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Exponential functionals and means of neutral-to-the-right priors

By ILENIA EPIFANI

Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, 20133 Milan, Italy

ileepi@mate.polimi.it

ANTONIO LIJOI and IGOR PRÜNSTER

Dipartimento di Economia Politica e Metodi Quantitativi, Università di Pavia, 27100 Pavia, Italy

lijoi@unipv.it   igor.pruenster@unipv.it

Summary

The mean of a random distribution chosen from a neutral-to-the-right prior can be represented as the exponential functional of an increasing additive process. This fact is exploited in order to give sufficient conditions for the existence of the mean of a neutral-to-the-right prior and for the absolute continuity of its probability distribution. Moreover, expressions for its moments, of any order, are provided. For illustrative purposes we consider a generalisation of the neutral-to-the-right prior based on the gamma process and the beta-Stacy process. Finally, by resorting to the maximum entropy algorithm, we obtain an approximation to the probability density function of the mean of a neutral-to-the-right prior. The arguments are easily extended to examine means of posterior quantities. The numerical results obtained are compared to those yielded by the application of some well-established simulation algorithms.

Some key words: Bayesian nonparametrics; Distribution of means of random probability measures; Exponential functional; Increasing additive process; Lévy measure; Neutral-to-the-right prior distribution.

1. Introduction

In recent years much attention has been paid to the study of the distributional properties of the exponential functional of a stochastic process ξ,

$$I_t(\xi) = \int_0^t e^{-\xi_s}\, ds \quad (t \geqslant 0),$$

where ξ is a Lévy process; see for example the deep and insightful works by Bertoin & Yor (2001), Carmona et al. (1997) and Urbanik (1992). The interest in the exponential functional results from its importance in a wide variety of applications: in mathematical finance it represents a key quantity in risk theory, and it also arises when facing the problems of pricing Asian options and of the determination of the law of a perpetuity, whereas in mathematical physics it appears in random dynamical systems and it is also essential in the study of a one-dimensional diffusion in a random Lévy environment. Further references about these applications are given in the above-mentioned papers.

As pointed out in Carmona et al. (2001), in order to determine the properties of $I_t(\xi)$ one need only consider the terminal value $I(\xi) := I_\infty(\xi)$, since the killing of ξ at an independent exponential time yields, through a Laplace transform inversion, the law at a fixed time t.

Throughout this paper we deal with the exponential functional of an increasing additive process over $\mathbb{R}^+$, that is an increasing and purely discontinuous Lévy process that is not necessarily homogeneous. Recently there has been growing interest in additive processes in view of their wide applicability, especially in finance; see for example Barndorff-Nielsen & Shephard (2001). Although all our results carry over, with slight modifications, to general, not necessarily increasing, additive processes with a continuous part, we prefer to consider the case of increasing additive processes in order to preserve a Bayesian nonparametric interpretation.

A very popular class of prior distributions in Bayesian nonparametric statistics, introduced by Doksum (1974), are the neutral-to-the-right random probability distributions. It is well known that a random distribution function, F, is neutral-to-the-right if and only if

$$F(t) = 1 - e^{-\xi_t},$$

for some transient increasing additive process ξ. Contributions to the theory of neutral-to-the-right priors include Ferguson (1974), Ferguson & Phadia (1979) and Walker & Muliere (1997).

In determining the distribution of a mean of a neutral-to-the-right prior, connections with $I(\xi)$ become clear. Indeed, it can easily be shown that a random mean of a neutral-to-the-right prior has the same distribution as the exponential functional of the corresponding increasing additive process. This fact was first noticed in a seminal paper by Cifarelli & Regazzini (1979), who studied the mean of a Dirichlet process by writing it as a neutral-to-the-right prior.

Despite the apparent statistical importance of means of random distribution functions, very few results have been obtained beyond the Dirichlet case (Cifarelli & Regazzini, 1979, 1990; Regazzini et al., 2002), mainly because of overwhelming technical difficulties. Recently however it has again become a very active area of research. For example, in a University of Oslo technical report, N. L. Hjort provided results for a generalised Dirichlet prior and Regazzini et al. (2003) gave the exact distribution of normalised random measures with independent increments. However, apart from the Dirichlet prior, nothing is known for measures of neutral-to-the-right random distribution functions.

When interpreting the exponential functional as a mean of a neutral-to-the-right prior, we may treat what we term generalised exponential functionals, that is

$$I^g(\xi) = \int_0^{+\infty} e^{-\xi_t} \, dg(t),$$

as simple exponential functionals of the corresponding time-changed increasing additive process for a suitable family of functions g. This is because the random mean $\int_0^{+\infty} g(t) \, dF(t)$ can be seen as a simple mean with respect to a reparameterised random distribution function. Hence, consideration of simple and generalised exponential functionals of increasing additive processes, instead of Lévy processes, allows us to deal with means of every possible neutral-to-the-right prior in a unified way.

In this paper we study some distributional properties of means of neutral-to-the-right priors. In order to do so, we need to extend known results in the theory on exponential functionals so that they are applicable to Bayesian nonparametrics. Moreover, we stress the additional insight provided by the interpretation of the exponential functional as a mean of a random probability measure. For instance, properties such as the reduction of the generalised exponential functional to the simple one should, we hope, raise the interest of probabilists in Bayesian nonparametrics and lead to fruitful interactions.

In pursuing these goals, we provide results for exponential functionals that encompass nonhomogeneous increasing additive processes, possibly with fixed points of discontinuity. These cases have not been covered before and are essential for the treatment of Bayesian nonparametric inferential problems. In particular, we obtain sufficient conditions for the existence of a mean and its moments of any order and for the absolute continuity of its distribution. Moreover, we determine an expression for the computation of moments of means of neutral-to-the-right priors. Finally, we show how these results turn out to be useful in applications related to survival analysis.

In § 2 the relationship between exponential functionals of increasing additive processes and means of neutral-to-the-right priors is established. In § 3 we state the main results. In § 4 we focus on two illustrative examples of neutral-to-the-right priors: the first is a generalisation of the neutral-to-the-right prior based on the gamma process, and the second is the beta-Stacy process introduced by Walker & Muliere (1997). Section 5 is devoted to an application to survival analysis. The exact law of the mean, i.e. the expected lifetime, corresponding to the beta-Stacy process, under prior and posterior conditions, is approximated by resorting to the maximum entropy estimator. A comparison with simulation techniques is also provided. Proofs are deferred to the Appendix.

2. Exponential functionals and means of random probability measures

A useful approach to defining random probability measures consists of taking suitable transformations of a class of independent increments processes that we concisely define as follows.

