# Characterizing Markov-Switching Rational Expectations Models: Technical Guide

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**Abstract**

Cho (2011) proposes a solution methodology and derives conditions for determinacy in a class of general Markov-switching Rational Expectations (MSRE) models. This document explains how to implement the methodology in Matlab.

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# 1 Introduction

This document explains how to implement in Matlab the methodology developed in "Characterizing Markov-Switching Rational Expectations (MSRE) models" by Cho (2011). The easiest way to learn how to use this matlab package would be to run examples. Run "Replicate.m" and "ReplicateQ.m" to replicate the results of the paper, or run "DL1.m" and "DL2.m" for models without predeterminate variables, or run "LWZ1.m" and "LWZ2.m" for models with predetermined variables.

To test determinacy/indeterminacy and obtain the general solutions, do the following. One should have a MSRE model, specify the necessary input arguments, run msres.m "msres.m" and follow the judgment below.

1. If the forward solution exists (forward convergence condition (FCC) holds),

   (a) If the forward solution is mean-square stable (MSS) and determinacy conditions hold, the model is determinate and the determinate solution is the forward solution. Stop.

   (b) If the forward solution is mean-square stable (MSS) and determinacy conditions do not hold, then run "Find_Min_R_Psi_LkL.m".

      i. If the indeterminacy conditions hold, the model is indeterminate. If one is interested in fundamental equilibria only, pick up the forward solution and stop. If one wants to construct an indeterminate solution, it is the forward solution plus the non-fundamental component, which is computed by "Find_Min_R_Psi_LkL.m". But whenever you run this code, you will get a different non-fundamental component. Stop.

      ii. If the indeterminacy conditions do not hold, the forward solution is the unique relevant MSS solution. The model may or may not be determinate technically, but all other MSV solutions violate the no-bubble condition (NBC). Stop.

   (c) If the forward solution is not mean-square stable, the model has no MSS solution satisfying the NBC. Stop.

2. If the FCC fails, there is no relevant equilibria because any MSV solution violates the NBC. Stop.

Section 2 introduces notations, key concepts and summarizes the main results of the paper. Section 3 illustrates how to implement the procedure described above in matlab language. Section 4 explains the main codes in detail. Section 5 explains the examples.

## 2 Summary of the Paper

We first present the MSRE models and the classes of the REEs. The general model is given by:

$$x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t, \tag{1}$$

$$z_t = Rz_{t-1} + \epsilon_t, \tag{2}$$

where

- $x_t$: $n \times 1$ vector of endogenous variables.
- $z_t$: $m \times 1$ vector of exogenous variables.
- $\epsilon_t$: $m \times 1$ vector of white noises.
- $s_t$: $S$-regime Markov chain.
- $P$: $S \times S$ transition matrix where $p_{ij} \equiv \Pr(s_t = j | s_{t-1} = i)$ is the $(i, j)$-th component of $P$.
- $A(\cdot), B(\cdot)$: $n \times n$ coefficient matrices.
- $C(\cdot)$: $n \times m$ coefficient matrix
- $R$: $m \times m$ stationary coefficient matrix.

**Proposition 1** *Any Rational Expectations solution to model (1) with (2) can be written as a sum of a MSV (fundamental) solution and a non-fundamental component, $w_t$ as:*

$$x_t = [\Omega(s_t)x_{t-1} + \Gamma(s_t)z_t] + w_t. \tag{3}$$

*The first two components of the right-hand side constitute a MSV solution, where $(\Omega(s_t), \Gamma(s_t))$ must satisfy the following conditions for all $s_t$ and $s_{t+1} = 1, 2, ..., S$:*

$$\Omega(s_t) = \Xi(s_t)^{-1}B(s_t), \tag{4a}$$

$$\Gamma(s_t) = \Xi(s_t)^{-1}C(s_t) + E_t[F(s_t, s_{t+1})\Gamma(s_{t+1})]R, \tag{4b}$$

*where* $\Xi(s_t)$ *and* $F(s_t, s_{t+1})$ *are defined as:*

$$\Xi(s_t) \;=\; (I - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})])\,, \tag{5}$$

$$F(s_t, s_{t+1}) \;=\; \Xi(s_t)^{-1}A(s_t, s_{t+1}), \tag{6}$$

*under the regularity condition that* $\Xi(s_t)$ *is non-singular for all* $s_t$*. The non-fundamental component* $w_t$ *must satisfy the following:*

$$w_t = E_t[F(s_t, s_{t+1})w_{t+1}]. \tag{7}$$

The ultimate goal of the paper is to derive determinacy and indeterminacy conditions. To complete our mission, we follow the four steps explained below.

1. We develop the forward method to solve for a particular fundamental equilibrium, referred to as forward solution, $x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t$ together with $F^*(s_t, s_{t+1})$, which does not require any information about stability and determinacy.

2. Next, we introduce the concept of mean-square stability and its properties, and show that the forward solution is mean-square stable if all the eigenvalues of a probability weighted matrix $\bar{\Psi}_{\Omega^*\otimes\Omega^*}$ (to be defined later) lie inside the unit circle.

3. It is proved that if a probability weighted matrix $\Psi_{F^*\otimes F^*}$ has all its eigenvalues lie inside or on the unit circle, there is no other mean-square stable process $w_t$. A key implication is that one does not need to obtain the full set of solutions of $w_t$ and examine stability of them.

4. Finally, the main proposition follows: if $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$ and $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$ where $r_\sigma(\cdot)$ represent the maximum absolute eigenvalue of the argument matrix, then there is no other mean-square stable MSV solutions. This big claim crucially depends on one key property of the forward method, plus some other important results we derived. This explains why we need to first develop the forward method. We additionally develop indeterminacy conditions as well.

Below we explain these four steps and we summarize the results at the end of this section.

## 2.1 Forward Method for MSRE Models

**Proposition 2** *Consider model (1) together with (2). For any initial regime* $s_t$*,* $x_t$*,* $x_{t-1}$ *and* $z_t$*, there exists a unique sequence of real-valued matrices* $(\Omega_k(s_t), \Gamma_k(s_t))$*,* $k = 1, 2, 3, ...$ *such that:*

$$x_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t, \tag{8}$$

*where* $\Omega_1(s_t) = B(s_t)$*,* $\Gamma_1(s_t) = C(s_t)$ *and for* $k = 2, 3, ...,$

$$\Omega_k(s_t) = \Xi_{k-1}(s_t)^{-1}B(s_t), \tag{9a}$$

$$\Gamma_k(s_t) = \Xi_{k-1}(s_t)^{-1}C(s_t) + E_t[F_{k-1}(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})]R, \tag{9b}$$

*with* $\Xi_{k-1}(s_t)$ *and* $F_{k-1}(s_t, s_{t+1})$ *given by:*

$$\Xi_{k-1}(s_t) = (I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]), \tag{10}$$

$$F_{k-1}(s_t, s_{t+1}) = \Xi_{k-1}(s_t)^{-1}A(s_t, s_{t+1}), \tag{11}$$

*if the following regularity condition is satisfied for all* $k > 1$ *and* $s_t = 1, 2, ..., S$*:*

$$|\Xi_{k-1}(s_t)| \neq 0. \tag{12}$$

We define some key concepts and results of the forward method.

