to employment.

While only suggestive and illustrative, the above analysis highlights some of the potential benefits that may derive from the use of fractionalization indices that do not simply rely on population shares, but also try to incorporate information on other dimensions along which individuals may differ.

## 6 Concluding remarks

The main purpose of this paper is to provide a theoretical foundation and an empirical illustration of a new measure of ethnic diversity. Unlike the most commonly used  $ELF$  index, our generalized version  $GELF$  makes use of a broader informational base. Instead of limiting the relevant variables to the population shares of predefined groups, we start out with a notion of similarity among individuals and calculate our index value accordingly. It is possible to derive a partition into groups endogenously, and the standard  $ELF$  index emerges as a special case when no partial similarity is allowed. The results of our empirical application suggest that accounting for the extent of similarity among individuals in observable dimensions other than race may indeed alter the picture of ethnic diversity in

the United States. In places like New England or Washington DC racial fractionalization is magnified when similarity in income, education or employment is taken into account; in places like California the opposite occurs.

Before concluding, we would like to stress an important methodological point. While in this paper we characterize *GELF* on the basis of similarities among individuals, our approach is silent on how these similarities should be defined. In particular, our approach is fully compatible with a setting in which the notion of continuous distance does not apply (that is, individuals are either fully similar or fully dissimilar, in which case our primitives  $s_{ij}$  take on the values zero or one only), as well as with a setting in which it is meaningful to think of similarity among individuals in a continuous way. In addition, our index allows to incorporate a multidimensional concept of similarity, as opposed to a single dimension. We view this flexibility as an advantage of our approach, and one that makes our index applicable in many different settings. Our choice in the empirical illustration was guided by the attempt to compare our results with well known patterns in the economics literature on ethnic fractionalization in the US. We chose as dimensions of similarity ethnicity, household income, education and employment status since we believe that these are important aspects of the US economy that could influence the behavior of individuals. However, the choice of variables to be employed in the measurement of similarity could include very different aspects and should be guided by the specific application that one has in mind.

Finally, the application of our index is not limited to studies involving ethno-linguistic fractionalization. The generalized index that we propose may be applied to various areas in economics, including for example industrial organization. *GELF* is an index of diversity, and the difference between one and the index value can be interpreted as an index of concentration. Embedding information on similarity among firms in a concentration

index may yield different results than the traditional Herfindahl index, which is purely based on market shares.

## Appendix

In this appendix, we illustrate that our characterization result is unchanged if the set of similarity matrices  $\mathcal{S}^n$  consists of all  $n \times n$  matrices  $S$  satisfying conditions (a) and (b) of section 2, but not necessarily (c). This is achieved by some straightforward modifications of the definitions used in the proof of theorem 1.

That any positive multiple of  $G$  satisfies the axioms on the larger domain as well is, again, straightforward to verify. Conversely, suppose  $D$  is a diversity measure defined on the larger domain satisfying normalization, anonymity, additivity and replication invariance. Let  $n \in \mathbb{N} \setminus \{1\}$ , and define the set  $\mathcal{X}^n \subseteq \mathbb{R}^{n(n-1)/2}$  by

$$\mathcal{X}^n = \left\{ x = (x_{ij})_{\substack{i \in \{1, \dots, n\} \\ j \in \{1, \dots, n\} \setminus \{i\}}} \mid \exists S \in \mathcal{S}^n \text{ such that } s_{ij} = x_{ij} \text{ for all } i \in \{1, \dots, n\} \right. \\ \left. \text{and for all } j \in \{1, \dots, n\} \setminus \{i\} \right\}.$$

Define the function  $F^n: \mathcal{X}^n \rightarrow \mathbb{R}$  by letting, for all  $x \in \mathcal{X}^n$ ,

$$(12) \quad F^n(x) = D(S) - D(I^n)$$

where  $S \in \mathcal{S}^n$  is such that  $s_{ij} = x_{ij}$  for all  $i \in \{1, \dots, n\}$  and for all  $j \in \{1, \dots, n\} \setminus \{i\}$ .

Because  $D$  is bounded below by zero, it follows that  $F^n$  is bounded below by  $-D(I^n)$ .

Furthermore, the additivity of  $D$  implies that  $F^n$  satisfies Cauchy's basic functional equation

$$(13) \quad F^n(x + y) = F^n(x) + F^n(y)$$

for all  $x, y \in \mathcal{X}^n$  such that  $(x + y) \in \mathcal{X}^n$ ; see Aczél (1966, section 2.1).

