ABSTRACT INTEGRATION OF SET-VALUED FUNCTIONS

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ABSTRACT. We develop an abstract notion of integration for Effros measurable correspondences whose values are weakly compact subsets of a separable Banach space. This notion is built on a basic monotonicity hypothesis and the simple requirements that the integral assigns at most one value to any single-valued correspondence and evaluates the constant functions in the obvious way; linearity of the integral is not required. These hypotheses alone guarantee that the abstract integral is relatively weakly compact-valued, and its closed convex hull decomposes into the abstract integrals of the measurable selections from that correspondence. We use this decomposition theorem to prove a Fatou-type lemma and a monotone convergence theorem, and to derive necessary and sufficient conditions for the linearity and parametric continuity of the abstract integral. In turn, we apply our main results to obtain simple characterizations of some classical set-valued integrals, and derive (possibly nonadditive) aggregation methods for correspondences. All in all, we find that abstract integration theory yields many results about particular integrals for set-valued maps in a unified manner, often with minimal recourse to measure-theoretic arguments.

1. INTRODUCTION

Integration of set-valued maps (= correspondences = multifunctions) is a fairly mature area within set-valued analysis, and arises as a matter of routine in a diverse set of fields such as control theory, mathematical economics, random set theory, and differential inclusions. The literature provides various ways of approaching this issue. Informally speaking, the majority of these approaches are of the following form: Given a measure space (Ω, Σ, μ) and a Banach space X, (i) decide on a set $\mathcal{F}$ of functions that map Ω into the set of all nonempty subsets of X; (ii) decide on a notion of integral I to use for Borel measurable functions from (Ω, Σ, μ) into X; and (iii) define the set-valued integral I as a function from $\mathcal{F}$ into 2^X by the formula

$$I(F) := \{I(f) : f \in \mathrm{Sel}(F) \text{ and } I(f) \text{ exists}\}, \tag{1}$$

where $\mathrm{Sel}(F)$ is the set of all Borel measurable selections from F. For lack of a better term, we refer to such a multivalued integral as a *selection integral* in what follows. In applications, it is desirable to work with closed and convex-valued integrals. This prompts looking also at integrals that are of the form

$$I(F) := \overline{\mathrm{co}}\{I(f) : f \in \mathrm{Sel}(F) \text{ and } I(f) \text{ exists}\};$$

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we call such an integral a *regularized selection integral.*

The most prominent example of a set-valued integral is surely the one of Aumann [7]. The original formulation of Aumann takes (Ω, Σ, μ) as the unit Lebesgue interval and X as $\mathbb{R}^n$, and chooses $\mathcal{F}$ as the set of all measurable correspondences from $[0,1]$ into $\mathbb{R}^n$. The *Aumann integral* of any $F \in \mathcal{F}$ is defined as

$$\mathrm{A}\int_{[0,1]} F := \left\{ \int_{[0,1]} f : f \in \mathrm{Sel}(F) \cap L^1([0,1]) \right\}$$

where the integrals on the right-hand side are Lebesgue. Obviously, the Aumann integral is a selection integral on $\mathcal{F}$. Besides, it is well-known that the Aumann integral is convex-valued. Thus

$$I(F) := \mathrm{cl}\left(\mathrm{A}\int_{[0,1]} F \right),$$

which is also a commonly used set-valued integral, is a regularized selection integral on $\mathcal{F}$. In addition, if $\mathcal{F}$ contains only the measurable, integrably bounded and closed-valued correspondences from $[0,1]$ to $\mathbb{R}^n$, the Aumann integral on $\mathcal{F}$ is compact and convex-valued, so it is itself a regularized selection integral (which may take $\varnothing$ as a value).

These observations can be substantially generalized. Let (Ω, Σ, μ) be a complete finite nonatomic measure space, X a separable Banach space, and let $\mathcal{F}$ consist of μ-integrably bounded, Effros measurable and closed-valued correspondences from Ω to X. For any $F \in \mathcal{F}$, define

$$I(F) := \mathrm{cl}\left\{ \int_{\Omega} f \,\mathrm{d}\mu : f \in \mathrm{Sel}(F) \right\}$$

where the integrals on the right-hand side are Bochner. Then, I is a regularized (nonempty-valued) selection integral on $\mathcal{F}$.[1]

The literature provides other forms of interesting integration concepts for set-valued maps. Some of these take on Aumann's approach, but replace the notion of integration for functions, adopting, for instance, the Lebesgue integral against a finitely additive measure (cf. [29]), or the Choquet integral against a capacity (cf. [46]). With suitable restrictions on their domains of integration, these set-valued integration theories yield selection integrals as well. There are also approaches that differ from that of Aumann at a deeper level. For instance, if we endow $ck(X)$, the set of all nonempty compact and convex subsets of a separable Banach space X, with the Hausdorff metric, we can use the Radström embedding theorem to isometrically embed the resulting metric space in a Banach space Y (as a convex cone), thereby identifying any $ck(X)$-valued map F on Ω with a function $\hat{F}$ from Ω into Y. We may then apply the Bochner integral to $\hat{F}$, and back out the "set-value" in X to be assigned as the integral of F. This is known as the *Debreu integral* (a special case of which was introduced in [13]). This integral is in general distinct from the Aumann integral, but it reduces to it, and hence it is a selection integral as well, under fairly general conditions (cf. [10] and [22]). Another example of note is the set-valued Pettis integral. While it is not at all evident at the level of its definition (see Section 6.3 below), it turns out that the Pettis integral is

[1]See [6, Section 8.6], [31, Section 2.1] and [45] for excellent reviews of the theory of Aumann and Aumann-Bochner integration.

also a regularized selection integral, provided $\mathcal{F}$ is the set of all weakly compact and convex-valued correspondences from Ω into a Banach space X that is either separable or has a weak*-separable dual. (See [11] and [16].)

The idea behind a selection integral is "natural," and the review above highlights that it is commonly adopted. It is, however, *ad hoc*. After all, it is not at all clear which sorts of set-valued integration principles justify defining the integral of a correspondence as the family of the integrals of all selections from that correspondence. The primary objective of the present paper is, in fact, to formulate such principles with the hope of understanding the overall structure of *any* selection integral. In contrast to the existing literature on set-valued integration, our approach is thus axiomatic. In particular, we do not start with a particular selection integral and study its properties, nor do we investigate if a particular integral can be realized as another type of a selection integral. Instead, informally speaking, for a given ideal $\mathcal{F}$ of Banach space-valued correspondences, we aim to identify the properties to be imposed on a correspondence I on $\mathcal{F}$ which would ensure I be a selection integral or a regularized selection integral on $\mathcal{F}$.

The basic outline of our contribution is in order. Let (Ω, Σ) be a measurable space, and X a separable Banach space. Let $\mathcal{F}$ be an ideal of weakly compact-valued (Effros) measurable correspondences from Ω to X. Our primitive is a correspondence of the form $\mathbf{I} : \mathcal{F} \rightrightarrows X$. And our primary questions are:

$*$ *When is* $\mathbf{I}$ *a selection integral?*

$*$ *When is it a regularized selection integral?*

Given that most set-valued integrals are either selection integrals or regularized selection integrals, the answers to these questions are bound to point to hitherto unseen commonalities between these integral concepts, and highlight what distinguishes them from other sorts of integrals.

The answers we provide to these two questions are pleasantly simple. First, we demand that $\mathbf{I}$ assign either $\varnothing$ or a singleton to any single-valued correspondence in $\mathcal{F}$ (allowing for the possibility that such a correspondence may not be $\mathbf{I}$-integrable). In addition, mainly to exclude trivialities, we posit that $\mathbf{I}$ of any constant single-valued correspondence in $\mathcal{F}$ is the singleton that consists of the value of that correspondence. These properties seem unimpeachable, and are satisfied by virtually all set-valued integration concepts in the literature. As a next step, it is tempting to impose some form of linearity or a monotonicity condition on $\mathbf{I}$. We do not follow the first route, because some important selection integrals, such as the Aumann-Choquet integral, is not additive. On the other hand, there is not an obvious way of tracing the second route simply because the image space X lacks an order structure in general. To circumvent this issue, we adopt a scalarization approach. For any $F, G \in \mathcal{F}$ and $\ell \in X^*$, let us say that F is *ℓ-larger than* G if for every selection g from G there is an $f \in \mathrm{Sel}(F)$ such that $\ell \circ f$ majorizes $\ell \circ g$. In turn, let us say that $\mathbf{I}$ is *ℓ-monotonic* if for every $y \in \mathbf{I}(G)$ there is an $x \in \mathbf{I}(F)$ with $\ell(x) \geq \ell(y)$, whenever F is ℓ-larger than G. Finally, we call $\mathbf{I}$ *scalarly monotonic* if it is ℓ-monotonic for every $\ell \in X^*$. This property is the main engine behind the entire development of the present work, and is discussed in fair bit of detail in Section 3.

We call $\mathbf{I}$ an *abstract integral* on $\mathcal{F}$ if it is scalarly monotonic, assigns at most a singleton to any single-valued correspondence in $\mathcal{F}$, and evaluates the constant single-valued correspondences in $\mathcal{F}$ in the obvious way. Our first main finding

(Theorem 4.1) is that an abstract integral comes close to being a selection integral in the sense that $\overline{\text{co}}$ of an abstract integral always equals the $\overline{\text{co}}$ of a selection integral. This gives a partial answer to the first question we posed above. As a corollary of this result (Theorem 4.3), we obtain an essentially complete answer to our second question: Every closed and convex-valued abstract integral on $\mathcal{F}$ is a regularized selection integral. Under rather mild conditions, the converse is true as well.

These observations, along with the fact that many interesting set-valued integrals (e.g., the Aumann-Bochner, the closed Aumann-Bochner, the Aumann-Choquet integrals) are abstract integrals on suitable domains, suggest that it may be worthwhile to explore the properties of abstract integrals. After deriving a Castaing-type representation for the abstract integral (Corollary 4.4), Section 5 is devoted to this issue. In that section, we show that closed and convex-valued abstract integrals are always weakly compact-valued. In addition, we prove that if it is closed and convex-valued, an abstract integral is linear iff it is linear for functions (Theorem 5.1), and it enjoys Fatou-type and monotone convergence properties (Theorems 5.4 and 5.5). In Section 5, we also show that Aumann identities hold for abstract integrals in great generality (Proposition 5.6), and derive a basic parametric continuity theorem for them (Theorem 5.9).

While they are of course not as sharp as the corresponding theorems for specific set-valued integrals, these results nonetheless unify quite a few findings of set-valued integration theory. At several instances of the exposition, we illustrate this by showing exactly how our general findings for the abstract integral read in the case of the Aumann-Bochner integral. In Section 5, we also show how to use abstract integrals to derive finitely additive set-valued measures, and provide conditions under which they are order-preserving when the image space X is a Banach lattice (Proposition 5.10).

The final section of the paper presents some applications. First, we use our main results to provide axiomatic characterizations of the Aumann and the Aumann-Bochner integrals.[2] Second, we present an alternative proof for the well-known characterization of the set-valued Pettis integral as a regularized selection integral. Third, we introduce a new set-valued integral in the context of probability spaces, which we dub *the convex integral*. This integral relates to the convex stochastic order, and closely resembles the Herer integral of random set theory. We prove that the convex integral is an abstract integral, and use our main decomposition theorem to show that it is actually a regularized selection integral. In our fourth, and final, application, we show that every (possibly non-additive) aggregator correspondence over the family of correspondences that map into the compact subsets of $[0, \infty)$, is an abstract integral. This allows us derive several results about aggregator correspondences (such as the Aumann-Choquet and the Aumann-Sugeno aggregators) in a unified manner.

[2] We are not aware of any such characterizations in the previous literature. One notable exception to this is the characterization of the closed Aumann integral by Ararat and Rudloff [2]. However, the approach of that paper is convex-analytic, and wholly different than the present one.

2. PRELIMINARIES

2.1. **Correspondences.** Let A and B be two nonempty sets. By $\Phi : A \rightrightarrows B$, we mean a set-valued map from A to B, that is, a map from A into the collection of all subsets of B. We henceforth refer to such a map as a **correspondence**, but note that some authors prefer the terminology of *multifunction* and *many-valued map* instead. By the **range** of Φ, we mean the set $\Phi(A) := \bigcup\{\Phi(a) : a \in A\}$. In general, any one value of Φ may be empty; the **effective domain** of Φ is defined as

$$\operatorname{dom}(\Phi) := \{a \in A : \Phi(a) \neq \varnothing\}.$$

If the domain and effective domain of Φ coincide, that is, $\Phi(a) \neq \varnothing$ for each $a \in A$, we refer to Φ as a **nonempty-valued** correspondence. The axiom of choice says that for any such correspondence Φ, there is a map $\phi : A \to B$ with $\phi(a) \in \Phi(a)$ for each $a \in A$. Such a map ϕ is said to be a **selection** from Φ.

If there is only one selection from Φ, we say that Φ is **single-valued**; in this case we may obviously identify Φ with its unique selection. Nevertheless, it will at times be convenient to have a notation that distinguishes the selection from a single-valued correspondence and that correspondence. Consequently, for any map $\phi : A \to B$, we denote by $\phi^{\{\}}$ the single-valued correspondence whose only selection is ϕ, that is, $\phi^{\{\}} : A \rightrightarrows B$ is defined by $\phi^{\{\}}(a) := \{\phi(a)\}$.

When B is a topological (linear) space, we say that a correspondence $\Phi : A \rightrightarrows B$ is **closed-valued** if $\Phi(a)$ is closed for each $a \in A$, and define (**weakly**) **compact-valued** correspondences analogously. When B is a linear space, we say that Φ is **convex-valued** if $\Phi(a)$ is convex for each $a \in A$.

For any correspondences Φ and Ψ from A to B, by $\Phi \sqsubseteq \Psi$ we simply mean that $\Phi(a) \subseteq \Psi(a)$ for each $a \in A$. The *union* and *intersection* of Φ and Ψ are the correspondences $\Phi \cup \Psi$ and $\Phi \cap \Psi$ from A to B defined by $(\Phi \cup \Psi)(a) := \Phi(a) \cup \Psi(a)$ and $(\Phi \cap \Psi)(a) := \Phi(a) \cap \Psi(a)$, respectively. These definitions are extended to the case of arbitrary unions and intersections in the obvious way. Finally, for any subset S of B, we denote the **constant correspondence** that maps every $a \in A$ to S by χ_S. That is, $\chi_S : A \rightrightarrows B$ is defined by

$$\chi_S(a) := S \qquad \text{for all } a \in A.$$

When $\Phi \sqsubseteq \chi_S$, we simply write $\Phi \sqsubseteq S$.

2.2. **Measurable Correspondences and Selections.** Let (Ω, Σ) be a measurable space and X a topological space. When we treat X itself as a measurable space, we always have the Borel σ-algebra $\mathcal{B}(X)$ in mind.

Let F be a correspondence from Ω to X. The **inverse image** of a subset S of X under F is defined as

$$F^{-1}(S) := \{\omega \in \Omega : F(\omega) \cap S \neq \varnothing\}.$$

In turn, we say that F is **Σ-measurable**, or **Effros measurable**, if $F^{-1}(O) \in \Sigma$ for every open subset O of X. We recall that Σ-measurability is preserved under countable unions. The intersection of two Σ-measurable correspondences from Ω to X need not be Σ-measurable (even when X is Polish), but if one of those correspondences is compact-valued, and the other closed-valued, then their intersection is Σ-measurable.

Remark 2.1. Some authors, such as Himmelberg [23], refer to Effros measurability as *weak measurability*, reserving the term "measurable" when $F^{-1}(C) \in \Sigma$ for every closed subset C of X. It is well-known that these two notions coincide when X is Polish, F is nonempty and closed-valued, and (Ω, Σ) is complete, that is, when there exists a complete σ-finite measure on Σ. Under these conditions, Σ-measurability of F is also equivalent to the graph of F, that is, $\bigcup_{\omega \in \Omega} \{\omega\} \times F(\omega)$, being a $\Sigma \otimes \mathcal{B}(X)$-measurable set. See [6, Theorem 8.1.4].

When X is a topological linear space, we denote its origin as 0_X. In this case, for any $A \subseteq \Omega$, we define the **restriction** of F to A as the correspondence $F \restriction_A : \Omega \rightrightarrows X$ where $F \restriction_A (\omega) := F(\omega)$ if $\omega \in A$ and $F \restriction_A (\omega) := \{0_X\}$ otherwise. (*Note.* $F \restriction_\varnothing = \chi_{\{0_X\}}$.) It is easy to verify that $F \restriction_A$ is Σ-measurable whenever $A \in \Sigma$. When this is the case, we refer to $F \restriction_A$ as a **measurable restriction** of F.

Notation. Throughout this paper, and when X is a Banach space, $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ stands for the set of all Σ-measurable, nonempty and weakly compact-valued correspondences from Ω to X. The set of all compact-valued members of $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ is, in turn, denoted as $\mathcal{K}(\Omega, \Sigma, X)$. These sets are identical when X is finite-dimensional.

By a **Σ-measurable selection** from F, we mean a selection f from F which is measurable as a map from (Ω, Σ) into X. In what follows, we denote the set of all Σ-measurable selections from F by $\mathrm{Sel}(F)$, assuming that the underlying measurable space is implicitly understood. The *Kuratowski-Ryll Nardzewski selection theorem* of [27] says that when X is a Polish space, we have $\mathrm{Sel}(F) \neq \varnothing$ for every nonempty and closed-valued Σ-measurable $F : \Omega \rightrightarrows X$. In particular, when X is a separable Banach space, $\mathrm{Sel}(F) \neq \varnothing$ for every $F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$.

A correspondence $F : \Omega \rightrightarrows X$ is said to admit a **Castaing representation** if

$$F(\omega) = \mathrm{cl}\{f_1(\omega), f_2(\omega), ...\} \qquad \text{for every } \omega \in \Omega$$

for some sequence (f_m) in $\mathrm{Sel}(F)$. It is well-known that, when X is a Polish space and F is nonempty and closed-valued, F is Σ-measurable iff it admits a Castaing representation.[3] In particular, when X is a separable Banach space, this representation is valid for any $F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$.

