Università Commerciale
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“Affine Term Structure Models”

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Overview

- Generalities on Bond Yields
- Bond Pricing in Continuous Time
- Partial Differential Equation for Bond Prices
- Affine Models
- Jumps
- Risk Adjustments with Jumps
- Some Famous Affine Models

Generalities: Why Care About Bond Yields?

Goal: describe affine term structure models and the general technique of pricing bonds in continuous time.

- Understanding what moves bond yields is important for at least four reasons:
  - Forecasting
  - Monetary policy
  - Debt policy
  - Derivative pricing and hedging
- Bond yield movements over time can be captured by simple vector autoregressions (VARs) in yields but some aspects set them apart from other variables typically used in VAR studies:

  (1) Yields are not normally distributed -- difficult to compute the risk-adjusted expectation of future short rates.

  (2) Bonds are assets and bonds with many different maturities are traded at the same time.

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Generalities

The advantage of affine models is tractability which though has to be paid with restrictive assumptions on the dynamics of x.

- However, bonds with long maturities are risky, and risk-averse investors demand a compensation.
- Arbitrage opportunities in these markets exist unless long yields are risk-adjusted expectations of future short rates.

- Movements in the cross section of yields are therefore closely tied together and these ties show up as cross-equation restrictions in a yield-VAR.
- Term structure models capture these aspects: they impose the cross-equation restrictions implied by no-arbitrage and allow yields to be non-normal. A special class of these models is affine term structure model: any arbitrage-free model in which yields are affine (constant plus linear) functions of a state variable x.
  - The yield $y^{(\tau)}$ of a τ-period bond is:

$$y^{(\tau)} = A(\tau) + \mathrm{B}(\tau)^{\mathrm{T}}\mathrm{x}$$

for coefficients $\mathrm{A}(\tau)$ and $\mathrm{B}(\tau)$ that depend on maturity τ.

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Bond Pricing in Continuous Time

Term structure modeling determines the price of zcbs.

- Buying a zcb at t and reselling it at $t+n$, generates a log holding period return of:

$$hpr_{t\to t+n}^{(\tau)} = logP_{t+n}^{(\tau-n)} - logP_t^{(\tau)}, \qquad n \leq \tau$$

  - The per-period holding period return is the yield-to-maturity:

$$y_t^{(\tau)} = hpr_{t\to t+n}^{(\tau)} / \tau = - logP_t^{(\tau)} / \tau$$

  - The short rate is the limit of yields as maturity approaches:

$$r_t = lim_{\tau\to 0} y_t^{(\tau)}$$

- Risk-neutral pricing: bonds are priced under a risk-neutral probability measure Q^*. Since the payoff of a zcb is 1 unit at maturity, its price is:

$$P_t^{(\tau)} = E_t^*[\exp(-\int_t^{t+\tau} r_u du)] \qquad (1)$$

  - Under Q^*, expected excess returns on bonds are zero, that is the expected return on bonds equals the riskless rate.

Bond Pricing in Continuous Time

The advantage of pricing bond in continuous time is Ito's Lemma

- The pricing relation shows that any yield-curve model consists of two ingredients:
  1. the change of measure from Q to Q^*;
  2. the dynamics of the short rate r under Q^*: r is a function R(x) of the state vector of factor x, and $x \in R^N$ is a time-homogeneous Markov process under Q^*.
- Then, the conditional expectation is some function F of the state x_t at time t and time-to-maturity τ:

$$P_t^{(\tau)} = F(x_t, \tau)$$

- Ito's Lemma says that smooth functions F of some Ito process x and time t are again Ito processes.
  - The lemma allows to turn the conditional expectation problem into one of solving a PDE for the bond price $F(x, \tau)$ (Feynman-Kac approach).
  - First define the local expectation hypothesis (LEH) and then derive the PDE for bond prices.

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Bond Pricing in Continuous Time

The LEH amounts to risk-neutral pricing: the data-generating measure Q and the risk-neutral Q^* coincide.

- The LEH states that the pricing relation (1) holds under the data-generating measure Q. Bond yields are thus given by:

$$y_t^{(\tau)} = \text{-}\ log\ E_t[\exp(-\int_t^{t+\tau} r_u du)]\ /\ \tau$$

- In continuos time, a Markov process x lives in some state space D$\subset R^N$ and solves the SDE:

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dz_t$$

  - z is an N-dimensional SBM under Q, μ_x: D $\rightarrow$ R^{N} is the drift of x, and σ_x: D $\rightarrow$ $\mathrm{R}^{\mathrm{N}\times\mathrm{N}}$ is its volatility.
  - The Markov process solving this SDE is time-homogenous as the functions μ_x and σ_x do not depend on time.
- Bond prices can be solved using the Feynman-Kac approach, under some regularity conditions such as the smoothness of $F(x,\tau)$ in order to apply Ito's Lemma.