Fix  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, n\} \setminus \{i\}$ , and define the function  $f_{ij}^n: [0, 1] \rightarrow \mathbb{R}$  by

$$f_{ij}^n(x_{ij}) = F^n(x_{ij}; \mathbf{0}^{n(n-1)-1})$$

for all  $x_{ij} \in [0, 1]$ , where the vector  $(x_{ij}; \mathbf{0}^{n(n-1)-1})$  is such that the component corresponding to  $ij$  is given by  $x_{ij}$  and all other entries (if any) are equal to zero. The function  $f_{ij}^n$  is bounded below because  $F^n$  is and, as an immediate consequence of (13), it satisfies the Cauchy equation

$$(14) \quad f_{ij}^n(x_{ij} + y_{ij}) = f_{ij}^n(x_{ij}) + f_{ij}^n(y_{ij})$$

for all  $x_{ij}, y_{ij} \in [0, 1]$  such that  $(x_{ij} + y_{ij}) \in [0, 1]$ . Because the domain of  $f_{ij}^n$  is an interval containing the origin and  $f_{ij}^n$  is bounded below, the only solutions to (14) are linear functions; see Aczél (1966, section 2.1). Thus, there exists  $c_{ij}^n \in \mathbb{R}$  such that

$$(15) \quad F^n(x_{ij}; \mathbf{0}^{n(n-1)-1}) = f_{ij}^n(x_{ij}) = c_{ij}^n x_{ij}$$

for all  $x_{ij} \in [0, 1]$ .

Let  $S \in \mathcal{S}^n$ . By additivity, the definition of  $F^n$  and (15),

$$F^n \left( (s_{ij})_{\substack{i \in \{1, \dots, n\} \\ j \in \{1, \dots, n\} \setminus \{i\}}} \right) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n F^n(s_{ij}; \mathbf{0}^{n(n-1)-1}) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n f_{ij}^n(s_{ij}) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n c_{ij}^n s_{ij}$$

and, defining  $d^n = D(F^n)$  and substituting into (12), we obtain

$$(16) \quad D(S) = \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n c_{ij}^n s_{ij} + d^n.$$

Now fix  $i, k \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, n\} \setminus \{i\}$  and  $\ell \in \{1, \dots, n\} \setminus \{k\}$ , and let  $S \in \mathcal{S}^n$  be such that  $s_{ij} = 1$  and all other off-diagonal entries of  $S$  are equal to zero. Let the bijection  $\pi \in \Pi^n$  be such that  $\pi(i) = k$ ,  $\pi(j) = \ell$ ,  $\pi(k) = i$ ,  $\pi(\ell) = j$  and  $\pi(h) = h$  for all  $h \in \{1, \dots, n\} \setminus \{i, j, k, \ell\}$ . By (16), we obtain

$$D(S) = c_{ij}^n + d^n \quad \text{and} \quad D(S_\pi) = c_{k\ell}^n + d^n,$$

and anonymity implies  $c_{ij}^n = c_{kl}^n$ . Therefore, there exists  $c^n \in \mathbb{R}$  such that  $c_{ij}^n = c^n$  for all  $i \in \{1, \dots, n\}$  and for all  $j \in \{1, \dots, n\} \setminus \{i\}$ , and substituting into (16) yields

$$D(S) = c^n \sum_{i=1}^{n-1} \sum_{j=i+1}^n s_{ij} + d^n$$

for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ .

Normalization requires

$$D(\mathbf{1}^n) = c^n n(n-1) + d^n = 0$$

and, therefore,  $d^n = -c^n n(n-1)$  for all  $n \in \mathbb{N} \setminus \{1\}$ . Using normalization again, we obtain

$$D(I^n) = -c^n n(n-1) > 0$$

which implies  $c^n < 0$  for all  $n \in \mathbb{N} \setminus \{1\}$ . Thus,

$$(17) \quad D(S) = c^n \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n s_{ij} - c^n n(n-1)$$

for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ .

Let  $n$  be an even integer greater than or equal to four. By replication invariance and (17),

$$D(R_{n/2}^2) = c^n n \left( \frac{n}{2} - 1 \right) - c^n n(n-1) = -c^2 = D(I^2).$$

Solving, we obtain

$$(18) \quad c^n = 2 \frac{c^2}{n^2}.$$

Now let  $n$  be an odd integer greater than or equal to three. Thus,  $q = 2n$  is even, and the above argument implies

$$(19) \quad c^q = 2 \frac{c^2}{q^2} = \frac{c^2}{2n^2}.$$

Furthermore, replication invariance requires

$$D(R_2^n) = D(R_2^{q/2}) = c^q q - c^q q(q-1) = -c^n n(n-1) = D(I^n).$$

Solving for  $c^n$  and using the equality  $q = 2n$ , it follows that  $c^n = 4c^q$  and, combined with (19), we obtain (18) for all odd  $n \in \mathbb{N} \setminus \{1\}$  as well.

Substituting into (17), simplifying and defining  $\gamma = -2c^2 > 0$ , it follows that, for all  $n \in \mathbb{N} \setminus \{1\}$  and for all  $S \in \mathcal{S}^n$ ,

$$\begin{aligned}
D(S) &= 2\frac{c^2}{n^2} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n s_{ij} - c^n n(n-1) \\
&= 2\frac{c^2}{n^2} \sum_{i=1}^n \sum_{j=1}^n s_{ij} - 2\frac{c^2}{n^2} n - 2\frac{c^2}{n^2} n(n-1) \\
&= -2c^2 \left[ 1 - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n s_{ij} \right] \\
&= \gamma G(S). \quad \blacksquare
\end{aligned}$$

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## References

- ACZÉL, J. (1966). *Lectures on Functional Equations and Their Applications*. New York: Academic Press.
- ALESINA, A., BAQIR, R. and EASTERLY, W. (1999). Public goods and ethnic divisions. *Quarterly Journal of Economics*, **114**, 1243–84.
- ALESINA, A., BAQIR, R. and HOXBY, C. (2004). Political jurisdictions in heterogeneous communities. *Journal of Political Economy*, **112**, 348–96.
- ALESINA, A., DEVLEESCHAUWER, A., EASTERLY, W., KURLAT, S. and WACZIARG, R. (2003). Fractionalization. *Journal of Economic Growth*, **8**, 155–94.
- ALESINA, A. and LA FERRARA, E. (2000). Participation in heterogeneous communities. *Quarterly Journal of Economics*, **115**, 847–904.
- ALESINA, A. and LA FERRARA, E. (2005). Ethnic diversity and economic performance. *Journal of Economic Literature*, **43**, 762–800.
- BISIN, A. and VERDIER, T. (2000). Beyond the melting pot: cultural transmission, marriage, and the evolution of ethnic and religious traits. *Quarterly Journal of Economics*, **115**, 955–88.
- BLACKORBY, C., PRIMONT, D. and RUSSELL, R.R. (1978). *Duality, Separability, and Functional Structure: Theory and Economic Applications*, Amsterdam: North-Holland.
- BOSSERT, W. (1990). An axiomatization of the single-series Ginis. *Journal of Economic Theory*, **50**, 82–92.



- BOSSERT, W., PATTANAIK, P.K. and XU, Y. (2003). Similarity of options and the measurement of diversity. *Journal of Theoretical Politics*, **15**, 405–21.
- CASELLI, F. and COLEMAN, W.J. (2006). On the theory of ethnic conflict. NBER: Working Paper 12125.
- DESMET, K., ORTUÑO-ORTÍN, I. and WEBER, S. (2005). Peripheral diversity and redistribution. CEPR: Discussion Paper 5112.
- DONALDSON, D. and WEYMARK, J.A. (1980). A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory*, **22**, 67–86.
- EASTERLY, W. and LEVINE, R. (1997). Africa's growth tragedy: policies and ethnic divisions. *Quarterly Journal of Economics*, **111**, 1203–50.
- ESTEBAN, J.-M. and RAY, D. (1994). On the measurement of polarization. *Econometrica*, **62**, 819–51.
- FEARON, J.D. (2003). Ethnic and cultural diversity by country. *Journal of Economic Growth*, **8**, 195–222.
- FERNANDEZ, R., FOGLI, A. and OLIVETTI, C. (2004). Mothers and sons: preference transmission and female labor force dynamics? *Quarterly Journal of Economics*, **119**, 1249–99.
- GINI, C. (1912). Variabilità e mutabilità. *Studi Economico-Giuridici della Facoltà di Giurisprudenza dell'Università di Cagliari*, **a.III**, parte II.
- GIULIANO, P. (2007). Living arrangements in western Europe: does cultural origin matter? *Journal of the European Economic Association*, **5**, 927–52.