Let $\{\nu_t : t \geqslant 0\}$ be a family of Lévy measures on the Borel σ-field of $\mathbb{R}^+$, $\mathscr{B}(\mathbb{R}^+)$; that is

(i) $\nu_0 \equiv 0$, $\int_{(0,+\infty)}(x \wedge 1)\nu_t(dx) < +\infty$ holds true for any $t > 0$;

(ii) $\nu_s(B) \leqslant \nu_t(B)$ for $s < t$ and $B \in \mathscr{B}(\mathbb{R}^+)$;

(iii) $\nu_s(B) \to \nu_t(B)$ as $s \to t$ in $[0, +\infty)$ with $B \in \mathscr{B}(\mathbb{R}^+)$ and $B \subset \{x : x > \varepsilon\}$, for some $\varepsilon > 0$.

According to Theorem 9.8 in Sato (1999), (i)–(iii) assure the existence of a stochastic process $\xi := \{\xi_t : t \geqslant 0\}$ defined on some probability space satisfying the following conditions:

(iv) for any choice of $n \geqslant 1$ and $0 \leqslant t_0 < t_1 < \ldots < t_n$, random variables ξ_{t_0}, $\xi_{t_1} - \xi_{t_0}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent;

(v) $\xi_0 = 0$ with probability 1;

(vi) ξ is stochastically continuous;

(vii) the mapping $t \mapsto \xi_t$ is almost surely increasing and right continuous.

Any process satisfying (iv)–(vii) is said to be an increasing additive process. Moreover, by virtue of the Lévy–Khinthcine representation we have, for any t,

$$E(e^{-\lambda \xi_t}) = \exp\left\{-\int_{[0+\infty)} (1 - e^{-\lambda x})\nu_t(dx)\right\}$$

for every $\lambda \geqslant 0$. For an exhaustive account of the theory of increasing additive processes see Sato (1999). For Bayesian applications, it is important to introduce an extension of increasing additive processes which takes into account the presence of fixed points of discontinuity. Let $J_1, J_2, \ldots$ be a sequence of independent and nonnegative random variables, which are assumed to have absolutely continuous distributions with respect to the Lebesgue measure, such that they are independent from ξ. Given a set of points $\{s_1, s_2, \ldots\}$, a process $\Phi = \{\Phi_t : t \geqslant 0\}$ defined by

$$\Phi_t = \xi_t + \sum_{k:s_k \leqslant t} J_k$$

is termed an increasing additive process with fixed points of discontinuity. According to such a definition Φ is completely characterised by the knowledge of the set of fixed points of discontinuity, the density functions of the jumps and the Lévy measure of ξ.

In the sequel, results concerning both the existence of a mean and of its moments and the absolute continuity of its distribution are stated just in terms of increasing additive processes, the extension to the case of fixed points of discontinuity being straightforward. When computing their moments, the two cases are dealt with separately.

Starting from increasing additive processes, one can, for instance, define a random probability measure by normalising the increments of the process; see Regazzini et al. (2003) for such an approach. Here we are interested in a transformation that gives rise to a neutral-to-the-right process prior. Indeed, it can be proved that a random distribution function F is neutral-to-the-right if and only if $F(t)$ has the same probability distribution as $1 - e^{-\xi_t}$ for some increasing additive process ξ such that $\xi_t \to +\infty$, almost surely, as t tends to $+\infty$.

An attractive feature of neutral-to-the-right priors is that they lead to simple posterior computations. In fact, if we deal with exchangeable observations, the posterior distribution of a neutral-to-the-right random probability measure is still neutral-to-the-right whether the observations are exact or right censored. Moreover, the posterior can be explicitly obtained by updating the Lévy measure in quite a straightforward way and by adding fixed points of discontinuity for the exact observations; see Ferguson (1974), Ferguson & Phadia (1979) and Walker & Muliere (1997) for details. In the following we consider only neutral-to-the-right prior distributions, having it in mind that analogous results may be obtained for posteriors by slight modifications.

If F is a random distribution function on $\mathbb{R}^+$, its mean can be written as

$$\int_0^{+\infty} \{1 - F(t)\}\, dt. \tag{1}$$

If it is assumed that F is neutral-to-the-right, (1) turns out to be the exponential functional $I(\xi) = \int_0^{+\infty} e^{-\xi_t}\, dt$. Hence, the combination of (1) and the definition of neutral-to-the-right prior suggests that the recently developed theory for exponential functionals of Lévy processes can be a useful tool for applications in Bayesian nonparametric inference. In fact, in order to cover the whole class of neutral-to-the-right priors, we would need results for exponential functionals of increasing additive processes; the next section is devoted to an attempt to adapt and extend known results for the Lévy case to our more general one.

From a statistical point of view we are interested in studying the distributional properties of means of the type

$$\int_0^{+\infty} g(t)\, dF(t), \tag{2}$$

where g is any nonnegative, continuous and increasing function such that $g(0) = 0$ and

$\lim_{t\to+\infty} g(t) = +\infty$. Obviously, (2) coincides with

$$\int_0^{+\infty} t \, dF\{g^{-1}(t)\},$$

which is the mean of the random distribution function $F(g^{-1}) = 1 - e^{-\zeta}$, ζ being an increasing additive process defined by $\zeta_t := \xi_{g^{-1}(t)}$. This leads to the following result.

LEMMA 1. *Let* $F = 1 - e^{-\xi}$ *be a neutral-to-the-right random distribution function. Then the generalised exponential functional*

$$I^g(\xi) := \int_0^{+\infty} e^{-\xi_t} \, dg(t)$$

has the same distribution as $\int_0^{+\infty} g(t) \, dF(t)$ *or equivalently as* $I(\zeta)$.

As a consequence of Lemma 1, we can confine ourselves to studying simple means, without loss of generality. The fact that the reduction of generalised to simple exponential functionals appears natural when considered within the framework of means of random distribution functions seems to provide some additional insight.

3. MAIN RESULTS

3·1. *Existence of the exponential functional of an increasing additive process*

In the present subsection, attention is focused on the problem of stating finiteness of the exponential functional of an increasing additive process. First we introduce some new terminology. According to the representation of the underlying Lévy measure, we distinguish three different types of process: the process ξ^H is a homogeneous increasing additive process if

$$\nu_t(dx) = t\nu(dx) \quad (t \in \mathbb{R}^+);$$

ξ^γ is a parameterised increasing additive process if

$$\nu_t(dx) = \gamma(t)\nu(dx) \quad (t \in \mathbb{R}^+),$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative, continuous and increasing function such that $\gamma(0) = 0$ and $\lim_{t\to+\infty} \gamma(t) = +\infty$; and ξ is a general increasing additive process if no restriction on the Lévy measure $\nu_t(dx)$ is specified.