Partial Differential Equation for Bond Prices

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- In continuos time, a Markov process x lives in some state space D$\subset R^N$ and solves the SDE:

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dz_t \qquad (2)$$

  - z is an N-dimensional SBM under Q, μ_x: D $\rightarrow$ R^N is the drift of x, and σ_x: D $\rightarrow$ $R^{N\times N}$ is its volatility.
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Partial Differential Equation for Bond Prices

The PDE can be obtained in four steps.

1. The bond's price at maturity equals its payoff → *F(x,0)=1* ∀x ∈ D.
2. The bond's price is the expected value of an exponential function, so $F(x,\tau)$ is strictly positive.
3. Ito's lemma implies that $F(x,\tau)$ is itself an Ito process with instantaneous expected bond's return:

$$\mu_F(x,\tau) = -\frac{F_\tau(x,\tau)}{F(x,\tau)} + \frac{F_x(x,\tau)^\top}{F(x,\tau)}\mu_x(x) + \frac{1}{2}tr\left[\sigma_x(x)\sigma_x(x)^\top \frac{F_{xx}(x,\tau)}{F(x,\tau)}\right]$$

4. LEH implies that the expected return is equal to the short rate:

$$\mu_F(x,\tau) = R(x)$$

- Bond prices can now be computed in different ways (e.g. using Monte-Carlo methods or solving the PDE numerically).
  - The alternative is to make strong functional form assumptions on μ_x, σ_x and R(x) so that the PDE has a closed-form solution.

Affine Models

Affine models make functional-form assumptions on the short-rate R(x) and the process x for the state vector under Q*.

- The broad class of exponential-affine solutions for $F(x, \tau)$ is called affine term structure models.

- The functional form is affine in both cases:
- Assumption 1 -- The short rate R(x) is affine and given by $r = R(x) = \delta_0 + \delta_1^T x,$ for $\delta_0 \in R$ and $\delta_1 \in R^N$. It usually serves as one of the factors in multidimensional models.
- Assumption 2 -- x is an affine diffusion since both the drift $\mu_x^*(x)$ and the variance matrix $\sigma_x^*(x)\sigma_x^*(x)^{\mathrm{T}}$ are affine. x solves $dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dz_t$ with coefficients:

$$\mu_x(x) = \kappa(\overline{x} - x)$$
$$\sigma_x(x) = \sum s(x)$$

  - $\mu_x(x_t)$ ensures that if $x_t > \overline{x}$, the change dx_t is likely to be negative as long as $\kappa > 0$, and viceversa.

Affine Models

- x_t is pulled back to its mean and the speed of such adjustment is given by k. If k=0, the process is nonstationary.
- Shocks dz_t disturb x_t from moving back to its mean. These shocks are normally distributed as N(0, dt). The effect of these shocks on x_t is determined by $\sigma_x(x_t)$.
- The volatility $s(x)$ is a diagonal NxN matrix.

- The coefficients $\mu_x(x)$ and $\sigma_x(x)$ must satisfy regularity requirements in order to guarantee:
  - that the solution does not explode (growth condition).
  - the existence of a unique solution to the SDE (Lipschitz condition). These solutions x are strong solutions, which means that any other Ito process that solves the SDE equals x almost everywhere.
  - However, these conditions restrict the admissible cross-correlations among state variables.

Affine Models

- For the univariate case, the SDE (2) for affine diffusions becomes:

$$x_t = \bar{x} + \exp\{-\kappa(t-s)\}\,[x_s - \bar{x}] + \int_s^t exp\{-\kappa(t-u)\}\Sigma s(x_u) dz_u$$

  - The same formula applies to the multivariate case, where exp{−κ(t−s)} is a matrix exponential.
- To compute bond prices, we add the assumption of risk-neutral pricing under Q.
- Assumption 3 -- The LEH holds.
- Under these 3 assumptions, Duffie and Kan (1996) guess a solution $F(x,\tau)$ for the PDE:

$$F(x,\tau)\text{=}\exp\{a(\tau) + b(\tau)^T x\}$$

  - where the coefficients $a(\tau)$ and $b(\tau)$ solve the following ODEs, starting at a(0)=0 and b(0)=0:

Affine Models

$$a'(\tau) = -\delta_0 + b(\tau)^T \kappa \overline{x} + \frac{1}{2}\sum_{i=1}^{N} [b(\tau)^T \Sigma]_i^2 s_{0i}$$

$$b'(\tau) = -\delta_1 + b(\tau)^T \kappa \overline{x} + \frac{1}{2}\sum_{i=1}^{N} [b(\tau)^T \Sigma]_i^2 s_{1i}$$

- Given the exponential-affine form $F(x,\tau)$, the instantaneous bond return is:

$$\mu_F(x,\tau) = -a'(\tau) - b'(\tau)^T x + b(\tau)^T \mu_x(x) + \frac{1}{2} b(\tau)^T \sigma_x(x) \sigma_x(x)^T b(\tau)$$

- The bond-price equation shows that the LEH together with a short rate which is affine implies that yields are given by:

$$y_t^{(\tau)} = -\log\mathrm{F}(x_t,\tau)/\tau = A(\tau) + B(\tau)^T x_t$$

for $A(\tau) = -a(\tau)/\tau$ and $B(\tau) = -b(\tau)/\tau$.