2.3. **Ordering of Sets.** Let X be a nonempty set and $\succsim$ a preorder (that is, a reflexive and transitive binary relation) on X. One natural way to use $\succsim$ in order to make 2^X a preordered set is by applying the containment ordering to the $\succsim$-decreasing closures of subsets of X. This leads to the **upper set-ordering $\succsim^\bullet$ induced by $\succsim$**. Put precisely, $\succsim^\bullet$ is the preorder on 2^X such that

$$A \succsim^\bullet B \quad \text{ iff } \quad \text{for every } y \in B \text{ there is an } x \in A \text{ with } x \succsim y.$$

(*Note.* $A \succsim^\bullet \varnothing$ for any $A \subseteq X$.)

[3] See [9, Theorem 6.6.8].

2.4. **Numerical Ordering of Sets.** Let X be a nonempty set and u a real-valued map on X. The **total preorder $\succsim_u$ on X induced by** u is defined as: $x \succsim_u y$ iff $u(x) \geq u(y)$. In accordance with the notation introduced above, the upper set-ordering induced by $\succsim_u$ on 2^X is denoted as $\succsim_u^\bullet$, that is,

$$A \succsim_u^\bullet B \quad \text{iff} \quad \text{for every } y \in B \text{ there is an } x \in A \text{ with } u(x) \geq u(y).$$

Clearly, for any $x, y \in X$, we have $\{x\} \succsim_u^\bullet \{y\}$ iff $u(x) \geq u(y)$, so $\succsim_u^\bullet$ may be regarded as an extension of $\succsim_u$ from X to 2^X.

We note that $A \succsim_u^\bullet B$ implies $\sup u(A) \geq \sup u(B)$. The converse may fail even in the simplest situations. For instance, $[0, 1) \succsim_{\mathrm{id}}^\bullet [0, 1]$ is false where id stands for the identity map on $\mathbb{R}$.

2.5. **Numerical Ordering of Correspondences.** Let (Ω, Σ) be a measurable space and X a Banach space. For any $u : X \to \mathbb{R}$, we preorder the maps from Ω into X according to their values under u by using the total preorder on X induced by u pointwise. That is, we define the preorder $\trianglerighteq_u$ on X^Ω as: $f \trianglerighteq_u g$ iff $u \circ f \geq u \circ g$ (i.e., $u(f(\omega)) \geq u(g(\omega))$ for every $\omega \in \Omega$). In turn, we use the upper set-ordering induced by $\trianglerighteq_u$, that is, $\trianglerighteq_u^\bullet$, to order the subsets of X^Ω. This, in turn, allows us make $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ a preordered set by comparing any two correspondences through their selections. Put precisely, we define the preorder $\blacktriangleright_u$ on $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ by

$$F \blacktriangleright_u G \quad \text{iff} \quad \mathrm{Sel}(F) \trianglerighteq_u^\bullet \mathrm{Sel}(G).$$

In other words, for any $F, G \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, we have

$$F \blacktriangleright_u G \quad \text{iff} \quad \text{for every } g \in \mathrm{Sel}(G) \text{ there is an } f \in \mathrm{Sel}(F) \text{ with } u \circ f \geq u \circ g.$$

Notice that, if f and g are Σ-measurable X-valued maps on (Ω, Σ), we have $f^{\{\}} \blacktriangleright_u g^{\{\}}$ iff $u \circ f$ majorizes $u \circ g$. The preorder $\blacktriangleright_u$ is thus a natural extension of the standard (coordinatewise) ordering of measurable maps from (Ω, Σ) into X on the basis of their u values.

3. SCALAR MONOTONICITY

3.1. **Scalarly Monotonic Evaluation of Correspondences.** Let (Ω, Σ) be a measurable space and X a Banach space. Informally speaking, by an *abstract integral* on a subset of $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, we think of a correspondence from that subset to X which

(i) behaves like a familiar integral for single-valued correspondences, and

(ii) acts "monotonically" in a suitable sense.

We will be rather permissive about modeling (i) here; we will only ask this correspondence to be at most single-valued on the set of all single-valued correspondences, and to evaluate constant single-valued correspondences in the obvious way. We do not impose any form of additivity at the outset.

The crux of the matter is to model (ii). This task is complicated by the fact that X does not have an inherent order structure. Our proposal in this regard is to capture the "monotonicity" of a set-valued integral by comparing the integrals of two correspondences through the sets of integrals of the ℓ-valuations of all selections from these correspondences (according to the upper set-ordering) for *every* continuous linear functional ℓ on X. The following definition introduces this property in precise terms.

Definition. Let (Ω,Σ) be a measurable space, X a Banach space, and $\mathcal{F}$ a nonempty subset of $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. We say that a correspondence $\mathbf{I}:\mathcal{F}\rightrightarrows X$ is **scalarly monotonic** if for every continuous linear functional ℓ on X, and every $F,G\in\mathcal{F}$,

$$F \blacktriangleright_{\ell} G \quad \text{implies} \quad \mathbf{I}(F) \succsim_{\ell}^{\bullet} \mathbf{I}(G). \tag{2}$$

Put more explicitly, $\mathbf{I}$ is scalarly monotonic iff for every $\ell\in X^*$ and $F,G\in\mathcal{F}$, the statement $F\blacktriangleright_{\ell} G$ (which means for every $g\in\mathrm{Sel}(G)$ there is an $f\in\mathrm{Sel}(F)$ with $\ell\circ f\geq\ell\circ g$) implies that either

- $\mathbf{I}(G)=\varnothing$ or;
- $\mathbf{I}(G)\neq\varnothing$ and for every $y\in\mathbf{I}(G)$ there is an $x\in\mathbf{I}(F)$ with $\ell(x)\geq\ell(y)$.

If (2) holds for every $\ell\in X^*$ and every *single-valued* $F,G\in\mathcal{F}$, we say that $\mathbf{I}$ is **scalarly monotonic for functions**. In other words, $\mathbf{I}$ is scalarly monotonic for functions iff for every $\ell\in X^*$ and Σ-measurable $f,g\in X^{\Omega}$ with $f^{\{\}},g^{\{\}}\in\mathcal{F}$, the inequality $\ell\circ f\geq\ell\circ g$ implies that either

- $\mathbf{I}(g^{\{\}})=\varnothing$ or;
- $\mathbf{I}(g^{\{\}})\neq\varnothing$ and for every $y\in\mathbf{I}(g^{\{\}})$ there is an $x\in\mathbf{I}(f^{\{\}})$ with $\ell(x)\geq\ell(y)$.

We next look at a few examples to illustrate this definition.

Example 3.1. Take any measurable space (Ω,Σ), and any nonempty family $\mathcal{M}$ of Σ-measurable real-valued functions on Ω. Put $\mathcal{F}:=\{f^{\{\}}:f\in\mathcal{M}\}$, and let $\mathbf{I}:\mathcal{F}\rightrightarrows\mathbb{R}$ be any single-valued correspondence. Then, $\mathbf{I}$ is scalarly monotonic iff

$$f\geq g \quad \text{implies} \quad \mathbf{I}(f^{\{\}})\geq\mathbf{I}(g^{\{\}})$$

for any $f,g\in\mathcal{M}$. Thus, the notion of scalar monotonicity is a generalization of the usual monotonicity property for real-valued integrals. $\square$

Example 3.2. For any $n\in\mathbb{N}$, put $N:=\{0,...,n\}$, and let $\mathcal{F}$ stand for the set of all nonempty compact-valued correspondences from $(N,2^N)$ to $\mathbb{R}$. Then, $\mathbf{I}_0:\mathcal{F}\rightrightarrows X$ defined by $\mathbf{I}_0(F):=-F(0)$, is not scalarly monotonic for functions. On the other hand, the correspondence $\mathbf{I}_1:\mathcal{F}\rightrightarrows X$ with $\mathbf{I}_1(F):=\left\{\sum_{i\in N}\min F(i)\right\}$, is scalarly monotonic for functions, but it is not scalarly monotonic. $\square$

As simple as they are, these examples suggest that scalar monotonicity is a reasonable monotonicity property for set-valued integrals that take values in a Banach space. We will next show that this property is satisfied by all the classical Banach space-valued integrals.

Notation. Throughout the paper, we denote the *Pettis* and the *Bochner integrals* of any Banach space-valued map f on a measure space (Ω,Σ,μ) by

$$\mathrm{P}\int_{\Omega} f\,\mathrm{d}\mu \quad \text{and} \quad \mathrm{B}\int_{\Omega} f\,\mathrm{d}\mu,$$

respectively, provided these integrals exist. However, when the Banach space under consideration is $\mathbb{R}^n$, either of these integrals (which then coincide) is denoted by $\int_{\Omega} f\,\mathrm{d}\mu$.

Example 3.3. Let (Ω,Σ,μ) be a finite measure space and X a Banach space. Let $\mathcal{P}$ stand for the set of all Pettis μ-integrable functions from Ω into X, and put $\mathcal{F}:=\{f^{\{\}}:f\in\mathcal{P}\}$. Finally, define $\mathbf{I}:\mathcal{F}\rightrightarrows X$ by setting $\mathbf{I}(f^{\{\}}):=\left\{\mathrm{P}\int_{\Omega} f\,d\mu\right\}$.

Then, $\mathbf{I}$ is scalarly monotonic. Indeed, for any $\ell \in X^*$ and $f, g \in \mathcal{P}$ with $\ell \circ f \geq \ell \circ g$, the very definition of the Pettis integral entails

$$\ell\left(\mathrm{P}\int_\Omega f\,\mathrm{d}\mu\right) = \int_\Omega (\ell \circ f)\,\mathrm{d}\mu \geq \int_\Omega (\ell \circ g)\,\mathrm{d}\mu = \ell\left(\mathrm{P}\int_\Omega g\,\mathrm{d}\mu\right),$$

that is, $\mathbf{I}(f^{\{\}}) \succsim_\ell^\bullet \mathbf{I}(g^{\{\}})$. $\square$

Notation. In what follows we will often simplify our notation by writing $\mathbf{I}(f)$ for $\mathbf{I}(f^{\{\}})$ for any $\mathbf{I} : \mathcal{F} \rightrightarrows X$ and Σ-measurable $f : \Omega \to X$ with $f^{\{\}} \in \mathcal{F}$.

Example 3.4. Let (Ω, Σ, μ) be a finite measure space and $X = (X, \|\cdot\|)$ a separable Banach space. The **Aumann-Bochner integral** is defined as the correspondence $\mathbf{I}_{\text{A-B}} : \mathcal{K}^{\text{w}}(\Omega, \Sigma, X) \rightrightarrows X$ with

$$\mathbf{I}_{\text{A-B}}(F) := \left\{\mathrm{B}\int_\Omega f\,\mathrm{d}\mu : f \in \mathrm{Sel}(F) \text{ and } f \text{ is Bochner } \mu\text{-integrable}\right\}.$$

In turn, we say that an $F \in \mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ is **Aumann-Bochner integrable** if $\mathbf{I}_{\text{A-B}}(F) \neq \varnothing$. When X is finite-dimensional, it is common to refer to $\mathbf{I}_{\text{A-B}}$ as the **Aumann integral**; in that case we use the notation $\mathbf{I}_{\text{A}}$. We say that a correspondence $F \in \mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ is **μ-integrably bounded** if there is a Bochner μ-integrable map $h : \Omega \to X$ with $\sup_{x \in F(\omega)} \|x\| \leq \|h(\omega)\|$ for every $\omega \in \Omega$.

Let $\mathcal{F}$ stand for the set of all μ-integrably bounded members of $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$. We claim that $\mathbf{I}_{\text{A-B}}$, restricted to $\mathcal{F}$, is scalarly monotonic. To see this, take any $\ell \in X^*$ and $F, G \in \mathcal{F}$ with $F \blacktriangleright_\ell G$, and pick an arbitrary $y \in \mathbf{I}_{\text{A-B}}(G)$. Then, $y = \mathrm{B}\int_\Omega g\,\mathrm{d}\mu$ for some Bochner μ-integrable $g \in \mathrm{Sel}(G)$. Since $F \blacktriangleright_\ell G$, there exists an $f \in \mathrm{Sel}(F)$ such that $\ell \circ f \geq \ell \circ g$, and since F is μ-integrably bounded, f is Bochner μ-integrable. Then, $x := \mathrm{B}\int_\Omega f\,\mathrm{d}\mu \in \mathbf{I}_{\text{A-B}}(F)$, while

$$\ell(x) = \int_\Omega (\ell \circ f)\,\mathrm{d}\mu \geq \int_\Omega (\ell \circ g)\,\mathrm{d}\mu = \ell(y).$$

In view of the arbitrary choice of y, this proves $\mathbf{I}_{\text{A-B}}(F) \succsim_\ell^\bullet \mathbf{I}_{\text{A-B}}(G)$. $\square$

3.2. **Ideals and Bornologies of Correspondences.** While the more general case of closed-valued correspondences is of obvious interest, the integration theory that we develop in this paper applies only to weakly-compact valued correspondences. Consequently, any *domain of integration* considered below will be a subclass of $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$. In general, we will only ask of such a subclass to contain the zero function and to be closed under taking subsets and finite unions. In a few instances, we will also need it to include all constant single-valued correspondences.

Definition. Let (Ω, Σ) be a measurable space, and X a Banach space (whose origin is denoted as 0_X). A nonempty subset $\mathcal{F}$ of $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ is said to be an **ideal in** $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ if $\chi_{\{0_X\}} \in \mathcal{F}$ and for any $F, G \in \mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$, we have (i) $F \in \mathcal{F}$ whenever $F \sqsubseteq G \in \mathcal{F}$, and (ii) $F \cup G \in \mathcal{F}$ whenever $F, G \in \mathcal{F}$. If, in addition, $\chi_{\{x\}} \in \mathcal{F}$ for every $x \in X$, we say that $\mathcal{F}$ is a **bornology in** $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$.[4]

As we shall see, an ideal in $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ has a sufficiently rich structure to serve as a domain for set-valued integrals, while most set-valued integrals are actually

[4] In point-set topology, a *bornology* on a nonempty set S is defined as a family of subsets of S that contains all singletons, and is closed under finite unions and taking subsets. Our terminology is patterned after this concept.

defined on a bornology in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. For the moment, we only point to the following obvious facts.

Lemma 3.1. *Let (Ω,Σ) and X be as in the definition above, and let $\mathcal{F}$ be an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. Then:*

a. *$\mathcal{F}$ is closed under measurable selections, that is, $f^{\{\}} \in \mathcal{F}$ for every $f \in \mathrm{Sel}(F)$ and $F \in \mathcal{F}$;*

b. *$\mathcal{F}$ is closed under measurable restrictions, that is, $F\restriction_A \in \mathcal{F}$ for any $F \in \mathcal{F}$ and $A \in \Sigma$.*

Proof. Part (a) is trivial. On the other hand, as $\mathcal{F}$ is an ideal, $\hat{F} := F \cup \chi_{\{0_X\}} \in \mathcal{F}$. Obviously, $F\restriction_A \sqsubseteq \hat{F}$ while it is readily checked that $F\restriction_A$ is Σ-measurable (whence $F\restriction_A \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$) for any $A \in \Sigma$. Since $\mathcal{F}$ is an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$, therefore, $F\restriction_A \in \mathcal{F}$. ■

As for examples, we note that the set of all Σ-measurable, nonempty finite-valued correspondences from Ω to X is a bornology in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. More generally, for any finite measure μ on Σ, the collection of all μ-integrably bounded members of $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$ is a bornology in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. (By contrast, in general, the set of all Aumann-Bochner integrable members of $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$ is not an ideal; it does not even have to be closed under measurable selections.) Finally, in the context of non-additive integration (Section 6.5), one often works with ideals like $\{F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,\mathbb{R}) : F \sqsubseteq [0,\infty)\}$ that are not bornologies.

3.3. A Weak Compactness Lemma.

A special case of the measurable maximum theorem – see [1, Theorem 18.19] – says that the correspondence $\omega \mapsto \{x \in F(\omega) : \varphi(x) = \sup \varphi(F(\omega))\}$ on Ω is Σ-measurable, where X is a Polish space, $\varphi \in \mathbf{C}(X)$, and $F \in \mathcal{K}(\Omega,\Sigma,X)$. In the sequel, we will need a version of this theorem in which X is a separable Banach space and F is known only to be weakly compact-valued. As we were unable to locate this type of a result in the literature, we provide a proof for it here (even though the argument closely follows that of [23] to deduce Filippov's implicit function theorem from Castaing's theorem, and the measurable maximum theorem from Filippov's theorem.)

Notation. Let $X = (X, \|\cdot\|)$ be a normed linear space, $y \in X$ and S a nonempty subset of X. In the rest of the paper, we denote the infimum distance between y and S by $\mathrm{dist}(y,S)$, that is,

$$\mathrm{dist}(y,S) := \inf_{x \in S} \|y - x\|.$$

Lemma 3.2. *Let (Ω,Σ) be a measurable space and X a separable Banach space. For any weakly continuous $\varphi : X \to \mathbb{R}$ and $F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$, the correspondence $H : \Omega \rightrightarrows X$ defined by*

$$H(\omega) := \operatorname*{arg\,max}_{x \in F(\omega)} \varphi(x),$$

belongs to $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$.

Proof. Take any φ and F as in the statement of the lemma, and note that H is a nonempty and weakly compact-valued correspondence by Weierstrass' extreme value theorem. Define $v : \Omega \to \mathbb{R}$ by $v(\omega) := \max\{\varphi(x) : x \in F(\omega)\}$. We claim that v is Σ-measurable. To prove this, apply Castaing's theorem to find a sequence (f_m) in $\mathrm{Sel}(F)$ such that $F(\omega) = \mathrm{cl}\{f_1(\omega), f_2(\omega), ...\}$ for every $\omega \in \Omega$. Then, since φ is continuous (being weakly continuous), we have $v(\omega) = \sup\{\varphi(x) : x \in \{f_1(\omega), f_2(\omega), ...\}\}$ for every $\omega \in \Omega$. Being the supremum of a countable collection of Σ-measurable real-valued maps, therefore, v is Σ-measurable.

For any $y \in X$, we define $d_y : \Omega \to [0, \infty)$ by

$$d_y(\omega) := \mathrm{dist}(y, H(\omega)).$$

We will show below that d_y is Σ-measurable for each $y \in X$. As this is equivalent to say that H is Σ-measurable, this will complete our proof. (See, for instance, [9, Proposition 6.5.8].)