- GIULIANO, P., SPILIMBERGO, A. and TONON, G. (2006). Genetic, cultural and geographical distances. IZA: Discussion Paper 2229.
- GREENBERG, J.H. (1956). The measurement of linguistic diversity. *Language*, **32**, 109–15.
- GUIO, L., SAPIENZA, P. and ZINGALES, L. (forthcoming). Cultural biases in economic exchange. *Quarterly Journal of Economics*.
- KOLENIKOV, S., and ANGELES, G. (2009). Socioeconomic status measurement with discrete proxy variables: is principal component analysis a reliable answer? *Review of Income and Wealth*, **55**, 128–65.
- LAITIN, D. (2000). What is a language community? *American Journal of Political Science*, **44**, 142–54.
- MAURO, P. (1995). Corruption and growth. *Quarterly Journal of Economics*, **110**, 681–712.
- MONTALVO, J.G. and REYNAL-QUEROL, M. (2005). Ethnic polarization, potential conflict, and civil wars. *American Economic Review*, **95**, 796–816.
- MONTALVO, J.G. and REYNAL-QUEROL, M. (2008). Discrete polarization with an application to the determinants of genocides. *Economic Journal*, **118**, 1835–65.
- MOODY, J. (2001). Race, school integration, and friendship segregation in America. *American Journal of Sociology*, **107**, 679–716.
- MOULIN, H. (1995). On additive methods to share joint costs. *Japanese Economic Review*, **46**, 303–32.

- MOULIN, H. (2002). Axiomatic cost and surplus sharing. In K.J. Arrow, A.K. Sen and K. Suzumura (eds.), *Handbook of Social Choice and Welfare, Vol. 1*. Amsterdam: Elsevier, 543–96.
- MOULIN, H. and SPRUMONT, Y. (2005). On demand responsiveness in additive cost sharing. *Journal of Economic Theory*, **125**, 1–35.
- MOULIN, H. and VOHRA, R. (2003). A representation of additive cost sharing methods. *Economics Letters*, **80**, 399–407.
- NEHRING, K. and PUPPE, C. (2002). A theory of diversity. *Econometrica*, **70**, 1155–98.
- PATTANAIK, P.K. and XU, Y. (2000). On diversity and freedom of choice. *Mathematical Social Sciences*, **40**, 123–30.
- PETERS, H. (1992). *Axiomatic Bargaining Game Theory*. Dordrecht: Kluwer.
- RAO, R.C. (1982). Diversity: its measurement, decomposition, apportionment and analysis. *Sankhyā*, **44 A**, 1–22.
- RAO, R.C. (1984). Convexity properties of entropy functions and analysis of diversity. In Y.L. Tong (ed.), *Inequalities in Statistics and Probability, IMS Lecture Notes*, **5**, 68–77.
- RAO, R.C. and NAYAK, T.K. (1985). Cross entropy, dissimilarity measures, and characterizations of quadratic entropy. *IEEE Transactions on Information Theory*, **IT-31**, 589–93.
- REYNAL-QUEROL, M. (2002). Ethnicity, political systems and civil war. *Journal of Conflict Resolution*, **46**, 29–54.

- SHAPLEY, L.S. (1953). A value for  $n$ -person games. In H.W. Kuhn and A.W. Tucker (eds.), *Contributions to the Theory of Games*, Princeton: Princeton University Press, 307–17.
- SIMPSON, E.H. (1949). Measurement of diversity. *Nature*, **163**, 688.
- SPOLAORE, E. and WACZIARG, R. (2009). The diffusion of development. *Quarterly Journal of Economics*, **124**, 469–529.
- VIGDOR, J.L. (2002). Interpreting ethnic fragmentation effects. *Economics Letters*, **75**, 271–76.
- WEITZMAN, M. (1992). On diversity. *Quarterly Journal of Economics*, **107**, 363–405.
- WEITZMAN, M. (1998). The Noah’s ark problem. *Econometrica*, **66**, 1279–98.
- WEYMARK, J.A. (1981). Generalized Gini inequality indices. *Mathematical Social Sciences*, **1**, 409–30.

Table 1: *GELF* and *ELF* in the US.