**Definition 1** *The MSRE model (1) is said to satisfy the forward convergence condition (FCC) if there exist* $(\Omega^*(s_t), \Gamma^*(s_t))$ *such that* $\Omega^*(s_t) = \lim_{k\to\infty} \Omega_k(s_t)$*,* $\Gamma^*(s_t) = \lim_{k\to\infty} \Gamma_k(s_t)$ *and* $F^*(s_t, s_{t+1}) = \lim_{k\to\infty} F_k(s_t, s_{t+1})$ *for every* $s_t$ *and* $s_{t+1}$*.*

**Definition 2** *The following forward solution to the model (1) is defined as:*

$$x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t. \tag{13}$$

*The general solution associated with this forward solution is given by* $x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t + w_t$ *where* $w_t$ *must satisfy*

$$w_t = E_t[F^*(s_t, s_{t+1})w_{t+1}]. \tag{14}$$

**Definition 3** *A rational expectations solution to the MSRE model (1) is said to satisfy* *the no-bubble Condition (NBC) if the expectation of the future endogenous variables converges to zero in the forward representation of model (8) when expectations are formed with that solution:*

$$\lim_{k\to\infty} E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] = 0_{n\times 1}. \tag{15}$$

**Proposition 3** *The forward solution (13) to the MSRE model (1) with (2) exists if and only if the model satisfies the FCC, and it is the unique MSV solution that satisfies the no-bubble condition.*

## 2.2 Mean-Square Stability

Consider the following $n \times 1$ process $y_{t+1}$:

$$y_{t+1} = G(s_t, s_{t+1})y_t + H(s_{t+1})\eta_{t+1}, \tag{16}$$

where $G(s_t, s_{t+1})$ and $H(s_{t+1})$ are $n \times n$, $n \times m$ matrices, respectively. $\eta_t$ is an arbitrary $m \times 1$ covariance-stationary (wide-sense stationary) process, independent of $s_t$.

**Definition 4**

$$\begin{aligned}
\Psi_G &= [p_{ij}G_{ij}] = \begin{bmatrix} p_{11}G_{11} & ... & p_{1S}G_{1S} \\ ... & ... & ... \\ p_{S1}G_{S1} & ... & p_{SS}G_{SS} \end{bmatrix}, \\
\bar{\Psi}_G &= [p_{ji}G_{ji}] = \begin{bmatrix} p_{11}G_{11} & ... & p_{S1}G_{S1} \\ ... & ... & ... \\ p_{1S}G_{1S} & ... & p_{SS}G_{SS} \end{bmatrix}, \\
\Psi_{G\otimes G} &= [p_{ij}G_{ij}\otimes G_{ij}] = \begin{bmatrix} p_{11}G_{11}\otimes G_{11} & ... & p_{1S}G_{1S}\otimes G_{1S} \\ ... & ... & ... \\ p_{S1}G_{S1}\otimes G_{S1} & ... & p_{SS}G_{SS}\otimes G_{SS} \end{bmatrix}, \\
\bar{\Psi}_{G\otimes G} &= [p_{ji}G_{ji}\otimes G_{ji}] = \begin{bmatrix} p_{11}G_{11}\otimes G_{11} & ... & p_{S1}G_{S1}\otimes G_{S1} \\ ... & ... & ... \\ p_{1S}G_{1S}\otimes G_{1S} & ... & p_{SS}G_{SS}\otimes G_{SS} \end{bmatrix}.
\end{aligned}$$

**Definition 5** *$r_\sigma : R^{n\times n} \to R$ is a spectral radius operator such that $r_\sigma(X) = \max_{1\le i\le n}\{|\lambda_i|\}$ where $\lambda_i$ is an eigenvalue of $n \times n$ matrix $X$.*

**Theorem 1** *For the process (16), if* $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$*, then* $r_\sigma(\bar{\Psi}_G) < 1$.

**Theorem 2** *The process (16) is mean-square stable if and only if* $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$.

By these definitions and concepts, we can say that the forward solution (13) is MSS if $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$.

### 2.3 Non-existence of MSS Non-Fundamental Components

**Proposition 4** *Any non-fundamental component* $w_t$ *in (7) can be written as:*

$$w_{t+1} = \Lambda(s_t, s_{t+1})w_t + V(s_{t+1})V(s_{t+1})'\eta_{t+1} \tag{17}$$

*where* $V(s_t)$ *is an* $n \times k(s_t)$ *matrix with orthonormal columns,* $0 \leq k(s_t) \leq n$ *and* $k(s_t) > 0$ *for some* $s_t$*.* $\eta_t$ *is an arbitrary* $n \times 1$ *covariance-stationary innovations such that* $E_t[V(s_{t+1})V(s_{t+1})'\eta_{t+1}] = 0_{n\times 1}$*,* $\Lambda(s_t, s_{t+1}) = V(s_{t+1})\Phi(s_t, s_{t+1})V(s_t)'$ *for some* $k(s_{t+1}) \times k(s_t)$ *matrix* $\Phi(s_t, s_{t+1})$ *such that*

$$\sum_{j=1}^{S} p_{ij}F_{ij}V_j\Phi_{ij} = V_i, \quad for \;\; 1 \leq i \leq S. \tag{18}$$

*where* $V_i = V(s_t = i)$*,* $\Phi_{ij} = \Phi(s_t = i, s_{t+1} = j)$*,* $F_{ij} = F(s_t = i, s_{t+1} = j)$*.*

**Lemma 3** *Consider two processes* $w_{t+1} = \Lambda(s_t, s_{t+1})w_t$ *and* $u_{t+1} = F'(s_t, s_{t+1})u_t$*. The following holds.*

1. *If* $r_\sigma(\bar{\Psi}_{F'\otimes F'}) < 1$ *and* $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$*, then* $r_\sigma(\bar{\Psi}_{\Lambda\otimes F'}) < 1$ *and* $w_{t+1} + u_{t+1}$ *is mean-square stable.*
2. *If* $r_\sigma(\bar{\Psi}_{F'\otimes F'}) \leq 1$ *and* $r_\sigma(\bar{\Psi}_{\Lambda\otimes F'}) \geq 1$*, then* $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1$*.*

**Lemma 4** *For any process (17) subject to (18), the following holds.*

1. $\bar{\Psi}_{\Lambda\otimes F'}$ *contains at least one root of 1, hence,* $r_\sigma(\bar{\Psi}_{\Lambda\otimes F'}) \geq 1$*.*
2. $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1/[r_\sigma(\Psi_{F\otimes F})]$ *for all* $\Lambda(s_t, s_{t+1})$*.*

Assertion 2 of Lemma 4 shows that $1/[r_\sigma(\Psi_{F\otimes F})]$ is the lower bound of $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ for all $\Lambda$.

