Probabilistic Sophistication, Second Order Stochastic Dominance, and Uncertainty Aversion*

Simone Cerreia-Vioglio[a] Fabio Maccheroni[a] Massimo Marinacci[a] Luigi Montrucchio[b]

[a]Department of Decision Sciences, Università Bocconi

[b]Collegio Carlo Alberto, Università di Torino

Revision of April 2012

Abstract

We study the interplay of probabilistic sophistication, second order stochastic dominance, and uncertainty aversion, three fundamental notions in choice under uncertainty. In particular, our main result, Theorem 2, characterizes uncertainty averse preferences that are probabilistically sophisticated, as well as uncertainty averse preferences that satisfy second order stochastic dominance. As a byproduct, Proposition 2 highlights a fundamental tension between probabilistic sophistication / second order stochastic dominance and uncertainty aversion in the presence of nontrivial unambiguous events.

JEL classification: D81

Keywords: Model Misspecification, Probabilistic Sophistication, Second Order Stochastic Dominance, Subjective Expected Utility, Unambiguous Events, Uncertainty Aversion.

*We thank for insightful comments and suggestions Peter Klibanoff, Bart Lipman, Marciano Siniscalchi, and Tomasz Strzalecki, as well as, the Coeditor Atsushi Kajii and two anonymous Referees. The financial support of the European Research Council (advanced grant, BRSCDP-TEA) is gratefully acknowledged.

1 Introduction

In recent years model uncertainty and its implications have been the object of extensive study in theoretical and applied economics. At the same time, this concept and related ideas made their way in policy debates where the inability to pin down the stochastic nature of problems is a cogent source of concern.[1]

Despite the fact that, typically, information may not be good enough to pin down a single probabilistic model, in many situations the existence of a reference model is assumed. This reference model is then the natural benchmark for other possible models that a decision maker (DM) considers relevant. In this paper we study a particular, but important, notion of what it means to have a model of reference: probabilistic sophistication. Specifically, in [5] we studied and characterized uncertainty averse preferences in an Anscombe-Aumann framework, that is, preferences that exhibit a negative attitude toward Knightian uncertainty. In this paper, we study the effect of adding a reference model that makes the DM probabilistically sophisticated. Roughly speaking, this means that the DM considers indifferent any two payoff profiles that share the same distribution with respect to the reference model.[2] Under this assumption, uncertainty aversion reduces to fear of misspecification of the reference model. For example, part of the analysis carried out in the robust approach in macroeconomics (see Hansen and Sargent [21]) can be set in the framework of the present paper. In particular, our setting admits a game theoretic interpretation in terms of zero-sum games against nature (see Section 2.2.2) that generalizes the one adopted in the robust approach to capture fear of model misspecification. As [21, p. 137] write "... Each game has a malevolent nature choose a model misspecification to hurt the decision maker ...".

Beyond the multiplier preferences and the constraint preferences used by Hansen and Sargent to capture fear of model misspecification, there are three other important classes of preferences that are at the same time uncertainty averse and probabilistically sophisticated:

- rank dependent preferences (with convex distortion) of Quiggin [32] and Yaari [41];
- divergence preferences of Maccheroni, Marinacci, and Rustichini [28], pioneered by Ben-Tal and Teboulle [2] and [3];
- second order expected utility preferences (with concave weighting) of Neilson [31], recently studied by Strzalecki [39] and Grant, Polak, and Strzalecki [20].

The first class of preferences is one of the most successful in economics and psychology. The latter two classes generalize, in different ways, both expected utility preferences and multiplier preferences. Their analytical tractability has been successfully exploited in financial applications. Moreover, these preferences are known to address Ellsberg's paradoxes in the Anscombe-Aumann setup (see [31] and [39]).

The main result of this paper, Theorem 2, explicitly derives the general representation of uncertainty averse and probabilistically sophisticated preferences. Theorem 2, also shows that the following facts are equivalent for an uncertainty averse DM:

[1] The references of Donald Rumsfeld (February 12, 2002) and Olivier Blanchard (January 29, 2009) to "unknown unknowns" quickly entered the public debate. On the other hand, both the *Stern Report on the Economics of Climate Change* and the *Basel Committee on Banking Supervision* suggested the adoption of decision criteria able to deal with model uncertainty.

[2] Probabilistic sophistication was introduced by Machina and Schmeidler [26]. Papers that study this notion and its implications include Grant [18], Machina and Schmeidler [27], Sarin and Wakker [35], Grant and Polak [19], Chew and Sagi [9] and [10], and Kopylov [23].

- she is probabilistically sophisticated with respect to the reference model q;
- her preferences are consistent with second order stochastic dominance (with respect to q) as far as payoff profiles are concerned;
- in the game against nature interpretation, she acts as if she thinks that nature's loss in choosing a model misspecification increases as the misspecified model gets more and more dispersed with respect to the reference model itself. In other words, the cost of inducing a small misspecification is lower than the cost of inducing an extreme one.

These findings substantially extend earlier results obtained by Maccheroni, Marinacci, and Rustichini [28] for variational preferences, a special class of uncertainty averse preferences. Notice, however, that the uncertainty averse preferences we consider – like variational preferences – are von Neumann-Morgenstern expected utility preferences on lotteries. This restriction is thus inherited by the class of probabilistically sophisticated preferences considered in Theorem 2 and, more in general, in all our results.

As a byproduct of the main result, in Proposition 2, we show that the presence of a nontrivial unambiguous event causes probabilistically sophisticated preferences to collapse to subjective expected utility preferences. This basic tension between probabilistic sophistication and uncertainty aversion (when there exists a nontrivial unambiguous event) was first identified by Marinacci [29] for the special multiple priors case of Gilboa and Schmeidler [17]. These results have been recently and independently extended by Strzalecki [39] to variational preferences.[3] Our findings extend much more generally to the uncertainty averse case including, for example, the smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji [22] and the confidence preferences of Chateauneuf and Faro [8]. This means that fear of model misspecification, as captured by uncertainty averse and probabilistically sophisticated preferences, refers to a pervasive form of ambiguity, since all nontrivial events are considered to be ambiguous despite the presence of a model of reference.

An important difference, relative to the cited [28], [29], [39], and [4], is that most of our results do not rely on the assumption that the reference model is adequate,[4] but they can be formulated for any reference model. This comes at the cost of imposing some stronger degree of probabilistic risk aversion in the form of second order stochastic dominance, as discussed in Section 3.

Mathematically, our results build on the theory of rearrangement invariant Banach spaces, first studied in the seminal paper of Luxemburg [25]. More precisely, Theorem 2 depends on a dual characterization of quasiconcave and rearrangement invariant functionals defined over the normed space of simple measurable functions. This characterization shares some of the techniques of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [7], where quasiconvex and rearrangement invariant functionals defined over L^∞ are studied and characterized. The present setting makes the derivation significantly more delicate.[5]

However, the analogies between the results in this paper and [7] are only at the formal level. The problems studied are conceptually different and so are the interpretation and the implications of the

[3]A partial result for preferences that are not uncertainty averse can be found in Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [4].

[4]That is, either nonatomic (on an infinite state space) or uniform (on a finite state space).

[5]The starting point here is a preference defined over acts on a measurable space rather than a functional over random variables on a probability space. Moreover, even after passing to the functional representation of preferences by means of a functional on the space $B_0(S,\Sigma)$, we still cannot directly use the arguments for $L^\infty(S,\Sigma,q)$. In fact, although $B_0(S,\Sigma)$ is dense in $L^\infty(S,\Sigma,q)$, the extension is not necessarily unique and closure operations do not obviously preserve the properties we are after. For this reason, we cannot rely on the results of [7], but we need to derive them *ex novo*.

results. The present paper studies the behavioral foundations of model misspecification and relates them to the representation of uncertainty averse preferences. On the other hand, [7] studies capital requirements (financial risk measures), particularly the ones that are law invariant. In that setting diversification, rather than fear of model misspecification, is the leading principle. The law invariance assumption is made for computational reasons since it allows to dispense with the description of a state space for the entire financial system. Finally, here the interpretation of the derived representation can be stated in terms either of games against nature or of separation between risk and uncertainty aversion. Instead, the representation in [7] captures the idea of scenario-dependent capital requirements.

2 Preliminaries

2.1 Mathematical Setup

We consider an Anscombe-Aumann setup [1]. Let S be a state space endowed with a σ-algebra Σ of events, and X a convex set of consequences (or outcomes). We denote by $\mathcal{F}$ the set of all simple acts $f : S \to X$, that is, the set of all Σ-measurable maps that take finitely many values. Given $A \in \Sigma$ and $f, g \in \mathcal{F}$, we denote by gAf the simple act that yields $g(s)$ if $s \in A$ and $f(s)$ if $s \notin A$.

$B_0 = B_0(S, \Sigma)$ is the set of simple Σ-measurable functions, $\varphi : S \to \mathbb{R}$, endowed with the supnorm. Denote by Δ the set of all finitely additive probabilities on Σ, endowed with the weak* topology. The subset of Δ consisting of all countably additive probabilities on Σ is denoted by Δ^σ. Given $q \in \Delta^\sigma$, denote by $\Delta^\sigma(q) = \{p \in \Delta^\sigma : p \ll q\}$ the set of all countably additive probabilities on Σ that are absolutely continuous with respect to (wrt, for short) q. Finally, when $q \in \Delta^\sigma$, we say that (S, Σ, q) is *adequate* if either q is nonatomic or S is finite and q is uniform.

Endow $\mathbb{R} \times \Delta$ with the product topology and define $\mathcal{L}(\mathbb{R}\times\Delta)$ as the class of functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ such that:

(i) $G(\cdot, p)$ is an increasing function for all $p \in \Delta$;

(ii) G is quasiconvex and lower semicontinuous;

(iii) $\min_{p\in\Delta} G(t, p) = t$ for all $t \in \mathbb{R}$.

A function $G \in \mathcal{L}(\mathbb{R} \times \Delta)$ is *linearly continuous* if the function $I : B_0 \to \mathbb{R}$ defined by

$$I(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) \tag{1}$$

is continuous. For example, [5] shows that $G \in \mathcal{L}(\mathbb{R}\times\Delta)$ is linearly continuous if $G(\cdot, p)$ is upper semicontinuous on $\mathbb{R}$ for each $p \in \Delta$.

2.2 Decision Theoretic Setup

2.2.1 Uncertainty Averse Preferences

We consider a binary relation $\succsim$ on $\mathcal{F}$ that satisfies the following classic axioms:

A 1 (Weak Order) *The binary relation $\succsim$ is nontrivial, complete, and transitive.*

A 2 (Monotonicity) *If* $f, g \in \mathcal{F}$ *and* $f(s) \succsim g(s)$ *for all* $s \in S$*, then* $f \succsim g$*.*

A 3 (Uncertainty Aversion) *If* $f, g \in \mathcal{F}$ *and* $\alpha \in (0,1)$*,* $f \sim g$ *implies* $\alpha f + (1-\alpha) g \succsim f$*.*

Following [5], a preference relation $\succsim$ that satisfies axioms A1-A3 is called *uncertainty averse.* As argued at length in [5], uncertainty averse preferences form a fundamental class of rational preferences on $\mathcal{F}$ that exhibit a negative attitude toward uncertainty in an Anscombe-Aumann setting. Notice however that A2 excludes state dependence of utility on outcomes. State dependence is very natural for "small" state spaces and it obviously does not exclude A3. Nevertheless, state dependence makes it difficult to identify uniquely the DM's probabilistic beliefs even in the expected utility case. For this reason, the assumption of existence of a reference model, which is central in this paper, becomes evanescent in a state dependent setup.