State name	State abb.	ELF	ELF rank	GELF	GELF rank	Difference
						(ELF rank-GELF rank)
Hawaii	HI	0.5245	1	0.0668	42	-41
District of Columbia	DC	0.5032	2	0.0767	1	1
Mississippi	MS	0.4344	3	0.0737	3	0
Louisiana	LA	0.4165	4	0.0731	4	0
California	CA	0.4042	5	0.0681	30	-25
Maryland	MD	0.3975	6	0.0709	12	-6
South Carolina	SC	0.3940	7	0.0722	6	1
Georgia	GA	0.3885	8	0.0716	9	-1
New York	NY	0.3644	9	0.0690	23	-14
Alabama	AL	0.3577	10	0.0717	7	3
Texas	TX	0.3534	11	0.0697	21	-10
North Carolina	NC	0.3425	12	0.0703	15	-3
New Mexico	NM	0.3332	13	0.0698	19	-6
Virginia	VA	0.3259	14	0.0717	8	6
Alaska	AK	0.3225	15	0.0738	2	13
Illinois	IL	0.3069	16	0.0688	24	-8
New Jersey	NJ	0.3005	17	0.0703	14	3
Delaware	DE	0.2904	18	0.0697	20	-2
Oklahoma	OK	0.2640	19	0.0696	22	-3
Michigan	MI	0.2591	20	0.0686	26	-6
Tennessee	TN	0.2566	21	0.0701	16	5
Arkansas	AR	0.2546	22	0.0688	25	-3
Arizona	AZ	0.2509	23	0.0679	34	-11
Florida	FL	0.2324	24	0.0677	35	-11
Nevada	NV	0.2248	25	0.0639	51	-26
Ohio	OH	0.2037	26	0.0686	27	-1
Connecticut	CT	0.1967	27	0.0700	17	10
Missouri	MO	0.1958	28	0.0698	18	10
Pennsylvania	PA	0.1821	29	0.0683	29	0
Colorado	CO	0.1815	30	0.0680	33	-3
Washington	WA	0.1637	31	0.0684	28	3
Indiana	IN	0.1574	32	0.0669	41	-9
Massachusetts	MA	0.1535	33	0.0710	11	22
Kansas	KS	0.1501	34	0.0669	40	-6
Kentucky	KY	0.1354	35	0.0728	5	30
Rhode Island	RI	0.1290	36	0.0712	10	26
Wisconsin	WI	0.1145	37	0.0671	39	-2
Oregon	OR	0.1054	38	0.0673	38	0
Utah	UT	0.1033	39	0.0664	45	-6
Montana	MT	0.1027	40	0.0665	44	-4
South Dakota	SD	0.1015	41	0.0660	47	-6
Nebraska	NE	0.0980	42	0.0659	49	-7
Wyoming	WY	0.0856	43	0.0661	46	-3
Idaho	ID	0.0797	44	0.0657	50	-6
Minnesota	MN	0.0788	45	0.0681	32	13
North Dakota	ND	0.0718	46	0.0659	48	-2
West Virginia	WV	0.0674	47	0.0708	13	34
Iowa	IA	0.0503	48	0.0666	43	5
New Hampshire	NH	0.0321	49	0.0675	37	12
Vermont	VT	0.0240	50	0.0677	36	14
Maine	ME	0.0238	51	0.0681	31	20

Table 2: *GELF* and *GroupedGelf* (Kolmogorov and Average) in the US.