**Proposition 5** *Consider equation (14). Suppose that the following condition holds:*

$$r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1 \tag{19}$$

*Then there is no stochastic MSS process $w_t$ satisfying (14).*

## 2.4 Determinacy

**Proposition 6** *Suppose that the MSRE model (1) satisfies the following properties.*

1. *The forward solution (13) exists.*
2. $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$
3. $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$,

*Then, there is no other mean-square stable MSV solution. Therefore, the is determinate in MSS sense and the forward solution is the determinate equilibrium.*

This proposition holds because of one key property embedded in the forward method. To see the intuition behind this result, we ask readers to refer to the paper. In the case of the model without predetermined variables, $\Omega^* = 0_{n\times n}$, then $F^*(s_t, s_{t+1})$ collapses to $A(s_t, s_{t+1})$.

Now, what if condition 3 does not hold? Then we need to search for $\Lambda$ minimizing $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$. Recall that $\min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1/[r_\sigma(\Psi_{F^*\otimes F^*})]$. This result makes the minimization problem much easier. In the paper, we showed that it is highly likely to find $\Lambda$ such that $\min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) = 1/[r_\sigma(\Psi_{F^*\otimes F^*})]$ for models where no state is absorbing. In any case, if the assertion 3 is replaced with $1/[r_\sigma(\Psi_{F^*\otimes F^*})] \leq \min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$, then the model is indeterminate. But if $1/[r_\sigma(\Psi_{F^*\otimes F^*})] < 1 \leq \min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$, then it is still the case that the forward solution is the unique MSS solution. In this case the model may or may not be determinate. But the important thing is that if there exist other MSS fundamental solutions, they are not economically relevant because they must violate the NBC. This procedure can be numerically done by using the code "Find_Min_R_Psi_LkL.m, which is explained in Section 4.

For a comparison of our results to those of other papers in the literature, we introduce the following concept.

**Definition 6** *The condition $r_\sigma(\Psi_{F^*}) \leq 1$ with appropriate positivity assumptions in the sense of Davig and Leeper (2007) is comparable to the LRTP.*

This condition may be referred to as the generalized LRTP. When $\Omega^* = 0_{n\times n}$, $F^*(s_t, s_{t+1}) = F^*(s_t) = A(s_t)$ and it is non-singular for all $s_t$, the LRTP is given by $r_\sigma(\Psi_{F^*}) < 1$.

## 2.5 Classification of the MSRE models and Characterization of the REEs

The following table summarizes the classification of the MSRE models based on FCC and characterizations of the economically relevant REEs based on NBC.

| | | If | | | Then |
|---|---|---|---|---|---|
| Case | FCC | MSS of MSV $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*})$ | non-MSS of $w_t$ $r_\sigma(\Psi_{F^*\otimes F^*})$ | the most MSS $w_t$ $\min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ | |
| **1** | o | $< 1$ | $\leq 1$ | | Determinate |
| **2** | o | $< 1$ | $> 1$ | $< 1$ | Indetermiante |
| 4 | o | $< 1$ | $> 1$ | $\geq 1$ | May be Determinate or Indeterminate |
| 5 | o | $\geq 1$ | | | No MSS FS |
| **3** | x | | | | Reject the Model |

Based on MSS and NBC, the forward solution is the unique relevant solution to the models belonging to Case 1 and 4. The forward solution plus the non-fundamental components associated with the forward solution are the set of relevant indeterminate equilibria to the models belonging to Case 2.

**Remark:** The researchers would face Case 1,2 and 3 in practice.

# 3 Implementation of the Methodology in Matlab

## 3.1 Introducing cell arrays in Matlab

Since the input and output arguments must be specified for all regimes, it is convenient to use "cell array"in Matlab. By using a cell array, one does not need to hassle with the dimension of the matrices in the model such as $n$ and $m$. For example, consider the case where $S = 2$ and $A$ depends only on the current regime as:

$$A(s_t = 1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A(s_t = 2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

In Matlab, $A$ is specified as:

```
A1=[1 0;0 1]; A2=[2 0;0 2];   A{1,1}=A1; A{2,1}=A2;
```

Here $A$ is a $S \times 1$ cell array. If $A$ depends on the current and future regimes such that $A = A(s_t = i, s_{t+1} = j)$, define A11,A12,A21,A22 and construct a cell array:

```
A{1,1}=A11; A{1,2}=A12; A{2,1}=A21; A{2,2}=A22;
```

To avoid confusion, we use the indexes $i$ and $j$ to denote the regimes for $s_t$ and $s_{t+1}$ throughout this document.

The output arguments will also be expressed as cell arrays. For instance, suppose that one defines GammaK to denote the FS for a given model, $\Gamma^*(s_t)$. Then, GammaK will be a $2 \times 1$ cell array such that GammaK{i,1} is $\Gamma^*(s_t = i)$.

## 3.2 The Set of Matlab Codes in the Package

### Main Code

- "msres.m": The main Matlab code. "msres.m"(1) identifies the forward convergence, (2) yields the forward solution if the model is forward-convergent, and (3) provides determinacy conditions. If these conditions are met ($r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*}) < 1$ and $r_\sigma(\Psi_{F^* \otimes F^*}) \leq 1$), the model is determinate in mean-square stability sense and the forward solution is the determinate equilibrium.

**Auxiliary codes used for the paper**

- "Find_Min_R_Psi_LkL.m": This code identifies existence of mean-square stable non-fundamental components and hence detect indeterminacy. Specifically, if the determinacy conditions are not satisfied from "msres.m"($r_\sigma(\Psi_{F^*\otimes F^*}) > 1$), this code finds a minimizer of $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$, $\Lambda_{\min}$. If $\min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$, the model becomes indeterminate and $w_t$ with $\Lambda_{\min}$ is one among the mean-square stable non-fundamental components. Therefore, this code constructs a general indeterminate equilibria. It is important to note that this code does not solve for ALL MSS $w_t$. Thus every time you run this code, you will get a different $\Lambda$ in general (mostly for multivariate models). The code is accompanied by "Find_Min_R_Psi_LkL_Q.m", the objective function and "Find_Min_R_Psi_LkL_NL.m", a set of non-linear constraints.

- "amsre.m": This solves for the unique MSV solution to a model without predetermined variables using the analytical formula given in the paper. We provide this code not because we recommend users to use this code, but because we want to show that this way of computing fundamental REEs may fail to detect non-FCC.

**Examples**

- "Replicate.m"and "ReplicateQ.m": These codes replicate all the results reported in the paper.

- "DL1.m"and "DL2.m": based on the works by Davig and Leeper (2007)

- "LWZ1.m"and "LWZ2.m": based on Liu, Waggoner, and Zha (2009)

## 3.3 Instruction for Users

What users need to do is as follows.

1. Specify the model in the form of Equation (1) and (2) in Section 2 and write a code, say "Ex.m". In "Ex.m", first, specify the transition matrix $P$ and $A(\cdot)$ for all regimes. Second, specify the optional input arguments $B(\cdot)$, $C(\cdot)$ and $R$ if necessary. One may also change other options such as the maximum number of forward iteration, tolerance level of forward convergence, forecasting horizon of impulse response analysis, etc.

2. Call "msres.m" to obtain the results such as the forward solution, determinacy conditions, impulse response functions, etc. To do so, type the following and run it.

```
[OmegaK,GammaK,FK,termK,R_BarPsi_OKkOK,R_Psi_FKkFK,...
        BarPsi_OKkOK, Psi_FKkFK,IRF]=msres(P,A,B,C,R);
```

3. Examine the results. Any model can be classified in the following several cases. But researchers would face only three cases (Case 1, 2 and 3 in the order of importance) in practice.