To derive a representation for uncertainty averse preferences we need some further mild axioms. The following axiom is peculiar to the Anscombe-Aumann setting and is a standard independence postulate on constant acts, that is, on acts that only involve risk and no state uncertainty.[6] An important consequence for our results is that this assumption together with the subsequent A5 implies that our DM, on constant acts, is a *von Neumann-Morgenstern expected utility maximizer.* This further restricts the overlap we study between uncertainty averse preferences and probabilistically sophisticated ones.

A 4 (Risk Independence) *If* $x, y, z \in X$ *and* $\alpha \in (0,1)$*,* $x \sim y$ *implies* $\alpha x + (1-\alpha) z \sim \alpha y + (1-\alpha) z$*.*

The next axioms are technical conditions that simplify the derivation and make the representation more tractable.

A 5 (Continuity) *If* $f, g, h \in \mathcal{F}$*, the sets* $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ *and* $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ *are closed.*

A 6 (Unboundedness) *There are* $x, y \in X$ *such that, for each* $\alpha \in (0,1)$*, there exist* $z, z' \in X$ *such that* $\alpha z + (1-\alpha) y \succ x \succ y \succ \alpha z' + (1-\alpha) x$*.*

A 7 (Monotone Continuity) *If* $f, g \in \mathcal{F}$*,* $x \in X$*,* $\{E_n\} \subseteq \Sigma$ *with* $E_n \downarrow \emptyset$*, then* $f \succ g$ *implies that there exists* $n_0 \in \mathbb{N}$ *such that* $x E_{n_0} f \succ g$*.*

If $\succsim$ satisfies axioms A1, A2, and A5, then each act $f \in \mathcal{F}$ has a certainty equivalent $x_f \in X$; i.e., $f \sim x_f$. Certainty equivalents play an important role in the following representation result for uncertainty averse preferences, proved in [5]. Here $\mathcal{U}(X)$ is the class of affine functions $u : X \to \mathbb{R}$.

Theorem 1 *Let* $\succsim$ *be a binary relation on* $\mathcal{F}$*. Then, the following conditions are equivalent:*

(i) $\succsim$ *satisfies axioms A1-A7;*

(ii) there exist $u \in \mathcal{U}(X)$*, with* $u(X) = \mathbb{R}$*, and* $G \in \mathcal{L}(\mathbb{R} \times \Delta)$ *linearly continuous with* $\operatorname{dom} G \subseteq \mathbb{R} \times \Delta^\sigma$*,*[7] *such that, for each* f *and* g *in* $\mathcal{F}$*,*

$$f \succsim g \Longleftrightarrow \min_{p \in \Delta} G\left(\int u(f)\, dp, p\right) \geq \min_{p \in \Delta} G\left(\int u(g)\, dp, p\right). \tag{2}$$

[6] This assumption is very common among nonexpected utility models in an Anscombe-Aumann framework (e.g., [8], [16], [17], [20], [22], [28], [31], [37], [38], [39]).

[7] Recall that $\operatorname{dom} G = \{(t,p) \in \mathbb{R} \times \Delta : G(t,p) < \infty\}$.

The function u is cardinally unique and, given u, the unique $G \in \mathcal{L}(\mathbb{R}\times\Delta)$ that satisfies (2) is

$$G(t,p) = \sup_{f\in\mathcal{F}} \left\{ u(x_f) : \int u(f)\, dp \leq t \right\}. \tag{3}$$

Observe that the technical axioms A5-A7 translate in the representation as follows: A5 guarantees the linear continuity of G, A6 corresponds to $u(X) = \mathbb{R}$, and A7 implies that $\operatorname{dom} G \subseteq \mathbb{R}\times\Delta^{\sigma}$.[8] Theorem 1 motivates the following definition.

Definition 1 *A pair $(u,G) \in \mathcal{U}(X) \times \mathcal{L}(\mathbb{R}\times\Delta)$ that represents a binary relation $\succsim$ in the sense of point (ii) of Theorem 1 is called an* uncertainty averse representation *of $\succsim$.*

Behaviorally, by Theorem 1, a binary relation admits an uncertainty averse representation if and only if it satisfies axioms A1-A7. Given an uncertainty averse representation (u,G), u describes preferences over consequences. In the original Anscombe-Aumann setup, these are objective lotteries, thus u captures risk aversion as well. On the other hand, the function G can be interpreted as an index of uncertainty aversion (see also [5]). Consider two preferences $\succsim_1$ and $\succsim_2$ with uncertainty averse representations (u_1, G_1) and (u_2, G_2). Preference $\succsim_1$ is more uncertainty averse than preference $\succsim_2$, in the sense of Ghirardato and Marinacci [15],[9] if and only if u_1 is cardinally equivalent to u_2 and, normalizing $u_1 = u_2$, we have $G_1 \leq G_2$.

In [5], we provide a complete characterization of the specific form taken by the index G for several important classes of uncertainty averse preferences. In particular, for variational preferences we show that

$$G(t,p) = t + c(p)$$

where c is a convex, grounded, and lower semicontinuous cost function. On the other hand, for second order expected utility preferences, we show that

$$G(t,p) = t + I_t(p||q)$$

where, for each t, $I_t(p||q)$ is a statistical distance function.[10]

2.2.2 Games against Nature

As anticipated in the introduction and discussed in detail in [5], our setting admits a game against nature interpretation where the DM views herself as playing a zero-sum game against (a malevolent) nature. In this case, f and p become, respectively, the strategies of the DM and of nature.

The new view on this old interpretation (see, e.g., Gilboa and Schmeidler [17]) is the fact that the "Waldean" solution to a decision problem under uncertainty perfectly describes the general uncertainty averse preferences. The interpretation goes as follows: for every uncertainty averse preference there exists a (suitably unique) game against nature

$$\Gamma(f,p) = G\left(\int u(f)\, dp, p\right) \qquad \forall (f,p) \in \mathcal{F}\times\Delta$$

[8] [5, Theorem 3] provides a more general representation result that does not rely on A6 and A7.

[9] They say that $\succsim_1$ is *more uncertainty averse than* $\succsim_2$ if, and only if, for all $f \in \mathcal{F}$ and $x \in X$, $f \succsim_1 x$ implies $f \succsim_2 x$. The intuition behind this is the following, $\succsim_1$ is more uncertainty averse than $\succsim_2$ if, whenever $\succsim_1$ is willing to take the chance of choosing an uncertain act f over a constant outcome x, then the same is true for $\succsim_2$.

[10] See Section 5 of [5] for details and other specifications. Here we privileged variational and second order expected utility classes of preferences since they contain the motivating examples appearing in the introduction. In fact, multiplier preferences, constraint preferences, rank dependent preferences, and divergence preferences are all specifications of variational preferences. Notice, however, that second order expected utility preferences are not, in general, variational.

such that the DM behaves as if she were playing this game against nature, and conversely for every such game there is a unique corresponding uncertainty averse preference. In this perspective, the reason why DMs prefer to randomize among indifferent acts (Axiom A3) is because this makes more difficult for nature (which has no control on the randomizing device) to best respond.

The structure of the game clearly reflects the existence of two sources of uncertainty in the Anscombe-Aumann model (see also Strzalecki [38]):

- $\int u(f)\,dp$ captures the objective risk preferences of the DM (once nature has chosen p, the expected utility of f is what matters to the DM); notice that, for fixed p, $G(\cdot, p)$ is an increasing transformation of the expected utility. This is the consequence of assumption A2.
- G captures the presence of subjective uncertainty and has the standard properties of convexity and continuity of a zero-sum game.
- The minimum – that is, the adoption of a maxmin strategy – captures the aversion of the DM to such uncertainty. This follows from assumption A3.

3 Sophistication, Dominance, and Uncertainty Aversion

In this section we state the paper's main result, Theorem 2, which characterizes uncertainty averse preferences that are probabilistically sophisticated.

Given a reference probability $q \in \Delta^\sigma$ and a preference $\succsim$ on $\mathcal{F}$:

(i) $\succsim$ is *probabilistically sophisticated* (wrt q) if, given any $f, g \in \mathcal{F}$,

$$q(\{s \in S : f(s) = x\}) = q(\{s \in S : g(s) = x\}) \ \ \forall x \in X \Longrightarrow f \sim g; \tag{4}$$

(ii) $\succsim$ satisfies *first order stochastic dominance* (wrt q) if, given any $f, g \in \mathcal{F}$,

$$q(\{s \in S : f(s) \succ x\}) \geq q(\{s \in S : g(s) \succ x\}) \ \ \forall x \in X \Longrightarrow f \succsim g, \tag{5}$$

or, equivalently,

$$\int \phi(u(f))\,dq \geq \int \phi(u(g))\,dq \quad \forall \phi \in \Phi_{mi} \Longrightarrow f \succsim g, \tag{6}$$

where Φ_{mi} is the set of all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$;

(iii) $\succsim$ satisfies *second order stochastic dominance* (wrt q) if, given any $f, g \in \mathcal{F}$,

$$\int \phi(u(f))\,dq \geq \int \phi(u(g))\,dq \quad \forall \phi \in \Phi_{icv} \Longrightarrow f \succsim g, \tag{7}$$

where Φ_{icv} is the set of all concave and increasing functions $\phi : \mathbb{R} \to \mathbb{R}$.[11]

These dominance notions can be interpreted in terms of the classic notions of stochastic dominance on lotteries. In fact, (6) is equivalent to require that the lottery induced by $u \circ f$ under q first order stochastically dominates the one induced by $u \circ g$, while (7) is equivalent to require that the first lottery second order stochastically dominates the second one.

[11] Here we assume that $\succsim$ restricted to X is represented by the affine utility function u. Moreover, since $\Phi_{icv} \subseteq \Phi_{mi}$, second implies first order stochatistic dominance. Finally, it can be shown that in (6) and (7) it is actually enough to consider strictly increasing functions.

When a reference model q is given, first order stochastic dominance is a very natural monotonicity condition. It says that, if for every outcome x the probability that act f outperforms x is greater than the probability that act g outperforms x, then f is preferred to g. While Rothschild and Stiglitz [33] and [34] have shown that one of the most compelling ways to formalize the statement "the payoff induced by f is less risky than the payoff induced by g", consists in requiring the first payoff second order stochastically dominates the second.

In terms of foundations for these requirements, one could consider Savage DMs in our Anscombe-Aumann setup and regard (6) and (7) as requirements of consistency with unanimous judgements; see Proposition 3 in the Appendix for details.