For each positive integer k, let us define $H_k : \Omega \rightrightarrows X$ by

$$H_k(\omega) := \left\{x \in F(\omega) : |\varphi(x) - v(\omega)| < \tfrac{1}{k}\right\},$$

and note that $H \sqsubseteq H_k$. Next, for any $(y, k) \in X \times \mathbb{N}$, define the real-valued map $d_{y,k}$ on Ω by

$$d_{y,k}(\omega) := \mathrm{dist}(y, H_k(\omega)),$$

and note that

$$\{d_{y,k} < t\} = \bigcup_{m=1}^{\infty} \left(\{\|y - f_m\| < t\} \cap \left\{|\varphi \circ f_m - v| < \tfrac{1}{k}\right\}\right).$$

It readily follows from this observation that $d_{y,k}$ is Σ-measurable, for each $(y, k) \in X \times \mathbb{N}$.

Now, $H(\omega) = H_1(\omega) \cap H_2(\omega) \cap \cdots$ for each $\omega \in \Omega$. Moreover, since $H \sqsubseteq \cdots \sqsubseteq H_2 \sqsubseteq H_1$, we have $d_{y,1} \leq d_{y,2} \leq \cdots \leq d_y$. Consequently, $\sup_{k \in \mathbb{N}} d_{y,k}(\omega) \leq d_y(\omega)$ for every $(y, \omega) \in X \times \Omega$. To derive a contradiction, suppose this inequality holds strictly for some $(y, \omega) \in X \times \Omega$. Then, there exists an $\varepsilon > 0$ with $d_{y,k}(\omega) < d_y(\omega) - \varepsilon$ for all $k \in \mathbb{N}$. On the other hand, for every positive integer k, there is an $x_k \in H_k(\omega)$ such that $\|y - x_k\| \leq d_{y,k}(\omega) + \varepsilon/2$, whence $\|y - x_k\| < d_y(\omega) - \varepsilon/2$. By definition of H_k, we have $x_k \in F(\omega)$ and $|\varphi(x_k) - v(\omega)| < 1/k$, for every $k \in \mathbb{N}$. Since $F(\omega)$ is weakly compact, the Eberlein-Šmulian theorem entails that there is a strictly increasing sequence (k_m) of positive integers such that (x_{k_m}) converges weakly to some $x \in F(\omega)$. Since φ is weakly continuous, we have $\varphi(x_{k_m}) \to \varphi(x)$, so as $|\varphi(x_{k_m}) - v(\omega)| < 1/k_m$ for each m, we must have $\varphi(x) = v(\omega)$, that is, $x \in H(\omega)$. It follows that $d_y(\omega) \leq \|y - x\|$. But then, since $\|\cdot\|$ is weakly lower semicontinuous – see, for instance, [1, Lemma 6.22] – we get

$$d_y(\omega) \leq \|y - x\| \leq \liminf \|y - x_{k_m}\| \leq d_y(\omega) - \tfrac{\varepsilon}{2},$$

a contradiction. Conclusion:

$$\sup_{k \in \mathbb{N}} d_{y,k}(\omega) = d_y(\omega) \qquad \text{for every } (y, \omega) \in X \times \Omega.$$

Since each $d_{y,k}$ is Σ-measurable, therefore, d_y is Σ-measurable, for every $y \in X$. As we have noted above, this means that H is Σ-measurable. ■

The following is the main result of this section.

Lemma 3.3. *Let (Ω, Σ) be a measurable space and X a separable Banach space. Let $\mathcal{F}$ be an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, and take any $\mathbf{I} : \mathcal{F} \rightrightarrows X$ such that $|\mathbf{I}(F)| \leq 1$ whenever F is single-valued. If $\mathbf{I}$ is scalarly monotonic for functions, then $\overline{\mathrm{co}} \bigcup \{\mathbf{I}(f) : f \in \mathrm{Sel}(F)\}$ is weakly compact for any $F \in \mathcal{F}$.*

Proof. Fix an arbitrary $F \in \mathcal{F}$. If $\mathbf{I}(f) = \varnothing$ for every $f \in \mathrm{Sel}(F)$, there remains nothing to prove, so we assume otherwise. For an arbitrarily fixed $\ell \in X^*$, we define the correspondence $H_{\ell,F} : \Omega \rightrightarrows X$ by $H_{\ell,F}(\omega) := \arg\max\{\ell(x) : x \in F(\omega)\}$. Since a linear functional is continuous iff it is weakly continuous, Lemma 3.2 applies: $H_{\ell,F} \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. In turn, by the Kuratowski-Ryll Nardzewski selection theorem, $\mathrm{Sel}(H_{\ell,F}) \neq \varnothing$.

Note that $\mathrm{Sel}(H_{\ell,F}) \subseteq \mathrm{Sel}(F)$, and pick any $\varphi \in \mathrm{Sel}(H_{\ell,F})$. By definition of $H_{\ell,F}$, we have $\ell(\varphi(\omega)) \geq \ell(x)$ for every $\omega \in \Omega$ and $x \in F(\omega)$. In particular, $\ell \circ \varphi \geq \ell \circ f$ for any $f \in \mathrm{Sel}(F)$. Since $F \in \mathcal{F}$ and $\mathcal{F}$ is closed under selections, $f^{\{\}} \in \mathcal{F}$ for each $f \in \mathrm{Sel}(F)$ and, in particular, $\varphi^{\{\}} \in \mathcal{F}$. As $\mathbf{I}$ is scalarly monotonic for functions, therefore, $\mathbf{I}(\varphi) \succsim_\ell^\bullet \mathbf{I}(f)$ for every $f \in \mathrm{Sel}(F)$. Since we are in the case where $\mathbf{I}(f) \neq \varnothing$ for some $f \in \mathrm{Sel}(F)$, it follows that $\mathbf{I}(\varphi) \neq \varnothing$ and $\ell(\mathbf{I}(\varphi)) \geq \ell(\mathbf{I}(f))$ for every $f \in \mathrm{Sel}(F)$ with $\mathbf{I}(f) \neq \varnothing$. Moreover, applying this reasoning with $-\ell$ playing the role of ℓ shows that there is a $\phi \in \mathrm{Sel}(F)$ such that $\mathbf{I}(\phi) \neq \varnothing$ and $\ell(\mathbf{I}(\phi)) \leq \ell(\mathbf{I}(f))$ for every $f \in \mathrm{Sel}(F)$ with $\mathbf{I}(f) \neq \varnothing$. Therefore, the supremum of $|\ell|$ on $\bigcup\{\mathbf{I}(f) : f \in \mathrm{Sel}(F)\}$ is attained. As ℓ is linear and continuous, the same is true for $\mathrm{cl}(\mathrm{co}(\bigcup\{\mathbf{I}(f) : f \in \mathrm{Sel}(F)\}))$. Since the latter set is also weakly closed and ℓ is arbitrary in X^*, we may invoke James' weak compactness theorem to conclude that $\overline{\mathrm{co}} \bigcup\{\mathbf{I}(f) : f \in \mathrm{Sel}(F)\}$ is weakly compact in X. ■

The following example specializes Lemma 3.3 to the case of the Aumann-Bochner integral (Example 3.4), thereby illustrating how the present approach may be useful in concrete situations.

Example 3.5. Let (Ω, Σ, μ) be a complete, finite and nonatomic measure space, and X a separable Banach space. Let $\mathcal{F}$ stand for the set of all μ-integrably bounded members of $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, and take an arbitrary $F \in \mathcal{F}$. We have seen in Example 3.4 that $\mathbf{I}_{\text{A-B}}|_{\mathcal{F}}$ is scalarly monotonic. By Lemma 3.3, therefore, $\overline{\mathrm{co}}(\mathbf{I}_{\text{A-B}}(F))$ is weakly compact. But it is well-known that $\mathrm{cl}(\mathbf{I}_{\text{A-B}}(F))$ is convex,[5] and this implies that $\overline{\mathrm{co}}(\mathbf{I}_{\text{A-B}}(F))$ equals $\mathrm{cl}(\mathbf{I}_{\text{A-B}}(F))$. Also for the same reason, $\mathrm{cl}(\mathbf{I}_{\text{A-B}}(F))$ equals the weak closure of $\mathbf{I}_{\text{A-B}}(F)$. Conclusion: $\mathbf{I}_{\text{A-B}}$ *is a relatively weakly compact-valued correspondence on* $\mathcal{F}$. □

The observation derived in Example 3.5 is by no means new. However, the typical way this result is proved in the literature is by first showing that $\mathrm{Sel}(F)$ is relatively weakly compact in $L^1(\mu, X)$ – see, for instance, [14] and [45] – and then exploiting the fact that $f \mapsto \int_\Omega f \,\mathrm{d}\mu$ is a weak-to-weak continuous map from this space into X. In contrast, the method afforded by Lemma 3.3 seems a bit simpler, and attacks the issue of weak compactness of $\mathbf{I}_{\text{A-B}}$ directly.

Example 3.6. Let (Ω, Σ, μ), X and $\mathcal{F}$ be as in the previous example. We define the **closed Aumann-Bochner integral** as the correspondence $\mathbf{I} : \mathcal{F} \rightrightarrows X$ with $\mathbf{I}(F) := \mathrm{cl}(\mathbf{I}_{\text{A-B}}(F))$. We wish to show that this integral is scalarly monotonic as

[5] See, for instance, [45, Theorem 6.2] where the assertion is obtained as an easy consequence of Uhl's theorem of [42].

well. To this end, take any $\ell \in X^*$ and $F, G \in \mathcal{F}$ with $F \blacktriangleright_\ell G$. To prove that $\mathbf{I}(F) \gtrsim_\ell^\bullet \mathbf{I}(G)$, pick any $y \in \mathbf{I}(G)$ which means $\mathrm{B}\int_\Omega g_m \,\mathrm{d}\mu \to y$ for some sequence (g_m) of Bochner μ-integrable maps in $\mathrm{Sel}(G)$. Since $F \blacktriangleright_\ell G$, for each m there is an $f_m \in \mathrm{Sel}(F)$ with $\ell \circ f_m \geq \ell \circ g_m$. Since F is μ-integrably bounded, each f_m is Bochner μ-integrable. But, by what we have seen in the previous example, $\mathbf{I}(F)$ is weakly compact. It then follows from the Eberlein-Šmulian theorem that there is a subsequence (f_{m_k}) of (f_m) such that

$$x := \text{weak-}\lim \mathrm{B}\int_\Omega f_{m_k} \,\mathrm{d}\mu \in \mathbf{I}(F).$$

In turn,

$$\ell(x) = \lim \ell\left(\mathrm{B}\int_\Omega f_{m_k} \,\mathrm{d}\mu\right) = \lim \int_\Omega (\ell \circ f_{m_k}) \,\mathrm{d}\mu \geq \lim \int_\Omega (\ell \circ g_{m_k}) \,\mathrm{d}\mu = \ell(y),$$

which proves our claim. □

3.4. **I-Integrability.** Let (Ω, Σ) be a measurable space and X a Banach space. For any nonempty $\mathcal{F} \subseteq \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ and $\mathbf{I} : \mathcal{F} \rightrightarrows X$, we say that a correspondence $F \in \mathcal{F}$ is **I-integrable** if $\mathbf{I}(f) \neq \varnothing$ for at least one $f \in \mathrm{Sel}(F)$. The following lemma shows that under the scalar monotonicity hypothesis, the set of all $\mathbf{I}$-integrable correspondences and the effective domain of $\mathbf{I}$ coincide.

Lemma 3.4. *Let (Ω, Σ) be a measurable space and X a separable Banach space. Let $\mathcal{F}$ be an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, and take any scalarly monotonic $\mathbf{I} : \mathcal{F} \rightrightarrows X$. Then, for any $F \in \mathcal{F}$ the following are equivalent*:

a. *F is $\mathbf{I}$-integrable*;
b. $F \in \mathrm{dom}(\mathbf{I})$;
c. $\mathbf{I}(f) \neq \varnothing$ *for every* $f \in \mathrm{Sel}(F)$.

Proof. Take any $F \in \mathcal{F}$. As $\mathcal{F}$ is an ideal, $f^{\{\}} \in \mathcal{F}$ for every $f \in \mathrm{Sel}(F)$. Suppose F is $\mathbf{I}$-integrable so that there is an $f \in \mathrm{Sel}(F)$ with $\mathbf{I}(f) \neq \varnothing$. For any $\ell \in X^*$ we trivially have $F \blacktriangleright_\ell f^{\{\}}$, so scalar monotonicity of $\mathbf{I}$ yields $\mathbf{I}(F) \gtrsim_\ell^\bullet \mathbf{I}(f) \neq \varnothing$, whence $\mathbf{I}(F) \neq \varnothing$. This proves that (a) implies (b). Next, take any $F \in \mathrm{dom}(\mathbf{I})$. Fix any $\ell \in X^*$, and proceed as in the proof of Lemma 3.3 to find two selections φ and ϕ of F such that $\ell \circ \varphi \geq \ell \circ f \geq \ell \circ \phi$ for every $f \in \mathrm{Sel}(F)$. Then, $\varphi^{\{\}} \blacktriangleright_\ell F$, so by scalar monotonicity of $\mathbf{I}$, we find $\mathbf{I}(\varphi) \gtrsim_\ell^\bullet \mathbf{I}(F) \neq \varnothing$, whence $\mathbf{I}(\varphi) \neq \varnothing$. But $-\ell \circ \phi \geq -\ell \circ \varphi$, so $\phi^{\{\}} \blacktriangleright_{-\ell} \varphi^{\{\}}$, and again by scalar monotonicity of $\mathbf{I}$, we find $\mathbf{I}(\phi) \neq \varnothing$. Finally, for any $f \in \mathrm{Sel}(F)$, since $\ell \circ f \geq \ell \circ \phi$, we may use scalar monotonicity of $\mathbf{I}$ one more time to get $\mathbf{I}(f) \neq \varnothing$. Thus: (b) implies (c). That (c) implies (a) is trivial. ■

4. THE ABSTRACT INTEGRAL

4.1. **The Abstract Integral.** The following definition introduces the central concept of this paper.

Definition. Let (Ω, Σ) be a measurable space, X a Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. We say that a correspondence $\mathbf{I} : \mathcal{F} \rightrightarrows X$ is an **abstract integral** if

(i) $|\mathbf{I}(F)| \leq 1$ whenever F is single-valued;

(ii) $\mathbf{I}$ is scalarly monotonic; and
(iii) $\mathbf{I}(\chi_{\{x\}}) = \{x\}$ for every $x \in X$ with $\chi_{\{x\}} \in \mathcal{F}$.
(*Note.* $\mathbf{I}(\chi_{\{0_X\}}) = \{0_X\}$ for any such $\mathbf{I}$.) If the restriction of a correspondence $\mathbf{I} : \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X) \rightrightarrows X$ to $\mathcal{F}$ is an abstract integral, we say that $\mathbf{I}$ *acts as* an abstract integral on $\mathcal{F}$.

The following are a few immediate examples. Others will be given in Section 6.

Example 4.1. In the context of the definition above, assume that X is not trivial, set $\mathcal{F} = \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, and fix an arbitrary point x_0 in $X \backslash \{0_X\}$. Define $\mathbf{I} : \mathcal{F} \rightrightarrows X$ and $\mathbf{J} : \mathcal{F} \rightrightarrows X$ by $\mathbf{I}(F) := \varnothing$ and $\mathbf{J}(F) := \{x_0\}$, respectively. These correspondences satisfy the requirements (i) and (ii), but they are not abstract integrals as they fail the requirement (iii). $\square$

Example 4.2. As in Example 3.2, put $N := \{0, ..., n\}$, and let $\mathcal{F}$ stand for the set of all nonempty compact-valued correspondences from $(N, 2^N)$ to $\mathbb{R}$, which is a bornology in $\mathcal{K}(N, 2^N, \mathbb{R})$. Consider the following correspondences from $\mathcal{F}$ to $\mathbb{R}$:

$$\mathbf{I}_1(F) := \left\{ \tfrac{1}{n+1} \sum_{i \in N} \min F(i) \right\}, \qquad \mathbf{I}_2(F) := \left\{ \tfrac{1}{n+1} \sum_{i \in N} \max F(i) \right\},$$

and $\mathbf{I}_3(F) := F(0) \cup F(1)$. None of these are abstract integrals; $\mathbf{I}_1$ and $\mathbf{I}_2$ are not scalarly monotonic while $\mathbf{I}_3$ fails the condition (i) above. By contrast, $\mathbf{I}_4(F) := \mathbf{I}_1(F) \cup \mathbf{I}_2(F)$ and $\mathbf{I}_5(F) := F(0)$ define abstract integrals on $\mathcal{F}$. $\square$

Example 4.3. Let (Ω, Σ, μ) be a complete probability space, X a separable Banach space, and set $\mathcal{F} := \{F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X) : F \text{ is } \mu\text{-integrably bounded}\}$. Then, the Aumann-Bochner integral acts as an abstract integral on $\mathcal{F}$ (Example 3.4). If μ is nonatomic, then the closed Aumann-Bochner integral is an abstract integral on $\mathcal{F}$ as well (Example 3.6). $\square$

Remark 4.1. Let (Ω, Σ), X and $\mathcal{F}$ be as in the definition above. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be an abstract integral. Then, $\mathrm{co}\mathbf{I}$ is an abstract integral on $\mathcal{F}$. (Here, of course, $\mathrm{co}\mathbf{I}$ is defined pointwise, that is, $\mathrm{co}\mathbf{I}(F) := \mathrm{co}(\mathbf{I}(F))$.) Since $\mathrm{co}\mathbf{I}$ agrees with $\mathbf{I}$ for single-valued members of $\mathcal{F}$, to prove this we only need to check its scalar monotonicity. To this end, take any $F, G \in \mathcal{F}$ and $\ell \in X^*$ with $F \blacktriangleright_\ell G$. If $y \in \mathrm{co}\mathbf{I}(G)$, then $y = \sum^k \lambda_i y_i$ for some $k \in \mathbb{N}$, $\lambda_1, ..., \lambda_k \geq 0$ with $\sum^k \lambda_i = 1$, and $y_1, ..., y_k \in \mathbf{I}(G)$. By scalar monotonicity of $\mathbf{I}$, for each $i \in \{1, ..., k\}$ there is an $x_i \in \mathbf{I}(F)$ with $\ell(x_i) \geq \ell(y_i)$, so $\ell(x) \geq \ell(y)$ holds for $x := \sum^k \lambda_i x_i \in \mathrm{co}\mathbf{I}(F)$. $\square$

Remark 4.2. Condition (iii) of the definition above suggests that an abstract integral is really an *averaging* operator. For instance, in the context of Example 4.3, but where (Ω, Σ, μ) is a finite measure space with $0 < \mu(\Omega) \neq 1$, the Aumann-Bochner integral is not an abstract integral on $\mathcal{F}$ while $\frac{1}{\mu(\Omega)} \mathbf{I}_{\text{A-B}}$ is. If one wishes to view an abstract integral as an *aggregation* operator, condition (iii) should be replaced with the following:
(iii') There exists a $\theta > 0$ such that $\mathbf{I}(\chi_{\{x\}}) = \{\theta x\}$ for every $x \in X$ with $\chi_{\{x\}} \in \mathcal{F}$.
The theory presented below would adjust easily to capture this more general situation, but with slightly more convoluted statements. For expositional purposes,

therefore, we will work with condition (iii) in what follows. As a consequence, the specific set-valued integrals we use will either be normalized (as in Example 4.2) or will be defined on a probability space. □

4.2. **Decomposition Theorems.** We proceed with our investigation of the abstract integral. As a first order of business, we show that the closed convex hull of the abstract integral of an Effros measurable correspondence decomposes into the closed convex hull of the abstract integral of the measurable selections from that correspondence.