   (a) (**Case 3**) If there is a warning sign, and if at least one element of OmegaK or GammaK in Figure 101 increases exponentially, the forward convergence condition (FCC) fails. Stop.

   (b) If there is no warning sign, the FCC is met and OmegaK,GammaK are the forward solution.

      i. If R_BarPsi_OKkOK $= r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*}) \geq 1$, the forward solution is not mean-square stable (MSS) and there is no MSS MSV solution that satisfies the No-bubble condition (NBC). Stop.

      ii. (**Case 1**) If R_BarPsi_OKkOK $\leq 1$ and R_Psi_FKkFK $= r_\sigma(\Psi_{F^* \otimes F^*}) \leq 1$, then the model is determinate. Stop.

      iii. If R_Psi_FKkFK $> 1$, go to step 4 in order to examine indeterminacy.

4. Write the following and run it.

```
[Lambda_min,V,Psi,k,R_BarPsi_LkL]=Find_Min_R_LkL(P,FK);
```

   (a) (**Case 2**) If R_BarPsi_LkL $= r_\sigma(\Psi_{\Lambda_{\min} \otimes \Lambda_{\min}}) < 1$, the model is indeterminate. And $w_t$ with $\Lambda_{\min}$ is one among the mean-square stable non-fundamental components.

   (b) If R_BarPsi_LkL $\geq 1$ the forward solution is the unique mean-square stable MSV solution and there is no mean-square stable $w_t$.

# 4 The Main Codes

## 4.1 msres.m

The function has the following form:

[OmegaK,varargout]=msres(P,A,varargin);

1. Input Arguments (Required)

   (a) P: $S \times S$ transition matrix $P$ where P(i,j)= $p_{ij} \equiv \Pr(s_t = j | s_{t-1} = i)$ is the $(i, j)$-th component of $P$.

   (b) A: $S \times S$ cell array of $A(s_t, s_{t+1})$ where A{i,j} is an $n \times n$ matrix $A(s_t = i, s_{t+1} = j)$ for all $i, j \in \{1, 2, ..., S\}$. If $A = A(s_t)$, then A is a $S \times 1$ cell array where A{i,1} is an $n \times n$ matrix $A(s_t = i)$ for all $i \in \{1, 2, ..., S\}$.

2. Input Arguments (Optional)

   (a) B: $S \times 1$ cell array, $B(s_t)$ where B{i,1} is an $n \times n$ matrix $B(s_t = i)$. When there are no lagged endogenous variables $x_{t-1}$, set B=[ ]; [Default: B=zeros(n,n);]

   (b) C: $S \times 1$ cell array, $C(s_t)$ where C{i,1} is an $n \times m$ matrix $C(s_t = i)$. If there is no $z_t$ in the model, Set C=[ ]; [Default: C=eye(n,n);]

   (c) R: $m \times m$ matrix. If $z_t = \epsilon_t$, set R=[ ]; [Default: R=zeros(m,m);]

   (d) Opt: Other optional input arguments.

      - Opt.maxK: maximum number of the forward iteration. [Default: 1000]
      - Opt.Warning: Plot $\Omega_k$ and $\Gamma_k$ against $k$ if $k$ reaches maxK. [Default: Opt.Warning=1 : Display Warning] Set Opt.Warning=0 if do not want to display Warning sign.
      - Opt.tolK: Tolerance level of forward convergence: The forward iteration stops if the maximum of the largest element of $\Omega_k(s_t) - \Omega_{k-1}(s_t)$ and the largest element of $\Gamma_k(s_t) - \Gamma_{k-1}(s_t)$ in absolute value is less than Opt.tolK. [Default: 0.000001]
      - Opt.IRsigma: $m \times 1$ vector of standard deviations of $\epsilon_t$. [Default: Impulse responses(IR) functions are generated for the initial shocks of size 1.] Setting "Opt.IRsigma=[sigma_1,...,sigma_m]' "will generate the IR to standard

deviation shocks where [sigma_1,...,sigma_m] is the vector of standard errors of $\epsilon_t$.

- Opt.IRT: Forecasting horizon for IR functions. [Default: 20]
- Opt.IRvs: The arrangement of IR functions. Refer to the IRS of the output argument below for details.

3. Output Arguments

(a) OmegaK: $S \times 1$ cell array, $\Omega_K(s_t)$ where OmegaK{i,1} is an $n \times n$ matrix $\Omega_K(s_t = i)$ at $K = termK$. (See below for termK.) If $B$ is not in the model, OmegaK{i,1} will be an $n \times n$ matrix of zeros.

(b) GammaK: $S \times 1$ cell array, $\Gamma_K(s_t)$ where GammaK{i,1} is an $n \times m$ matrix $\Gamma_K(s_t = i)$ at $K = termK$.

(c) FK: $S \times S$ cell array, $F_K(s_t, s_{t+1})$ where FK{i,j} is an $n \times n$ matrix $F_K(s_t = i, s_{t+1} = j)$ at $K = termK$. If $B$ is not in the model, FK{i,j} is A{i,j}.

(d) termK: Number of iteration. If the forward solution exists, then $termK$ denotes the number of forward iteration at which the iteration stops and $termK < maxK$. If the forward solution does not exist, then $termK = maxK$.

(e) R_BarPsi_OKkOK: $r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*})$

(f) R_Psi_FKkFK: $r_\sigma(\Psi_{F^* \otimes F^*})$

(g) BarPsi_OKkOK: $\bar{\Psi}_{\Omega^* \otimes \Omega^*}$

(h) Psi_FKkFK: $\Psi_{F^* \otimes F^*}$

(i) IRS: $S \times 1$ cell array of impulse-response functions of $x_t$ where IRS{i,1} is a $T \times nm$ matrix when the initial state is $s_t = i$. By default, it is arranged such that the first $m$ columns are the responses of the first variable to the $m$ innovations. The next $m$ columns are those of the second variable to the $m$ innovations, and so on. Set Opt.IRvs=1 in the input argument if one wants to arrange IRS{i,1} such that the first $n$ columns are the responses of $n$ variables to the first shock, ..., the responses of $n$ variables to the last shock.

**Remark:** When one wants to compute the forward solution for the Fixed-Regime (FR) counterpart, simply assign an identity matrix of size $S$ to the transition matrix $P$. That is, set "P=eye(S); ".

**Remark:** The output arguments depend on the forward convergence. There are two cases.

- Case 1. When the FCC holds and thus the FS exists, no message will pop up. $(\Omega_K(\cdot), \Gamma_K(\cdot))$ is the forward solution for $K = termK < maxK$ where $maxK$ is the maximum number of iteration.