Our results use the convex order, a classic stochastic order. Specifically, the *convex order* $\succsim_{cx}$ on $L^1(q) = L^1(S, \Sigma, q)$ is defined by

$$\varphi \succsim_{cx} \psi \Longleftrightarrow \int \ell(\varphi)\, dq \geq \int \ell(\psi)\, dq \quad \forall \ell \in \Phi_{cx}, \tag{8}$$

where Φ_{cx} is the set of all convex functions $\phi : \mathbb{R} \to \mathbb{R}$. Notice that this order can be also defined over $\Delta^\sigma(q)$ by

$$p \succsim_{cx} p' \Longleftrightarrow \frac{dp}{dq} \succsim_{cx} \frac{dp'}{dq},$$

where dp/dq and dp'/dq in $L^1(q)$ are the Radon-Nikodym derivatives of p and p', respectively. In this case, the symmetric part of $\succsim_{cx}$ coincides with the identical distribution of the densities wrt q. Intuitively, see Rothschild and Stiglitz [33], $p \succsim_{cx} p'$ means that the "masses" $dp(s)$ are more scattered than the masses $dp'(s)$, with respect to $dq(s)$. In particular, $p \sim_{cx} p'$ means that these masses are symmetrically sparse.

Example 1 If $S = \{s_1, s_2, ..., s_N\}$ and q is uniform, then $p \succsim_{cx} p'$ if and only if

$$\begin{aligned} p_{[1]} &\geq p'_{[1]} \\ p_{[1]} + p_{[2]} &\geq p'_{[1]} + p'_{[2]} \\ &\cdots \\ p_{[1]} + p_{[2]} + \dots + p_{[N]-1} &\geq p'_{[1]} + p'_{[2]} + \dots + p'_{[N]-1} \end{aligned}$$

where $p_{[i]}$ (resp., $p'_{[i]}$) is the probability of the i-th most likely state for p (resp., p'). This simply means that p is more dispersed than p'. Clearly, the Dirac probabilities are the maximally dispersed probabilities with respect to the uniform q, while the minimally dispersed one is q itself. Moreover, $p \sim_{cx} p'$ if and only if there is a permutation $\pi : S \to S$ such that $p' = p \circ \pi$.[12] ▲

A function $T : \Delta \to (-\infty, \infty]$, with $\operatorname{dom} T \subseteq \Delta^\sigma(q)$, is

(i) *rearrangement invariant* (wrt q) if $p \sim_{cx} p' \Longrightarrow T(p) = T(p')$;

(ii) *Schur convex* (wrt q) if $p \succsim_{cx} p' \Longrightarrow T(p) \geq T(p')$.

A final piece of notation: given $\varphi \in L^1(q)$, its inverse distribution function $F_\varphi^{-1} : [0,1] \to [-\infty, \infty]$ is defined by $F_\varphi^{-1}(\omega) = \inf\{x \in \mathbb{R} : q(\{s \in S : \varphi(s) \leq x\}) \geq \omega\}$ for all $\omega \in [0,1]$. We are ready to state our main result.

[12] We refer the interested reader to Marshall and Olkin [30] for a complete treatment of the convex order (also known as majorization).

Theorem 2 *Let $\succsim$ be a binary relation with uncertainty averse representation (u, G). Then, the following conditions are equivalent (wrt q):*

(i) $\succsim$ satisfies second order stochastic dominance;

(ii) $G(t, \cdot)$ is Schur convex on Δ for all $t \in \mathbb{R}$.

In this case,

$$\min_{p\in\Delta} G\left(\int u(f)\, dp, p\right) = \min_{p\in\Delta^{\sigma}(q)} G\left(\int_0^1 F_{u\circ f}^{-1}(\omega)\, F_{\frac{dp}{dq}}^{-1}(1-\omega)\, d\omega, p\right) \quad \forall f \in \mathcal{F} \tag{9}$$

and

$$G(t,p) = \begin{cases} \sup\left\{u(x_f) : \int_0^1 F_{u\circ f}^{-1}(\omega)\, F_{\frac{dp}{dq}}^{-1}(1-\omega)\, d\omega < t\right\} & \text{if } (t,p) \in \mathbb{R}\times\Delta^{\sigma}(q) \\ \infty & \text{else.} \end{cases} \tag{10}$$

Moreover, if (S, Σ, q) is adequate, then (i) and (ii) are equivalent to:

(iii) $\succsim$ satisfies first order stochastic dominance;

(iv) $\succsim$ is probabilistically sophisticated;

(v) $G(t, \cdot)$ is rearrangement invariant on Δ for all $t \in \mathbb{R}$.

To fix ideas, consider the important case where (S, Σ, q) is adequate. This case is arguably the most relevant in terms of foundations of subjective (and classical probability). In fact, starting with the seminal works of de Finetti [13] and Savage [36] on the numerical representation of subjective qualitative probabilities, the nonatomic case arises naturally from the assumption of equidivisibility of the space (Savage's P6). This assumption was kept in the more robust definition of subjective probability of Machina and Schmeidler [26], where probabilistic sophistication was introduced. On the other hand, the finite uniform case corresponds to the definition of classical probability stated by Laplace in 1814. Still nowadays, the adequate case lies at the heart of the behavioral foundations of probability (see, e.g., Chew and Sagi [9] on probabilistic sophistication).[13]

Theorem 2 has a few important features in the adequate case. First, it substantiates the interpretation of probabilistically sophisticated uncertainty averse preferences as describing fear of model misspecification. Arguably, the Subjective Expected Utility (SEU) preference $\succsim_{u,q}$, represented by $\int u(f)\, dq$, describes a DM whose reference model is q and whose behavior reveals full trust in it. By Theorem 2, it follows that any uncertainty averse $\succsim$ whose reference model is q is more uncertainty averse than $\succsim_{u,q}$, "because" she does not fully trust q. In fact, formally, Theorem 2 implies the following:

Corollary 1 *Let $\succsim$ be a binary relation with uncertainty averse representation (u, G) and let q be adequate. If $\succsim$ is probabilistically sophisticated (wrt q), then $\succsim$ is more uncertainty averse than $\succsim_{u,q}$.*

This consistency check is not a tautology and indeed it relates the absolute definition of uncertainty aversion of Schmeidler (Axiom A3) with the comparative foundation of Ghirardato and Marinacci [15].[14]

[13] They also show that if there exists an adequate q such that the decision maker is probabilistically sophisticated with respect to it, such q is unique (see also Kopylov [23]).

[14] In the general case, when q is not necessarily adequate, it sufficies to replace probabilistic sophistication with second order stochastic dominance to obtain the same implication.

Second, Theorem 2 characterizes the class of preferences that belong to the intersection of the classes of uncertainty averse preferences and probabilistically sophisticated preferences, and it explicitly provides computational formulas for this case.[15] The intersection between the two classes coincides with the family of preferences which admit an uncertainty averse representation (u, G) where G is Schur convex. In a game theoretic perspective, this amounts to say that the probabilistically sophisticated DM depicts her fictitious opponent as having q as a default action. Deviations (misspecified models p) from this action have a cost which increases as the dispersion (of the misspecified model p) with respect to q increases.

Third, Theorem 2 establishes the equivalence of probabilistic sophistication, first order stochastic dominance, and second order stochastic dominance under uncertainty aversion (in the adequate case). The relations among these properties and their relevance in portfolio selection were first investigated by Dekel [14] in a setup of choice under risk. He also shows a "converse implication", that is, under what conditions second order stochastic dominance implies uncertainty aversion. Loosely speaking, in our Anscombe-Aumann setup, this amounts to say that preference for *ex-post* randomization (Axiom A3) is implied by second order stochastic dominance and preference for *ex-ante* randomization. One way to express preference for ex-ante randomization in our setting is

$$f \sim g \text{ implies } fAg \succsim f$$

provided $f, g \in \mathcal{F}$ and $A \in \Sigma$ are independent with respect to q. Recall that Axiom A3 requires

$$f \sim g \text{ implies } \alpha f + (1 - \alpha) g \succsim f$$

provided $f, g \in \mathcal{F}$ and $\alpha \in (0, 1)$, and notice that independence ex-post is embedded in the convex structure of the consequence space.

Taken together, all these features of Theorem 2 establish for the adequate case a complete characterization of uncertainty averse preferences that are probabilistically sophisticated. Moreover, the equivalence of (i) and (ii) and the computational formulas (9) and (10) constitute the first treatment in the literature of the non-adequate case. In fact, the results of [29] and [28] for multiple priors and variational preferences only apply to the adequate case.

4 Unambiguous Events

Marinacci [29] pointed out a possible tension between probabilistic sophistication, which is based on a single reference probability, and the multiple priors representation, which instead relies on several possible probabilities, in the presence of a nontrivial unambiguous event. Thanks to Theorem 2, in this section we show that this possible tension holds, much more generally, among probabilistic sophistication and uncertainty averse representations.

In order to do so, we first extend to our setting the notion of nontrivial unambiguous event of [29]. Consider a multiple priors representation à la Gilboa and Schmeidler [17]

$$V(f) = \min_{p \in C} \int u(f)\, dp \quad \forall f \in \mathcal{F}, \tag{11}$$

[15] Notably, formulas (9) and (10) dispense with the specification of a state space since their inputs are distribution functions alone.

where C is a weak* closed set of Δ. An event is nontrivial and unambiguous if and only if $0 < p(A) = p'(A) < 1$ for all $p, p' \in C$. To generalize this notion to the present setting, consider the revealed unambiguous preference of Ghirardato, Maccheroni, and Marinacci [16], defined as

$$f \succsim^* g \Longleftrightarrow \lambda f + (1-\lambda) h \succsim \lambda g + (1-\lambda) h \quad \forall h \in \mathcal{F}, \forall \lambda \in (0,1]. \tag{12}$$

In [5, Theorem 10] we show that for preferences with an uncertainty averse representation it holds

$$f \succsim^* g \Longleftrightarrow \int u(f)\, dp \geq \int u(g)\, dp \quad \forall p \in \operatorname{dom}_\Delta G, \tag{13}$$

where $\operatorname{dom}_\Delta G = \{p \in \Delta : G(t,p) < \infty \text{ for some } t \in \mathbb{R}\}$. This motivates the following definition.

Definition 2 *Let $\succsim$ be an uncertainty averse preference. An event A in Σ is* nontrivial and unambiguous *if there exist $x, y, z \in X$ such that $x \succ z \succ y$ and $xAy \sim^* z$.*

In other words, an event A is unambiguous if the act xAy is unambiguously indifferent to a constant act z. Clearly, constant acts are unambiguous since their outcomes are independent of the underlying state space realizations. Moreover, the condition $x \succ z \succ y$ rules out the possibility that A is "unambiguous" because either A or its complement are deemed null with respect to $\succsim$.

Proposition 1 *Let $\succsim$ be a binary relation with uncertainty averse representation (u, G). Then, the following properties are equivalent:*

(i) A is nontrivial and unambiguous;

(ii) for each $x, y \in X$ such that $x \succ y$ there exists $z \in X$ such that $x \succ z \succ y$ and $xAy \sim^* z$;

(iii) $0 < p(A) = p'(A) < 1$ for all $p, p' \in \operatorname{dom}_\Delta G$.

Remark Strzalecki [39] implicitly provides different notions of unambiguous events for unbounded variational preferences. By (13) and Proposition 1-(iii), it follows that our notion gives a behavioral foundation and a generalization of the notion contained in his Assumption 2.