Theorem 4.1. *Let (Ω, Σ) be a measurable space, X a separable Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. Then, for any abstract integral $\mathbf{I} : \mathcal{F} \rightrightarrows X$ and $F \in \mathcal{F}$, we have*

$$\overline{\mathrm{co}}(\mathbf{I}(F)) = \overline{\mathrm{co}}\left(\bigcup_{f \in \mathrm{Sel}(F)} \mathbf{I}(f)\right). \tag{3}$$

In what follows, for any normed linear space X, we view the dual space X^* as a normed linear space relative to the operator norm (which we denote by $\|\cdot\|^*$), and let B_X stand for the closed unit ball of X. The following elementary fact of convex analysis will facilitate the proof of Theorem 4.1.

Lemma 4.2. *Let X be a normed linear space and L a dense subset of X^*. Let A and B be two subsets of X such that $\mathrm{cl}(\ell(A)) = \ell(B)$ for every $\ell \in L$. If $L = X^*$, or B is bounded, we have $\overline{\mathrm{co}}(A) = \overline{\mathrm{co}}(B)$.*

Proof. As the proof of the case where $L = X^*$ is easier, we only concentrate on the case where B is bounded. If either A or B is empty, then both of these sets are empty, so our claim holds trivially. We thus assume that both A and B are nonempty. Take an element x of $X \backslash \overline{\mathrm{co}}(B)$. Then, by the separating hyperplane theorem, we can find real numbers a and $\varepsilon > 0$, and an $\ell_0 \in X^*$, such that

$$\sup \ell_0(\overline{\mathrm{co}}(B)) \leq a - \varepsilon < a + \varepsilon < \ell_0(x).$$

Now set $C := \overline{\mathrm{co}}(B) \cup \{x\}$, and note that C is a bounded subset of X, so there is a real number $\lambda > 0$ such that $\lambda C \subseteq B_X$. Moreover, as L is dense in X^*, there exists an $\ell \in L$ with $\|\ell - \ell_0\|^* < \lambda\varepsilon$. Then,

$$\sup_{z \in C} |\ell(z) - \ell_0(z)| = \tfrac{1}{\lambda} \sup_{y \in \lambda C} |\ell(y) - \ell_0(y)| \leq \tfrac{1}{\lambda} \|\ell - \ell_0\|^* < \varepsilon,$$

and it follows that $\sup \ell(\overline{\mathrm{co}}(B)) < a < \ell(x)$. As $\ell(A) \subseteq \ell(B)$ by hypothesis, therefore, x is not in A. Conclusion: $A \subseteq \overline{\mathrm{co}}(B)$, and hence, $\overline{\mathrm{co}}(A) \subseteq \overline{\mathrm{co}}(B)$. This, in particular, shows that A is bounded. Conversely, suppose there is an element x in $\overline{\mathrm{co}}(B) \backslash \overline{\mathrm{co}}(A)$. Then, using again the separating hyperplane theorem along with the previous density argument (which applies because A is bounded), we find a real number a and an $\ell \in L$ such that $\sup \ell(\overline{\mathrm{co}}(A)) < a < \ell(x)$. Put $\delta := a - \sup \ell(\overline{\mathrm{co}}(A))$. Since $x \in \mathrm{cl}(\mathrm{co}(B))$, we have $x_m \to x$ for some sequence (x_m) in $\mathrm{co}(B)$. Moreover,

$$\ell(\mathrm{co}(B)) = \mathrm{co}(\ell(B)) = \mathrm{co}(\mathrm{cl}(\ell(A))) \subseteq \overline{\mathrm{co}}(\ell(A)) = \mathrm{cl}(\ell(\mathrm{co}(A))),$$

so for every m, there is a $y_m \in \mathrm{co}(A)$ with $|\ell(y_m) - \ell(x_m)| < \delta/2$. As $\ell(y_m) \leq a - \delta$ for each m, therefore, $\ell(x_m) < a - \delta/2$, so, since $x_m \to x$, we find $\ell(x) \leq a - \delta/2$, a contradiction. We conclude that $\overline{\mathrm{co}}(A) = \overline{\mathrm{co}}(B)$. ■

Proof of Theorem 4.1. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be an abstract integral, and take any $F \in \mathcal{F}$. If F is not $\mathbf{I}$-integrable, Lemma 3.4 implies (3) with both sides being $\varnothing$. Assume, then, F is $\mathbf{I}$-integrable. It then follows from Lemma 3.4 that $\mathbf{I}(F) \neq \varnothing$, and $\mathbf{I}(f) \neq \varnothing$ for every $f \in \mathrm{Sel}(F)$. Since $\mathbf{I}$ is an abstract integral, therefore, for every $f \in \mathrm{Sel}(F)$ there is an $x_f \in X$ with $\mathbf{I}(f) = \{x_f\}$.

Let us fix an arbitrary $\ell \in X^*$, and proceed as in the proof of Lemma 3.3 to find a $\varphi \in \mathrm{Sel}(F)$ with $\ell \circ \varphi \geq \ell \circ f$ for all $f \in \mathrm{Sel}(F)$. Then, $\varphi^{\{\}} \blacktriangleright_\ell F$ while $\varphi^{\{\}} \in \mathcal{F}$ (Lemma 3.1). So, by scalar monotonicity of $\mathbf{I}$, we find $\mathbf{I}(\varphi) \succsim_\ell^\bullet \mathbf{I}(F) \neq \varnothing$. Thus: $\ell(x_\varphi) \geq \sup \ell(\mathbf{I}(F))$. But, as $\varphi \in \mathrm{Sel}(F)$, we trivially have $F \blacktriangleright_\ell \varphi^{\{\}}$, so, again by scalar monotonicity of $\mathbf{I}$, $\sup \ell(\mathbf{I}(F)) \geq \ell(x_\varphi)$. Thus: $\ell(x_\varphi) = \sup \ell(\mathbf{I}(F))$. Moreover, applying this reasoning with $-\ell$ playing the role of ℓ yields $\ell(x_\phi) = \inf \ell(\mathbf{I}(F))$ for some $\phi \in \mathrm{Sel}(F)$. Since $\overline{\mathrm{co}}(\ell(\mathbf{I}(F)))$, which equals $\mathrm{cl}(\ell(\mathrm{co}(\mathbf{I}(F))))$, is a closed interval, therefore,

$$\mathrm{cl}(\ell(\mathrm{co}(\mathbf{I}(F)))) = [\ell(x_\phi), \ell(x_\varphi)]. \tag{4}$$

On the other hand, by the choice of ϕ and φ, we have $\ell \circ \varphi \geq \ell \circ f \geq \ell \circ \phi$ for every $f \in \mathrm{Sel}(F)$. Thus, by scalar monotonicity of $\mathbf{I}$ on $\mathcal{F}$, we have $\ell(x_\varphi) \geq \ell(x_f) \geq \ell(x_\phi)$ for every $f \in \mathrm{Sel}(F)$. So, by (4) and because ϕ and φ belong to $\mathrm{Sel}(F)$,

$$\mathrm{cl}(\ell(\mathrm{co}(\mathbf{I}(F)))) \supseteq \ell\left(\{x_f : f \in \mathrm{Sel}(F)\}\right) \supseteq \{\ell(x_\phi), \ell(x_\varphi)\}.$$

Using (4) again, then,

$$\mathrm{cl}(\ell(\mathrm{co}(\mathbf{I}(F)))) = \mathrm{co}\left(\ell(\{x_f : f \in \mathrm{Sel}(F)\})\right) = \ell(\mathrm{co}\{x_f : f \in \mathrm{Sel}(F)\}).$$

Since $\ell \in X^*$ is arbitrary here, we may then apply Lemma 4.2 (with $A := \mathrm{co}(\mathbf{I}(F))$ and $B := \mathrm{co}\{x_f : f \in \mathrm{Sel}(F)\}$) to conclude that

$$\overline{\mathrm{co}}(\mathrm{co}(\mathbf{I}(F))) = \overline{\mathrm{co}}(\mathrm{co}\{x_f : f \in \mathrm{Sel}(F)\}).$$

Since the closed convex hull of the convex hull of a set is equal to the closed convex hull of that set, this completes our proof. ■

Remark 4.3. Let (Ω, Σ), X and $\mathcal{F}$ be as in Theorem 4.1, and let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be an abstract integral. We have noted in Remark 4.1 that $\mathrm{co}\mathbf{I}$ is an abstract integral on $\mathcal{F}$. We now show that this is also true for $\mathbf{J} := \overline{\mathrm{co}}\mathbf{I}$ (defined pointwise). Since $\mathbf{J}$ agrees with $\mathbf{I}$ for single-valued members of $\mathcal{F}$, we only need to verify its scalar monotonicity. To this end, take any $F, G \in \mathcal{F}$ and $\ell \in X^*$ with $F \blacktriangleright_\ell G$, and any $y \in \mathbf{J}(G)$. Then $\lim y_m = y$ for some sequence (y_m) in $\mathrm{co}\mathbf{I}(G)$. By scalar monotonicity of $\mathrm{co}\mathbf{I}$, for each m there is an $x_m \in \mathrm{co}\mathbf{I}(F)$ with $\ell(x_m) \geq \ell(y_m)$. Moreover, applying Theorem 4.1 to $\mathrm{co}\mathbf{I}$ yields $\mathbf{J}(F) = \overline{\mathrm{co}} \bigcup_{f \in \mathrm{Sel}(\mathrm{co}F)} \mathbf{I}(f)$, and hence, by Lemma 3.3, we may conclude that $\mathbf{J}(F)$ is weakly compact. By the Eberlein-Šmulian theorem, therefore, there is a subsequence (x_{m_k}) of (x_m) such that weak-$\lim x_{m_k} = x$ for some $x \in \mathbf{J}(F)$. But then $\ell(x) = \lim \ell(x_{m_k}) \geq \lim \ell(y_{m_k}) = \ell(y)$, and we are done. □

For closed and convex-valued correspondences, Theorem 4.1 turns into a characterization result.

Theorem 4.3. *Take (Ω, Σ), X, and $\mathcal{F}$ as in Theorem 4.1, and let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be a closed and convex-valued correspondence such that $|\mathbf{I}(F)| \leq 1$ whenever F is* *single-valued. Then,* **I** *is an abstract integral if, and only if, it is scalarly monotonic for functions,* $\mathbf{I}(\chi_{\{x\}}) = \{x\}$ *for all* $x \in X$ *with* $\chi_{\{x\}} \in \mathcal{F}$*, and*

$$\mathbf{I}(F) = \overline{\mathrm{co}}\left(\bigcup_{f\in \mathrm{Sel}(F)} \mathbf{I}(f)\right) \tag{5}$$

for every $F \in \mathcal{F}$*.*

Proof. In view of Theorem 4.1, we only need to prove the "if" part of the assertion. To this end, assume that **I** is scalarly monotonic for functions, and that (5) holds. The theorem will be proved if we can show that **I** is scalarly monotonic. To this end, take any $\ell \in X^*$ and $F, G \in \mathcal{F}$ with $F \blacktriangleright_\ell G$. If $\mathbf{I}(G) = \varnothing$, we trivially have $\mathbf{I}(F) \succsim_\ell^\bullet \mathbf{I}(G)$, so assume otherwise. Then, by (5), $\mathbf{I}(g) \neq \varnothing$ (whence $|\mathbf{I}(g)| = 1$) for some $g \in \mathrm{Sel}(G)$.

Let us define $S := \{x : \{x\} = \mathbf{I}(f)$ for some $f \in \mathrm{Sel}(F)\}$ and $T := \{x : \{x\} = \mathbf{I}(g)$ for some $g \in \mathrm{Sel}(G)\}$. We know that T is nonempty. In addition, since $F \blacktriangleright_\ell G$ and **I** is scalarly monotonic for functions, S is nonempty and $\sup \ell(S) \geq \sup \ell(T)$. By linearity of ℓ, then,

$$\sup \ell(\mathrm{co}(S)) \geq \sup \ell(\mathrm{co}(T)). \tag{6}$$

Moreover, by Lemma 3.3, $\mathrm{cl}(\mathrm{co}(S))$ is weakly compact in X. As continuity and weak continuity are equivalent for any linear functional on a normed linear space, it follows that $\ell(\mathrm{cl}(\mathrm{co}(S)))$ is compact, and this implies $\ell(\mathrm{cl}(\mathrm{co}(S))) \supseteq \mathrm{cl}(\ell(\mathrm{co}(S)))$. As the converse of this containment follows readily from the continuity of ℓ, therefore, we have $\ell(\overline{\mathrm{co}}(S)) = \mathrm{cl}(\ell(\mathrm{co}(S)))$, and similarly, $\ell(\overline{\mathrm{co}}(T)) = \mathrm{cl}(\ell(\mathrm{co}(T)))$. It then follows from (6) that $\max \ell(\overline{\mathrm{co}}(S)) \geq \max \ell(\overline{\mathrm{co}}(T))$. As $\overline{\mathrm{co}}(S) = \mathbf{I}(F)$ and $\overline{\mathrm{co}}(T) = \mathbf{I}(G)$ by (5), we find $\mathbf{I}(F) \succsim_\ell^\bullet \mathbf{I}(G)$, as desired. ■

Theorems 4.1 and 4.3 are our answers to the two basic questions posed in the Introduction. In the framework of Theorem 4.1, if **I** is an "integral" on $\mathcal{F}$ that assigns to any single-valued correspondence $\varnothing$ or a singleton, evaluates constant single-valued correspondences in the obvious way, and is scalarly monotonic, it comes very close to being a *selection integral.* Put precisely, under these conditions, even if **I** may fail to be a selection integral, $\overline{\mathrm{co}}(\mathbf{I})$ is sure to be the closed and convex hull of a selection integral. On the other hand, the axiomatization of the structure of *regularized selection integrals* is complete. Theorem 4.3 says that **I** is a regularized selection integral provided that it is closed and convex-valued, scalarly monotonic, and assigns to any single-valued correspondence a unique value.

4.3. **A Castaing-Type Representation for the Abstract Integral.** When X is a reflexive and separable Banach space (or more generally, a Banach space with a separable dual), it is possible to obtain a representation of the abstract integral that has a flavor of the Castaing representation of Effros measurable correspondences that map to a Polish space. In particular, in that case, we can express a closed and convex-valued abstract integral of a correspondence as the closed convex hull of the union of the abstract integrals of countably many selections from that correspondence.

Corollary 4.4. *Let* (Ω, Σ) *be a measurable space and* X *a reflexive and separable Banach space. Let* $\mathcal{F}$ *be an ideal in* $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$*, and* $\mathbf{I} : \mathcal{F} \rightrightarrows X$ *an abstract* *integral. Then, for every $F \in \mathcal{F}$ there is a sequence (f_m) in* $\mathrm{Sel}(F)$ *with*

$$\overline{\mathrm{co}}(\mathbf{I}(F)) = \overline{\mathrm{co}}\left(\mathbf{I}(f_1) \cup \mathbf{I}(f_2) \cup \cdots\right). \tag{7}$$

Proof. Take any $F \in \mathcal{F}$, and assume F is $\mathbf{I}$-integrable, for otherwise (7) holds trivially. Then, by Lemma 3.4, $\mathbf{I}(F) \neq \varnothing$, and for every $f \in \mathrm{Sel}(F)$ there is an $x_f \in X$ with $\mathbf{I}(f) = \{x_f\}$. Since X is a reflexive and separable Banach space, X^* is separable.[6] Let L be a countable dense set in X^*, and enumerate it as $\{\ell_1, \ell_2, ...\}$. For any $m \in \mathbb{N}$, we proceed exactly as in the last paragraph of the proof of Theorem 4.1 to find two maps φ_m and ϕ_m in $\mathrm{Sel}(F)$ such that

$$[\ell_m(x_{\phi_m}), \ell_m(x_{\varphi_m})] = \mathrm{cl}(\ell_m(\mathrm{co}(\mathbf{I}(F)))) = \ell_m(\mathrm{co}\{x_f : f \in \mathrm{Sel}(F)\}).$$

Therefore, for the sequence (f_m) with $f_{2i-1} := \varphi_i$ and $f_{2i} := \phi_i$ for each $i = 1, 2, ...$, we have $\mathrm{cl}(\ell(\mathrm{co}(\mathbf{I}(F)))) = \ell(\mathrm{co}\{x_{f_1}, x_{f_2}, ...\})$ for every $\ell \in L$. Note also that, by Lemma 3.3, $\mathrm{co}\{x_{f_1}, x_{f_2}, ...\}$ is relatively weakly compact in X, and hence it is bounded. We may then apply Lemma 4.2 (with $A := \mathrm{co}(\mathbf{I}(F))$ and $B := \mathrm{co}\{x_{f_1}, x_{f_2}, ...\}$) to conclude that $\overline{\mathrm{co}}(\mathbf{I}(F)) = \overline{\mathrm{co}}\{x_{f_1}, x_{f_2}, ...\}$. ■

5. PROPERTIES OF THE ABSTRACT INTEGRAL

The decomposition results of Section 4.2 allow us identify the general properties of an abstract integral which it inherits from its single-valued version. Many of the results we present in this section are inheritance theorems of this form.