<i>State</i>	<i>GELF</i>	<i>GELF rank</i>	<i>GroupedGELF_d</i>	<i>GroupedGELF_d rank</i>	<i>GroupedGELF</i>	<i>GroupedGELF rank</i>
HI	0.0668	42	0.0917	6	0.0588	13
DC	0.0767	1	0.2306	1	0.1864	1
MS	0.0737	3	0.1161	2	0.1061	2
LA	0.0731	4	0.1070	3	0.0951	3
CA	0.0681	30	0.0793	9	0.0586	14
MD	0.0709	12	0.0675	14	0.0564	16
SC	0.0722	6	0.0974	4	0.0879	4
GA	0.0716	9	0.0850	7	0.0758	6
NY	0.0690	23	0.0701	12	0.0617	10
AL	0.0717	7	0.0810	8	0.0727	7
TX	0.0697	21	0.0783	10	0.0646	8
NC	0.0703	15	0.0701	13	0.0613	12
NM	0.0698	19	0.0656	17	0.0614	11
VA	0.0717	8	0.0752	11	0.0637	9
AK	0.0738	2	0.0966	5	0.0875	5
IL	0.0688	24	0.0664	15	0.0576	15
NJ	0.0703	14	0.0661	16	0.0541	18
DE	0.0697	20	0.0545	19	0.0510	20
OK	0.0696	22	0.0365	26	0.0303	26
MI	0.0686	26	0.0558	18	0.0516	19
TN	0.0701	16	0.0432	24	0.0356	25
AR	0.0688	25	0.0543	20	0.0475	21
AZ	0.0679	34	0.0440	23	0.0558	17
FL	0.0677	35	0.0448	22	0.0388	23
NV	0.0639	51	0.0363	27	0.0285	29
OH	0.0686	27	0.0392	25	0.0364	24
CT	0.0700	17	0.0481	21	0.0407	22
MO	0.0698	18	0.0291	31	0.0252	31
PA	0.0683	29	0.0324	29	0.0294	27
CO	0.0680	33	0.0348	28	0.0293	28
WA	0.0684	28	0.0257	35	0.0184	37
IN	0.0669	41	0.0284	32	0.0241	32
MA	0.0710	11	0.0310	30	0.0256	30
KS	0.0669	40	0.0261	34	0.0216	33
KY	0.0728	5	0.0202	41	0.0146	42
RI	0.0712	10	0.0247	36	0.0197	36
WI	0.0671	39	0.0272	33	0.0213	34
OR	0.0673	38	0.0168	44	0.0125	45
UT	0.0664	45	0.0222	39	0.0180	39
MT	0.0665	44	0.0222	38	0.0184	38
SD	0.0660	47	0.0243	37	0.0201	35
NE	0.0659	49	0.0182	43	0.0148	41
WY	0.0661	46	0.0189	42	0.0157	40
ID	0.0657	50	0.0214	40	0.0145	43
MN	0.0681	32	0.0157	46	0.0114	46
ND	0.0659	48	0.0164	45	0.0128	44
WV	0.0708	13	0.0123	47	0.0097	47
IA	0.0666	43	0.0085	48	0.0058	48
NH	0.0675	37	0.0052	49	0.0040	49
VT	0.0677	36	0.0044	50	0.0020	51
ME	0.0681	31	0.0043	51	0.0030	50

Table 3: *GELF* and *GroupedGelf*(income, education, employment) in the US.