We now state the main result of this section, which, as discussed in the Introduction, can be viewed as a stark implication of Theorem 2.

Proposition 2 *Let $\succsim$ be a binary relation with uncertainty averse representation (u, G). If there exists a nontrivial unambiguous event, then:*

(i) $\succsim$ satisfies second order stochastic dominance (wrt q) if and only if $\succsim$ is the SEU preference $\succsim_{u,q}$.

(ii) $\succsim$ is probabilistically sophisticated (wrt q) if and only if $\succsim$ is the SEU preference $\succsim_{u,q}$, provided (S, Σ, q) is adequate.

Point (ii) generalizes the main results of [29], for multiple priors preferences, and [39], for variational preferences, to the present general setting. Point (i) shows that, even when (S, Σ, q) is not adequate, second order stochastic dominance and uncertainty aversion can be both satisfied only by a SEU preference as soon as there exists at least one nontrivial unambiguous event.

By Theorem 2, probabilistic sophistication and second order stochastic dominance are equivalent properties in the adequate case. Thus, point (i) shows that the tension originally identified by [29] among probabilistic sophistication and multiple priors, in the presence of a nontrivial unambiguous event, holds much more generally among second order stochastic dominance and uncertainty aversion. Since second order stochastic dominance is a widely used property in applications, this is an important novel insight of Proposition 2. Along with its substantially greater generality, this insight is what makes Proposition 2 a significant advance relative to the analysis of [29] and [39]. [16]

We reiterate that Proposition 2 should not lead to think that the class of uncertainty averse preferences that satisfy second order stochastic dominance is either small or irrelevant. On the contrary, this is the class where two of the most economically relevant notions of aversion to uncertainty, *convexity* (Debreu [12] and Schmeidler [37]) and *risk aversion* (Rothschild and Stiglitz [33]) coexist. This class contains some important preferences used in economic and financial applications such as the rank dependent preferences of Quiggin [32] and Yaari [41], the constraint, and the multiplier preferences of Hansen and Sargent [21]. Therefore, the message of Proposition 2 is that these preferences capture pervasive uncertainty about probabilistic scenarios, since no event is considered nontrivial and unambiguous at the same time.

A Proofs and Related Analysis

A.1 Savage Expected Utility in an Anscombe-Aumann Setting

Here we discuss a representation result that shows what the classic axioms of Savage [36] imply in the present Anscombe-Aumann setting, thus providing a behavioral foundation for the definitions of (first and) second order stochastic dominance used in the main text. It is an essentially known result, studied for example by Neilson [31] and, more recently, by Strzalecki [38]. For completeness, we report a proof since we could not find it in the literature.

Proposition 3 *Let $\succsim$ be a binary relation on $\mathcal{F}$. Then, the following conditions are equivalent:*

(i) $\succsim$ satisfies Savage's axioms P1-P6 and axioms A4-A5;

(ii) there exist a nonatomic probability measure q, a nonconstant affine $u : X \to \mathbb{R}$, and a strictly increasing and continuous function $\phi : u(X) \to \mathbb{R}$ such that, for each f and g in $\mathcal{F}$,

$$f \succsim g \iff \int \phi(u(f))\, dq \geq \int \phi(u(g))\, dq. \tag{14}$$

The probability q is unique, u is cardinally unique, and ϕ is cardinally unique given u.[17] *Moreover, ϕ is concave if and only if $\succsim$ satisfies A3, and $q \in \Delta^\sigma$ if and only if $\succsim$ satisfies A7.*

Proof. (i) implies (ii). By Savage's Expected Utility Theorem,[18] there are a nonconstant $v : X \to \mathbb{R}$ and a nonatomic probability q on Σ such that $V : \mathcal{F} \to \mathbb{R}$ given by $V(f) = \int v(f)\, dq$ represents $\succsim$. In particular, $\succsim$ satisfies A1 and A2, which together with A4 and A5, guarantee that:

- There exists a nonconstant affine function $u : X \to \mathbb{R}$ and a function $I : B_0(u(X)) \to \mathbb{R}$ normalized, monotone, and continuous such that $f \succsim g \iff I(u(f)) \geq I(u(g))$. Moreover, u is cardinally unique, and, given u, there is a unique normalized $I : B_0(u(X)) \to \mathbb{R}$ that represents $\succsim$ in the above sense (see [5, Lemma 57]).
- For each $f \in \mathcal{F}$ there exists $x_f \in X$ such that $f \sim x_f$.

[16] It is still an open question whether the identified tension between second order stochastic dominance (resp., probabilistic sophistication) and uncertainty aversion persists when preferences are nonexpected utility over lotteries.

[17] See (17) below.

[18] Notice that we are assuming that Σ is a σ-algebra (see, e.g., Wakker [40, Observation 2]).

Since u is affine, $u(X) = K$ is an interval. Since both u and v represent $\succsim$ on X, there exists a strictly increasing $\phi : u(X) \to \mathbb{R}$ such that $v = \phi \circ u$. It only remains to show that ϕ is continuous. For all $\psi \in B_0(K)$, let $f \in \mathcal{F}$ and x_f in X be such that $\psi = u(f)$ and $x_f \sim f$. Then

$$\int \phi(\psi)\, dq = \int \phi(u(f))\, dq = V(f) = v(x_f) = \phi(u(x_f)),$$

and so $\int \phi(\psi)\, dq \in \operatorname{Im}\phi$.[19] Now, for each $t_1 = \phi(k_1), t_2 = \phi(k_2) \in \operatorname{Im}\phi$ and $\alpha \in (0,1)$, take $A \in \Sigma$ such that $q(A) = \alpha$ (this is possible since q is nonatomic). Then

$$\alpha t_1 + (1-\alpha)t_2 = \alpha\phi(k_1) + (1-\alpha)\phi(k_2) = \int \phi(k_1 1_A + k_2 1_{A^c})\, dq \in \operatorname{Im}\phi.$$

Therefore, $\operatorname{Im}\phi$ is convex and ϕ is continuous (ϕ is increasing).

(ii) implies (i). Clearly, (14) is equivalent to $f \succsim g \iff \int \phi(u(f))\, dq \geq \int \phi(u(g))\, dq$. Thus, P1-P6 hold. Moreover, A4 follows from the fact that $V : X \to \mathbb{R}$ given by $V(x) = \phi(u(x))$ represents $\succsim$ on X, with ϕ is strictly increasing and u affine.

It remains to show that $\psi \mapsto \int \phi(\psi)\, dq$ is continuous on $B_0(K)$, which in turn implies A5. Let ψ_n be a sequence in $B_0(K)$ that supnorm converges to $\psi \in B_0(K)$. For each $\delta > 0$, eventually

$$|\psi_n(s) - \psi(s)| \leq \delta \qquad \forall s \in S. \tag{15}$$

Moreover, ψ_n is supnorm bounded and so it is easy to check that there are $a, b \in \mathbb{R}$ such that $[a,b] \subseteq K$ and, eventually, $\psi_n, \psi \in B_0([a,b])$ for all $n \geq 1$. But, being continuous, ϕ is also uniformly continuous on $[a,b]$. Thus, for all $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$t, r \in [a,b] \text{ and } |t - r| \leq \delta_\varepsilon \Longrightarrow |\phi(t) - \phi(r)| \leq \varepsilon$$

Then, eventually $|\psi_n(s) - \psi(s)| \leq \delta_\varepsilon$ for all $s \in S$, and $|\phi(\psi_n(s)) - \phi(\psi(s))| \leq \varepsilon$ for all $s \in S$. That is, $B_0(\phi(K)) \ni \phi(\psi_n) \to \phi(\psi)$ and $\int \phi(\psi_n)\, dq \to \int \phi(\psi)\, dq$, as wanted. Let $f, g, h \in \mathcal{F}$, and $\{\alpha_n\} \in [0,1]$ be such that $\alpha_n f + (1-\alpha_n) g \succsim h$ for all $n \geq 1$, and assume $\alpha_n \to \alpha$. Then

$$\int \phi(u(\alpha_n f + (1-\alpha_n) g))\, dq = V(\alpha_n f + (1-\alpha_n) g) \geq V(h) \qquad \forall n \geq 1. \tag{16}$$

But, $u(\alpha_n f + (1-\alpha_n) g) = \alpha_n u(f) + (1-\alpha_n) u(g) = u(g) + \alpha_n (u(f) - u(g)) \to u(\alpha f + (1-\alpha) g)$ in the supnorm. Thus, passing to the limits in (16),

$$V(\alpha f + (1-\alpha) g) = \int \phi(u(\alpha f + (1-\alpha) g))\, dq \geq V(h),$$

which immediately delivers A5.

As to uniqueness, we show that $(\bar{q}, \bar{u}, \bar{\phi})$ represents $\succsim$ in the sense of (14) if and only if $\bar{q} = q$ and there exist $\alpha, \beta, \eta, \kappa \in \mathbb{R}$ with $\alpha, \eta > 0$ such that for all $x \in X$ and $t \in \bar{u}(X)$:

$$\bar{u}(x) = \frac{u(x) - \kappa}{\eta} \qquad \text{and} \qquad \bar{\phi}(t) = \alpha(\phi(\eta t + \kappa)) + \beta. \tag{17}$$

By Savage's Expected Utility Theorem, $\bar{q} = q$ and there are $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\bar{\phi} \circ \bar{u} = \alpha(\phi \circ u) + \beta$. By the von Neumann-Morgenstern's Expected Utility Theorem, there are $\eta > 0$ and $\kappa \in \mathbb{R}$ such that $\bar{u} = \eta^{-1}(u - \kappa)$. Therefore, $\bar{\phi}(\bar{u}(x)) = \alpha(\phi(u(x))) + \beta = \alpha(\phi(\eta \bar{u}(x) + \kappa)) + \beta$ for all $x \in X$, and $\bar{\phi}(t) = \alpha(\phi(\eta t + \kappa)) + \beta$ for all $t \in \bar{u}(X)$. The converse is easily checked.

[19] In particular, $\phi^{-1}\left(\int \phi(\psi)\, dq\right) = u(x_f) = I(u(x_f)) = I(u(f)) = I(\psi)$.