Processes of the first type are also known as subordinators. Parameterised increasing additive processes are designed, in the Bayesian literature, as those processes that give rise to the so-called homogeneous neutral-to-the-right priors. A remarkable example of a general increasing additive process, not included in either of the two previous classes, is represented by the beta-Stacy process, whose properties are studied in more depth in the following sections.

Turning to the main problem of existence of the exponential functional of any of the three types of increasing additive process considered above, we should mention that sufficient conditions, valid for the homogeneous case, are explicitly stated in Carmona et al. (1997).

In dealing with the parameterised case, one can reduce the problem to the homogeneous case and then specify suitable choices of γ that ensure finiteness of the exponential function and of the corresponding mean.

Proposition 1. *Let ξ^{γ} be a parameterised increasing additive process. If there exist $a \geqslant 0$ and $0 \leqslant c < +\infty$ such that $a < \int_0^{+\infty} \nu(x, +\infty)\, dx < +\infty$ and*

$$\lim_{t \to +\infty} e^{-at}\gamma^{-1}(t) = c \tag{3}$$

then $I(\xi^{\gamma}) < +\infty$ almost surely.

So far as general increasing additive processes are concerned, the existence of the mean of a neutral-to-the-right process is deduced from the existence of its first moment. Apart from the Dirichlet case, we know of no other sufficient condition for the existence of the mean of a neutral-to-the-right prior based on a general increasing additive process. When F is a Dirichlet process with parameter α, it is well known that a necessary and sufficient condition for finiteness of its mean is that $\int \log(1 + |x|)\alpha(dx) < +\infty$; see Feigin & Tweedie (1989) and Cifarelli & Regazzini (1990). However, such a condition was obtained by those authors without interpreting the Dirichlet process as a neutral-to-the-right prior.

3·2. *Absolute continuity of the exponential functional of an increasing additive process*

After checking that conditions for the existence of $I(\xi)$ are fulfilled, one might be interested in dealing with the issue of absolute continuity of the probability distribution of $I(\xi)$. Such a topic, which is of interest from a theoretical point of view, also has a relevant practical implication in this paper. Indeed, the numerical algorithm that is employed to approximate the distribution of $I(\xi)$ is based on the existence of a probability density function of the exponential functional itself.

It is worth recalling that, if $\xi = \xi^H$, the problem has been solved by Carmona et al. (1997). They prove that $I(\xi^H)$ has absolutely continuous distribution function and provide an integral equation

$$k(y) = \int_y^{+\infty} \int_{\log(z/y)}^{+\infty} \nu(dx)k(z)\, dz, \tag{4}$$

whose solution coincides with the probability density function of $I(\xi^H)$. Unfortunately, the explicit solution of (4) cannot be easily determined, apart from some special cases.

Here, we establish conditions under which the probability distribution of $I(\xi)$ is absolutely continuous whenever ξ coincides either with a parameterised increasing additive process ξ^{γ} or with a general increasing additive process.

Consider first the case in which $\xi = \xi^{\gamma}$ and γ is a diffeomorphism, that is γ is differentiable and has differentiable inverse, and let $g := (\gamma^{-1})'$ the derivative of γ^{-1}. One therefore has

$$I(\xi^{\gamma}) = \int_0^{+\infty} e^{-\xi_t}\, dt = \int_0^{+\infty} e^{-\zeta_t} g(t)\, dt,$$

where $\zeta = \{\zeta_t : t \geqslant 0\} = \{\xi_{\gamma^{-1}(t)} : t \geqslant 0\}$ is a homogeneous increasing additive process. The representation of $I(\xi)$ in terms of a suitable exponential functional of a homogeneous increasing additive process ζ allows us to prove the following result.

Proposition 2. *If γ is a diffeomorphism and satisfies conditions stated in Proposition* 1, *then the probability distribution of $I(\xi^{\gamma})$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

The conditions imposed on γ are not restrictive in our setting since prior specification usually leads to choices of γ that satisfy the requirements stated in Proposition 2. This point is further detailed in § 4·1.

With reference to a general increasing additive process ξ, we simply provide a sufficient condition for the absolute continuity of the distribution function of the corresponding exponential functional. Its main advantage is that it can be easily checked.

PROPOSITION 3. *Let ξ be a general increasing additive process such that $I(\xi)$ exists, almost surely. Define $\nu_{s,t} := \nu_t - \nu_s$, for $0 \leqslant s < t < +\infty$. Then the distribution function of $I(\xi)$ is absolutely continuous if*

$$\int_{(0,+\infty)} \frac{1}{1+x^2} \nu_{s,t}(dx) = +\infty \tag{5}$$

for all $0 \leqslant s < t < +\infty$.

Remark 1. Whenever (5) holds true, the absolute continuity of the distribution of $F(b) - F(a)$ is ensured, for any $0 \leqslant a < b < +\infty$.

3·3. *Moments of the exponential functional of an increasing additive process*

In view of the statistical applications to be presented in § 5, determination of the moments of $I(\xi)$ is relevant. First, a sufficient condition for the existence of the nth moment

$$E(I^n) = E\left[\left\{\int_0^{+\infty} t\, dF(t)\right\}^n\right]$$

is stated. Secondly, formulae for its calculation are provided.

PROPOSITION 4. *If ξ is a general increasing additive process and*

$$\int_0^{+\infty} \exp\left\{-\int_0^{+\infty} (1 - e^{-x}) \nu_{t^{1/n}}(dx)\right\} dt < +\infty, \tag{6}$$

then $E(I^n) < +\infty$.

The previous formula admits a straightforward interpretation in terms of prior beliefs concerning F. Indeed, if F_0 is the prior guess at the shape of F, (6) is tantamount to requiring the existence of the moment of order n at F_0.

When ξ is a parameterised increasing additive process, (6) reduces to

$$\int_0^{+\infty} e^{-\gamma(t^{1/n})\psi(t)}\, dt < +\infty$$

with $\psi(\lambda) := \int_0^{+\infty} (1 - e^{-\lambda x}) \nu(dx)$. If it is further assumed that $\gamma(t) = t$, that is $\xi = \xi^H$, the condition for the existence of moments of any order coincides with $n!/\psi(1)^n < +\infty$ for any $n \geqslant 1$. Hence, we immediately recover a result proved in Carmona et al. (1997) as a special case of our more general treatment.

Remark 2. Not all the distributional properties of exponential functionals of a homogeneous increasing additive process extend to the parameterised case. For instance, Proposition 3.3 in Carmona et al. (1997) states that, if $I(\xi^H)$ exists, then all the moments exist as well. One can see that, if $\gamma(t) = \log(1+t)$, and $\nu(dx) = (1 - e^{-1\cdot 5x})(1 - e^{-x})^{-1} x^{-1} e^{-x}\, dx$, we have $1 < E(\xi_1) < +\infty$ and $\lim_{t \to +\infty} e^{-t} \gamma^{-1}(t) = 1$. Hence, by Proposition 1, $I(\xi^\gamma) < +\infty$. On the other hand, by observing that

$\int_0^{+\infty}(1-e^{-x})\nu(dx)=\log(2\cdot 5)$, we obtain

$$E\{I(\xi^\gamma)\} = \int_0^{+\infty} (1+t)^{-\log(2\cdot 5)}\, dt = +\infty.$$

Thus, the exponential functional exists but it does not possess moments of any order.