5.1. **Basic Properties of the Abstract Integral.** Throughout this subsection, (Ω, Σ), X, and $\mathcal{F}$ are as in Theorem 4.1, and $\mathbf{I} : \mathcal{F} \rightrightarrows X$ is a closed and convex-valued abstract integral.

Property 1. (Weak Compactness) $\mathbf{I}$ is weakly compact-valued.

Property 2. (Monotonicity) $\mathbf{I}(F) \subseteq \mathbf{I}(G)$ for any $F, G \in \mathcal{F}$ with $F \sqsubseteq G$.

Property 3. (Homogeneity) Given any $\lambda \in \mathbb{R}$, if $\mathbf{I}(\lambda F) = \lambda \mathbf{I}(F)$ for every single-valued $F \in \mathcal{F}$ with $\lambda F \in \mathcal{F}$, then $\mathbf{I}(\lambda F) = \lambda \mathbf{I}(F)$ for every $F \in \mathcal{F}$ with $\lambda F \in \mathcal{F}$.

Property 1 is an immediate consequence of Lemma 3.3 and Theorem 4.3, while the other two readily follow from Theorem 4.3 alone. In passing, we note that Property 2 entails

$$\mathbf{I}(F \cap G) \subseteq \mathbf{I}(F) \cap \mathbf{I}(G) \subseteq \mathbf{I}(F) \cup \mathbf{I}(G) \subseteq \mathbf{I}(F \cup G)$$

for any $F, G \in \mathcal{F}$ such that $F \cap G \in \mathcal{F}$.

In the statements of the following two properties, we assume that $\mathcal{F}$ is a bornology in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, that is, it is an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ that includes all constant single-valued correspondences. The domain of integration in most (but not all; see Section 6.5) applications are of this form.

Property 4. (Range of the Abstract Integral) $\mathbf{I}(F) \subseteq \overline{\mathrm{co}}(F(\Omega))$ for every $F \in \mathcal{F}$ with a relatively weakly compact range.

To see this, take any $F \in \mathcal{F}$ such that $K := \text{weak-}\mathrm{cl}(F(\Omega))$ is weakly compact. Next, let f be any Σ-measurable selection from F. We claim that $\mathbf{I}(f) \subseteq \overline{\mathrm{co}}(f(\Omega))$.

[6] See [30, Corollary 1.12.12].

To derive a contradiction, suppose this is false. In that case $\mathbf{I}(f) = \{x\}$ for some $x \in X \backslash S$ where $S := \overline{\text{co}}(f(\Omega))$. By the separating hyperplane theorem, there is an $\ell \in X^*$ with $\ell(x) > \sup \ell(S)$. On the other hand, as weak-cl$(f(\Omega))$ is weakly compact (because it is contained in K) and ℓ is weakly continuous, there exists a $y \in$ weak-cl$(f(\Omega))$ with $\ell(y) \geq \ell(z)$ for all $z \in$ weak-cl$(f(\Omega))$. Since weak-cl$(f(\Omega)) \subseteq S$, we have $\ell(x) > \ell(y)$. But $\chi_{\{y\}} \blacktriangleright_\ell f$ so

$$\{y\} = \mathbf{I}(\chi_{\{y\}}) \succsim_\ell^\bullet \mathbf{I}(f) = \{x\},$$

that is, $\ell(y) \geq \ell(x)$, a contradiction.

We have just proved that $\mathbf{I}(f) \subseteq \overline{\text{co}}(f(\Omega))$, whence $\mathbf{I}(f) \subseteq \overline{\text{co}}(F(\Omega))$, for every $f \in \text{Sel}(F)$. By Theorem 4.3, therefore, $\mathbf{I}(F) \subseteq \overline{\text{co}}(F(\Omega))$, as desired.

Property 5. (Integration of Constants) $\mathbf{I}(\chi_S) = S$ for any nonempty, convex and weakly compact $S \subseteq X$ with $\chi_S \in \mathcal{F}$.

By Theorem 4.3,

$$\mathbf{I}(\chi_S) = \overline{\text{co}} \bigcup_{f \in \text{Sel}(\chi_S)} \mathbf{I}(f) \supseteq \overline{\text{co}} \bigcup_{x \in S} \mathbf{I}(\chi_{\{x\}}) = \overline{\text{co}}(S) = S$$

for any such $S \subseteq X$. Conversely, Property 4 implies $\mathbf{I}(\chi_S) \subseteq S$.

5.2. **On the Linearity of the Abstract Integral.** An abstract integral need not be additive in general; see Section 6.5 for some concrete examples. Indeed, an obvious necessary condition for this is that the integral act additively across single-valued correspondences. Somewhat surprisingly, it turns out that this condition is also sufficient in most situations of interest.

Let us first clarify the notion of *additivity* we adopt here. Let (Ω, Σ) be a measurable space, X a Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$. For any abstract integral $\mathbf{I} : \mathcal{F} \rightrightarrows X$, we say that $\mathbf{I}$ is **additive** if

$$\mathbf{I}(F + G) = \mathbf{I}(F) + \mathbf{I}(G) \tag{8}$$

for all $F, G \in \mathcal{F}$ with $F + G \in \mathcal{F}$. We say that it is **additive for functions** if (8) holds for all single-valued $F, G \in \mathcal{F}$ with $F + G \in \mathcal{F}$.[7]

Theorem 5.1. *Let (Ω, Σ) be a measurable space, X a separable Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be a closed and convex-valued abstract integral. Then, $\mathbf{I}$ is additive if, and only if, it is additive for functions.*

We recall the following elementary lemma from convex analysis.

Lemma 5.2. *Let A and B be two nonempty sets in a normed linear space X such that $\overline{\text{co}}(A)$ is weakly compact. Then, $\overline{\text{co}}(A + B) = \overline{\text{co}}(A) + \overline{\text{co}}(B)$.*

Proof. An elementary convexity argument yields $\text{co}(A + B) = \text{co}(A) + \text{co}(B)$. It is also readily checked that $\text{cl}(\text{co}(A + B)) \supseteq \text{cl}(\text{co}(A)) + \text{cl}(\text{co}(B))$. To prove the converse containment, observe that $\text{cl}(\text{co}(B))$ is a closed and convex, and hence weakly closed, subset of X. Since $\overline{\text{co}}(A)$ is weakly compact, and the sum of a closed and a compact set is closed in a normed linear space, we find that $\text{cl}(\text{co}(A)) +$

[7] Here $F+G$ is understood as pointwise defined Minkowski sums, which is to say $F+G : \Omega \rightrightarrows X$ is defined as $(F + G)(\omega) := F(\omega) + G(\omega)$.

cl(co(B)) is a weakly closed and convex set in X. This set is thus a closed set that contains co$(A+B)$, so we have cl(co$(A+B)$) $\subseteq$ cl(co(A))+ cl(co(B)). ■

Proof of Theorem 5.1. Take any $F, G \in \mathcal{F}$ with $F+G \in \mathcal{F}$. We claim:

$$\mathrm{Sel}(F+G) = \mathrm{Sel}(F) + \mathrm{Sel}(G). \tag{9}$$

The $\supseteq$ part of this assertion is trivially true. To see the converse containment, take any $h \in \mathrm{Sel}(F+G)$. Define the correspondence $\Psi : \Omega \rightrightarrows X \times X$ by

$$\Psi(\omega) := \{(x,y) \in X \times X : x + y = h(\omega)\}.$$

Since X is separable, every open subset of $X \times X$ can be written as the union of countably many sets each of which is the product of two open subsets of X. But $\Psi^{-1}(O \times U) = h^{-1}(O+U) \in \Sigma$ for any open $O, U \subseteq X$. It follows that Ψ is Σ-measurable. Given this fact, it is readily checked that $(F \times G) \cap \Psi$ is a nonempty closed-valued Σ-measurable correspondence from Ω to $X \times X$. By the Kuratowski-Ryll Nardzewski selection theorem, therefore, there exists a $\psi \in \mathrm{Sel}((F \times G) \cap \Psi)$. If we denote the first and second component functions of ψ by f and g, respectively, we get $f \in \mathrm{Sel}(F)$, $g \in \mathrm{Sel}(G)$ and $f + g = h$, so $h \in \mathrm{Sel}(F)$+$\mathrm{Sel}(G)$, as we sought.

The additivity assertion is now readily proved. Note first that by Theorem 4.3, (9), and additivity for functions,

$$\mathbf{I}(F+G) = \overline{\mathrm{co}} \bigcup_{h \in \mathrm{Sel}(F+G)} \mathbf{I}(h) = \overline{\mathrm{co}} \bigcup_{\substack{f \in \mathrm{Sel}(F) \\ g \in \mathrm{Sel}(G)}} \mathbf{I}(f+g) = \overline{\mathrm{co}} \bigcup_{\substack{f \in \mathrm{Sel}(F) \\ g \in \mathrm{Sel}(G)}} (\mathbf{I}(f) + \mathbf{I}(g)).$$

Consequently, by Lemma 3.3 and Lemma 5.2,

$$\mathbf{I}(F+G) = \overline{\mathrm{co}} \left(\bigcup_{f \in \mathrm{Sel}(F)} \mathbf{I}(f) + \bigcup_{g \in \mathrm{Sel}(G)} \mathbf{I}(g) \right) = \overline{\mathrm{co}} \bigcup_{f \in \mathrm{Sel}(F)} \mathbf{I}(f) + \overline{\mathrm{co}} \bigcup_{g \in \mathrm{Sel}(G)} \mathbf{I}(g).$$

In view of Theorem 4.3, we conclude that $\mathbf{I}(F+G) = \mathbf{I}(F) + \mathbf{I}(G)$. ■

When (Ω, Σ), X, $\mathcal{F}$ and $\mathbf{I}$ are as in Theorem 5.1, the notions of $\mathbf{I}$ being *linear* and *linear for functions* are defined in the obvious way. In turn, combining Theorem 5.1 and Property 3 of the previous section, we see that $\mathbf{I}$ is linear iff it is linear for functions.

5.3. **Set-Valued Measures Induced by Abstract Integrals.**

This subsection digresses briefly from the theory of abstract integration. Its objective is to demonstrate that we can define a set-valued measure from an abstract integral much the same way this concept is defined through the Aumann integral.

Let (Ω, Σ) be a measurable space and X a separable Banach lattice. As usual, we denote the positive cone of X by X_+. In turn, we order the dual space X^* by means of the so-called dual cone $X_+^* := \{\ell \in X^* : \ell(x) \geq 0$ for every $x \in X_+\}$; it is well-known that this makes X^* itself a Banach lattice. Besides, $x \in X_+$ iff $\ell(x) \geq 0$ for every $\ell \in X_+^*$.

Recall that a nonempty-valued correspondence $M : \Sigma \rightrightarrows X$ is said to be *finitely additive* if $M(A \sqcup B) = M(A) + M(B)$ for any disjoint $A, B \in \Sigma$. Following Artstein [3], we say that M is a **finitely additive set-valued measure** on Σ if M is finitely additive, $M(\varnothing) = \{0_X\}$, and $M \sqsubseteq X_+$. In this context, a correspondence $F : \Omega \rightrightarrows X$ is called **positive** if $F \sqsubseteq X_+$.

The following result builds a bridge between abstract integration and set-valued measures.

Proposition 5.3. *Let (Ω, Σ) be a measurable space and X a separable Banach lattice. Let $\mathcal{F}$ be an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. Take any closed and convex-valued abstract integral $\mathbf{I} : \mathcal{F} \rightrightarrows X$, and any positive $F \in \mathcal{F}$. If $\mathbf{I}$ is additive for functions, then $M_{\mathbf{I},F} : \Sigma \rightrightarrows X$, defined by $M_{\mathbf{I},F}(A) := \mathbf{I}(F \restriction_A)$, is a finitely additive set-valued measure on Σ.*

Proof. By Lemma 3.1, $M_{\mathbf{I},F}$ is well-defined. Let us then fix an arbitrary $A \in \Sigma$, and take any $f \in \mathrm{Sel}(F \restriction_A)$. By positivity of F, we have $f(\omega) \in X_+$ for all $\omega \in \Omega$, so $\ell \circ f \geq 0$ for every $\ell \in X_+^*$. By scalar monotonicity of $\mathbf{I}$, and because $\mathbf{I}(\chi_{\{0_X\}}) = \{0_X\} \neq \varnothing$, we have $\mathbf{I}(f) \neq \varnothing$, that is, $\mathbf{I}(f) = \{x\}$ for some $x \in X$, and $\ell(x) \geq \ell(0_X) = 0$ for every $\ell \in X_+^*$. As we have noted above, this implies $x \in X_+$. Conclusion: $\mathbf{I}(f) \subseteq X_+$ for every $f \in \mathrm{Sel}(F \restriction_A)$, and $M_{\mathbf{I},F}$ is nonempty-valued. Combining this observation with Theorem 4.3 shows that $M_{\mathbf{I},F}(A) = \mathbf{I}(F \restriction_A) \subseteq X_+$ for every $A \in \Sigma$. As it is plain that $M_{\mathbf{I},F}(\varnothing) = \mathbf{I}(F \restriction_\varnothing) = \mathbf{I}(\chi_{\{0_X\}}) = \{0_X\}$, and Theorem 5.1 implies the finite additivity of $M_{\mathbf{I},F}$, we are done. ■

5.4. **Convergence Theorems for the Abstract Integral.** For any sequence (S_m) of nonempty subsets of X, the (Kuratowski-Painlevé) *lower limit* of (S_m) is defined as

$$\liminf S_m := \{x \in X : \mathrm{dist}(x, S_m) \to 0\}.$$

It is easy to check that $\liminf S_m$ is precisely the set of limits of all sequences (x_m) where $x_m \in S_m$ for each m. Since $\mathrm{dist}(\cdot, S)$ is convex iff S is a convex set in X, the lower limit of (S_m) is sure to be convex, provided that each S_m is convex. Moreover,

$$\liminf S_m = \bigcap_{(m_k) \in \mathbf{m}} \mathrm{cl} \bigcup_{k \geq 1} S_{m_k}$$

where $\mathbf{m}$ is the set of all strictly increasing sequences of positive integers ([9, Proposition 5.2.2]). This is a convenient formulation to infer the fact that the lower limit of (S_m) is always closed. Thus, $\liminf S_m$ is a closed and convex set whenever each S_m is convex. If, in addition, (S_m) is an increasing sequence in the sense that $S_1 \subseteq S_2 \subseteq \cdots$, the formula above reduces to $\liminf S_m = \mathrm{cl}(\bigcup_{m \geq 1} S_m)$. In that case, we write

$$\lim S_m = \mathrm{cl} \bigcup_{m \geq 1} S_m,$$

and say that S_m *converges to* $\mathrm{cl}(S_1 \cup S_2 \cup \cdots)$.[8]

Now let (Ω, Σ) be a measurable space, X a separable Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be an abstract integral. We say that a correspondence $F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ is **I-integrably bounded** if

$$\sup_{x \in F(\omega)} \|x\| \leq \|g(\omega)\| \qquad \text{for all } \omega \in \Omega$$

[8] This is a special case of the standard Kuratowski-Painlevé convergence criterion. The *upper limit* of (S_m) is defined as $\limsup S_m := \{x \in X : \liminf \mathrm{dist}(x, S_m) = 0\}$. In turn, (S_m) is said to be *convergent* if $\liminf S_m = \limsup S_m$, and the common limit is denoted as $\lim S$. It is easily verified that when $S_1 \subseteq S_2 \subseteq \cdots$, (S_m) is convergent, and $\lim S_m$ equals the closure of $S_1 \cup S_2 \cup \cdots$.

for some Σ-measurable function $g : \Omega \to X$ such that $g^{\{\}}$ is $\mathbf{I}$-integrable. In turn, following Khan and Sagara [25], we say that a sequence (F_m) in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ is **I-well-dominated** if there is an $\mathbf{I}$-integrably bounded $F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ such that $F_m \sqsubseteq F$ for each $m \geq 1$. Plainly, these concepts are patented after the notion of integrable boundedness of a correspondence with respect to a measure.

Finally, we say that $\mathbf{I}$ satisfies the **dominated convergence property for functions** if for any sequence (f_m) in $\mathcal{F}$ such that (i) $(\lim f_m)^{\{\}} \in \mathcal{F}$; and (ii) there is a Σ-measurable function $g : \Omega \to X$ such that $g^{\{\}}$ is $\mathbf{I}$-integrable and $\|f_m(\omega)\| \leq \|g(\omega)\|$ for all $m \in \mathbb{N}$ and $\omega \in \Omega$, we have

$$\lim \mathbf{I}(f_m) = \mathbf{I}(\lim f_m).$$

For example, the Aumann-Bochner integral satisfies the dominated convergence property for functions where we take $\mathcal{F}$ as the set of all μ-integrably bounded members of $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, with μ being any finite measure on Ω. This is a classical result of Bochner integration theory.

The following is a "Fatou Lemma" for the abstract integral.

Theorem 5.4. *Let (Ω, Σ) be a complete measurable space, X a separable Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be a closed and convex-valued abstract integral that satisfies the dominated convergence property for functions. Then, for any $\mathbf{I}$-well-dominated sequence (F_m) in $\mathcal{F}$,*

$$\mathbf{I}(\liminf F_m) \subseteq \liminf \mathbf{I}(F_m) \tag{10}$$

provided that $\liminf F_m \in \mathcal{F}$.[9]

Proof. Let (F_m) be as in the statement of the theorem. Take any $f \in \mathrm{Sel}(\liminf F_m)$. Let us momentarily fix $m \in \mathbb{N}$. Since F_m is closed-valued and $(\omega, x) \mapsto \|f(\omega) - x\|$ is a Carathéodory function on $\Omega \times X$, the correspondence $G_m : \Omega \rightrightarrows X$ defined by

$$G_m(\omega) := \operatorname*{arg\,min}_{x \in F_m(\omega)} \|f(\omega) - x\|$$

is Σ-measurable ([6, Theorem 8.2.11]). In addition, since $F_m(\omega)$ is weakly compact, and $x \mapsto \|f(\omega) - x\|$ is weakly lower semicontinuous on X, we have $G_m(\omega) \neq \varnothing$, for all $\omega \in \Omega$. As one can readily verify that G_m is closed-valued, we may then apply the Kuratowski-Ryll Nardzewski selection theorem to find an $f_m \in \mathrm{Sel}(G_m) \subseteq \mathrm{Sel}(F_m)$ such that $\|f(\omega) - f_m(\omega)\| = \mathrm{dist}(f(\omega), F_m(\omega))$ for all $\omega \in \Omega$. On the other hand, since $\liminf F_m \in \mathcal{F}$, we have $f^{\{\}} \in \mathcal{F}$ (Lemma 3.1), while $f(\omega) \in \liminf F_m(\omega)$ means $\mathrm{dist}(f(\omega), F_m(\omega)) \to 0$, for every $\omega \in \Omega$. In particular,

$$\|f(\omega) - f_m(\omega)\| \to 0 \quad \text{for all } \omega \in \Omega. \tag{11}$$

Since (F_m) is $\mathbf{I}$-well-dominated, there exists an $\mathbf{I}$-integrably bounded $F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ such that $\bigcup_{m \geq 1} F_m \sqsubseteq F$. Let $g : \Omega \to X$ be a Σ-measurable function such that $g^{\{\}}$ is $\mathbf{I}$-integrable and $\sup_{x \in F(\omega)} \|x\| \leq \|g(\omega)\|$ for all $\omega \in \Omega$. Then, $\sup_{x \in F_m(\omega)} \|x\| \leq \|g(\omega)\|$ for all $\omega \in \Omega$ and $m \geq 1$, whence

$$\sup_{m \geq 1} \|f_m(\omega)\| \leq \|g(\omega)\| \quad \text{for all } \omega \in \Omega.$$

[9] Here $\liminf F_m$ is defined pointwise, that is, $(\liminf F_m)(\omega) := \liminf F_m(\omega)$ for every $\omega \in \Omega$.