Next we show that A3 implies concavity of ϕ and A7 implies $q \in \Delta^\sigma$, the converse implications being trivial. Assume per contra that A3 holds and that ϕ is not concave. Since ϕ is continuous, there are $r, t \in K$ such that $\phi\left(2^{-1}t + 2^{-1}r\right) < 2^{-1}\phi(t) + 2^{-1}\phi(r)$. Let $H \in \Sigma$ be such that $q(H) = 2^{-1}$, and $x, y \in X$ be such that $u(x) = r$ and $u(y) = t$. Then

$$\begin{aligned} V(xHy) &= \int \phi(u(xHy))\,dq = \int \phi(u(x))\,1_H + \phi(u(y))\,1_{H^c}dq = \frac{1}{2}\phi(r) + \frac{1}{2}\phi(t) \\ &= \frac{1}{2}\phi(t) + \frac{1}{2}\phi(r) = V(yHx) \end{aligned}$$

and we get the following violation of A3:

$$V\left(\frac{1}{2}xHy + \frac{1}{2}yHx\right) = V\left(\frac{1}{2}x + \frac{1}{2}y\right) = \phi\left(u\left(\frac{1}{2}x + \frac{1}{2}y\right)\right) = \phi\left(\frac{r}{2} + \frac{t}{2}\right) < V(xHy).$$

Suppose A7 holds and let $\Sigma \ni E_n \searrow \emptyset$. Choose $z \succ y$ and consider the sequence $z_m = \left(1 - m^{-1}\right)z + m^{-1}y$ for all $m \geq 1$. We have $u(z_m) = u(z) - \frac{1}{m}(u(z) - u(y)) < u(z)$. For all $m \geq 1$, $z \succ z_m$ and there is $n_m \geq 1$ such that $yE_{n_m}z \succ z_m$, i.e.,

$$q(E_{n_m})\,\phi(u(y)) + (1 - q(E_{n_m}))\,\phi(u(z)) > \phi(u(z_m)). \tag{18}$$

Wlog set $\phi(u(y)) = 0 = 1 - \phi(u(z))$. Thus, $w_n = \phi(u(z_m)) \to 1$. By (18), for all $m \geq 1$ there is $n_m \geq 1$ such that $1 - q(E_{n_m}) > w_m$, i.e., $q(E_{n_m}) < 1 - w_m$. But, $q(E_k)$ is a decreasing sequence, therefore $0 \leq \lim_k q(E_k) \leq q(E_{n_m}) < 1 - w_m$ for all $m \geq 1$. Thus, $\lim_k q(E_k) = 0$ and $q \in \Delta^\sigma$. ■

A.2 Proof of Theorem 2

In this appendix we prove the main result of Section 3. Let $(u, G) \in \mathcal{U}(X) \times \mathcal{L}(\mathbb{R} \times \Delta)$ be an uncertainty averse representation of a preference $\succsim$ in the sense of Definition 1 and set

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \quad \forall \varphi \in B_0. \tag{19}$$

By [5, Theorem 50] there exists at least one $q \in \Delta^\sigma$ such that $\operatorname{dom} G \subseteq \mathbb{R} \times \Delta^\sigma(q)$, and hence

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int \varphi dp, p\right) \quad \forall \varphi \in B_0.$$

Notice that, by Theorem 1,

$$G(t, p) = \sup\left\{I(\varphi) : \int \varphi dp \leq t\right\} \quad \forall (t, p) \in \mathbb{R} \times \Delta.$$

In the study of rearrangement invariance it is useful to consider some important stochastic orders. We already introduced in (8) the convex order $\succsim_{cx}$ on $L^1(q)$. The *increasing convex order* $\succsim_{icx}$, the *first order stochastic dominance (fsd)*, and the *second order stochastic dominance (ssd)* are defined analogously by replacing the set of convex functions Φ_{cx} with that of increasing convex functions Φ_{icx}, increasing functions Φ_{mi}, and increasing concave functions Φ_{icv}. Notice that $\varphi \succsim_{icx} \psi$ if and only if $-\varphi \precsim_{ssd} -\psi$, and that the preorders $\succsim_{cx}$, $\succsim_{icx}$, $\succsim_{fsd}$, and $\succsim_{ssd}$ all share the same symmetric part $\sim_d$, which is the *identical distribution* relation wrt q.[20]

A function J defined on a subset of $L^1(q)$ with values in $(-\infty, \infty]$ is:

1. *rearrangement invariant* if $\varphi \sim_d \psi \Longrightarrow J(\varphi) = J(\psi)$;

[20] See Chong (1974) for this fact and for alternative characterizations of some of these stochastic orders.

2. *Schur convex* if $\varphi \succsim_{cx} \psi \Longrightarrow J(\varphi) \geq J(\psi)$

Moreover, J *preserves first* (resp., *second*) *order stochastic dominance* if $\varphi \succsim_{fsd} \psi$ (resp., $\varphi \succsim_{ssd} \psi$) implies $J(\varphi) \geq J(\psi)$.

Theorem 3 *Let I be the function defined by (19) and $q \in \Delta^\sigma$ be such that* $\operatorname{dom} G \subseteq \mathbb{R} \times \Delta^\sigma(q)$*. The following conditions are equivalent (wrt q):*

(i) I preserves second order stochastic dominance on B_0;

(ii) $G(t, \cdot)$ is Schur convex on Δ for all $t \in \mathbb{R}$.

In this case,

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int_0^1 F_\varphi^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1-\omega)\, d\omega, p\right) \quad \forall \varphi \in B_0 \tag{20}$$

and

$$G(t,p) = \begin{cases} \sup\left\{ I(\psi) : \int_0^1 F_\psi^{-1}(\omega) F_{\frac{dp}{dq}}^{-1}(1-\omega)\, d\omega < t \right\} & \text{if } (t,p) \in \mathbb{R} \times \Delta^\sigma(q) \\ \infty & \text{else.} \end{cases} \tag{21}$$

Moreover, if (S, Σ, q) is adequate, then (i) and (ii) are equivalent to:

(iii) I preserves first order stochastic dominance on B_0;

(iv) I is rearrangement invariant on B_0;

(v) $G(t, \cdot)$ is rearrangement invariant on Δ for all $t \in \mathbb{R}$.

For all $\varphi \in L^1(q)$ and all $\omega \in [0,1]$, set

$$\delta_\varphi(\omega) = \inf\{x \in \mathbb{R} : q(\{s \in S : \varphi(s) > x\}) \leq \omega\} \quad \left(= \inf\{x \in \mathbb{R} : F_\varphi(x) \geq 1-\omega\} = F_\varphi^{-1}(1-\omega)\right).$$

Proof. The proof relies on the theory of rearrangement invariant Banach spaces developed by Luxemburg [25] and Chong and Rice [11].

Step 1. If $\psi \in B_0$ and $p \in \Delta^\sigma(q)$, then

$$\left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \precsim_{cx} p \right\} = \left[\int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(1-\omega)\, d\omega, \int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(\omega)\, d\omega \right]. \tag{22}$$

Moreover, if (S, Σ, q) is adequate, then

$$\int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(1-\omega)\, d\omega = \min\left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p \right\} \text{ and} \tag{23}$$

$$\int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(\omega)\, d\omega = \max\left\{ \int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p \right\}. \tag{24}$$

Proof. [11, 10.2, 13.4, and 13.8] guarantee that, if $\varphi, \psi \in L^1(q)$ and $\delta_{|\psi|}\delta_{|\varphi|} \in L^1([0,1], \mathcal{B}, \lambda) = L^1(\lambda)$, then

$$\left\{ \int \psi\varphi' dq : L^1(q) \ni \varphi' \precsim_{cx} \varphi \right\} = \left[\int_0^1 \delta_\psi(\omega)\, \delta_\varphi(1-\omega)\, d\omega, \int_0^1 \delta_\psi(\omega)\, \delta_\varphi(\omega)\, d\omega \right]. \tag{25}$$

Moreover, if (S, Σ, q) is adequate, then

$$\int_0^1 \delta_\psi(\omega)\, \delta_\varphi(1-\omega)\, d\omega = \min\left\{\int \psi\varphi' dq : L^1(q) \ni \varphi' \sim_d \varphi\right\} \text{ and} \tag{26}$$

$$\int_0^1 \delta_\psi(\omega)\, \delta_\varphi(\omega)\, d\omega = \max\left\{\int \psi\varphi' dq : L^1(q) \ni \varphi' \sim_d \varphi\right\}. \tag{27}$$

Notice that, the condition $\delta_{|\psi|}\delta_{|\varphi|} \in L^1(\lambda)$ is implied by $\delta_{|\psi|} \in L^\infty(\lambda)$ and $\delta_{|\varphi|} \in L^1(\lambda)$, which is implied by $\psi \in B_0$ and $\varphi \in L^1(q)$ [11, 4.3].

If, in addition, φ is a probability density (p.d.) and $\varphi' \precsim_{cx} \varphi$, then $\operatorname{essinf} \varphi' \geq 0$ [11, 10.2] and $\int \varphi' dq = \int \varphi dq = 1$, i.e., φ' is a probability density.

Finally, if $\psi \in B_0$ and $p \in \Delta^\sigma(q)$, then

$$\begin{aligned}
\left\{\int \psi dp' : \Delta^\sigma(q) \ni p' \precsim_{cx} p\right\} &= \left\{\int \psi\varphi' dq : \varphi' \text{ is a p.d. and } \varphi' \precsim_{cx} \frac{dp}{dq}\right\} \\
&= \left\{\int \psi\varphi' dq : L^1(q) \ni \varphi' \precsim_{cx} \frac{dp}{dq}\right\} \\
&= \left[\int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(1-\omega)\, d\omega, \int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(\omega)\, d\omega\right].
\end{aligned}$$

Moreover, if (S, Σ, q) is adequate, then

$$\begin{aligned}
\int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(1-\omega)\, d\omega &= \min\left\{\int \psi\varphi' dq : L^1(q) \ni \varphi' \sim_d \frac{dp}{dq}\right\} \\
&= \min\left\{\int \psi\varphi' dq : \varphi' \text{ is a p.d. and } \varphi' \sim_d \frac{dp}{dq}\right\} \\
&= \min\left\{\int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p\right\}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \delta_\psi(\omega)\, \delta_{\frac{dp}{dq}}(\omega)\, d\omega &= \max\left\{\int \psi\varphi' dq : L^1(q) \ni \varphi' \sim_d \frac{dp}{dq}\right\} \\
&= \max\left\{\int \psi\varphi' dq : \varphi' \text{ is a p.d. and } \varphi' \sim_d \frac{dp}{dq}\right\} \\
&= \max\left\{\int \psi dp' : \Delta^\sigma(q) \ni p' \sim_d p\right\}
\end{aligned}$$

as wanted. □

The next step is essentially due to Hardy.

Step 2. Let $r = \infty$ and $\bar{r} = 1$ or viceversa, $\varphi, \varphi' \in L^r(q)$ and $\psi \in L^{\bar{r}}(q)$.

(a) $\varphi \precsim_{cx} \varphi'$ implies $\int_0^1 \delta_\varphi(\omega)\, \delta_\psi(\omega)\, d\omega \leq \int_0^1 \delta_{\varphi'}(\omega)\, \delta_\psi(\omega)\, d\omega$.

(b) $\varphi \precsim_{cx} \varphi'$ implies $\int_0^1 \delta_\varphi(\omega)\, \delta_\psi(1-\omega)\, d\omega \geq \int_0^1 \delta_{\varphi'}(\omega)\, \delta_\psi(1-\omega)\, d\omega$.

(c) $\varphi \precsim_{icx} \varphi'$ and $\psi \geq 0$ (q-a.e.) implies $\int_0^1 \delta_\varphi(\omega)\, \delta_\psi(\omega)\, d\omega \leq \int_0^1 \delta_{\varphi'}(\omega)\, \delta_\psi(\omega)\, d\omega$.