- Case 2. When the FCC does not hold, then the code will still produce $(\Omega_K(\cdot), \Gamma_K(\cdot))$ where $K = maxK$. But in this case, a warning sign will be displayed and a graph will be shown. The $(1, i)$-th panel plots all the elements of $vec(\Omega_k(s_t))$ at $s_t = i$ against $k = 1, .., maxK$. The $(2, i)$-th panel displays all the elements of $vec(\Gamma_k(s_t))$. By looking at the figure, one can visibly see whether the $(\Omega_k(\cdot), \Gamma_k(\cdot))$ would converge if a larger $maxK$ is allowed or the FS does not exist because at least one element explodes.[1]

## 4.2 Find_Min_R_LkL.m

The function has the following form:

[LambdaMin,varargout]=Find_Min_R_LkL(P,F,Opt);

1. Input Arguments (Required)

   (a) P: $S \times S$ transition matrix $P$ where P(i,j)= $p_{ij} \equiv \Pr(s_t = j | s_{t-1} = i)$ is the $(i, j)$-th component of $P$.

   (b) F: $S \times S$ cell array of $A(s_t, s_{t+1})$ where F{i,j} is an $n \times n$ matrix $A(s_t = i, s_{t+1} = j)$ for all $i, j \in \{1, 2, ..., S\}$. If $A = A(s_t)$, then A is a $S \times 1$ cell array where F{i,1} is an $n \times n$ matrix $A(s_t = i)$ for all $i \in \{1, 2, ..., S\}$.

2. Input Arguments (Optional)

[1] For most of numerical examples in this note and all the experiments we have conducted, the iteration stops at $K < 100$ whenever the FS exists. Thus, it is very unlikely that the convergence is extremely slow and one needs to set a larger $maxK$.

(a) Opt.ks: $S \times 1$ vector of $[k_1, .., k_S]'$. [Default: $k_i = rank$ of $F(i,i)$].

(b) Opt.V : $S \times 1$ cell array of $V(s_t)$ where $V(s_t = i)$ is an $n \times k_i$ matrix with orthornormal columns. [Default: Columns of $V_i$ are $k_i$ random orthonormal vectors.]

(c) Opt.Psi : $S \times S$ cell array of $\Phi(s_t, s_{t+1})$. [Default: $\Phi_{ij}$ is $k_j \times k_i$ matrix of random normal elements.]

3. Output Arguments

(a) LambdaMin: $S \times S$ cell array, $\Lambda(s_t, s_{t+1})$ that minimizes $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$.

(b) V: $S \times 1$ cell array, $V(s_t)$ such that $\Lambda(s_t, s_{t+1}) = V(s_{t+1})\Phi(s_t, s_{t+1})V(s_t)'$

(c) Psi:$S \times S$ cell array, $\Phi(s_t, s_{t+1})$ such that $\Lambda(s_t, s_{t+1}) = V(s_{t+1})\Phi(s_t, s_{t+1})V(s_t)'$.

(d) k: $S \times 1$ vector of $[k_1, .., k_S]'$.

(e) R_BarPsi_LkL: $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$

(f) R_BarPsi_L: $r_\sigma(\bar{\Psi}_{\Lambda})$

(g) R_Psi_FkF: $r_\sigma(\Psi_{F \otimes F})$

(h) R_Psi_F: $r_\sigma(\Psi_F)$

(i) Psi_LtkF: $r_\sigma(\Psi_{\Lambda' \otimes F})$

**Remark:** It is important to note that this procedure starts from randomized initial values of $V(s_t)$ and $\Phi(s_t, s_{t+1})$, hence, it yields different $\Lambda_{\min} = \arg\min r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$ whenever implemented in general. Also one may have to run many times to find $\min r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$.

## 4.3 amsre.m

The Matlab function "amsre.m"computes the fundamental solution to a MSRE model when $B = 0_{n \times n}$, using the analytical formula described below. The function can be written as:

AGamma=amsre(P,A,C,R);

The input arguments $P$, $A$, $C$ and $R$ are the same as those in "msres.m". The output argument is $\Gamma(s_t)$.

**The formula**

Consider the following MSRE model without predetermined variables:

$$
\begin{aligned}
x_t &= E_t[A(s_t, s_{t+1})x_{t+1}] + C(s_t)z_t, &(20)\\
z_t &= Rz_{t-1} + \epsilon_t. &(21)
\end{aligned}
$$

Guess the fundamental solution of the form as:

$$x_t = \Gamma(s_t)z_t. \tag{22}$$

Then, using (22) and (21), the expectational term in (20) can be computed as:

$$
\begin{aligned}
E_t[A(s_t, s_{t+1})x_{t+1}] &= E_t[A(s_t, s_{t+1})\Gamma(s_{t+1})z_{t+1}]\\
&= E_t[A(s_t, s_{t+1})\Gamma(s_{t+1})]Rz_t,
\end{aligned}
$$

where we have used the independence of $s_{t+1}$ and $\epsilon_{t+1}$. Therefore, for every $s_t = i$, $i = 1, 2, ..., S$, the following must hold:

$$
\begin{aligned}
\Gamma(i) &= E_t[A(i, s_{t+1})\Gamma(s_{t+1})]R + C(i)\\
&= \sum_{j=1}^{S} p_{ij}A(i, j)\Gamma(j)R + C(i).
\end{aligned}
$$

In order to solve for $\Gamma(i)$, we vectorize this equation:

$$vec(\Gamma(i)) = \sum_{j=1}^{S} p_{ij}\left(R' \otimes A(i, j)\right) vec(\Gamma(j)) + vec(C(i)).$$

By stacking $vec(\Gamma(i))$ and $vec(C(i))$ for all $i$, we have the following:

$$\Gamma^v = \Psi_{R'\otimes A}\Gamma^v + C^v,$$

where, $\Psi_{R'\otimes A} = [p_{ij}R' \otimes A(i, j)]$. Therefore, the solution for $\Gamma^v$ is given by:

$$\Gamma^v = (I_{nmS} - \Psi_{R'\otimes A})^{-1}C^v.$$

From this equation, one can recover the solution for $vec(\Gamma(i))$ and therefore $\Gamma(i)$.

# 5 Examples

We present several numerical examples. Simply running each code in Matlab will identify the relevant set of equilibria and determinacy. Whenever a given model satisfies the FCC, the code will produce the forward solution for the MSRE. Additionally, the code will plot the impulse response functions. There are currently 4 examples and one example that replicates all the results of Cho (2011).

| Code name | Predetermined Variable | # of Variables | References |
|---|---|---|---|
| DL1.m | No | Univariate | Davig and Leeper (2007) |
| DL2.m | No | Multivariate | Farmer, Waggoner, and Zha (2009) |
| LWZ1.m | Yes | Multivariate | Liu, Waggoner, and Zha (2009) |
| LWZ2.m | Yes | Multivariate | Liu, Waggoner, and Zha (2009) |
| Replicate.m | | | Cho (2011) |

### DL1.m: Fisherian Model of Inflation Determination

This example is based on Davig and Leeper (2007). It is a univariate MSRE model without predetermined variables. We consider 4 sets depending on the parameter values. Vary "Set=i;" for $i = 1, .., 4$, at the beginning of the code and run it. "Set=1;" corresponds to determinacy (Case 1), "Set=2" and "Set=3" to indeterminacy (Case 2). But the LRTP holds for the former and does not for the latter. "Set=4" belongs to the case that the FCC fails to hold(Case 3).