Moreover, (11) says $\lim f_m = f$, whence $(\lim f_m)^{\{\}} \in \mathcal{F}$. So, as $\mathbf{I}$ satisfies the dominated convergence property for functions, we have $\mathbf{I}(f_m) \to \mathbf{I}(f)$. Since $\mathbf{I}(f_m) \subseteq \mathbf{I}(F_m)$ for each m, therefore,

$$\operatorname{dist}(\mathbf{I}(f), \mathbf{I}(F_m)) \leq \|\mathbf{I}(f) - \mathbf{I}(f_m)\| \to 0.$$

By definition of the lower limit of $(\mathbf{I}(F_m))$, we thus find $\mathbf{I}(f) \subseteq \liminf \mathbf{I}(F_m)$. So, in view of the arbitrary choice of f and Theorem 4.3, we conclude:

$$\mathbf{I}(\liminf F_m) = \overline{\mathrm{co}} \bigcup_{f \in \mathrm{Sel}(\liminf F_m)} \mathbf{I}(f) \subseteq \overline{\mathrm{co}}(\liminf \mathbf{I}(F_m)).$$

Since each $\mathbf{I}(F_m)$ is convex by hypothesis, $\liminf \mathbf{I}(F_m)$ is a closed and convex set, and the theorem is proved. ■

Remark 5.1. In applications, it may be easier to verify the condition $\liminf F_m \in \mathcal{F}$ from the primitives. To wit, let (Ω, Σ), X, and $\mathcal{F}$ be as in Theorem 5.4. Under this hypothesis, $\liminf F_m$ is Σ-measurable ([6, Theorem 8.2.5]). Now take any sequence (F_m) in $\mathcal{F}$ for which there is an $\mathbf{I}$-integrably bounded $F \in \mathcal{F}$ with $\bigcup_{m \geq 1} F_m \sqsubseteq F$. We next present three scenarios in which $\liminf F_m \in \mathcal{F}$ holds automatically.

(i) Suppose each F_m is convex-valued, and $F \in \mathcal{F}$. Then, $\liminf F_m(\omega)$ is a closed and convex, hence weakly closed, subset of the weakly compact set $F(\omega)$, for all $\omega \in \Omega$. It follows that $\liminf F_m \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, provided that $\liminf F_m$ is nonempty-valued. But, since $F_m \sqsubseteq F$ for all m, we have $\liminf F_m \sqsubseteq F \in \mathcal{F}$. As $\mathcal{F}$ is an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, we get $\liminf F_m \in \mathcal{F}$.

(ii) Suppose X has the Schur property (such as $\mathbb{R}^n$ or ℓ^1). Then, closedness of $\liminf F_m$ entails the weak-closedness of this set, and we apply the previous argument to get $\liminf F_m \in \mathcal{F}$, provided that $\liminf F_m$ is nonempty-valued.

(iii) Suppose X has a separable dual and (F_m) converges *in the sense of Mosco.*[10] Fix any $\omega \in \Omega$. Since $F(\omega)$ is weakly compact, the weak topology restricted on $F(\omega)$ is metrizable. As observed by Hess [21, p. 232], this implies that $\text{weak-}\limsup F(\omega)$ is weakly closed. Since (F_m) convergent in the sense of Mosco, $\text{weak-}\limsup F(\omega)$ equals $\liminf F(\omega)$, so we again find $\liminf F_m(\omega)$ is weakly closed, which is enough to conclude $\liminf F_m \in \mathcal{F}$. □

Theorem 5.4 easily leads to a "monotone convergence theorem" for the abstract integral.

Theorem 5.5. *Let (Ω, Σ) be a complete measurable space, X a separable Banach space, and $\mathcal{F}$ an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be a closed and convex-valued abstract integral that satisfies the dominated convergence property for functions. Then, for any sequence (F_m) of convex-valued correspondences in $\mathcal{F}$ with $F_1 \sqsubseteq F_2 \sqsubseteq \cdots$, we have*

$$\mathbf{I}\left(\mathrm{cl} \bigcup_{m \geq 1} F_m\right) = \mathbf{I}(\lim F_m) = \lim \mathbf{I}(F_m) = \mathrm{cl} \bigcup_{m \geq 1} \mathbf{I}(F_m),$$

provided that there is an $\mathbf{I}$-integrably bounded $F \in \mathcal{F}$ with $F_m \sqsubseteq F$ for each $m \geq 1$.

[10] Let $S_0, S_1, S_2, \ldots$ be closed subsets of X. By $\text{weak-}\limsup_m = S_0$, we mean the set of all $x \in X$ for which there exists a strictly increasing $(m_k) \in \mathbb{N}^\infty$ such that x is the weak limit of some $(x_{m_k}) \in S_{m_1} \times S_{m_2} \times \cdots$. In turn, (F_m) being convergent *in the sense of Mosco* means $\liminf F_m(\omega) = \text{weak-}\limsup F_m(\omega)$ for all $\omega \in \Omega$.

Proof. Let (F_m) and F be as in the statement of the theorem. It is routine to verify that $\mathrm{cl}(\bigcup_{m\geq 1} F_m) \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. Since $\mathrm{cl}(\bigcup_{m\geq 1} F_m) \sqsubseteq F$ and $\mathcal{F}$ is an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$, we thus have $\mathrm{cl}(\bigcup_{m\geq 1} F_m) \in \mathcal{F}$. Besides, $\lim F_m(\omega) = \mathrm{cl}(\bigcup_{m\geq 1} F_m(\omega))$ for all $\omega \in \Omega$, whence $\mathbf{I}(\lim F_m) = \mathbf{I}(\mathrm{cl}(\bigcup_{m\geq 1} F_m))$. In turn, by monotonicity of $\mathbf{I}$ (Property 2 of Section 5.1), $\mathbf{I}(F_k) \subseteq \mathbf{I}(\lim F_m)$ for all $k \geq 1$. Since $\mathbf{I}(\lim F_m)$ is a closed set by hypothesis, therefore, $\mathrm{cl}(\bigcup_{m\geq 1} \mathbf{I}(F_m)) \subseteq \mathbf{I}(\lim F_m)$. On the other hand, since $\mathbf{I}(F_1) \subseteq \mathbf{I}(F_2) \subseteq \cdots$, we have $\lim \mathbf{I}(F_m) = \mathrm{cl}(\bigcup_{m\geq 1} \mathbf{I}(F_m))$. Therefore,

$$\mathbf{I}\left(\mathrm{cl}\bigcup_{m\geq 1} F_m\right) = \mathbf{I}(\lim F_m) \supseteq \lim \mathbf{I}(F_m) = \mathrm{cl}\bigcup_{m\geq 1} \mathbf{I}(F_m).$$

By Theorem 5.4, the containment in this statement holds as an equality. ■

5.5. **Aumann Identities for the Abstract Integral.** Let (Ω,Σ) be a measurable space and X a separable Banach space. It is well-known that if $F : \Omega \rightrightarrows X$ is a Σ-measurable, nonempty and closed-valued correspondence, then $\overline{\mathrm{co}}F$, the correspondence on Ω that maps any $\omega \in \Omega$ to the closed convex hull of $F(\omega)$, is Σ-measurable.[11] Thus, by the Krein-Šmulian weak compactness theorem, $\overline{\mathrm{co}}F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$ for every $F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$. It thus makes sense to ask under what conditions the abstract integrals of F and $\overline{\mathrm{co}}F$ coincide. The following result provides an answer to this query.

Proposition 5.6. *Let (Ω,Σ) be a measurable space and X a separable Banach space. Let $\mathcal{F}$ be an ideal in $\mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,X)$, and $\mathbf{I} : \mathcal{F} \rightrightarrows X$ an abstract integral. Then, for every $F \in \mathcal{F}$ with $\overline{\mathrm{co}}F \in \mathcal{F}$,*

$$\overline{\mathrm{co}}(\mathbf{I}(F)) = \overline{\mathrm{co}}\left(\mathbf{I}\left(\overline{\mathrm{co}}F\right)\right). \tag{12}$$

In particular, if $\mathbf{I}$ is closed and convex-valued, we have $\mathbf{I}(F) = \mathbf{I}(\overline{\mathrm{co}}F)$ for every $F \in \mathcal{F}$ with $\overline{\mathrm{co}}F \in \mathcal{F}$.

Proof. Take any $F \in \mathcal{F}$ such that $\overline{\mathrm{co}}F \in \mathcal{F}$. By Remark 4.3, $\overline{\mathrm{co}}\mathbf{I}$ is an abstract integral on $\mathcal{F}$, so it is monotonic (Property 2 of Section 5.1). We thus only need to prove the $\supseteq$ part of (12). To derive a contradiction, suppose there is an x in $\overline{\mathrm{co}}\left(\mathbf{I}\left(\overline{\mathrm{co}}F\right)\right)$ that does not belong to $\overline{\mathrm{co}}(\mathbf{I}(F))$. Then, by the separating hyperplane theorem, there is an $\ell \in X^*$ such that

$$\sup \ell\left(\overline{\mathrm{co}}\left(\mathbf{I}\left(\overline{\mathrm{co}}F\right)\right)\right) \geq \ell(x) > \sup \ell\left(\overline{\mathrm{co}}\left(\mathbf{I}\left(F\right)\right)\right) = \sup \ell\left(\mathbf{I}\left(F\right)\right), \tag{13}$$

where the equality follows from the linearity and continuity of ℓ. Now define $H_{\ell,F}$ as in the proof of Lemma 3.3, note that $\varnothing \neq \mathrm{Sel}(H_{\ell,F}) \subseteq \mathrm{Sel}(F)$, and pick any $f \in \mathrm{Sel}(H_{\ell,F})$. Clearly, $\ell(f(\omega)) \geq \ell(z)$ for every $\omega \in \Omega$ and $z \in F(\omega)$. Since ℓ is linear and continuous, then, $\ell(f(\omega)) \geq \ell(z)$ for every $\omega \in \Omega$ and $z \in \overline{\mathrm{co}}(F(\omega))$. In particular, $\ell \circ f \geq \ell \circ g$ for any $g \in \mathrm{Sel}(\overline{\mathrm{co}}F)$, whence $F \blacktriangleright_\ell \overline{\mathrm{co}}F$. By scalar monotonicity of $\mathbf{I}$, then, $\mathbf{I}\left(F\right) \succsim_\ell^\bullet \mathbf{I}\left(\overline{\mathrm{co}}F\right)$, and since ℓ is linear and continuous,

$$\sup \ell\left(\mathbf{I}\left(F\right)\right) \geq \sup \ell\left(\mathbf{I}\left(\overline{\mathrm{co}}F\right)\right) = \sup \ell\left(\overline{\mathrm{co}}\left(\mathbf{I}\left(\overline{\mathrm{co}}F\right)\right)\right),$$

which contradicts (13). ■

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[11] See [9, Proposition 6.6.9].

This result generalizes the related invariance theorems for the Aumann-Bochner integral, which are sometimes called *Aumann identities.* To illustrate, let (Ω, Σ, μ) be a complete nonatomic probability space, X a separable Banach space and $\mathcal{F}$ the set of all μ-integrably bounded correspondences in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. In the case where (Ω, Σ, μ) is the unit Lebesgue interval and $X = \mathbb{R}^n$, $\mathbf{I}_{\mathrm{A}} : \mathcal{F} \rightrightarrows X$ is a convex and compact-valued abstract integral (Example 3.3, and [7, Theorems 1 and 4]), so Proposition 5.6 says that the Aumann integrals of F and $\mathrm{co}F$ are the same for any $F \in \mathcal{F}$. More generally, when (Ω, Σ, μ) is a Loeb space, $\mathbf{I}_{\text{A-B}} : \mathcal{F} \rightrightarrows X$ is a convex- and weakly compact-valued abstract integral (Example 3.3 and [40, Theorems 1 and 3]). Proposition 5.6 then entails that the Aumann-Bochner integrals of F and $\overline{\mathrm{co}}F$ coincide for any μ-integrably bounded $F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. This observation is precisely Theorem 4 of [40].

Abstract integration theory is useful in pointing to the commonalities across many different types of set-valued integrals, but it cannot be expected to yield the sharpest results for any particular integral. For instance, when all we know is that (Ω, Σ, μ) is a nonatomic probability space, and X is an arbitrary separable Banach space, we cannot be sure that $\mathbf{I}_{\text{A-B}} : \mathcal{F} \rightrightarrows X$ is closed-valued, so the second part of Proposition 5.6 does not apply to $\mathbf{I}_{\text{A-B}}$. In this case, all we can do is to apply the result to the *closed* Aumann-Bochner integral to get

$$\mathrm{cl}(\mathbf{I}_{\text{A-B}}(F)) = \mathrm{cl}\left(\mathbf{I}_{\text{A-B}}\left(\overline{\mathrm{co}}F\right)\right)$$

for every $F \in \mathcal{F}$ with $\overline{\mathrm{co}}F \in \mathcal{F}$. This is not a first-best result. It is known that we actually have $\mathrm{cl}(\mathbf{I}_{\text{A-B}}(F)) = \mathbf{I}_{\text{A-B}}\left(\overline{\mathrm{co}}F\right)$ for any such F (cf. [45, Theorem 6.3]), a result which is afforded by the special structure of $\mathbf{I}_{\text{A-B}}$.

5.6. **Parametric Continuity of the Abstract Integral.** Under suitable boundedness conditions, Aumann integration of an integrably bounded correspondence (from a finite measure space to $\mathbb{R}^n$) which depends on a parameter continuously yields a correspondence that depends on that parameter upper hemicontinuously (see [8] and [39]). This property of the Aumann integral is found useful in applications to mathematical economics and optimal control, and it extends to the case of the Aumann-Bochner integral (see [34] and [44]). In this section we establish this sort of a parametric continuity property for the abstract integral, thereby extending some of the related results on Aumann-Bochner integration to our nonlinear context.

For any topological spaces X and Y, a correspondence $\Gamma : Y \rightrightarrows X$ is said to be **upper hemicontinuous** if $\Gamma^{-1}(C)$ is closed in Y for every closed $C \subseteq X$, and **lower hemicontinuous** if $\Gamma^{-1}(O)$ is open in Y for every open $O \subseteq X$. We say that Γ is **continuous** if it is both upper and lower hemicontinuous. In turn, when (Ω, Σ) is a measurable space, and X and Y are separable metric spaces, a correspondence $F : \Omega \times Y \rightrightarrows X$ is said to be a **Carathéodory correspondence** if $F(\cdot, y)$ is Σ-measurable for every $y \in Y$, and $F(\omega, \cdot)$ is continuous for every $\omega \in \Omega$.

Lemma 5.7. *Let (Ω, Σ) be a measurable space, and X and Y separable metric spaces. If $F : \Omega \times Y \rightrightarrows X$ is a nonempty and closed-valued Carathéodory correspondence, then it is $\Sigma \otimes \mathcal{B}(Y)$-measurable.*[12]

[12] There are many variants of this result. See, for example, [35, Theorem 3.3].

Proof. Fix an arbitrary $z \in X$, and define the real map d_z on $\Omega \times Y$ by $d_z(\omega, y) := \text{dist}(z, F(\omega, y))$. Since $F(\cdot, y)$ is Σ-measurable, so is $d_z(\cdot, y)$, for every $y \in Y$ ([9, Proposition 6.5.8]). On the other hand, since $F(\omega, \cdot)$ is continuous, so is $d_z(\omega, \cdot)$, for every $\omega \in \Omega$ ([9, Corollary 6.2.10]). It follows that d_z is a Carathéodory function; as such, given that Y is separable, it is $\Sigma \otimes \mathcal{B}(Y)$-measurable ([1, Lemma 4.51]). As this conclusion is valid for every $z \in X$, applying [9, Proposition 6.5.8] one more time shows that F is $\Sigma \otimes \mathcal{B}(Y)$-measurable. ∎

Lemma 5.8. *Let (Ω, Σ) be a measurable space, X a separable Banach space, and Y a separable metric space. Let $F : \Omega \times Y \rightrightarrows X$ be a nonempty and weakly compact-valued Carathéodory correspondence. Then, for every $\ell \in X^*$ there exist $\varphi, \psi \in \text{Sel}(F)$ such that*

$$\ell \circ \varphi(\omega, y) = \max_{x \in F(\omega, y)} \ell(x) \quad \text{and} \quad \ell \circ \psi(\omega, y) = \min_{x \in F(\omega, y)} \ell(x)$$

for all $(\omega, y) \in \Omega \times Y$. Moreover, $\ell \circ \varphi(\omega, \cdot)$ is upper semicontinuous, and $\ell \circ \psi(\omega, \cdot)$ is lower semicontinuous, for any $\omega \in \Omega$.