We now give a general formula for the computation of moments of the exponential functional of an increasing additive process.

Proposition 5. *If ξ is a general increasing additive process with Lévy measure ν_t and the moment of order n of the corresponding exponential functional exists, then*

$$E(I^n) = n! \int_0^{+\infty} \ldots \int_{t_{n-1}}^{+\infty} \prod_{j=1}^{n} \exp\left\{ - \int_0^{+\infty} (1-e^{-x}) e^{-(n-j)x} \nu_{t_j}(dx) \right\} dt_n \ldots dt_1. \quad (7)$$

Moreover, if Φ is a general increasing additive process with fixed points of discontinuity characterised by the Lévy measure ν_t of the associated increasing additive process ξ, by the set of discontinuities $\{s_1, s_2, \ldots\}$ and by the jumps $J_1, J_2, \ldots$, then

$$E(I^n) = n! \int_0^{+\infty} \ldots \int_{t_{n-1}}^{+\infty} \prod_{i=1}^{n} \exp\left\{ - \int_0^{+\infty} (1-e^{-x}) e^{-(n-i)x} \nu_{t_i}(dx) \right\} \times \prod_{\{k: s_k \leqslant t_i\}} \frac{E(e^{-(n+1-i)J_k})}{E(e^{-(n-i)J_k})}\, dt_n \ldots dt_1 \quad (8)$$

provided $t_0 = 0$.

Hence it is possible to obtain the moment of order n by multiple integration of n Laplace transforms whose Lévy measures are updated at each step of integration by a factor of e^{-x}. It can be easily shown that, in the homogeneous case, the previous expression reduces to the simple and appealing formula provided in Carmona et al. (1997) and Urbanik (1992),

$$E(I^n) = \frac{n!}{\prod_{j=1}^{n} \psi(j)},$$

where, as previously, $\psi(j) := \int_0^{+\infty}(1-e^{-jx})\nu(dx)$. Moreover, if we set $\mu_0 \equiv 1$ and, for any $n \geqslant 1$,

$$\mu_n(t) := \int_t^{+\infty} \exp\left\{ - \int_0^{+\infty} (1-e^{-x}) \nu_s^{(n-1)}(dx) \right\} \mu_{n-1}(s)\, ds$$

we can rewrite (7) as

$$E(I^n) = n!\, \mu_n(0). \quad (9)$$

In cases in which μ_n can be explicitly determined, (9) provides a recursive formula for the computation of the moments.

In § 4 we analyse some illustrative examples of neutral-to-the-right priors, which correspond to particular choices of ν_t.

4. Examples

4·1. *The generalised gamma neutral-to-the-right prior*

The aim of this section is to introduce a new neutral-to-the-right prior based on a family of infinitely divisible probability distributions on $[0, +\infty)$ proposed by Tweedie (1984) and Hougaard (1986), which includes the gamma distribution. In particular, we deal with a subclass of these distributions, namely those corresponding to the natural exponential family generated by the positive stable distributions. We employ such measures to build up a parameterised increasing additive process, which is characterised by the Lévy measure

$$v_t(dx) = \gamma(t)\{\Gamma(1-\sigma)\}^{-1} e^{-\tau x} x^{-(1+\sigma)}\, dx \quad (0 \leqslant \sigma < 1, \tau > 0).$$

The neutral-to-the-right random distribution function based on this increasing additive process reduces to the one based on the gamma process, introduced by Ferguson & Phadia (1979), by taking $\sigma = 0$. Hence we may call it a generalised gamma neutral-to-the-right prior. It seems worth stressing that, with a little computational effort, we have a more general prior at hand which implies a greater flexibility in terms of prior specification.

Since $\int_0^{+\infty} v(x, +\infty)\, dx = \tau^{-(1-\sigma)} < +\infty$, the mean exists if γ satisfies (3) in Proposition 1. If, further, γ fulfils conditions in Proposition 2, then we are able to assert absolute continuity of the distribution function of $I(\xi^\gamma)$.

In the homogeneous case the mean exists and it is absolutely continuous. Moreover, its density, k, satisfies (4), which reduces to

$$k(y) = \frac{\tau^\sigma}{\Gamma(1-\sigma)} \int_y^{+\infty} \Gamma\left\{-\sigma, \tau \log\left(\frac{z}{y}\right)\right\} k(z)\, dz,$$

where $\Gamma(.,.)$ denotes the incomplete gamma function. Nonetheless it seems impossible to solve this integral equation.

For generalised gamma neutral-to-the-right priors the condition for the existence of the moment of order n reduces to $\int_0^{+\infty} \exp[-\gamma(t^{1/n})\sigma^{-1}\{(\tau+1)^\sigma - \tau^\sigma\}]\, dt < +\infty$. The moment of order n is given by

$$E(I^n) = n! \int_0^{+\infty} \cdots \int_{t_{n-1}}^{+\infty} \prod_{j=1}^{n} \exp[-\gamma(t_j)\sigma^{-1}\{(\tau+n+1-j)^\sigma - (\tau+n-j)^\sigma\}]\, dt_n \ldots dt_1.$$

If it is further supposed that $\gamma(t) = t$, we have

$$E(I^n) = \frac{n!\sigma^n}{\prod_{j=1}^n \{(\tau+j)^\sigma - \tau^\sigma\}}.$$

For the special case of a gamma neutral-to-the-right prior, analogous formulae for the moments are given in a technical report by F. Ruggeri from the National Research Council, Milan.

The choice of the function γ and of the parameters τ and σ can be made in order to reflect the experimenter's prior opinion about the survival function $S(t) := 1 - F(t)$. The prior guess at the shape of S is given by $E\{S(t)\} = S_0(t)$. As suggested in Ferguson & Phadia (1979), it seems reasonable to choose γ according to our prior guess; that is, for any τ and σ,

$$\gamma(t) = -\frac{\sigma}{(\tau+1)^\sigma - \tau^\sigma} \log\{S_0(t)\}.$$

Unlike Ferguson & Phadia (1979), we do not interpret τ as 'strength of belief'. Indeed, our main concern is that as τ tends to $+\infty$ then ξ_t tends to 0, almost surely, and the definition of $S(t)$ becomes vacuous. As an alternative, we suggest using σ as a measure of how confident we are about our prior beliefs. By observing that $\text{var}\{S(t)\} \to 0$, as $\sigma \to 1$, we may argue that, if one is sure about one's prior guess, then σ must be chosen close to 1.