Proof. See [11, 9.1] □

Step 3. If either $G(t, \cdot)$ is Schur convex on Δ for all $t \in \mathbb{R}$, or (S, Σ, q) is adequate and $G(t, \cdot)$ is rearrangement invariant on Δ for all $t \in \mathbb{R}$, then

$$I(\varphi) = \min_{p \in \Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega)\, \delta_{\frac{dp}{dq}}(1-\omega)\, d\omega, p\right) \quad \forall \varphi \in B_0. \tag{28}$$

Proof. Let $\varphi \in B_0$. Then, by (22), $\int \varphi dp \geq \int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega$ for all $p \in \Delta^\sigma(q)$. Thus, monotonicity of G in the first component implies

$$I(\varphi) = \min_{p\in\Delta^\sigma(q)} G\left(\int \varphi dp, p\right) \geq \inf_{p\in\Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega, p\right).$$

Conversely, by (22), for any $p \in \Delta^\sigma(q)$ there exists $p' \precsim_{cx} p$ (resp., by (23) there exists $p' \sim_d p$) such that

$$\int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega = \int \varphi dp'.$$

Thus,

$$G\left(\int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega, p\right) = G\left(\int \varphi dp', p\right) \geq G\left(\int \varphi dp', p'\right) \geq I(\varphi)$$

by Schur convexity (resp., rearrangement invariance).

Therefore,

$$\inf_{p\in\Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega, p\right) \geq I(\varphi)$$

and the infimum is attained. □

Step 4. (ii) implies (i) and (20), also (v) implies (i) and (20) provided (S, Σ, q) is adequate.

Proof. By Step 3, (ii) guarantees that (28) holds and the same is true for (v) if (S, Σ, q) is adequate. But, for all $\varphi \in B_0$ and $p \in \Delta^\sigma(q)$,

$$\int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega = \int_0^1 \delta_\varphi(1-\omega)\,\delta_{\frac{dp}{dq}}(\omega)\,d\omega = \int_0^1 F_\varphi^{-1}(\omega)\,F_{\frac{dp}{dq}}^{-1}(1-\omega)\,d\omega$$

which plugged in (28) delivers (20).

Moreover, $\varphi \succsim_{ssd} \psi$ if and only if $-\varphi \precsim_{icx} -\psi$. Thus, Step 2.c implies $\int_0^1 \delta_{-\varphi}(\omega)\,\delta_{dp/dq}(\omega)\,d\omega \leq \int_0^1 \delta_{-\psi}(\omega)\,\delta_{dp/dq}(\omega)\,d\omega$ for all $p \in \Delta^\sigma(q)$, but $\delta_{-\varphi}(\omega) = -\delta_\varphi(1-\omega)$ (λ-a.e.) [11, 4.4] and the same is true for ψ. This implies that $\int_0^1 -\delta_\varphi(1-\omega)\,\delta_{dp/dq}(\omega)\,d\omega \leq \int_0^1 -\delta_\psi(1-\omega)\,\delta_{dp/dq}(\omega)\,d\omega$ and hence $\int_0^1 \delta_\varphi(\omega)\,\delta_{dp/dq}(1-\omega)\,d\omega \geq \int_0^1 \delta_\psi(\omega)\,\delta_{dp/dq}(1-\omega)\,d\omega$ for all $p \in \Delta^\sigma(q)$. By (28), monotonicity of G allows to conclude that

$$I(\varphi) = \min_{p\in\Delta^\sigma(q)} G\left(\int_0^1 \delta_\varphi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega, p\right) \geq \min_{p\in\Delta^\sigma(q)} G\left(\int_0^1 \delta_\psi(\omega)\,\delta_{\frac{dp}{dq}}(1-\omega)\,d\omega, p\right) = I(\psi).$$

Therefore, I preserves second order stochastic dominance and, in particular, it is rearrangement invariant. □

Step 5. If $\varphi \in L^\infty(q)$ then there exists $\{\varphi_n\} \subseteq B_0$ such that φ_n is the conditional expectation of φ on a finite σ-algebra for all $n \in \mathbb{N}$ and $\varphi_n \overset{\|\cdot\|_\infty}{\to} \varphi$. In particular, $\varphi \succsim_{cx} \varphi_n$ for all $n \in \mathbb{N}$.

Proof. Let $\varphi \in L^\infty(q)$ and wlog take a bounded version of φ. There exists $\{\psi_n\} \subseteq B_0$ that uniformly converges to φ. Set, for each $n \in \mathbb{N}$, $d_n = \|\varphi - \psi_n\|$, $\psi_n^o = \psi_n - d_n$, $\psi_n' = \psi_n + d_n$, $\Sigma_n = \sigma(\psi_n) = \sigma(\psi_n^o) = \sigma(\psi_n')$. It is immediate to see that $\psi_n^o \leq \varphi \leq \psi_n'$ for all $n \in \mathbb{N}$. Moreover, both $\{\psi_n^o\}$ and $\{\psi_n'\}$ converge uniformly to φ, and, for each $n \in \mathbb{N}$, there exist suitable versions of $\mathbb{E}(\psi_n^o|\Sigma_n)$, $\mathbb{E}(\varphi|\Sigma_n)$, and $\mathbb{E}(\psi_n'|\Sigma_n)$ such that $\psi_n = \mathbb{E}(\psi_n^o|\Sigma_n) \leq \mathbb{E}(\varphi|\Sigma_n) \leq \mathbb{E}(\psi_n'|\Sigma_n) = \psi_n'$. Define $\varphi_n = \mathbb{E}(\varphi|\Sigma_n)$ for all $n \in \mathbb{N}$. Clearly, $\varphi_n \in B_0$ and it uniformly converges to φ.

Finally, observe that for all convex functions $\ell : \mathbb{R} \to \mathbb{R}$, by Jensen's inequality, we have that q-a.e.

$$\ell(\varphi_n) = \ell(\mathbb{E}(\varphi|\Sigma_n)) \leq \mathbb{E}(\ell(\varphi)|\Sigma_n) \quad \forall n \in \mathbb{N}.$$

Then, by integrating both sides, for all convex $\ell : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}\left(\ell\left(\varphi_n\right)\right) \leq \mathbb{E}\left(\mathbb{E}\left(\ell\left(\varphi\right)|\Sigma_n\right)\right) = \mathbb{E}\left(\ell\left(\varphi\right)\right) \quad \forall n \in \mathbb{N}.$$

$\square$

Step 6. Let $\psi \in B_0$ and $p \in \Delta^\sigma(q)$. Then,

$$\mathrm{cl}_{L^\infty(q)}\left(\left\{\varphi \in B_0 : \varphi \precsim_{cx} \psi\right\}\right) = \left\{\varphi \in L^\infty(q) : \varphi \precsim_{cx} \psi\right\}. \tag{29}$$

In particular,

$$\inf\left\{\int \varphi dp : B_0 \ni \varphi \precsim_{cx} \psi\right\} = \min\left\{\int \varphi dp : L^\infty(q) \ni \varphi \precsim_{cx} \psi\right\} = \int_0^1 \delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega. \tag{30}$$

Moreover, if (S, Σ, q) is adequate then

$$\min\left\{\int \varphi dp : B_0 \ni \varphi \sim_d \psi\right\} = \min\left\{\int \varphi dp : L^\infty(q) \ni \varphi \sim_d \psi\right\} = \int_0^1 \delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega. \tag{31}$$

Proof. It is easy to verify that $\{\varphi \in L^\infty(q) : \varphi \precsim_{cx} \psi\}$ is closed wrt $\|\cdot\|_\infty$. Therefore,

$$\mathrm{cl}_{L^\infty(q)}\left(\left\{\varphi \in B_0 : \varphi \precsim_{cx} \psi\right\}\right) \subseteq \left\{\varphi \in L^\infty(q) : \varphi \precsim_{cx} \psi\right\}.$$

Conversely, by Step 5, for all $\varphi \in L^\infty(q)$ such that $\varphi \precsim_{cx} \psi$ there exists $\{\varphi_n\} \subseteq B_0$ such that $\varphi_n \overset{\|\cdot\|_\infty}{\to} \varphi$ and $\varphi_n \precsim_{cx} \varphi \precsim_{cx} \psi$ for all $n \in \mathbb{N}$. Hence, (29) follows.

Moreover, [11, 4.3, 10.2, and 13.8] guarantee that, if $\psi \in B_0$ and $p \in \Delta^\sigma(q)$ then $\delta_{|\psi|}\delta_{\left|\frac{dp}{dq}\right|} \in L^1(\lambda)$ and

$$\left\{\int \varphi \frac{dp}{dq} dq : L^\infty(q) \ni \varphi \precsim_{cx} \psi\right\} = \left[\int_0^1 \delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega, \int_0^1 \delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(\omega)\,d\omega\right]. \tag{32}$$

By (29) and (32), (30) follows since $\int \cdot dp$ is a continuous linear functional on $L^\infty(q)$.

If (S, Σ, q) is adequate, [11, 4.3, 10.2, and 13.4] guarantee that, if $\psi \in B_0$ and $p \in \Delta^\sigma(q)$ then $\delta_{|\psi|}\delta_{\left|\frac{dp}{dq}\right|} \in L^1(\lambda)$ and

$$\min\left\{\int \varphi \frac{dp}{dq} dq : L^\infty(q) \ni \varphi \sim_d \psi\right\} = \int_0^1 \delta_{\frac{dp}{dq}} \delta_\psi(1-\omega)(\omega)\,d\omega.$$

But, notice that if ψ is simple and $L^\infty(q) \ni \varphi \sim_d \psi$, then there exists a version of φ which is simple too, thus proving (31). $\square$

Step 7. If either I preserves second order stochastic dominance or if (S, Σ, q) is adequate and I is rearrangement invariant, then, for all $(t, p) \in \mathbb{R} \times \Delta^\sigma(q)$,

$$G(t, p) = \sup\left\{I(\psi) : \int \psi dp < t\right\} = \sup\left\{I(\psi) : \int_0^1 \delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega < t\right\}. \tag{33}$$

Proof. Observe that the set $\left\{\psi \in B_0 : \int \psi dp \leq t\right\}$ is the closure of $\left\{\psi \in B_0 : \int \psi dp < t\right\}$. The fact that I is continuous implies the first equality. In what follows $\varphi, \psi \in B_0$. If I preserves second order stochastic dominance, then I is Schur concave. For, if $\varphi \succsim_{cx} \psi$ then $-\varphi \succsim_{cx} -\psi$ and $-\varphi \succsim_{icx} -\psi$. Hence, $\psi \succsim_{ssd} \varphi$. It follows that $I(\psi) \geq I(\varphi)$. Let I be Schur concave (resp., rearrangement invariant). Then,

$$\sup\left\{I(\varphi) : \int \varphi dp < t\right\} = \sup\left\{I(\psi) : \text{there exists } \varphi \precsim_{cx} \psi \text{ s.t. } \int \varphi dp < t\right\}$$

$$(\text{resp., } = \sup\left\{I(\psi) : \text{there exists } \varphi \sim_d \psi \text{ s.t. } \int \varphi dp < t\right\}).$$

But,

$$\sup\left\{I(\psi):\int\varphi dp<t\text{ for some }\varphi\precsim_{cx}\psi\right\}=\sup\left\{I(\psi):\inf\left\{\int\varphi dp:B_0\ni\varphi\precsim_{cx}\psi\right\}<t\right\}$$

$$\text{(resp., }\sup\left\{I(\psi):\int\varphi dp<t\text{ for some }\varphi\sim_d\psi\right\}=\sup\left\{I(\psi):\inf\left\{\int\varphi dp:B_0\ni\varphi\sim_d\psi\right\}<t\right\}\text{).}$$

By Step 6, (33) follows. □

Step 8. (i) implies (ii) and (21), also (iv) implies (ii) and (21) provided (S,Σ,q) is adequate.