Proof. For any fixed $\ell \in X^*$, define the correspondence $H_{\ell,F} : \Omega \times Y \rightrightarrows X$ by $H_{\ell,F}(\omega, y) := \arg\max\{\ell(x) : x \in F(\omega, y)\}$. By Lemma 5.7, $F \in \mathcal{K}^{\text{w}}(\Omega \times Y, \Sigma \otimes \mathcal{B}(Y), X)$, so we may argue exactly as in Lemma 3.3 to find some $\varphi \in \text{Sel}(F)$ with $\ell(\varphi(\omega, y)) \geq \ell(x)$ for every $(\omega, y) \in \Omega \times Y$ and $x \in F(\omega, y)$. Moreover, for any $\omega \in \Omega$, $F(\omega, \cdot)$ is (weakly) upper hemicontinuous, and nonempty and weakly compact-valued, so [1, Lemma 17.30] entails that $\ell \circ \varphi(\omega, \cdot)$ is upper semicontinuous. Applying the same argument to $-\ell$ proves the remaining parts of the lemma. ∎

Let $\mathcal{F}$ be a bornology in $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$. In what follows, we refer to $\mathcal{F}$ as **bounded** if every $F \in \mathcal{F}$ has a relatively weakly compact range. Moreover, we say that an abstract integral $\mathbf{I} : \mathcal{F} \rightrightarrows X$ satisfies the **Fatou condition** if for every single-valued $f, f_1, f_2, ... \in \mathcal{F}$ such that $f_1(\Omega) \cup f_2(\Omega) \cup \cdots$ is relatively weakly compact, and for every $\ell \in X^*$,

$$\liminf \ell \circ f_m \geq \ell \circ f \quad \text{implies} \quad \liminf \ell(\mathbf{I}(f_m)) \geq \ell(\mathbf{I}(f)).$$

The following is the main result of this subsection:

Theorem 5.9. *Let (Ω, Σ) be a measurable space, X a reflexive and separable Banach space, and Y a separable metric space. Let $\mathcal{F}$ be a bounded bornology in $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$, and $\mathbf{I} : \mathcal{F} \rightrightarrows X$ an abstract integral that satisfies the Fatou condition. For any nonempty and weakly compact-valued Carathéodory correspondence $F : \Omega \times Y \rightrightarrows X$, define the correspondence $\Gamma_F : Y \rightrightarrows X$ by $\Gamma_F(y) := \overline{\text{co}}(\mathbf{I}(F(\cdot, y)))$. If $F(\Omega \times Y)$ is relatively weakly compact and $F(\cdot, y)$ is $\mathbf{I}$-integrable for each $y \in Y$, then Γ_F is weakly upper hemicontinuous. In particular, under these conditions, $y \mapsto \mathbf{I}(F(\cdot, y))$ is a weakly upper hemicontinuous correspondence on Y, provided that $\mathbf{I}$ is closed and convex-valued.*

Proof. Put $K := \overline{\text{co}}(F(\Omega \times Y))$, which is a weakly compact and convex subset of X that contains the range of F. Since X is a reflexive and separable Banach space, the relative topology induced by the weak topology on K is metrizable. Besides, as $\overline{\text{co}}\mathbf{I}$ is an abstract integral on $\mathcal{F}$ (Remark 4.3), we have $\overline{\text{co}}(\mathbf{I}(F)) \subseteq \overline{\text{co}}(F(\Omega))$ for every $F \in \mathcal{F}$ (Property 4 of Section 5.1). We can thus regard Γ_F as a correspondence from Y to K. Take any $(x_m) \in X^\infty$ and $(y_m) \in Y^\infty$ such that $x_m \in \Gamma_F(y_m)$ for each m, and $y_m \to y$ for some $y \in Y$. So, by the Eberlein-Šmulian theorem, there exist an $x \in K$ and a strictly increasing sequence (m_k) of positive integers such that $x_{m_k} \xrightarrow{\text{w}} x$. To derive a contradiction, let us assume that x does not belong to $\Gamma_F(y)$. Then, by the separating hyperplane theorem, there exists an $\ell \in X^*$ such that $\inf \ell(\Gamma_F(y)) > \ell(x)$.

Now let φ and ψ be as found in Lemma 5.8. Arguing exactly as in the proof of Lemma 3.3, we see that both $\varphi(\cdot, y)$ and $\psi(\cdot, y)$ are in $\mathrm{Sel}(F(\cdot, y))$, and we have

$$\ell \circ \varphi(\cdot, y) \geq \ell \circ g \geq \ell \circ \psi(\cdot, y)$$

for every $g \in \mathrm{Sel}(F(\cdot, y))$. By scalar monotonicity of $\mathbf{I}$ and Lemma 3.4, therefore,

$$\ell(\mathbf{I}(\varphi(\cdot, y))) \geq \ell(\mathbf{I}(g)) \geq \ell(\mathbf{I}(\psi(\cdot, y)))$$

for every $g \in \mathrm{Sel}(F(\cdot, y))$. By Theorem 4.1, this implies that $\ell(\Gamma_F(y))$ equals the interval $[\ell(\mathbf{I}(\psi(\cdot, y))), \ell(\mathbf{I}(\varphi(\cdot, y)))]$. Since $\inf \ell(\Gamma_F(y)) > \ell(x)$, then, $\ell(\mathbf{I}(\psi(\cdot, y))) > \ell(x)$. On the other hand, replacing y with y_{m_k} here, we find that $\ell(\Gamma_F(y_{m_k}))$ equals the interval $[\ell(\mathbf{I}(\psi(\cdot, y_{m_k}))), \ell(\mathbf{I}(\varphi(\cdot, y_{m_k})))]$, and hence

$$\ell(\mathbf{I}(\varphi(\cdot, y_{m_k}))) \geq \ell(x_{m_k}) \geq \ell(\mathbf{I}(\psi(\cdot, y_{m_k})))$$

for each k. But, since $\ell \circ \psi(\omega, \cdot)$ is lower semicontinuous for all $\omega \in \Omega$, we have $\liminf \ell(\psi(\cdot, y_{m_k})) \geq \ell(\psi(\cdot, y))$. Since $\psi(\Omega, y_{m_1}) \cup \psi(\Omega, y_{m_2}) \cup \cdots \subseteq K$ and $\mathbf{I}$ satisfies the Fatou condition, therefore, $\liminf \ell(\mathbf{I}(\psi(\cdot, y_{m_k}))) \geq \ell(\mathbf{I}(\psi(\cdot, y)))$. So, given that $x_{m_k} \xrightarrow{\text{w}} x$, we find

$$\ell(x) = \liminf \ell(x_{m_k}) \geq \liminf \ell(\mathbf{I}(\psi(\cdot, y_{m_k}))) \geq \ell(\mathbf{I}(\psi(\cdot, y))) > \ell(x)$$

a contradiction. ■

Let (Ω, Σ, μ) be a probability space, X a reflexive and separable Banach space and $\mathcal{F}$ is the set of all μ-integrably bounded members of $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ with relatively weakly compact range. Then, $\mathcal{F}$ is a bounded bornology in $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$, and the Aumann-Bochner integral $\mathbf{I}_{\text{A-B}}$ on $\mathcal{F}$ is an abstract integral that satisfies the Fatou condition. This allows deducing parametric continuity theorems for this integral by means of Theorem 5.9 without recourse to dominated convergence arguments.

Example 5.1. Consider the unit Lebesgue interval $([0, 1], \mathcal{B}[0, 1], \mu)$, and let Y be a separable metric space. Let $F : [0, 1] \times Y \rightrightarrows \mathbb{R}^n$ be a nonempty and compact-valued Carathéodory correspondence such that $\sup\{\|F(\omega, y)\| : \omega \in [0, 1] \text{ and } y \in Y\} < \infty$. As the Aumann integral is closed and convex-valued on the set of μ-integrably bounded correspondences, therefore, Theorem 5.9 readily yields that $y \mapsto \mathbf{I}_{\text{A}}(F(\cdot, y))$ is an upper hemicontinuous correspondence from Y to $\mathbb{R}^n$. □

5.7. **The Order-Monotonicity of the Abstract Integral.** Let (Ω, Σ) be a measurable space and X a normed Riesz space whose partial order is denoted by $\geq_X$. There is a natural way of ordering the functions from Ω into X by using $\geq_X$ pointwise. By an abuse of notation, we denote this ordering by $\geq_X$ as well, that is, write $f \geq_X g$ for any $f, g \in X^\Omega$ whenever $f(\omega) \geq_X g(\omega)$ for each $\omega \in \Omega$. In turn, we say that a map Φ from a subset of X^Ω into X is *order-preserving* with respect to $\geq_X$ if $f \geq_X g$ implies $\Phi(f) \geq_X \Phi(g)$.

This notion can be extended to the case of correspondences from a subset of $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ to X. To this end, we first extend $\geq_X$ to a preorder on $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ through using the measurable selections of correspondences. Put precisely, we define the preorder $\succsim_X$ on $\mathcal{K}^{\text{w}}(\Omega, \Sigma, X)$ by setting $F \succsim_X G$ iff for every $g \in \mathrm{Sel}(G)$ there is an $f \in \mathrm{Sel}(F)$ with $f \geq_X g$. In turn, we refer to a correspondence $\mathbf{I}$ from a subset $\mathcal{F}$ of $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$ to X as **order-preserving** if

$$F \succsim_X G \quad \text{implies} \quad \mathbf{I}(F) \geq_X^{\bullet} \mathbf{I}(G),$$

that is, when $F \succsim_X G$ implies that for every $y \in \mathbf{I}(G)$ there is an $x \in \mathbf{I}(F)$ with $x \geq_X y$.

In the development of the abstract integral, we have sidestepped the lack of order structure on the image space X by using a scalarization approach, and adopting scalar monotonicity as our main building block. However, when X happens to be an ordered space, such as a Banach lattice, an equally appealing approach would be to require the abstract integral be order-preserving. The final result of this section shows that these two approaches are duly compatible for any closed and convex-valued abstract integral whose values are closed under joins.

Proposition 5.10. *Let (Ω, Σ) be a measurable space, X a separable Banach lattice, and $\mathcal{F}$ a bornology in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$. Let $\mathbf{I} : \mathcal{F} \rightrightarrows X$ be a closed, convex and sub-$\vee$-semilattice-valued abstract integral. Then, $\mathbf{I}$ is order-preserving.*

Proof. Take any $F, G \in \mathcal{F}$ with $F \succsim_X G$, and assume $\mathbf{I}(F) \neq \varnothing$. We shall first show that $\bigvee \mathbf{I}(F)$ exists in X and $\bigvee \mathbf{I}(F) \in \mathbf{I}(F)$. Note first that $\mathbf{I}(F)$ is separable, because so is X. Let $\{x_1, x_2, ...\}$ be a countable dense subset of $\mathbf{I}(F)$, and define $z_m := x_1 \vee \cdots \vee x_m$ for each positive integer m. Clearly, (z_m) is a sequence in $\mathbf{I}(F)$, and $\cdots \geq_X z_2 \geq_X z_1$. We know that $\mathbf{I}(F)$ is weakly compact (Property 1 in Section 5.1). By the Eberlein-Šmulian theorem, then, there is a subsequence (z_{m_k}) of (z_m) that weakly converges to some $z \in \mathbf{I}(F)$. Then, for any positive linear functional ℓ on X, we have $\ell(z_{m_k}) \to \ell(z)$ while $(\ell(z_m))$ is an increasing sequence of real numbers. Thus: $\ell(z_m) \to \ell(z)$ for every $\ell \in X_+^*$. As $X^* = X_+^* - X_+^*$, it follows that (z_m) converges to z weakly. Moreover, z is the least upper bound for the set $\{z_1, z_2, ...\}$. But for any $x \in \mathbf{I}(F)$, there is a self-map π on $\mathbb{N}$ such that $x_{\pi(m)} \to x$. Besides, $z \geq_X z_{\pi(m)} \geq_X x_{\pi(m)}$, so letting $m \to \infty$ yields $z \geq_X x$. This shows that $z = \bigvee \mathbf{I}(F)$ which proves our claim.

We next prove that $\mathbf{I}(F) \geq_X^{\bullet} \mathbf{I}(G)$. If $\mathbf{I}(G) = \varnothing$, this is trivial, so assume otherwise. Let us then fix an arbitrary $g \in \mathrm{Sel}(G)$. By Lemma 3.4, $\mathbf{I}(g) \neq \varnothing$. Besides, since $F \succsim_X G$, there is an $f \in \mathrm{Sel}(F)$ with $f \geq_X g$. Then, for any positive linear functional ℓ on X, $\ell \circ f$ majorizes $\ell \circ g$, so, by scalar monotonicity of $\mathbf{I}$ and Lemma 3.4, we have $\mathbf{I}(f) \neq \varnothing$, $\ell(\mathbf{I}(f)) \geq \ell(\mathbf{I}(g))$, and, in particular, $\mathbf{I}(F) \neq \varnothing$. By the previous part of the proof and Theorem 4.3, we can then conclude that

$$\ell(\bigvee \mathbf{I}(F)) \geq \ell(\mathbf{I}(g)) \quad \text{for all } \ell \in X_+^* \text{ and } g \in \mathrm{Sel}(G).$$

This implies

$$\bigvee \mathbf{I}(F) \geq_X \mathbf{I}(g) \quad \text{for all } g \in \mathrm{Sel}(G).$$

But, as $\geq_X$ is a vector order, we have $\bigvee \mathbf{I}(F) \geq_X y$ for any convex linear combination of finitely many elements of $\{\mathbf{I}(g) : g \in \mathrm{Sel}(G)\}$, that is, $\bigvee \mathbf{I}(F) \geq_X y$ for every $y \in \mathrm{co}\{\mathbf{I}(g) : g \in \mathrm{Sel}(G)\}$. As X_+ is closed, it follows from this that $\bigvee \mathbf{I}(F) \geq_X y$ for every $y \in \overline{\mathrm{co}}\{\mathbf{I}(g) : g \in \mathrm{Sel}(G)\}$. By Theorem 4.3, then, $\bigvee \mathbf{I}(F) \geq_X y$ for every $y \in \mathbf{I}(G)$. As $\bigvee \mathbf{I}(F) \in \mathbf{I}(F)$, this shows that $\mathbf{I}(F) \geq_X^{\bullet} \mathbf{I}(G)$, as sought. ■

6. APPLICATIONS

Throughout this section (Ω, Σ, μ) stands for a complete probability space and X a Banach space. We also adopt the following notation:

$$\mathcal{F}(\mu, X) := \{F \in \mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X) : F \text{ is } \mu\text{-integrably bounded}\},$$

and

$$\mathcal{F}_{\text{s-v}}(\mu, X) := \{F \in \mathcal{F}(\mu, X) : F \text{ is single-valued}\}.$$

Either of these collections is a bornology in $\mathcal{K}^{\mathrm{w}}(\Omega, \Sigma, X)$, and hence, they are closed under measurable selections and restrictions (Lemma 3.1).

6.1. **The Aumann Integral, Revisited.** We have seen in Example 4.3 that the Aumann integral is an abstract integral. The following result improves this observation by providing an axiomatic characterization of the Aumann integral.

Proposition 6.1. *Assume (Ω, Σ, μ) is nonatomic, and X is finite-dimensional. Let $\mathbf{I} : \mathcal{K}(\Omega, \Sigma, X) \rightrightarrows X$ be a correspondence. Then, $\mathbf{I}$ is closed and convex-valued on $\mathcal{F}(\mu, X)$, acts as an abstract integral on $\mathcal{F}(\mu, X)$, and agrees with the Lebesgue integral on $\mathcal{F}_{\text{s-v}}(\mu, X)$ if, and only if, $\mathbf{I}(F) = \mathbf{I}_A(F)$ for every $F \in \mathcal{F}(\mu, X)$.*

It is well-known that $\mathbf{I}_A(F)$ is a compact and convex set for any $F \in \mathcal{F}(\mu, X)$. Proposition 6.1 obtains easily by combining this fact with Theorem 4.3.

6.2. **The Aumann-Bochner Integral, Revisited.** When X is infinite-dimensional, extending the characterization above to the case of the Aumann-Bochner integral requires an amendment in the hypotheses. For, due to the failure of Lyapunov's Theorem for vector measures with values in an infinite-dimensional space, the Aumann-Bochner integral is, in general, neither closed- nor convex-valued. (See Rustichini [38].) One needs to strengthen the nonatomicity hypothesis on μ to resurrect these properties, and quite a bit is known as to how to do this. In particular, it was proved by Sun [40] that if (Ω, Σ, μ) is a nonatomic finite Loeb measure space, then $\mathbf{I}_{\text{A-B}}$ is weakly compact and convex-valued on $\mathcal{F}(\mu, X)$. It was shown later in [36] and [41] that this is the case iff (Ω, Σ, μ) is nonatomic and the Maharam spectrum of μ consists only of uncountable cardinals. Following Hoover and Keisler [24], we refer to any such probability space (Ω, Σ, μ) as **saturated**.[13]

We have then the following characterization of the Aumann-Bochner integral.

Proposition 6.2. *Assume (Ω, Σ, μ) is saturated, X is separable, and let $\mathbf{I} : \mathcal{F}(\mu, X) \rightrightarrows X$ be a correspondence. Then, $\mathbf{I}$ is a closed and convex-valued abstract integral that agrees with the Bochner integral on $\mathcal{F}_{\text{s-v}}(\mu, X)$ if, and only if, $\mathbf{I} = \mathbf{I}_{\text{A-B}}$.*

In view of the remarks above, Proposition 6.2 is proved by using Theorem 4.3 in exactly the same way as Proposition 6.1.

[13]It is known that a nonatomic probability measure space (Ω, Σ, μ) is saturated iff for every Polish metric spaces Z and W, every Borel probability measure τ on $Z \times W$, and every Z-valued random variable f on Ω whose distribution equals the marginal of τ on Z, there exists a W-valued random variable g on Ω such that the distribution of (f, g) is τ.

6.3. **The Pettis Integral.** For any $F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,\mu)$ and weakly continuous $\varphi : X \to \mathbb{R}$, we define the map $s_{F,\varphi} : \Omega \to \mathbb{R}$ by

$$s_{F,\varphi}(\omega) := \max_{z \in F(\omega)} \varphi(z).$$

This notation will be used in the remainder of the paper.

We say that a correspondence $F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,\mu)$ is said to be **Pettis integrable** if $s_{\overline{\mathrm{co}}(F),\ell}$ is (Lebesgue) integrable, and there exists a nonempty, weakly compact and convex subset C_F of X such that

$$\max_{x \in C_F} \ell(x) = \int_\Omega s_{\overline{\mathrm{co}}(F),\ell} \,\mathrm{d}\mu$$

for every $\ell \in X^*$. If it exists, C_F is unique; it is called the **Pettis integral** of F which we denote by $\mathbf{I}_{\mathrm{P}}(F)$.