<i>State</i>	<i>GELF rank</i>	<i>GroupedGELF_income</i>	<i>rank</i>	<i>GroupedGELF_edu</i>	<i>rank</i>	<i>GroupedGELF_empl</i>	<i>rank</i>
<i>DC</i>	1	0.2880	1	0.2436	1	0.1426	29
<i>AK</i>	2	0.0936	10	0.0825	8	0.2261	5
<i>MS</i>	3	0.1181	2	0.1048	2	0.1748	23
<i>LA</i>	4	0.1181	3	0.0836	6	0.1960	16
<i>KY</i>	5	0.0249	36	0.0063	46	0.1169	34
<i>SC</i>	6	0.1054	5	0.0905	4	0.1943	17
<i>AL</i>	7	0.0937	9	0.0618	14	0.2023	11
<i>VA</i>	8	0.0938	8	0.0675	12	0.2100	9
<i>GA</i>	9	0.1161	4	0.0690	11	0.2021	12
<i>RI</i>	10	0.0285	33	0.0190	35	0.1151	36
<i>MA</i>	11	0.0401	25	0.0259	26	0.1332	32
<i>MD</i>	12	0.1028	6	0.0581	16	0.2117	8
<i>WV</i>	13	0.0135	44	0.0039	49	0.0631	47
<i>NJ</i>	14	0.0936	11	0.0596	15	0.2140	7
<i>NC</i>	15	0.0827	13	0.0568	18	0.2081	10
<i>TN</i>	16	0.0555	22	0.0236	29	0.1845	20
<i>CT</i>	17	0.0709	17	0.0435	21	0.1615	25
<i>MO</i>	18	0.0353	30	0.0174	37	0.1575	26
<i>NM</i>	19	0.0625	19	0.0751	9	0.2300	4
<i>DE</i>	20	0.0735	16	0.0533	19	0.2004	14
<i>TX</i>	21	0.0840	12	0.0830	7	0.2346	3
<i>OK</i>	22	0.0396	26	0.0254	28	0.2013	13
<i>NY</i>	23	0.0967	7	0.0674	13	0.2349	2
<i>IL</i>	24	0.0778	15	0.0576	17	0.2156	6
<i>AR</i>	25	0.0564	20	0.0389	22	0.1849	19
<i>MI</i>	26	0.0629	18	0.0356	23	0.1917	18
<i>OH</i>	27	0.0475	24	0.0257	27	0.1617	24
<i>WA</i>	28	0.0230	37	0.0209	34	0.1411	30
<i>PA</i>	29	0.0396	27	0.0229	30	0.1499	28
<i>CA</i>	30	0.0779	14	0.0851	5	0.2522	1
<i>ME</i>	31	0.0028	51	0.0025	51	0.0233	51
<i>MN</i>	32	0.0169	40	0.0067	45	0.0737	45
<i>CO</i>	33	0.0361	29	0.0330	25	0.1531	27
<i>AZ</i>	34	0.0556	21	0.0733	10	0.1965	15
<i>FL</i>	35	0.0552	23	0.0480	20	0.1790	22
<i>VT</i>	36	0.0031	50	0.0034	50	0.0235	50
<i>NH</i>	37	0.0047	49	0.0051	48	0.0313	49
<i>OR</i>	38	0.0129	46	0.0156	39	0.0963	38
<i>WI</i>	39	0.0264	35	0.0153	40	0.1033	37
<i>KS</i>	40	0.0265	34	0.0209	33	0.1304	33
<i>IN</i>	41	0.0304	32	0.0178	36	0.1337	31
<i>HI</i>	42	0.0325	31	0.0940	3	0.1166	35
<i>IA</i>	43	0.0054	48	0.0056	47	0.0482	48
<i>MT</i>	44	0.0175	39	0.0131	41	0.0935	40
<i>UT</i>	45	0.0168	41	0.0211	32	0.0945	39
<i>WY</i>	46	0.0150	43	0.0171	38	0.0796	43
<i>SD</i>	47	0.0188	38	0.0084	43	0.0922	41
<i>ND</i>	48	0.0131	45	0.0073	44	0.0674	46
<i>NE</i>	49	0.0162	42	0.0119	42	0.0899	42
<i>ID</i>	50	0.0117	47	0.0224	31	0.0745	44
<i>NV</i>	51	0.0394	28	0.0330	24	0.1811	21