Proof. By Step 7, (i) guarantees that (33) holds, and the same is true for (iii) if (S,Σ,q) is adequate. But, $\int_0^1\delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega=\int_0^1F_\psi^{-1}(\omega)\,F_{\frac{dp}{dq}}^{-1}(1-\omega)\,d\omega$ for all $\psi\in B_0$ and for all $p\in\Delta^\sigma(q)$. Hence, (33) implies (21). By Step 2.b and (33), it descends the following chain of implications

$$\begin{aligned}
&p\precsim_{cx}p'\Longrightarrow\int_0^1\delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega\geq\int_0^1\delta_{\frac{dp'}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega\text{ for all }\psi\in B_0\\
&\Longrightarrow\left\{\psi\in B_0:\int_0^1\delta_{\frac{dp}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega<t\right\}\subseteq\left\{\psi\in B_0:\int_0^1\delta_{\frac{dp'}{dq}}(\omega)\,\delta_\psi(1-\omega)\,d\omega<t\right\}\quad\forall t\in\mathbb{R}\\
&\Longrightarrow G(t,p)\leq G(t,p')\quad\forall t\in\mathbb{R}.
\end{aligned}$$

Hence, $G(t,\cdot)$ is Schur convex for all $t\in\mathbb{R}$. □

Step 9. (i) implies (iii), (iii) implies (iv), and (ii) implies (v).

Proof. The step is proved by a routine argument. □

Finally, Steps 4 and 8 guarantee that (i)⇔(ii). In this case (20) and (21) hold. Moreover, if (S,Σ,q) is adequate, the same steps and Step 9 deliver both (v)⇒(i)⇒(iii)⇒(iv)⇒(ii)⇒(v). ■

Before proving Theorem 2, we prove two ancillary results.

Lemma 1 *Let (u,G) be an uncertainty averse representation for $\succsim$. If $\succsim$ is probabilistically sophisticated wrt $q\in\Delta^\sigma$, then* $\operatorname{dom}G\subseteq\mathbb{R}\times\Delta^\sigma(q)$.

Proof. By definition, $\operatorname{dom}G\subseteq\mathbb{R}\times\Delta^\sigma$. Next, we show that $G(t,p)=\infty$ for all $(t,p)\in\mathbb{R}\times\Delta^\sigma\backslash\Delta^\sigma(q)$. Fix $t\in\mathbb{R}$ and $p\in\Delta^\sigma\backslash\Delta^\sigma(q)$. It follows that there exists $A\in\Sigma$ such that $p(A)>0$ and $q(A)=0$. Since $u(X)=\mathbb{R}$, there exist $\{x_n\},\{y_n\}\subseteq X$ such that $u(x_n)=\sqrt{n}$ and $u(y_n)=-n$. Define $f_n=y_nAx_n$ for all $n\in\mathbb{N}$. By probabilistic sophistication, it follows that $f_n\sim x_n$ for all $n\in\mathbb{N}$. Hence, $u(x_{f_n})=u(x_n)=\sqrt{n}$ for all $n\in\mathbb{N}$. But, for each $n\in\mathbb{N}$

$$\int u(f_n)\,dp=-np(A)+\sqrt{n}p(A^c)\to-\infty\text{ as }n\to\infty.$$

It follows that eventually $f_n\in\left\{f\in\mathcal{F}:\int u(f)\,dp\leq t\right\}$ and $\sqrt{n}\in\left\{u(x_f):\int u(f)\,dp\leq t\right\}$. We conclude that $G(t,p)=\infty$. ■

Lemma 2 *Let (u,G) be an uncertainty averse representation for $\succsim$ and let I be defined as in (19). The following statements are true wrt $q\in\Delta^\sigma$.*

(a) $\succsim$ is probabilistically sophisticated if and only if I is rearrangement invariant on B_0;

(b) $\succsim$ satisfies first order stochastic dominance if and only if I preserves first order stochastic dominance on B_0;

(c) $\succsim$ satisfies second order stochastic dominance if and only if I preserves second order stochastic dominance on B_0.

Proof. First notice that $B_0 = \{u(f) : f \in \mathcal{F}\}$. By definition,

$$V(f) = I(u(f)) \quad \forall f \in \mathcal{F}$$

represents $\succsim$.

(a) "Only if." Consider $\varphi, \psi \in B_0$ such that $\varphi \sim_d \psi$, then

$$q(\{s \in S : \varphi(s) = t\}) = q(\{s \in S : \psi(s) = t\}) \quad \forall t \in \mathbb{R}.$$

Since $u(X) = \mathbb{R}$, for each $t \in \mathbb{R}$ choose $x_t \in X$ such that $u(x_t) = t$. Since φ and ψ are simple, it follows that $\varphi(S) = \{t_1, ..., t_n\}$ and $\psi(S) = \{t'_1, ..., t'_{n'}\}$. Define

$$A_i = \{s \in S : \varphi(s) = t_i\} \text{ and } B_j = \left\{s \in S : \psi(s) = t'_j\right\}.$$

Finally, define f and g such that $f(s) = x_{t_i}$ if $s \in A_i$ and $g(s) = x_{t'_j}$ if $s \in B_j$. It follows that $\varphi = u(f)$, $\psi = u(g)$ and f and g satisfy (4).[21] Thus, $f \sim g$. Therefore, $I(\varphi) = I(u(f)) = I(u(g)) = I(\psi)$.

"If." Suppose that f and g satisfy (4). Define $\varphi = u \circ f$ and $\psi = u \circ g$. Since φ and ψ are simple, it is immediate to see that $\varphi \sim_d \psi$. Then, since I is rearrangement invariant, it follows that $I(u(f)) = I(\varphi) = I(\psi) = I(u(g))$, which implies that $f \sim g$.

(b) "Only if." Consider $\varphi, \psi \in B_0$ such that $q(\{s \in S : \varphi(s) \leq t\}) \leq q(\{s \in S : \psi(s) \leq t\})$ for each $t \in \mathbb{R}$. Define $f, g \in \mathcal{F}$ to be such that $\varphi = u(f)$ and $\psi = u(g)$. It follows that

$$\begin{aligned} q(\{s \in S : f(s) \precsim x\}) &= q(\{s \in S : u(f(s)) \leq u(x)\}) = q(\{s \in S : \varphi(s) \leq u(x)\}) \\ &\leq q(\{s \in S : \psi(s) \leq u(x)\}) = q(\{s \in S : u(g(s)) \leq u(x)\}) \\ &= q(\{s \in S : g(s) \precsim x\}) \quad \forall x \in X. \end{aligned}$$

Therefore, it is clear that f and g satisfy (5). It follows that $f \succsim g$, and so $I(\varphi) = I(u(f)) \geq I(u(g)) = I(\psi)$.

"If." Suppose that f and g satisfy (5). Define $\varphi = u \circ f$ and $\psi = u \circ g$. It follows that

$$\begin{aligned} q(\{s \in S : \varphi(s) \leq t\}) &= q(\{s \in S : u(f(s)) \leq u(x_t)\}) = q(\{s \in S : f(s) \precsim x_t\}) \\ &\leq q(\{s \in S : g(s) \precsim x_t\}) = q(\{s \in S : u(g(s)) \leq u(x_t)\}) \\ &= q(\{s \in S : \psi(s) \leq t\}) \quad \forall t \in \mathbb{R}. \end{aligned}$$

It follows that $\varphi \succsim_{fsd} \psi$. Then, since I preserves first order stochastic dominance, it follows that $I(u(f)) = I(\varphi) \geq I(\psi) = I(u(g))$, which implies $f \succsim g$.

The proof of (c) is analogous. ■

Proof of Theorem 2. Consider the uncertainty averse representation (u, G) for $\succsim$. Then, define the functional $I : B_0 \to \mathbb{R}$ as in (19).

[21] Notice that $q\left(f^{-1}(x_t)\right) = q\left(\varphi^{-1}(t)\right) = q\left(\psi^{-1}(t)\right) = q\left(g^{-1}(x_t)\right)$ for all $t \in \mathbb{R}$, while $q\left(f^{-1}(x)\right) = 0 = q\left(g^{-1}(x)\right)$ if $x \notin \{x_t\}_{t \in \mathbb{R}}$.

(i) implies (ii). Since $\succsim$ satisfies second order stochastic dominance (wrt q), it is probabilistically sophisticated. By Lemma 1, $\operatorname{dom} G \subseteq \mathbb{R} \times \Delta^{\sigma}(q)$. By Lemma 2, I preserves second order stochastic dominance. Hence, Theorem 3 guarantees that (ii) holds.

(ii) implies (i). Since $\operatorname{dom} G \subseteq \mathbb{R} \times \Delta^{\sigma}(q)$, by Theorem 3 I preserves second order stochastic dominance. By Lemma 2, $\succsim$ satisfies second order stochastic dominance.

Furthermore, assume (i) or (ii) hold. By Theorem 3, I satisfies (20) and (21). Hence, (9) and (10) follow from the observation that $I(u(f)) = \min_{p\in\Delta} G\left(\int u(f)\,dp, p\right) = u(x_f)$ for each $f \in \mathcal{F}$.

Assume that (S, Σ, q) is adequate.

(i) implies (iii) and (iii) implies (iv). The statement is proved by a routine argument.

(iv) implies (v). Since $\succsim$ is probabilistically sophisticated, by Lemma 1, $\operatorname{dom} G \subseteq \mathbb{R} \times \Delta^{\sigma}(q)$, and by Lemma 2, I is rearrangement invariant. By Theorem 3, (v) holds.

By Theorem 3, (v) implies (ii), which concludes the proof. ■

A.3 Proofs of Corollary 1, Proposition 1, and Proposition 2

Let $I : B_0 \to \mathbb{R}$ be defined as in (19). For each $\varphi \in B_0$ the normalized Greenberg-Pierskalla superdifferential of I at φ is the set

$$\partial I(\varphi) = \left\{ p \in \Delta : \int \varphi dp \geq \int \psi dp \Rightarrow I(\varphi) \geq I(\psi) \right\}.$$

Proposition 4 *Let (u, G) be an uncertainty averse representation for $\succsim$. The following conditions are equivalent for $\bar{p} \in \Delta$:*

(i) $\succsim$ is more uncertainty averse than $\succsim_{u,\bar{p}}$;

(ii) $G(t, \bar{p}) = \min_{p\in\Delta} G(t, p)$ for all $t \in \mathbb{R}$;

(iii) $\bar{p} \in \bigcap_{t\in\mathbb{R}} \partial I(t)$.