Without LEH

Whenever the LEH does not hold, we link the risk-neutral dynamics of the state vector to its data-generating process by using Girsanov's theorem.

- The volatility of the state vector is the same under both measures $\sigma_x(x) = \sigma_x^*(x)$ (diffusion invariance principle).
- Only the drift changes:

$$\mu_x(x) = \mu_x^*(x) - \sigma_x(x)\sigma_\xi(x)^T$$

  - where ξ stands for the density and is a strictly positive martingale.

- Under Q*, the process x solves $dx_t = \mu_x^* dt + \sigma_x^*(x_t) dz_t^*$ with coefficients:

$$\mu_x^*(x) = \kappa^*(\overline{x}^* - x)$$
$$\sigma_x^*(x) = \Sigma^* s^*(x)$$

- To obtain exponential-affine bond-price solutions, the risk-neutral drift and variance-covariance matrix need to be affine.

Jumps

Large movements in yields are modeled as discontinuous moves, or jumps, in the state vector x.

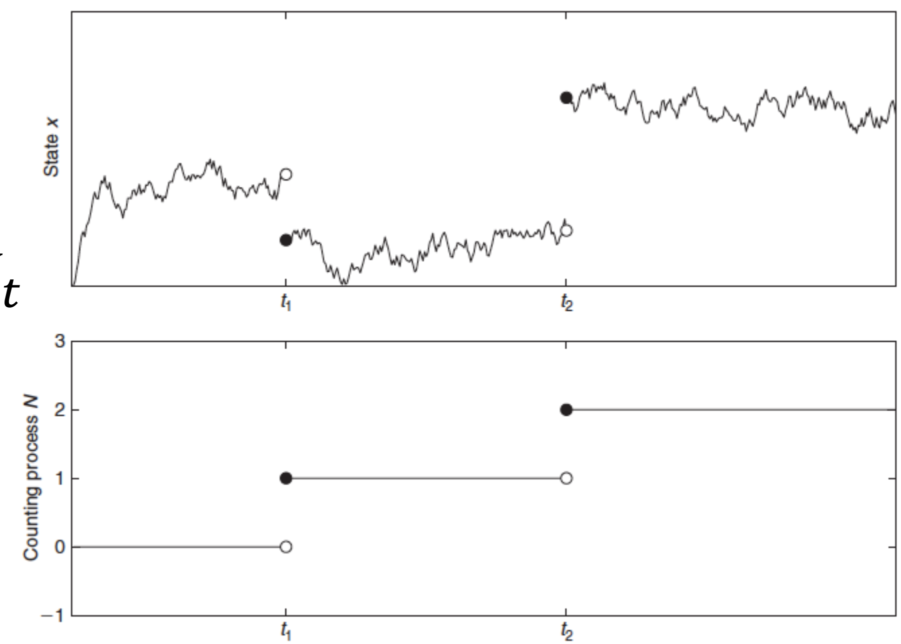
- The jumps occur at arrival times $t_1, \ldots, t_n$ and the counting process starts at 0 and records the number of jumps.
  - The jump in x at t is:

$$\Delta x_t = x_t - x_{t^-}$$

- Jump-diffusions x solve

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dz_t + dJ_t$$

where J is a pure jump process.



- Functional-form assumptions are needed for the jump intensities and the distribution of jump sizes conditional on information "right before" the jump.

Jumps

J can be either caused by a Poisson process N^P with stochastic intensity λ, or jumps may happen at deterministic points in time, recorded by a deterministic counting process N^D.

- Assumption 4

  ① The stochastic intensity λ of the Poisson process is affine:

  $$\lambda(x) = \lambda_0 + \lambda_1^T x, \quad \text{with } \lambda_0 \in R \text{ and } \lambda_1 \in R^N$$

  ② Given a Poisson jump at a stopping time t, the distribution of the jump size Δx_t is independent of x_{t^-}.