Now, the map $s_{\overline{\mathrm{co}}(F),\ell}$ is Σ-measurable for any $F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,\mu)$ and $\ell \in X^*$ (cf. [6, Theorem 8.2]), and it is integrable for any μ-integrably bounded $F \in \mathcal{K}^{\mathrm{w}}(\Omega,\Sigma,\mu)$. Thus, the first requirement of Pettis integrability holds for any $F \in \mathcal{F}(\mu,X)$. Furthermore, [16, Theorem 3.7] says that if at least one element of $\mathrm{Sel}(\overline{\mathrm{co}}F)$ is Bochner integrable, then F is Pettis integrable. Since $\mathrm{Sel}(F) \subseteq \mathrm{Sel}(\overline{\mathrm{co}}F)$, and every element of $\mathrm{Sel}(F)$ is Bochner integrable, therefore, we conclude that any $F \in \mathcal{F}(\mu,X)$ is Pettis integrable.

We claim that $\mathbf{I}_{\mathrm{P}}$ is scalarly monotonic on $\mathcal{F}(\mu,X)$. To see this, take an arbitrary $\ell \in X^*$ and $F,G \in \mathcal{F}(\mu,X)$, and assume $F \blacktriangleright_\ell G$, which implies $\overline{\mathrm{co}}F \blacktriangleright_\ell \overline{\mathrm{co}}G$. We define $H_{\ell,\overline{\mathrm{co}}G}$ as in the proof of Lemma 3.3, and note that $\varnothing \neq \mathrm{Sel}(H_{\ell,\overline{\mathrm{co}}G}) \subseteq \mathrm{Sel}(\overline{\mathrm{co}}G)$. Let us pick any $g \in \mathrm{Sel}(H_{\ell,\overline{\mathrm{co}}G})$. Then, $g \in \mathrm{Sel}(\overline{\mathrm{co}}G)$ and $\ell \circ g = s_{\overline{\mathrm{co}}G,\ell}$. But, by hypothesis, there is an $f \in \mathrm{Sel}(\overline{\mathrm{co}}F)$ with $\ell \circ f \geq \ell \circ g$, whence

$$s_{\overline{\mathrm{co}}F,\ell} \geq \ell \circ f \geq \ell \circ g = s_{\overline{\mathrm{co}}G,\ell}.$$

It follows that $\max \ell(\mathbf{I}_{\mathrm{P}}(F)) \geq \max \ell(\mathbf{I}_{\mathrm{P}}(G))$, and our claim is proved.

It is an easy exercise to show that $\mathbf{I}_{\mathrm{P}}(f^{\{\}}) = \left\{\mathrm{B}\int_\Omega f \,\mathrm{d}\mu\right\}$ for any Bochner μ-integrable $f : \Omega \to X$. Therefore, $\mathbf{I}_{\mathrm{P}}$ acts as an abstract integral on $\mathcal{F}(\mu,X)$. We may thus use Theorem 4.3 to obtain:

Proposition 6.3. $\mathbf{I}_{\mathrm{P}}(F) = \overline{\mathrm{co}}(\mathbf{I}_{\text{A-B}}(F))$ *for every* $F \in \mathcal{F}(\mu,X)$.

This result is, essentially, the combination of Corollary 3.10 and Proposition 3.12 of El Amri and Hess [16]. It is obtained here by means of an easy application of abstract integration theory.

6.4. **The Convex Integral.** Let us denote the set of all convex real-valued functions on $\mathbb{R}^n$ by $\mathrm{Conv}(\mathbb{R}^n)$. As is well-known, $\mathrm{Conv}(\mathbb{R}^n) \subseteq C(\mathbb{R}^n)$, where the latter set stands for the set of all continuous real-valued maps on $\mathbb{R}^n$. We define the **convex integral** as the correspondence $\mathbf{I}_{\mathrm{conv}} : \mathcal{F}(\mu,\mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ with

$$\mathbf{I}_{\mathrm{conv}}(F) := \left\{x \in \mathbb{R}^n : \int_\Omega s_{F,\varphi}\,\mathrm{d}\mu \geq \varphi(x) \text{ for all } \varphi \in \mathrm{Conv}(\mathbb{R}^n)\right\}.$$

This integral relates closely to the theory of stochastic orders. For instance, when $n = 1$, we have

$$\mathbf{I}_{\mathrm{conv}}(f^{\{\}}) = \left\{x \in \mathbb{R} : f \succsim_{\mathrm{conv}} \chi_{\{x\}}\right\} \qquad \text{for all } f \in L^1(\Omega,\Sigma,\mu),$$

where $\succsim_{\mathrm{conv}}$ is the classical *convex order* which is widely used in applied probability and related fields to make variability comparisons across random variables.

A bit less obvious is the fact that the convex integral is none other than the standard expectation operator for single-valued correspondences. Indeed, for any μ-integrable $f : \Omega \to \mathbb{R}^n$, we have $s_{f^{\{\}},\varphi} = \varphi \circ f$, so Jensen's inequality says $\int_\Omega f \, \mathrm{d}\mu \in \mathbf{I}_{\mathrm{conv}}(f^{\{\}})$. Conversely, if $y \in \mathbf{I}_{\mathrm{conv}}(f^{\{\}})$, then $\ell\left(\int_\Omega f \, \mathrm{d}\mu\right) = \int_\Omega \ell \circ f \, \mathrm{d}\mu \geq \ell(y)$ for every linear functional on $\mathbb{R}^n$, which is possible only if $y = \int_\Omega f \, \mathrm{d}\mu$. Thus:

$$\mathbf{I}_{\mathrm{conv}}(f^{\{\}}) = \left\{ \int_\Omega f \, \mathrm{d}\mu \right\}.$$

As a special case of this observation, we see also that $\mathbf{I}_{\mathrm{conv}}(\chi_{\{x\}}) = \{x\}$ for every $x \in \mathbb{R}^n$.

We next show that $\mathbf{I}_{\mathrm{conv}}$ is scalarly monotonic. To this end, take any $F, G \in \mathcal{F}$ and $\ell \in X^*$, and suppose $F \blacktriangleright_\ell G$. Pick an arbitrary $y \in \mathbf{I}_{\mathrm{conv}}(G)$. We need to find an $x \in \mathbf{I}_{\mathrm{conv}}(F)$ with $\ell(x) \geq \ell(y)$. Since $H_{\ell,G}$ (defined as in Lemma 3.3) is Σ-measurable and closed-valued, there is a $g \in \mathrm{Sel}(H_{\ell,G}) \subseteq \mathrm{Sel}(G)$. We choose any $f \in \mathrm{Sel}(F)$ with $\ell \circ f \geq \ell \circ g$, and put $x := \int_\Omega f \, \mathrm{d}\mu$. Then,

$$\ell(x) = \int_\Omega \ell \circ f \, \mathrm{d}\mu \geq \int_\Omega \ell \circ g \, \mathrm{d}\mu = \int_\Omega s_{G,\ell} \, \mathrm{d}\mu \geq \ell(y).$$

Besides $s_{F,\varphi} \geq \varphi \circ f$ for every $\varphi \in \mathrm{Conv}(\mathbb{R}^n)$, so $\{x\} = \mathbf{I}_{\mathrm{conv}}(f^{\{\}}) \subseteq \mathbf{I}_{\mathrm{conv}}(F)$, as desired.

We have established that $\mathbf{I}_{\mathrm{conv}}$ is an abstract integral. Now fix an arbitrary $F \in \mathcal{F}(\mu, \mathbb{R}^n)$. Then $\mathbf{I}_{\mathrm{conv}}(F)$ is a convex set, because for any $x, y \in \mathbf{I}_{\mathrm{conv}}(F)$ and $0 < \lambda < 1$,

$$\int_\Omega s_{F,\varphi} \, \mathrm{d}\mu \geq \max\{\varphi(x), \varphi(y)\} \geq \lambda\varphi(x) + (1-\lambda)\varphi(y) \geq \varphi(\lambda x + (1-\lambda)y)$$

whenever $\varphi \in \mathrm{Conv}(\mathbb{R}^n)$. Finally, $\mathbf{I}_{\mathrm{conv}}(F)$ is closed, because for any sequence (x_m) in $\mathbf{I}_{\mathrm{conv}}(F)$ with $x_m \to x$ for some $x \in \mathbb{R}^n$, we have $\int_\Omega s_{F,\varphi} \, \mathrm{d}\mu \geq \varphi(x_m) \to \varphi(x)$ for every convex (hence continuous) $\varphi : \mathbb{R}^n \to \mathbb{R}$. We proved:

Proposition 6.4. $\mathbf{I}_{\mathrm{conv}} : \mathcal{F}(\mu, \mathbb{R}^n) \rightrightarrows \mathbb{R}^n$ *is a closed and convex-valued abstract integral.*

Given this observation, we can apply Theorem 4.3 to obtain the following characterization: For every $F \in \mathcal{F}(\mu, \mathbb{R}^n)$,

$$\mathbf{I}_{\mathrm{conv}}(F) = \overline{\mathrm{co}} \left\{ \int_\Omega f \, \mathrm{d}\mu : f \in \mathrm{Sel}(F) \right\}.$$

Thus, perhaps unexpectedly, we find that the convex integral is a regularized selection integral. This illustrates how abstract integration theory may yield novel insights in the context of particular set-valued integrals.

6.5. **Aggregator Correspondences.** So far the particular notions of integrals we have used for functions were additive. In this section we demonstrate that abstract set-valued integration works also with non-additive integrals.

Throughout this application, we work with the class of all positive Σ-measurable bounded correspondences on (Ω, Σ) that map to the set of all nonempty compact subsets of the reals. We denote this class by $\mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$, that is,

$$\mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma) := \left\{ F \in \mathcal{K}(\Omega, \Sigma, \mathbb{R}) : F(\Omega) \subseteq [0, \infty) \text{ and } \sup_{\omega \in \Omega} \max F(\omega) < \infty \right\},$$

which is an ideal in $\mathcal{K}(\Omega, \Sigma, \mathbb{R})$. The set of all positive bounded Σ-measurable real maps is, as usual, denoted by $B^{+}(\Omega, \Sigma)$. (We metrize this space by the sup-metric.) Obviously, $\mathrm{Sel}(F) \subseteq B^{+}(\Omega, \Sigma)$ for every $F \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$.

By an **aggregation function** on $B^{+}(\Omega, \Sigma)$, we mean any continuous and increasing map $\mathfrak{a} : B^{+}(\Omega, \Sigma) \to \mathbb{R}$ such that $\mathfrak{a}(k 1_{\Omega}) = k$ for every real number $k \geq 0$. Numerous examples of aggregation functions arise in a variety of applied fields such as probability theory, decision theory, operations research, pattern recognition and mathematical psychology. We refer the reader to the excellent account provided in [18] for the theory and applications of such functions (with finite Ω).

Example 6.1. A **capacity** on Σ is an $\supseteq$-increasing map $\nu : \Sigma \to [0, 1]$ with $\nu(\varnothing) = 0$ and $\nu(\Omega) = 1$. The *Choquet integral* of any $f \in B^{+}(\Omega, \Sigma)$ with respect to a capacity ν on Σ is defined as

$$\mathrm{C} \int_{\Omega} f \, \mathrm{d}\nu := \int_{0}^{\infty} \nu\{f \geq x\} \, \mathrm{d}x.$$

The map $f \mapsto \mathrm{C}\int_{\Omega} f \, \mathrm{d}\nu$, which we naturally call the **Choquet integral**, is an example of a non-additive aggregation function on $B^{+}(\Omega, \Sigma)$. (See [26] for a nice survey on the Choquet and related integrals.) In addition, this map is 1-Lipschitz on $B^{+}(\Omega, \Sigma)$. $\square$

The notion of an aggregation function can be extended to the context of correspondences in the same way Aumann extended the Lebesgue integral to that context. We thus say that a correspondence $\mathsf{A} : \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma) \rightrightarrows \mathbb{R}$ is an **aggregator correspondence** if there is an aggregation function $\mathfrak{a}$ on $B^{+}(\Omega, \Sigma)$ such that

$$\mathsf{A}(F) = \{\mathfrak{a}(f) : f \in \mathrm{Sel}(F)\} \qquad \text{for every } F \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma).$$

It is plain that $|\mathsf{A}(F)| = 1$ if F is single-valued and $\mathsf{A}(F) \neq \varnothing$ for every $F \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$.

Proposition 6.5. *Every aggregator correspondence A is an abstract integral on $\mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$ such that $\mathsf{A}(\mathrm{co}F) = \mathrm{co}(\mathsf{A}(F))$ for any $F \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$. In fact, $\mathsf{A}(F)$ is a compact interval whenever $F \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$ is convex-valued.*

Proof. Take any linear $\ell : \mathbb{R} \to \mathbb{R}$ and arbitrary $F, G \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$ with $F \blacktriangleright_{\ell} G$. Since $\mathfrak{a}$ is increasing, it is readily checked that $\ell \circ f \geq \ell \circ g$ implies $\ell(\mathfrak{a}(f)) \geq \ell(\mathfrak{a}(g))$ for any $f, g \in B^{+}(\Omega, \Sigma)$, so $F \blacktriangleright_{\ell} G$ surely implies $\mathsf{A}(F) \succsim_{\ell}^{\bullet} \mathsf{A}(G)$. That A satisfies the other two requirements of being an abstract integral is obvious.

Next, notice that $\mathrm{co}F \in \mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$. Where we take ℓ as the identity function on $\mathbb{R}$, let φ and ϕ be defined as in the proof of Lemma 3.3. Then, $\phi, \varphi \in \mathrm{Sel}(F)$ and $\varphi(\omega) \geq x \geq \phi(\omega)$ for all $\omega \in \Omega$ and $x \in F(\omega)$, which implies $\varphi \geq f \geq \phi$ for all $f \in \mathrm{Sel}(\mathrm{co}F)$. Since $\mathrm{Sel}(F) \subseteq \mathrm{Sel}(\mathrm{co}F)$ and $\mathfrak{a}$ is increasing, this gives $[\mathfrak{a}(\phi), \mathfrak{a}(\varphi)] \supseteq$

$\mathsf{A}(\mathrm{co}F) \supseteq \mathsf{A}(F) \supseteq \{\mathfrak{a}(\phi), \mathfrak{a}(\varphi)\}$, and in particular, $[\mathfrak{a}(\phi), \mathfrak{a}(\varphi)] = \mathrm{co}(\mathsf{A}(F)) = \mathrm{co}(\mathsf{A}(\mathrm{co}F))$. By the intermediate value theorem, and since $\mathfrak{a}$ is monotonic and continuous, $\{\mathfrak{a}(\lambda\phi + (1-\lambda)\varphi)\}_{\lambda\in[0,1]} = [\mathfrak{a}(\phi), \mathfrak{a}(\varphi)]$. As $\lambda\phi + (1-\lambda)\varphi \in \mathrm{Sel}(\mathrm{co}F)$ for all $\lambda \in [0,1]$, we obtain $\mathrm{co}(\mathsf{A}(\mathrm{co}F)) \supseteq \mathsf{A}(\mathrm{co}F) \supseteq \{\mathfrak{a}(\lambda\phi + (1-\lambda)\varphi)\}_{\lambda\in[0,1]} = [\mathfrak{a}(\phi), \mathfrak{a}(\varphi)] = \mathrm{co}(\mathsf{A}(\mathrm{co}F))$. Thus:

$$\mathsf{A}(\mathrm{co}F) = [\mathfrak{a}(\phi), \mathfrak{a}(\varphi)] = \mathrm{co}(\mathsf{A}(F)),$$

as we sought. ■

In particular, we see that an aggregator correspondence A is compact interval-valued on the set of all positive Σ-measurable compact interval-valued correspondences. For instance, apparently, the correspondence

$$F \mapsto \left\{ \mathrm{C}\int_{\Omega} f \,\mathrm{d}\nu : f \in \mathrm{Sel}(F) \right\}$$

on $\mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$ maps any convex-valued F to a compact interval.

As another application, we consider the parametric continuity of aggregator correspondences:

Proposition 6.6. *Let A be an aggregator correspondence such that*

$$\lim \mathsf{A}(f_m) = \mathsf{A}(f),$$

for every $f, f_1, ... \in B^{+}(\Omega, \Sigma)$ with either $f_m \uparrow f$ or $f_m \downarrow f$. Let Y be a separable metric space and $F : \Omega \times Y \rightrightarrows \mathbb{R}_{+}$ a nonempty closed interval-valued Carathéodory correspondence such that $F(\Omega \times Y)$ is bounded. Then, $y \mapsto \mathsf{A}(F(\cdot, y))$ is an upper hemicontinuous correspondence on Y.

Proof. This is a straigthforward application of Theorem 5.9 with one caveat: $\mathcal{K}_{\mathrm{b}}^{+}(\Omega, \Sigma)$ is not a bornology in $\mathcal{K}(\Omega, \Sigma, \mathbb{R})$ since it does not contain every constant single-valued correspondence from Ω to $\mathbb{R}$, but only those with positive values. However, if we set $K := [0, M]$ with $M > 0$ such that $F(\Omega \times Y) \subseteq K$, then the argument we gave for Property 4 in Section 5.1 yields $\mathbf{I}(F(\cdot, y)) = \overline{\mathrm{co}}(\mathbf{I}(F(\cdot, y))) \subseteq K$ for all $y \in Y$. This ensures that the conclusion of Theorem 5.9 still holds in the current setting. ■

It is well-known that the monotone convergence theorem (as stated in the hypotheses of Proposition 6.6) is valid for the Choquet integral against a capacity which is continuous (from both below and above); see, for instance, Murofushi and Sugeno [33, Proposition 3.2]. Consequently, Proposition 6.6 applies in the context of such a Choquet integral. Put more formally, for any separable metric space Y, bounded interval J, and a nonempty closed interval-valued Carathéodory correspondence $F : \Omega \times Y \rightrightarrows J$,

$$y \mapsto \left\{ \mathrm{C}\int_{\Omega} f(\cdot, y) d\nu : f \in \mathrm{Sel}(F(\cdot, y)) \right\}$$

is an upper hemicontinuous correspondence on Y, under this continuity hypothesis. Even though there is some literature on the Choquet integral for $2^{[0,\infty)}$-valued maps – see, for instance, [19] and [46] – this seems to be a new finding.

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