4·2. *The beta-Stacy process*

The beta-Stacy process (Walker & Muliere, 1997) can be motivated by two main reasons: it represents a straightforward generalisation of the Dirichlet process allowing for a more flexible elicitation of prior beliefs, and it is conjugate to both exact and right-censored observations. The latter property makes the beta-Stacy prior suitable for applications to survival analysis. Here, we shed light on some distributional properties of the mean of a beta-Stacy process, which we believe to be of relevance for statistical applications.

We give conditions for the existence of the moments, of any order, of the mean of a beta-Stacy process and determine a formula for computing them. Such a result, combined with the absolute continuity of the distribution of the mean, enables us to implement a numerical algorithm for deriving an approximation of the corresponding probability density function.

A beta-Stacy process is a neutral-to-the-right prior based on the log-beta process whose Lévy measure coincides with

$$v_t(dx) = \frac{1}{1 - e^{-x}} \int_{(0,t)} e^{-x\beta(s)} \alpha(ds)\, dx,$$

where β is a positive function, α is a measure concentrated on $\mathbb{R}^+$ which is absolutely continuous with respect to the Lebesgue measure and $\int_{(0,+\infty)} \{\beta(s)\}^{-1} \alpha(ds) = +\infty$. The assumption of absolute continuity of α is equivalent to requiring that no fixed point of discontinuity is present. However, such discontinuities appear when dealing with the posterior.

The Dirichlet process is a special case with α such that $\alpha(\mathbb{R}) < +\infty$ and $\beta(s) = \alpha\{(s, +\infty)\}$. Another special case is the simple homogeneous process defined in Ferguson & Phadia (1979), which arises when β is constant.

The condition for the existence of the moment of order n reduces to

$$\int_{(0,+\infty)} \exp\left[- \int_{(0,t^{1/n})} \{\beta(s)\}^{-1} \alpha(ds) \right] dt < +\infty. \tag{10}$$

If (10) holds then the moment of order n of the mean $I(\xi)$ is

$$n! \int_0^{+\infty} \cdots \int_{t_{n-1}}^{+\infty} \prod_{j=1}^{n} \exp\left\{ - \int_{(0,t_j)} \frac{\alpha(ds)}{\beta(s) + n - j} \right\} dt_n \ldots dt_1. \tag{11}$$

The absolute continuity of the distribution function of $I(\xi)$ follows from a straightforward application of Proposition 3.

Let $y^{(N)} = (y_1, \ldots, y_N)$ denote exchangeable observations and let $s^{(r)} = (s_1, \ldots, s_r)$ be the r exact observations among the y_j's. For simplicity, we assume that the observations are all distinct. The posterior increasing additive process, Φ, is a log-beta process, with fixed points of discontinuities, whose Laplace transform is

$$E(e^{-\lambda\Phi_t}) = \exp\left\{ - \int_{(0,+\infty)} (1 - e^{-\lambda x}) v_t^*(dx) \right\} \prod_{\{k: s_k \leqslant t\}} E(e^{-\lambda J_k}) \tag{12}$$

with

$$\nu_t^*(dx) = \frac{1}{1-e^{-x}} \int_{(0,\ t)} e^{-x\{\beta(s)+M(s)\}} \alpha(ds)\, dx,$$

where $M(.) := \sum_{i=1}^{N} \mathbb{I}_{[.,+\infty)}(y_i)$ and $\mathbb{I}_B$ stands for the indicator function of set B. Moreover, $s_1, \ldots, s_r$ are the fixed points of discontinuity corresponding to the exact observations, the random jump J_i being exponentially distributed with parameter $\tau_i := \beta(s_i) + M(s_i) - 1$ for $i = 1, \ldots, r$. For details, see Walker & Muliere (1997).

Hence, previous arguments for the prior mean apply to positive means. From (8), one obtains

$$n! \int_0^{+\infty} \int_{t_1}^{+\infty} \cdots \int_{t_{n-1}}^{+\infty} \prod_{j=1}^{n} \exp\left\{ - \int_{(0,t_j)} \frac{\alpha(ds)}{\beta(s)+M(s)+n-j} \right\} \times \left(\prod_{\{i: s_i \leqslant t_j\}} \frac{\tau_i + n - j}{\tau_i + n - j + 1} \right) dt_n \ldots dt_1 \qquad (13)$$

as an expression for the posterior moments of $I(\xi)$, with the proviso that, if $\{i : s_i \leqslant t_j\} = \varnothing$ for some j, then $\prod_{\varnothing} := 1$.

5. Numerical approximation of the exact distribution

5·1. *An example from survival analysis*

Our theoretical results also have some practical relevance. We can use the moments of $I(\xi)$ in order to approximate the density function of the mean of a neutral-to-the-right random distribution function on $[0, T]$, where $T < +\infty$.

As an illustrative example, we consider the Kaplan & Meier (1958) dataset, already extensively examined in the Bayesian nonparametric literature; see for example Susarla & Van Ryzin (1976), Ferguson & Phadia (1979), Damien et al. (1995) and Walker & Damien (1998). The data consist of exact observed failures at 0·8, 3·1, 5·4 and 9·2 months and of right-censored observations at 1·0, 2·7, 7·0 and 12·1 months. In order to model the random distribution function F of the survival times, we resort to the beta-Stacy process prior with

$$\beta(s) = \alpha\{(s,\ +\infty)\}, \quad \alpha(dx) = 0{\cdot}1 e^{-0{\cdot}1x} \mathbb{I}_{[0,T]}(x)(1 - e^{-0{\cdot}1T})^{-1}\, dx \quad (T = 100).$$

Usual approaches confine themselves to considering Bayes estimates of F. Here we wish to stress the importance of making inference also on the distribution of the mean of F, which takes on the interpretation of expected lifetime. For instance, when comparing the effects of two different treatments and the Bayes estimates of the corresponding survival functions intersect, one might hope to gain some further insight by examining the distribution of expected lifetimes.

In order to approximate the density function of the mean of F, we adopt two different methods. Both rely upon the knowledge of moments and the absolute continuity of the distribution of $I(\xi)$.