The model is given by:

$$\begin{aligned} \alpha(s_t)\pi_t &= E_t\pi_{t+1} + r_t \\ r_t &= \rho r_{t-1} + \epsilon_t \end{aligned}$$

where $\pi_t$ and $r_t$ are inflation and the real interest rate, respectively. $\alpha(s_t)$ captures a monetary policy stance: When $\alpha(s_t) < (>)1$, monetary policy is passive (active). The model can be cast into the general model (1) as:

$$x_t = A(s_t)E_t x_{t+1} + C(s_t)z_t$$

where $x_t = \pi_t$, $z_t = r_t$, $A(s_t) = C(s_t) = 1/\alpha(s_t)$, $B(s_t) = 0$ and $R = \rho$. The forward solution will have the following form if it exists:

$$x_t = \Gamma^*(s_t) z_t$$

**Set=1:** $\alpha(1) = 1/1.05 < 1$, $\alpha(2) = 1/0.9 > 1$, $\rho = 0.6$, $p_{11} = 0.75$ and $p_{22} = 0.75$. One may write the code as:

```
S=2; p11=0.75; p22=0.75; p12=1-p11; p21=1-p22; P=[p11 p12;p21 p22];
rho=0.6; alpha{1}=1/1.05; alpha{2}=1/0.9;
for j=1:S, A{j,1}=1/alpha{j}; C{j,1}=1/alpha{j}; end,    B=[ ]; R=rho;
[OmegaK,GammaK,FK,termK,R_BarPsi_OKkOK,R_Psi_FKkFK,...
              BarPsi_OKkOK, Psi_FKkFK,IRF]=msres(P,A,B,C,R);
AGamma=amsre(P,A,C,R);
```

The first three lines specify the number of regimes, transition matrix and $A$, $B$, $C$ and $R$. In the 4th line, the function "msres.m" takes these input arguments and produces the forward solution, "GammaK" as a cell array. Typing "GammaK{1}" and "GammaK{2}" will display numerical solution $\Gamma^*(s_t)$ for $s_t = 1$ and $s_t = 2$. Since there is no predetermined variable, if there exists a fundamental solution, it is at most one and it can be solved analytically. "AGamma" in the 6th line denotes the solution obtained by "amsre.m" where the analytical formula is explained in the appendix.

The example DL1.m produces the following output. First, the forward solution exists: $\Gamma^*(1)$=2.6196 and $\Gamma^*(2)$=2.1070. Since there is no predetermined variable, "OmegaK" will be 0 for all regimes. Second, R_Psi_FKkFK=$r_\sigma(\Psi_{F^*\otimes F^*})$=0.9777 $< 1$. Hence, the model is determinate. Here $F^* = A$. Therefore, one can compute the LRTP condition $r_\sigma(\Psi_{F^*})$. From Theorem 1, $r_\sigma(\Psi_{F^*})$ should be less than 1. Indeed $r_\sigma(\Psi_{F^*}) = 0.9807$. Third, the code computes the MSV solution using the formula in Davig and Leeper (2007) or the one in Section 4. In this case, the MSV solution coincides with the forward solution. Fourth, the impulse response function of inflation to a real rate shock of size 1 conditional on the initial state.

**Set=2: Same as Set=1 except** $\alpha(1) = 1/1.08$.

Here, the monetary policy is more passive in the regime 1. The forward solution exists and it is MSS because $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) = 0$. However, $r_\sigma(\Psi_{F^*\otimes F^*})$ =1.0185. Since this is greater than 1, we need to search for $\Lambda = \arg\min\ r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ subject to (18) and $V_i'V_i = I_{k_i}$ for all $i = 1, 2, ..., S$. Find_Min_R_Psi_LkK.m together with the constraints and objective functions (Find_Min_R_Psi_LkK_NL.m, Find_Min_R_Psi_LkK_Q.m) does this job. To implement this procedure, type [Lambda,V,Phi]=Find_Min_R_Psi_LkL(P,FK); We found that min $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ =0.9818=1/1.0185, which confirms Lemma 4. Therefore, the model is indeterminate. We also find that $r_\sigma(\Psi_{F^*}) = 0.9981$. This case illustrates that the LRTP is not sufficient for determinacy of the model.

**Set=3: Same as Set=2 except** $\alpha(1) = 1/1.5$.

Here, the monetary policy is much more passive in the regime 1. Nevertheless the forward solution still exists and MSS. $r_\sigma(\Psi_{F^*})$ =1.2674 and $r_\sigma(\Psi_{F^*\otimes F^*})$ =1.7843. We find that $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$, Hence, the model is indeterminate and the LRTP does not hold.

**Set=4: Same as Set=1 except** $\alpha(1) = 1/1.5$ and $\rho = 0.9$.

This case is indeterminate, precisely because of the same reason as that in **Set=3**: $r_\sigma(\Psi_{F^*\otimes F^*})$ =1.7843 is exactly the same as that in Case 3 because it does not depend on $\rho$. However, $\rho$ does affect forward convergence of the model. In this case, the FCC fails to hold because $\Gamma_k(s_t)$ explodes as $k$ goes to infinity. The code will produce a warning sign and plots $\Gamma_k(s_t)$ over $k = 1, ..., maxK$. From the figure, one can see that the elements $\Gamma_k$ explode as $k$ increases. The analytical solution $\Gamma(s_t)$ still exists, but the coefficients are all negative, which makes no sense. The forward method detects this case by the fact that the model fails to satisfy the FCC and the analytical solution must violates the NBC. This example shows the importance of checking forward convergence. The following table summarizes the results for the model.

| Set | Case | LRTP | FS | Analytical MSV Solution (NBC) |
|---|---|---|---|---|
| Set=1 | 1. Determinate | Yes | Exists | Exists, same as the FS (Yes) |
| Set=2 | 2. Indeterminate | Yes | Exists | Exists, same as the FS (Yes) |
| Set=3 | 2. Indeterminate | No | Exists | Exists, same as the FS (Yes) |
| Set=4 | 3. FCC fails | No | Not Exist | Exists (**No**) |

## DL2.m: Regime-Switching Monetary Policy in a canonical New Keynesian Model

This is a canonical New Keynesian model consisting of Phillips curve, intertemporal IS equation and monetary policy equation considered by Davig and Leeper (2007), and Lubik and Schorfheide (2004). This is a multivariate MSRE model without predetermined variables. Here the analytical solution can be obtained. We consider four sets of the model analogous to the example DL1.m. The model is given by:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa y_t + z_{S,t}, \\
y_t &= E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + z_{D,t}, \\
i_t &= \phi_\pi(s_t)\pi_t + \phi_y(s_t) y_t,
\end{aligned}$$

where $\pi_t$ and $y_t$ are inflation and the output gap, respectively. $\phi_\pi(s_t)$ captures a monetary policy stance against inflation, which is active (passive) if $\phi_\pi(s_t) > 1$ ($\phi_\pi(s_t) \leq 1$). The exogenous disturbances $z_{i,t}$ for $i = S, D$ follows an AR(1) process as:

$$z_{i,t} = \rho_i z_{i,t-1} + \epsilon_{i,t},$$

where $\epsilon_{i,t}$ is an $i.i.d.$ process, independent of the regime $s_t$. The model can be written in a matrix form as:

$$\begin{aligned}
B_1(s_t) x_t &= A_1 E_t x_{t+1} + C_1 z_t, \\
z_t &= R z_{t-1} + \epsilon_t, \\
B_1(s_t) &= \begin{bmatrix} 1 & -\kappa & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ -\phi_\pi(s_t) & -\phi_y(s_t) & 1 \end{bmatrix}, \ A_1 = \begin{bmatrix} \beta & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
C_1(s_t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ R = \begin{bmatrix} \rho_S & 0 \\ 0 & \rho_D \end{bmatrix},
\end{aligned}$$

where $x_t = [\pi_t \ y_t \ i_t]'$, $z_t = [z_{S,t} \ z_{D,t}]'$ and $\epsilon_t = [\epsilon_{S,t} \ \epsilon_{D,t}]'$. Therefore, we have the following:

$$x_t = A(s_t) E_t x_{t+1} + C(s_t) z_t,$$

where $A(s_t) = B_1^{-1}(s_t) A_1$ and $C(s_t) = B_1^{-1}(s_t) C_1$. The forward solution will have the following form if it exists:

$$x_t = \Gamma^*(s_t) z_t.$$

The parameter values are summarized in the following table.

| Common Parameters | $\beta = 0.99,\ \kappa = 0.27,\ \sigma = 2.84,\ \rho_S = 0.75, p_{11} = 0.95$ |
|---|---|
| Set=1 | $p_{22} = 0.89, \alpha(1) = 2.19,\ \alpha(2) = 0.89,\ \gamma(1) = 0.3, \gamma(2) = 0.15, \rho_D = 0.75$ |
| Set=2 | same as Case 1 except $p_{22} = 0.9,$ |
| Set=3 | $p_{22} = 0.9, \alpha(1) = 1.5,\ \alpha(2) = 0.5,\ \gamma(1) = \gamma(2) = 0.25, \rho_D = 0.75$ |
| Set=4 | same as Case 1 except $\rho_D = 95$ |

The results are qualitatively similar to those of example, DL1.m.

## LWZ1.m: Regime-Switching Monetary Policy in an extended New Keynesian Model

This is a more general New Keynesian model featuring endogenous persistence of inflation, the output gap and the interest rate considered by Liu, Waggoner, and Zha (2009). This is a multivariate MSRE model with predetermined variables. Here analytical solutions cannot be obtained but a numerical solution can be obtained using the method proposed by Farmer, Waggoner, and Zha (forthcoming). We examine determinacy of their model. Again, there are four sets. The model is given by:

$$\begin{aligned}
\Delta_t \pi_t &= \beta\psi_1(s_t, s_{t-1}) E_t \pi_{t+1} + \gamma(s_{t-1})\pi_{t-1} + \psi_2(s_{t-1}) \left( \frac{\xi+1}{\alpha} + \frac{b}{\lambda - b} \right) y_t \\
&\quad - \psi_2(s_{t-1}) \frac{b}{\lambda - b} y_{t-1} + \psi_2(s_{t-1}) \mu_{w,t} + \psi_2(s_{t-1}) \frac{b}{\lambda - b} v_t, \\
y_t &= \frac{\lambda}{\lambda + b} E_t y_{t+1} + \frac{b}{\lambda + b} y_{t-1} - \frac{\lambda - b}{\lambda + b} (i_t - E_t \pi_{t+1}) + \frac{(\lambda - b)(1 - \rho_a)}{\lambda + b} a_t - \frac{b - \rho_v \lambda}{\lambda + b} \nu_t, \\
i_t &= \rho_r(s_t) i_{t-1} + (1 - \rho_r(s_t)) \left( \phi_\pi(s_t)\pi_t + \phi_y(s_t) y_t \right) + u_{rt},
\end{aligned}$$

where

$$\begin{aligned}
\psi_1(s_t, s_{t-1}) &= \frac{\bar{\eta}}{\eta(s_{t-1})} \frac{1 - \eta(s_{t-1})}{1 - \eta(s_t)}, \\
\psi_2(s_{t-1}) &= \frac{(1 - \beta\bar{\eta})(1 - \eta(s_{t-1}))}{\eta(s_{t-1})} \frac{1}{1 + \theta_p (1 - \alpha)/\alpha}, \\
\Delta_t(s_t, s_{t-1}) &= 1 + \beta\psi_1(s_t, s_{t-1})\gamma(s_t).
\end{aligned}$$

The exogenous variables are given by:

$$\begin{aligned}
u_{rt} &= \epsilon_{rt}, \\
a_t &= \rho_a a_{t-1} + \epsilon_{a,t}, \\
\mu_{w,t} &= \rho_w \mu_{w,t-1} + \epsilon_{w,t}, \\
\nu_t &= \rho_v v_{t-1} + \epsilon_{v,t}.
\end{aligned}$$

Here we have redefined $\epsilon_{rt}$ as $u_{rt}$ for convenience. Let $x_t = [\pi \; y_t \; i_t]'$, $z_t = [u_{rt} \; a_t \; \mu_{w,t} \; \nu_t]'$. Then, the model can be written in matrix form as:

$$B_1(s_t, s_{t-1}) x_t = A_1(s_t, s_{t-1}) E_t x_{t+1} + B_2(s_t, s_{t-1}) x_{t-1} + C_1(s_t, s_{t-1}) z_t,$$

$$B_1(s_t, s_{t-1}) = \begin{bmatrix} \Delta_t & -\psi_2(s_{t-1})\left(\frac{\xi+1}{\alpha} + \frac{b}{\lambda-b}\right) & 0 \\ 0 & 1 & \frac{\lambda-b}{\lambda+b} \\ -(1-\rho_r(s_t))\,\phi_\pi(s_t) & -(1-\rho_r(s_t))\,\phi_y(s_t) & 1 \end{bmatrix},$$

$$A_1(s_t, s_{t-1}) = \begin{bmatrix} \beta\psi_1(s_t, s_{t-1}) & 0 & 0 \\ \frac{\lambda-b}{\lambda+b} & \frac{\lambda}{\lambda+b} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_2(s_t, s_{t-1}) = \begin{bmatrix} \gamma(s_{t-1}) & -\psi_2(s_{t-1})\frac{b}{\lambda-b} & 0 \\ 0 & \frac{b}{\lambda+b} & 0 \\ 0 & 0 & \rho_r(s_t) \end{bmatrix},$$

$$C_1(s_t, s_{t-1}) = \begin{bmatrix} 0 & 0 & \psi_2(s_{t-1}) & \psi_2(s_{t-1})\frac{b}{\lambda-b} \\ 0 & \frac{(\lambda-b)(1-\rho_a)}{\lambda+b} & 0 & -\frac{b-\rho_v\lambda}{\lambda+b} \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_a & 0 & 0 \\ 0 & 0 & \rho_w & 0 \\ 0 & 0 & 0 & \rho_v \end{bmatrix}.$$

By inverting $B_1(s_t, s_{t-1})$, we have the following:

$$x_t = A(s_t, s_{t-1})E_t x_{t+1} + B(s_t, s_{t-1})x_{t-1} + C(s_t, s_{t-1})z_t, \tag{23}$$

where $A(s_t, s_{t-1}) = B_1(s_t, s_{t-1})^{-1}A_1(s_t, s_{t-1})$, $B(s_t, s_{t-1}) = B_1(s_t, s_{t-1})^{-1}B_2(s_t, s_{t-1})$ and $C(s_t, s_{t-1}) = B_1(s_t, s_{t-1})^{-1}C_1(s_t, s_{t-1})$.