Figure 1: *GELF* and *ELF* in the US

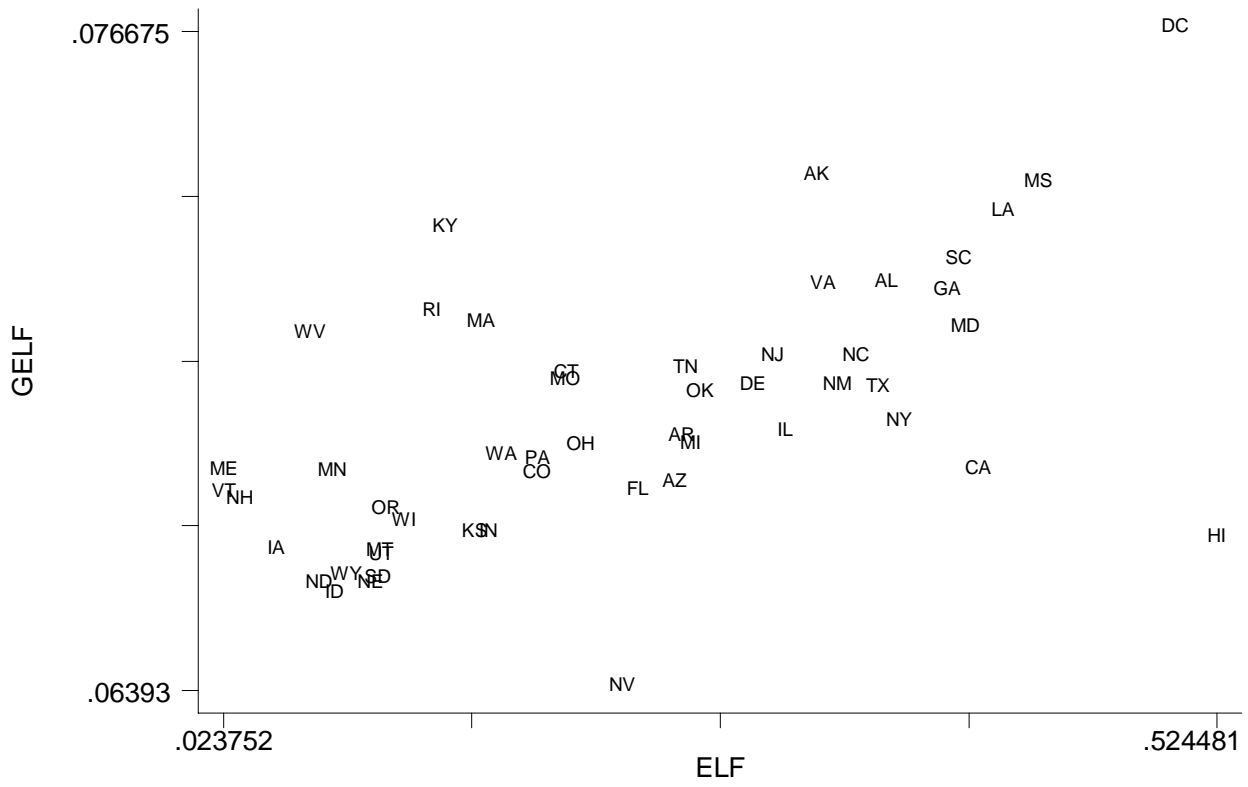






Figure 2, Panel B: *GroupedGELF* (education) and *ELF* in the US

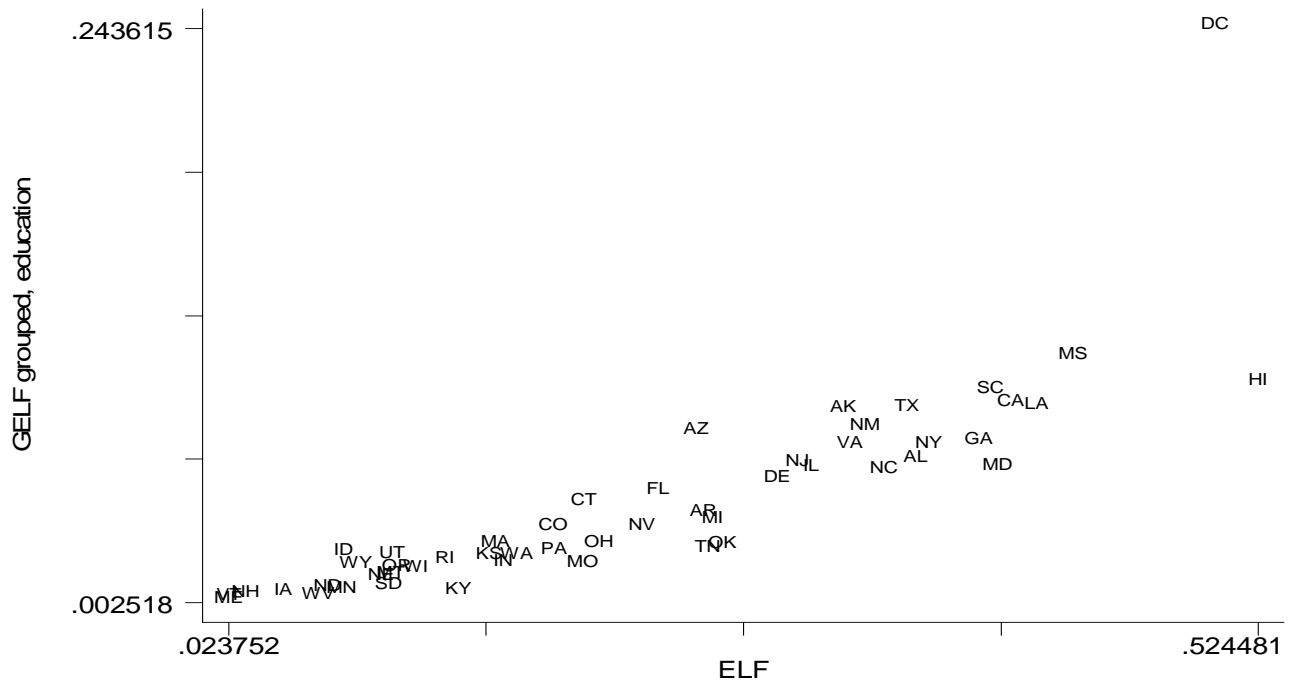


Figure 2, Panel C: *GroupedGELF*(employment) and *ELF* in the US