5·2. *Maximum entropy method*

In the specific framework in which we are examining the Kaplan–Meier dataset the moments uniquely determine the corresponding probability distribution. Hence, we use the so-called maximum entropy method, an algorithm which recovers the density function using the moments as inputs. If f denotes the density function of $I(\xi)$, then, for any $n \geqslant 1$, an approximation to f is the function f_n that maximises the Boltzmann–Shannon entropy $H(g) = -\int g(x) \log\{g(x)\}\, dx$, for all densities g satisfying the $n+1$ constraints

$$\int_0^T x^j g(x)\, dx = E(I^j) \quad (j = 0, 1, \ldots, n).$$

The integer n is said to be the order of the estimator and f_n can be expressed as $e^{-\sum_{j=0}^{n} \lambda_j x^j}$, $\lambda_0, \ldots, \lambda_n$ being the Lagrange multpliers. Note that f_n exists for any n, since $I(\xi)$ is absolutely continuous with respect to the Lebesgue measure and then the moment sequence $\{E(I^n), n \geqslant 0\}$ is strictly completely monotonic, meaning that

$$\sum_{m=0}^{k} (-1)^m \binom{k}{m} E(I^{n+m}) > 0$$

for any nonnegative integers k and n. For an exhaustive treatment of maximum entropy techniques, the reader is referred to Csiszár (1975) and Mead & Papanicolaou (1984).

The exact moments of the prior and of the posterior mean can be computed via (11) and (13), respectively, and in Table 1 the first six moments are reported.

Table 1: *Maximum entropy method. Prior and posterior moments of the expected lifetime*

n	Prior moments	Posterior moments
1	9·9955	9·8877
2	$1{\cdot}4968 \times 10^2$	$1{\cdot}1423 \times 10^2$
3	$3{\cdot}1427 \times 10^3$	$1{\cdot}6000 \times 10^3$
4	$8{\cdot}8403 \times 10^4$	$2{\cdot}8174 \times 10^4$
5	$3{\cdot}1847 \times 10^6$	$6{\cdot}3517 \times 10^5$
6	$1{\cdot}4021 \times 10^8$	$1{\cdot}8120 \times 10^7$

Thus, one obtains an approximation to the prior and posterior densities whose graphs are sketched in Fig. 1(a). Computations were carried out with a Fortran code kindly provided by A. Tagliani. As may be seen from Fig. 1(b), the prior density for $n = 6$ is stable with respect to increase or decrease of the number of employed moments.

Remark 3. Previous analyses of the Kaplan–Meier dataset in a Bayesian nonparametric framework focused on the estimation of posterior probabilities of certain intervals. However, these treatments of the problem did not consider the issue of absolute continuity of the laws of the random quantities of interest. Moreover, one can obtain the moments of $F(b) - F(a)$ for any $0 \leqslant a < b < +\infty$ and an application of the maximum entropy algorithm provides an approximation to its probability density function. Clearly, by construction, the Bayes estimate of $F(b) - F(a)$, under quadratic loss, provides the exact value.

5·3. *The inverse Lévy measure algorithm*

An alternative approach to the approximation problem uses an algorithm for simulating samples from neutral-to-the-right random distribution functions. Here we compare the maximum entropy estimator with the approximation arising from work of Wolpert & Ickstadt (1998a).

We approximate $\int_{(\cdot, +\infty)} \nu_s(dx)$ by an incomplete beta function, as suggested in Wolpert

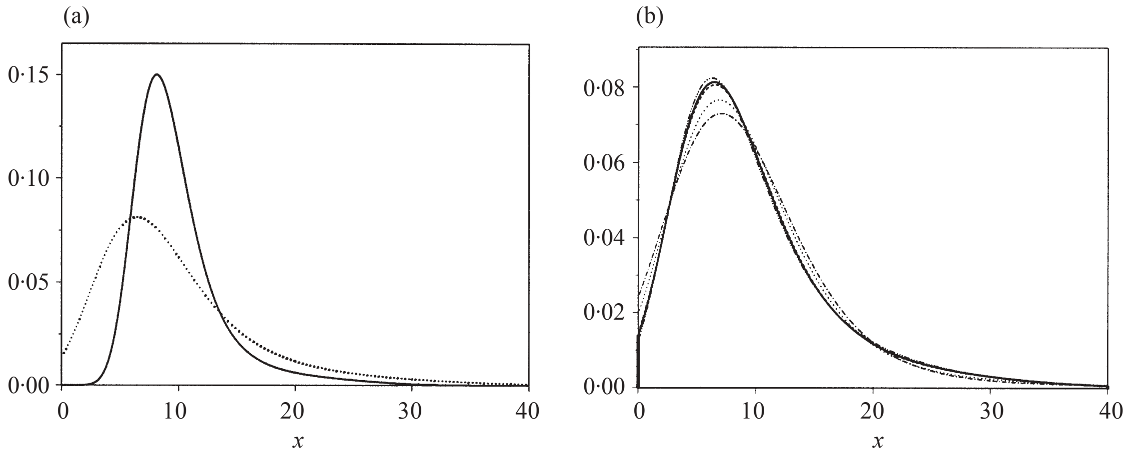

Fig. 1: Survival analysis example. (a) Maximum entropy prior (····) and posterior (——) densities of the mean of a beta-Stacy process based on the first six moments. (b) Maximum entropy prior densities of the mean of a beta-Stacy process based on 4 (-·-·), 5 (····), 6 (——), 7 (- - -) and 8 (-····-) moments.