To examine this model with Markov-switching monetary policy only, fix the parameters $\eta$ and $\gamma$. Then, the model does not depend on $s_{t-1}$ and can be written as:

$$\begin{aligned} x_t &= A(s_t)E_t x_{t+1} + B(s_t)x_{t-1} + C(s_t)z_t, \\ z_t &= Rz_{t-1} + \epsilon_t. \end{aligned}$$

The parameter values of the four cases are given in the following table.

| | |
|---|---|
| Common Parameters | $\beta = 0.9952,\ \xi = 2,\ b = 0.75,\ \alpha = 0.7,\ \lambda = 1.005,\ \theta_p = 10$<br>$\sigma_r = 0.1,\ \sigma_a = 0.1,\ \sigma_w = 0.1,\ \sigma_v = 0.1,\ \phi_y(1) = 0.5,\ \phi_y(2) = 0.5$<br>$\rho_a = 0.9,\ \rho_r = 0.55,\ \rho_w = 0.9,\ \rho_v = 0.2,\ p_{11} = p_{22} = 0.95$<br>$\eta = 0.66, \gamma = 1$ |
| Set=1 | $\phi_\pi(1) = 0.9,\ \phi_\pi(2) = 2.5$ |
| Set=2 | $\phi_\pi(1) = 0.9,\ \phi_\pi(2) = 1.5$ |
| Set=3 | $\phi_\pi(1) = 0.9,\ \phi_\pi(2) = 1.15$ |
| Set=4 | $\phi_\pi(1) = 0.75,\ \phi_\pi(2) = 1.5$ |

The results are qualitatively similar to those of example DL1.m or DL2.m.

## LWZ2.m: Regime-Switching Phillips Curve and Monetary Policy in an extended New Keynesian Model

This is the model we present in "LWZ1.m". Here both the private sector and monetary policy depend on a common regime. In the appendix, we show how to represent the model (23) into the form of (1). The only difference from "LWZ1.m" is that the parameters $\eta$ and $\gamma$ are regime-dependent and their values are specified as $\eta(1) = 0.66,\ \eta(2) = 0.75,\ \gamma(1) = 1,\ \gamma(2) = 0$. We take the same parameter values in Set=1 through Set=4 from "LWZ1.m". We compute $r_\sigma(\Psi_{\Omega^*\otimes\Omega^*})$ and $r_\sigma(\Psi_{F^*\otimes F^*})$ for the 4 cases in "LWZ1.m" and "LWZ2.m" as follows.

| | | "LWZ1.m" | | "LWZ2.m" | |
|---|---|---|---|---|---|
| Model | FCC | $r_\sigma(\Psi_{\Omega^*\otimes\Omega^*})$ | $r_\sigma(\Psi_{F^*\otimes F^*})$ | $r_\sigma(\Psi_{\Omega^*\otimes\Omega^*})$ | $r_\sigma(\Psi_{F^*\otimes F^*})$ |
| Set=1 | Yes | 0.4556 | 0.9698 | 0.4654 | 1.0046 |
| Set=2 | Yes | 0.4678 | 1.0107 | 0.4731 | 1.0358 |
| Set=3 | Yes | 0.4778 | 1.0399 | 0.4765 | 1.0502 |
| Set=4 | No | 0.6719 | 1.4200 | 0.6419 | 1.3765 |

What's striking is that whereas the parameter values of Set=1 in "LWZ1.m" belong to determinacy region, they become indeterminate in "LWZ2.m" because $r_\sigma(\Psi_{F^*\otimes F^*}) > 1$.

# Appendix

## A Impulse Response Functions of MSRE Models

The forward solution and the exogenous process $z_t$ are given by:

$$\begin{aligned} x_t &= \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t, \\ z_t &= Rz_{t-1} + \epsilon_t. \end{aligned}$$

The one-step ahead prediction of $x_{t+1}$ conditional on time $t$ information including $s_t$ is given by:

$$E_t x_{t+1} = H(s_t, 1)x_t + G(s_t, 1)z_t, \tag{24}$$

where

$$\begin{aligned} H(s_t, 1) &= E[\Omega^*(s_{t+1})|s_t], \\ G(s_t, 1) &= E[\Gamma^*(s_{t+1})|s_t]R. \end{aligned}$$

The $k$-step ahead prediction of $x_t$ is then, given by:

$$E_t x_{t+k} = H(s_t, k)x_t + G(s_t, k)z_t, \tag{25}$$

where,

$$\begin{aligned} H(s_t, k) &= E[H(s_{t+1}, k-1)\Omega^*(s_{t+1})|s_t], \\ G(s_t, k) &= E[(G(s_{t+1}, k-1) + H(s_{t+1}, k-1)\Gamma^*(s_{t+1}))|s_t]R, \end{aligned}$$

for $k \geq 2$. We may define $H(s_t, 0) = I_n$ and $G(s_t, 0) = 0_{n\times m}$. Then, the impulse responses of $x_{t+k}$ to the $l-$th innovation at time $t$ conditional on $s_t$ is given by:

$$IR(s_t, k) = (H(s_t, k)\Gamma^*(s_t) + G(s_t, k))e_l, \tag{26}$$

for $k = 0, 1, 2, ..$ where $e_l$ is an indicator vector of which $l$-th element is 1 and 0 elsewhere.

# B More General MSRE Models

Suppose that the model contains the regime variable $s_{t-1}$. For instance, consider the following model:

$$x_t = E_t[A(s_{t-1}, s_t, s_{t+1})x_{t+1}] + B(s_{t-1}, s_t)x_{t-1} + C(s_{t-1}, s_t)z_t. \tag{27}$$

This kind of extended model can be written in the form of (1) and analyzed. Define the extended regime variable $\hat{s}_t = (s_{t-1}, s_t)$ such that:

| $\hat{s}_t$ | 1 | 2 | ... | $S$ | $S+1$ | $S+2$ | ... | $2S$ | ... | $(S-1)S+1$ | $(S-1)S+2$ | ... | $S^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s_{t-1}$ | 1 | 1 | ... | 1 | 2 | 2 | ... | 2 | ... | $S$ | $S$ | ... | $S$ |
| $s_t$ | 1 | 2 | ... | $S$ | 1 | 2 | ... | $S$ | ... | 1 | 2 | ... | $S$ |

The corresponding transition matrix is defined as $\hat{P} = (i_S \otimes I_S \otimes i'_S)diag(vec(P'))$ where $i_S$ is an $S \times 1$ column vector of ones. For instance, when $S = 2$, $\hat{P}$ is given by:

$$\hat{P} = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} \\ p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} \end{bmatrix}.$$

Therefore, the model (27) can be written in the form of (1) such that

$$x_t = E_t[A(\hat{s}_t, \hat{s}_{t+1})x_{t+1}] + B(\hat{s}_t)x_{t-1} + C(\hat{s}_t)z_t.$$

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