1. Calendar Time Does Not Matter
   - Let's assume that LEH holds. Ito's Lemma for the case with Poisson jumps implies that the bond price is itself an Ito process and it is again of the exponential-affine form.
   - Bond returns now also compensate for jumps in the state vector and the jump in returns is:

$$J_F^P(\Delta x, \tau) = exp(b(\tau)^T \Delta x) - 1$$

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$$\overline{\mu}_F(x,\tau) = \mu_F(x,\tau) + \lambda(x) E[J_F^P(\Delta x, \tau)]$$

Jumps

Bond returns now also compensate for jumps in the state vector and the jump in returns is:

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- The bond-price coefficients solve the ODEs starting at a(τ)=0 and b(τ)=0:

$$d'(\tau) = -\delta_0 + b(\tau)^\top \kappa \bar{x} + \frac{1}{2}\sum_{i=1}^{N}\left[b(\tau)^\top \Sigma\right]_i^2 s_{0i} + \lambda_0 E[J_F^P(\Delta x, \tau)]$$

$$b'(\tau) = -\delta_1 - \kappa^\top b(\tau) + \frac{1}{2}\sum_{i=1}^{N}\left[b(\tau)^\top \Sigma\right]_i^2 s_{1i} + \lambda_1 E[J_F^P(\Delta x, \tau)]$$

- When $\lambda_0 = 0$ and $\lambda_1 = 0$, these equations collapse to the ODEs for the case without jumps.

2. Calendar Time Matters
   - $P_t^{(T)} = F(x, t, \tau) = \exp(\mathrm{a(t, T)} + \mathrm{b(t, T)^T x})$ now denotes the price at time t for a bond that matures at T.

Jumps

LEH does not hold anymore.

- The computation of a(t, T) and b(t, T) proceeds recursively, starting at time to maturity with boundary condition a(T, T) = 0 and b(T, T) = 0.
- For every t, the price $P^{(\tau)}$ is exponential affine.

- We rely again on Girsanov's theorem.
- Changes of measure with jumps have effects on the jump intensity and jump size distribution.
- The risk adjustments involves a density ξ which is a strictly positive martingale.
  - The jump intensity λ^* under the risk-neutral measure is:

$$\lambda_t^* = \lambda_t E_{t-}\left(1 + J_\xi^P(\Delta x_t)\right),$$

which is well defined as J_ξ^P>-1.

Risk Adjustments with Jumps

LEH does not hold anymore.

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Some Famous Affine Models

1. Vasicek-type models: x is Gaussian.

- Key features:
  - R(x)=x
  - $\sigma_x(x)$=Σ
  - $\sigma_\xi(x)$ = q
- Inserting these coefficients into $\mu_x(x) = \mu_x^*(x) - \sigma_x(x)\sigma_\xi(x)^T$, we see that the speed of mean reversion k=k* in x (and thus the short rate) is the same under both probability measures.
  - Only the long-run mean differs.
- The market price of risk q is usually estimated to be negative.
  - Intuitively, this means that yields are expected values of average future short rates, which are on average higher $\overline{r}^* > \overline{r}$ than their historical average.
  - This is therefore an implicit form of risk adjustment.

Some Famous Affine Models

2. Cox–Ingersoll–Ross (CIR)-type models: x consists of independent square-root processes.

- Key features:
  - R(x)=x
  - $\sigma_x(x)=\Sigma\sqrt{x}$
  - $\sigma_\xi(x) = q\sqrt{x}$
- Here, the change of measure affects not only the long-run mean but also the speed of mean reversion.
  - A negative q implies that under Q*, x mean reverts more slowly to a higher mean.
- Vasicek and CIR models are first-generation affine models, for which the state is an affine diffusion under both Q* and Q measures.
- These early models were one-factor models: the factor was the "short rate".

Some Famous Affine Models

Duffie and Kan (1996) paved the way for a second-generation of mixture models.

3. Mixture models: x consists of possibly correlated affine processes, and are built from the two basic building blocks.

- Factor models need to specify what their factors stand for.
  - Duffie and Kan (1996) proposed to explain yields with latent factors, that is the econometrician does not observe x directly but may be able to infer x from yields.
  - The state x can in this case be thought of as consisting of yields.
- Most papers with latent factors try to give their variables intuitive labels, for instance:

  1. Labels based on fundamentals: yield curves depend on state variables easily interpretable in terms of fundamentals. Then, the model could be estimated using observations on both macro variables and yields.

Some Famous Affine Models

Duffie and Kan (1996) paved the way for a second-generation of mixture models.

② Labels based on moments of the short rate: the short rate is not Markov under Q* so that other variables (in addition to r_t) help in forecasting the short rate and thus to compute bond yields. Examples :

- Stochastic mean models, where x = (r, θ) and the short rate r reverts quickly to a time-varying mean θ, which reverts slowly to its long-run (unconditional) mean.
- Stochastic volatility models, which take $x = (r, v)^T$, where v is the volatility of the short rate. To keep it positive, it is specified to be a square-root process.