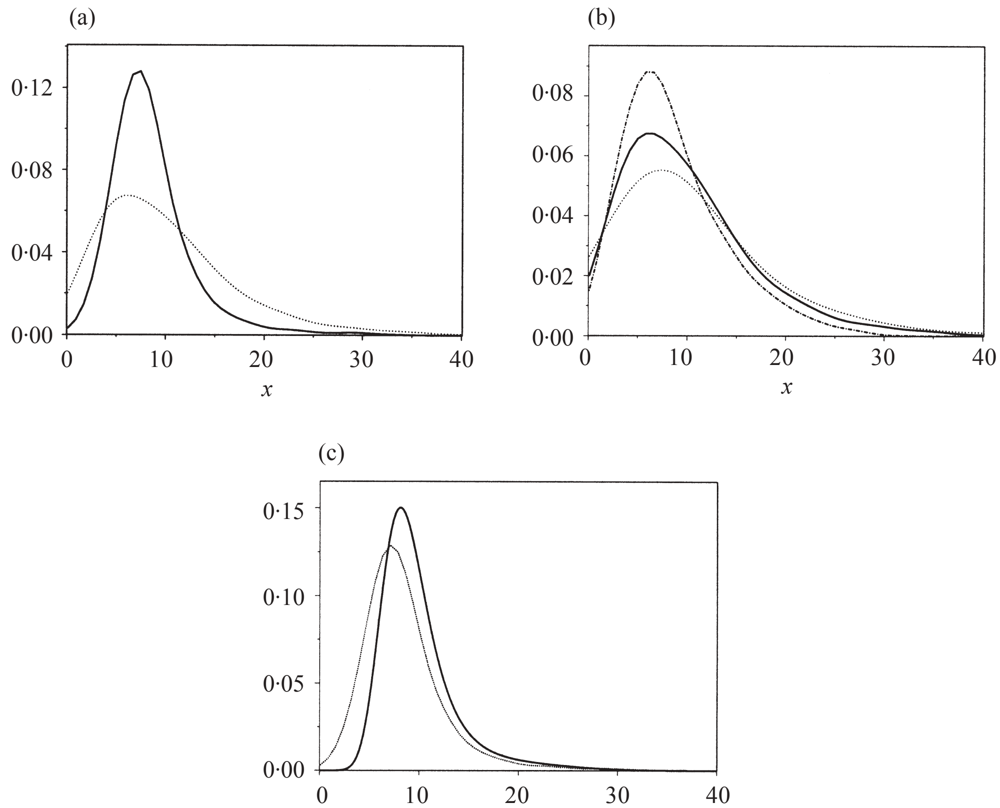

Fig. 2. (a) Prior (····) and posterior (——) densities of the mean of a beta-Stacy process simulated via the inverse Lévy measure algorithm. (b) Prior densities of the mean of a beta-Stacy process, simulated via the inverse Lévy measure algorithm, with: $M = 2000$, $\varepsilon = 10^{-10}$ (——); $M = 1000$, $\varepsilon = 10^{-10}$ (····); $M = 2000$, $\varepsilon = 10^{-5}$ (-·-·). (c) Posterior densities of the mean of a beta-Stacy process obtained by maximum entropy (——) and inverse Lévy measure (····) algorithms.

& Ickstadt (1998b) with reference to the simple homogeneous process. In order to carry out simulations, one needs to fix in advance the value of two parameters, M and ε, in accordance with notation employed in Wolpert & Ickstadt (1998b). Figure 2(a) displays the kernel density estimates of the prior and posterior means, calculated over 1000 trajectories, with $M=2000$ and $\varepsilon=10^{-10}$. These values of M and ε are based on the knowledge of moments of $I(\xi)$. We performed a post-simulation check to verify whether or not the sample moments matched the true moments. Indeed, we notice that as M increases the sample moments decrease, for fixed ε, and as ε decreases the sample moments increase, for fixed M. Figure 2(b) plots the prior densities of the mean, still based on 1000 simulated trajectories, for different values of M and ε. Note that an increase of ε, for fixed M, leads one to underestimate the right tail of the expected lifetime, a serious mistake when facing problems in survival analysis. Table 2 summarises the deviations in percentages of the sample moments from the actual ones. For the prior moments the match obtained for $M=2000$ and $\varepsilon=10^{-10}$ is extremely good, whereas the other choices lead to systematic over- or underestimation. However, the moment-match does not seem to carry over to the posterior case. As a consequence, the posterior density, recovered by implementation of the inverse Lévy measure algorithm, deviates significantly from the one obtained by the maximum entropy approximation as may be seen in Fig. 2(c).

Table 2. *Percentage deviations of prior and posterior sample moments, simulated via the inverse Lévy measure algorithm, from the true moments*

	$M=2000,\ \varepsilon=10^{-10}$		$M=1000,\ \varepsilon=10^{-10}$		$M=2000,\ \varepsilon=10^{-5}$	
n	Prior	Posterior	Prior	Posterior	Prior	Posterior
1	0·90	−14·65	4·22	−14·47	−13·04	−20·70
2	1·99	−23·38	15·08	−25·08	−31·08	−35·77
3	2·11	−28·12	43·33	−34·45	−50·63	−46·85
4	0·16	−31·54	120·39	−43·90	−68·26	−54·30
5	5·20	−37·41	307·82	−53·80	−81·55	−58·90
6	14·47	−47·70	687·57	−63·79	−90·09	−62·49

Hence, in our opinion, the maximum entropy approach is to be preferred. Nonetheless, if one is willing to implement a simulation algorithm, a matching procedure such as the one sketched above seems to be necessary before assessing any inference. Once it is guaranteed that sample and theoretical moments match, up to a certain order, one is quite confident that the samples are drawn from the correct distribution.

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Appendix

Proofs

Proof of Lemma 1. Starting from $\int_0^{+\infty} g(t)\,dF(t)$ we have the equalities

$$\int_0^{+\infty} g(t)\,dF(t)=\int_0^{+\infty}\int_0^{t} dg(s)\,dF(t)=\int_0^{+\infty}\int_s^{+\infty} dF(t)\,dg(s)=\int_0^{+\infty} e^{-\xi_s}\,dg(s),$$

having employed Fubini's theorem and the definition of neutral-to-the-right random distribution function. On the other hand, by definition of neutral-to-the-right random distribution function we also have that

$$\int_0^{+\infty} g(t)\,dF(t) = \int_0^{+\infty} t\,dF\{g^{-1}(t)\} = \int_0^{+\infty} \exp\{-\xi_{g^{-1}}(t)\}\,dt,$$

as required. □

Proof of Proposition 1. Set $\zeta_t = \xi_{\gamma^{-1}(t)}$ for any $t \geqslant 0$. Hence $\zeta = \{\zeta_t : t \geqslant 0\}$ is a homogeneous increasing additive process and, by Lemma 1,

$$I(\xi^\gamma) = \int_0^{+\infty} e^{-\zeta_t}\,d\gamma^{-1}(t).$$

Moreover, $E(\zeta_1) = \int_0^{+\infty} \nu(x, +\infty)\,dx := \mu_1 \in (a, +\infty)$. The Strong Law of Large Numbers implies that there exists a set Ω_0 with probability 1 such that, for any $\varepsilon > 0$ and $\omega \in \Omega_0$, for a suitable $t^* = t^*(\varepsilon, \omega)$ one has

$$\frac{\zeta_t(\omega)}{t} \geqslant \mu_1 - \varepsilon,$$

for all $t \geqslant t^*(\varepsilon, \omega)$. One can also determine a $T = T(\varepsilon) > 0$ such that $\gamma^{-1}(t)e^{-at} < c + \varepsilon$ for every $t \geqslant T$. Consequently, if $M = M(\varepsilon, \omega) = \max\{t^*(\varepsilon, \omega), T(\varepsilon)\}$ for any $\omega \in \Omega_0$, the following inequalities hold:

$$\begin{aligned}\int_0^{+\infty} e^{-\zeta_t}\,d\gamma^{-1}(t) &\leqslant \int_{(0,M)} e^{-\zeta_t}\,d\gamma^{-1}(t) + \int_{[M,+\infty)} e^{-(\mu_1-\varepsilon)t}\,d\gamma^{-1}(t)\\ &\leqslant \gamma^{-1}(M) + \int_{[M,+\infty)} e^{-(\mu_1-\varepsilon)t}\,d\gamma^{-1}(t).\end{aligned}$$

In order to prove that the last integral in the right-hand side above is bounded, one fixes $\varepsilon < \mu_1 - a$ and integrates by parts to obtain

$$\begin{aligned}\int_{[M,+\infty)} e^{-(\mu_1-\varepsilon)t}\,d\gamma^{-1}(t) &= \gamma^{-1}(t)e^{-(\mu_1-\varepsilon)t}\Big|_{M(\varepsilon)}^{+\infty} + (\mu_1-\varepsilon)\int_{[M,+\infty)} \gamma^{-1}(t)e^{-(\mu_1-\varepsilon)t}\,dt\\ &\leqslant K_\varepsilon + (\mu_1-\varepsilon)\int_{[M,+\infty)} \gamma^{-1}(t)e^{-at}e^{-(\mu_1-\varepsilon-a)t}\,dt\\ &\leqslant K'_\varepsilon < +\infty.\end{aligned}$$

□

Proof of Proposition 2. For any $n \geqslant 1$, let $0 = t_{0,n} < t_{1,n} < \ldots < t_{k_n,n} < +\infty$ be a grid of points chosen in such a way that

$$\lim_{n\to+\infty} \max_{1\leqslant j\leqslant k_n} (t_{j,n} - t_{j-1,n}) = 0, \quad \lim_{n\to+\infty} t_{k_n,n} = +\infty.$$

Correspondingly, a sequence of functions $(g_n)_{n\geqslant 1}$ is defined by

$$g_n = \sum_{j=1}^{k_n} g_{j,n}\mathbb{I}_{[t_{j-1,n},t_{j,n})} + g_{k_n+1,n}\mathbb{I}_{[t_{k_n,n},+\infty)},$$

where $\mathbb{I}$ denotes the indicator function,

$$g_{j,n} = \min_{t_{j-1,n}\leqslant t\leqslant t_{j,n}} g(t) \quad (j = 1, \ldots, k_n), \quad g_{k_n+1,n} = \inf_{t\geqslant t_{k_n,n}} g(t).$$

It is easy to check that $g_n \uparrow g$ pointwise as n tends to $+\infty$ and, by monotone convergence,

$$\int_0^{+\infty} e^{-\zeta_t}\,g_n(t)\,dt \uparrow \int_0^{+\infty} e^{-\zeta_t}g(t)\,dt. \tag{A1}$$

On the other hand

$$\int_0^{+\infty} e^{-\zeta_t} g_n(t)\,dt = \sum_{j=1}^{k_n} g_{j,n} \int_{t_{j-1,n}}^{t_{j,n}} e^{-\zeta_t}\,dt + g_{k_n+1,n} \int_{t_{k_n,n}}^{+\infty} e^{-\zeta_t}\,dt,$$

and each of the integrals appearing in the right-hand side above has a probability distribution that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$; see Carmona et al. (1997). Hence, the distribution of $\int_0^{+\infty} e^{-\zeta_t} g_n(t)\,dt$ is absolutely continuous. Finally, by virtue of (A1), which in turn implies monotonicity of the corresponding sequence of distribution functions with respect to n, absolute continuity of the probability distribution of $I(\xi^\gamma)$ is established. □

Proof of Proposition 3. First observe that $I(\xi)$ can be obtained as the limit of

$$Z_n := \frac{1}{2^n} \sum_{j=1}^{n2^n} \exp(-\xi_{j2^{-n}}). \tag{A2}$$

For each $n \geqslant 1$, the sum appearing in (A2) is a function of the random vector.

$$\left(\exp(-\xi_{2^{-n}}), \exp(-\xi_{2^{-n}}) \exp\{-(\xi_{2^{-n+1}} - \xi_{2^{-n}})\}, \ldots, \exp\left\{ -\sum_{j=1}^{n2^n} (\xi_{j2^{-n}} - \xi_{(j-1)2^{-n}}) \right\} \right),$$

and, if $\xi_t - \xi_s$ has an absolutely continuous distribution function, for every $0 \leqslant s < t < +\infty$, then the distribution function of Z_n is also absolutely continuous. This, together with monotonicity of Z_n, allows us to conclude that the probability distribution of $I(\xi)$ is absolutely continuous. Hence we are left with proving the absolute continuity of the distribution function of each increment $\xi_t - \xi_s$. According to Theorem 9.7 in Sato (1999), the probability distribution of $\xi_t - \xi_s$ is infinitely divisible with Lévy measure $\nu_{s,t}$. One can therefore use a criterion due to Tucker (1962) in order to check the absolute continuity of the distribution function of $\xi_t - \xi_s$. In our setting, the criterion reduces to

$$\int_{(0,+\infty)} \frac{1}{1+x^2}\,\nu_{s,t}(dx) = +\infty. \qquad \square$$

Proof of Proposition 4. Note that, by Lemma 1,

$$I^n = \left\{ \int_0^{+\infty} t\,dF(t) \right\}^n \leqslant \int_0^{+\infty} t^n\,dF(t) = \int_0^{+\infty} \exp(-\xi_{t^{1/n}})\,dt.$$

Thus, we obtain the desired inequality,

$$E(I^n) \leqslant \int_0^{+\infty} \exp\left\{ -\int_0^{+\infty} (1-e^{-x})\nu_{t^{1/n}}(dx) \right\} dt. \qquad \square$$

Proof of Proposition 5. Following Carmona et al. (1997) we can write

$$\begin{aligned} E(I^n) &= n!E\left[\int_0^{+\infty} \cdots \int_{t_{n-1}}^{+\infty} \prod_{j=1}^{n} \exp\{-(n+1-j)(\xi_{t_j} - \xi_{t_{j-1}})\}\,dt_n \ldots dt_1 \right] \\ &= n! \int_0^{+\infty} \cdots \int_{t_{n-1}}^{+\infty} \prod_{j=1}^{n} \frac{E[\exp\{-(n+1-j)\xi_{t_j}\}]}{E[\exp\{-(n-j)\xi_{t_j}\}]}\,dt_n \ldots dt_1 \end{aligned} \tag{A3}$$

where $t_0 := 0$. However, one can easily see that

$$\frac{E[\exp\{-(n+1-j)\xi_{t_j}\}]}{E[\exp\{-(n-j)\xi_{t_j}\}]} = \exp\left\{ -\int_0^{+\infty} (1-e^{-x})e^{-(n-j)x}\nu_{t_j}(dx) \right\},$$

and (7) is proved.

Finally (8) can be obtained from the last equality in (A3) by exploiting the representation of Φ

as the sum of two independent components, the increasing additive process ξ and the sum of jumps J_i at fixed discontinuities.

Note that an alternative proof can be given by resorting to the technique proposed in Cifarelli & Regazzini (1979). □

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