Proof. (i) implies (ii). Since $\succsim_{u,\bar{p}}$ is a SEU preference, setting

$$\bar{G}(t, p) = \begin{cases} t & \text{if } (t, p) = (t, \bar{p}) \\ \infty & \text{otherwise} \end{cases} \tag{34}$$

for all $(t, p) \in \mathbb{R} \times \Delta$, it follows that the pair $(u, \bar{G})$ is an uncertainty averse representation of $\succsim_{u,\bar{p}}$. By [5, Proposition 6], the fact that $\succsim$ is more uncertainty averse that $\succsim_{u,\bar{p}}$ translates into

$$t \leq G(t, p) \leq \bar{G}(t, p) \quad \forall (t, p) \in \mathbb{R} \times \Delta. \tag{35}$$

Substituting $p = \bar{p}$ in (35) delivers

$$G(t, \bar{p}) = t = \min_{p\in\Delta} G(t, p) \quad \forall t \in \mathbb{R}$$

where the last equality follows from $G \in \mathcal{L}(\mathbb{R} \times \Delta)$.

(ii) implies (iii). Since (u, G) is an unbounded uncertainty averse representation,

$$G(t, p) = \sup\left\{ u(x_f) : \int u(f)\,dp \leq t \right\} = \sup\left\{ I(u(f)) : \int u(f)\,dp \leq t \right\} = \sup\left\{ I(\psi) : \int \psi dp \leq t \right\} \tag{36}$$

for all $(t,p) \in \mathbb{R} \times \Delta$. Let $\psi \in B_0$ be such that $\int \psi d\bar{p} \le t$, then, by (36), $I(\psi) \le G(t,\bar{p})$. But (ii) implies that $G(t,\bar{p}) = t$. Thus $I(\psi) \le t = I(t)$ implies that if $\int \psi dp \le \int t d\bar{p}$ then $I(\psi) \le I(t)$ and we conclude that $\bar{p} \in \partial I(t)$.

(iii) implies (i). Let $f \in \mathcal{F}$ and $x \in X$. Since $\bar{p} \in \bigcap_{t \in \mathbb{R}} \partial I(t)$, it follows that $\bar{p} \in \partial I(u(x))$. Therefore,

$$\int u(f)\, d\bar{p} \le \int u(x)\, d\bar{p} \Rightarrow I(u(f)) \le I(u(x))$$

that is

$$x \succsim_{u,\bar{p}} f \Rightarrow x \succsim f.$$

But the latter condition can be easily seen to be equivalent to $\succsim$ being more uncertainty averse that $\succsim_{u,\bar{p}}$.[22] ■

Proof of Corollary 1. For each $p \in \Delta^{\sigma}(q)$, by Jensen's inequality,

$$\int \ell\left(\frac{dp}{dq}\right) dq \ge \ell\left(\int \frac{dp}{dq} dq\right) = \ell(1) = \int \ell\left(\frac{dq}{dq}\right) dq$$

for all convex functions $\ell : \mathbb{R} \to \mathbb{R}$. This implies that $p \succsim_{cx} q$ for all $p \in \Delta^{\sigma}(q)$. By Theorem 2, $G(t,\cdot)$ is Schur convex for all $t \in \mathbb{R}$. Therefore, for each $t \in \mathbb{R}$, $G(t,p) \ge G(t,q)$ if $p \in \Delta^{\sigma}(q)$ and $G(t,p) = \infty$ if $p \notin \Delta^{\sigma}(q)$. That is, $q \in \arg\min G(t,\cdot)$ for all $t \in \mathbb{R}$. By Proposition 4, the statement follows. ■

Let $u : X \to \mathbb{R}$ be a nonconstant affine function and C a nonempty subset of Δ. Set

$$f \succsim^* g \Longleftrightarrow \int u(f)\, dp \ge \int u(g)\, dp \quad \forall p \in C. \tag{37}$$

Notice that $\succsim^*$ is complete (and represented by u) on X, hence the definition of nontrivial unambiguous event can be naturally extended to this more general setting.[23] Next we prove that the equivalence among points (i)-(iii) of Proposition 1 holds more in general for any relation $\succsim^*$ defined as above (the special case is obtained by setting $C = \mathrm{dom}_{\Delta}\, G$ and observing that $\succsim$ and $\succsim^*$ coincide on X).

Proof of Proposition 1. (ii) implies (i). This implication follows immediately from the definition of unambiguous event and the fact that $\succsim^*$ is nontrivial on X.

(i) implies (iii). By definition, $f \succsim^* g$ if and only if $\int u(f)\, dp \ge \int u(g)\, dp$ for all $p \in C$. By (i), there exist $x, y, z \in X$ such that $x \succ^* z \succ^* y$ and $xAy \sim^* z$. It follows that $u(x)\, p(A) + (1 - p(A))\, u(y) = u(z)$ for all $p \in C$. Since $u(x) > u(z) > u(y)$, we can conclude that

$$0 < p(A) = \frac{u(z) - u(y)}{u(x) - u(y)} < 1, \qquad \forall p \in C,$$

as desired.

[22] Indeed, consider two preorders $\succsim_1$ and $\succsim_2$ on $\mathcal{F}$. Assume that $\succsim_i$ over X is represented by an affine nonconstant function u_i and for each $f \in \mathcal{F}$ there exists $x_f^i \sim_i f$ $(i = 1, 2)$. Then, the following conditions are equivalent:

(i) For each $f \in \mathcal{F}$ and each $x \in X$, $f \succsim_1 x \Rightarrow f \succsim_2 x$;

(ii) $\succsim_1$ coincides with $\succsim_2$ on X and $x_f^2 \succsim_1 x_f^1$ for all $f \in \mathcal{F}$;

(iii) For each $f \in \mathcal{F}$ and each $x \in X$, $f \succ_1 x \Rightarrow f \succ_2 x$ (i.e. $x \succsim_2 f \Rightarrow x \succsim_1 f$).

[23] An event A in Σ is nontrivial and unambiguous if there exist $x, y, z \in X$ such that $x \succ^* z \succ^* y$ and $xAy \sim^* z$.

(iii) implies (ii). Consider $x, y \in X$ such that $x \succ^* y$. By (iii), it follows that there exists $\alpha \in \mathbb{R}$ such that $\alpha = \int u(xAy)\, dp$ for all $p \in C$ and $u(x) > \alpha > u(y)$. Since $u(X)$ is an interval, it follows that there exists $z \in X$ such that $u(z) = \alpha = \int u(xAy)\, dp$ for all $p \in C$. It follows that $x \succ^* z \succ^* y$ and, by (37), that $xAy \sim^* z$. ■

A subset C of $\Delta^\sigma(q)$ is Schur convex (wrt $q \in \Delta^\sigma$) if and only if $\{p \in \Delta^\sigma(q) : p \precsim_{cx} p'\} \subseteq C$ for each $p' \in C$.

Proposition 5 *Let $q \in \Delta^\sigma$ and $\succsim^*$ be defined as in (37). If A is a nontrivial unambiguous event for $\succsim^*$ and C is Schur convex (wrt q), then $C = \{q\}$.*

Proof. Wlog $p(A) = \alpha \in (0, 1/2]$ for all $p \in C$. Let $\bar{p} \in C$, then, by Step 1 of the proof of Theorem 3,

$$\max\left\{\int 1_A dp : \Delta^\sigma(q) \ni p \precsim_{cx} \bar{p}\right\} = \int_0^1 \delta_{1_A}(\omega)\, \delta_{\frac{d\bar{p}}{dq}}(\omega)\, d\omega$$
$$\min\left\{\int 1_A dp : \Delta^\sigma(q) \ni p \precsim_{cx} \bar{p}\right\} = \int_0^1 \delta_{1_A}(\omega)\, \delta_{\frac{d\bar{p}}{dq}}(1-\omega)\, d\omega.$$

Since $q \in \{p \in \Delta^\sigma(q) : p \precsim_{cx} \bar{p}\} \subseteq C$ and A is a nontrivial unambiguous event, then $\left\{\int 1_A dp : \Delta^\sigma(q) \ni p \precsim_{cx} \bar{p}\right\} = \{q(A)\} = \{\alpha\}$, hence,

$$\int_0^1 \delta_{1_A}(\omega)\, \delta_{\frac{d\bar{p}}{dq}}(\omega)\, d\omega = \int_0^1 \delta_{1_A}(\omega)\, \delta_{\frac{d\bar{p}}{dq}}(1-\omega)\, d\omega.$$

As well known,

$$\delta_{1_A}(\omega) = \begin{cases} 1 & \omega \in (0, \alpha) \\ 0 & \omega \in [\alpha, 1). \end{cases}$$

Therefore,

$$\int_0^\alpha \delta_{\frac{d\bar{p}}{dq}}(\omega)\, d\omega = \int_0^\alpha \delta_{\frac{d\bar{p}}{dq}}(1-\omega)\, d\omega \tag{38}$$

but $\delta_{\frac{d\bar{p}}{dq}} : (0,1) \to [0,\infty)$ is decreasing and $\int_0^1 \delta_{\frac{d\bar{p}}{dq}}(\omega)\, d\omega = 1$ [11, 4.3]. Therefore, by standard arguments, (38) implies $\delta_{\frac{d\bar{p}}{dq}} = 1$ (λ-a.e.). It follows that $\frac{d\bar{p}}{dq} = 1$ (q-a.e.) [11, 2.8], and $\bar{p} = q$. ■

Proof of Proposition 2. Let A be a nontrivial unambiguous event for $\succsim$.

(i) Sufficiency is immediate. As to necessity, notice that $\succsim^*$ is represented by (13) and, by Theorem 2, $\mathrm{dom}_\Delta\, G$ is a Schur convex subset of $\Delta^\sigma(q)$. Proposition 5 delivers $\mathrm{dom}_\Delta\, G = \{q\}$. By definition of $\mathrm{dom}_\Delta\, G$ and since $G \in \mathcal{L}(\mathbb{R} \times \Delta)$, it follows that for each $(t, p) \in \mathbb{R} \times \Delta$

$$G(t, p) = \begin{cases} t & \text{if } (t,p) = (t,q) \\ \infty & \text{otherwise.} \end{cases}$$

The statement follows.

(ii) Assume that (S, Σ, q) is adequate. Sufficiency is trivial. As for necessity, notice that, by Theorem 2, $\succsim$ satisfies second order stochastic dominance and the statement follows. ■

References

[1] F. J. Anscombe and R. J. Aumann, A definition of subjective probability, *Annals of Mathematical Statistics*, 34, 199–205, 1963.

[2] A. Ben-Tal and M. Teboulle, Expected utility, penalty functions and duality in stochastic nonlinear programming, *Management Science*, 32, 1445–1466, 1986.

[3] A. Ben-Tal and M. Teboulle, Penalty functions and duality in stochastic programming via ϕ-divergence functionals, *Mathematics of Operations Research*, 12, 224–240, 1987.

[4] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 341–375, 2011.

[5] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, *Journal of Economic Theory*, 146, 1275–1330, 2011.

[6] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Complete monotone quasiconcave duality, *Mathematics of Operations Research*, 36, 321–339, 2011.

[7] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Risk measures: Rationality and diversification, *Mathematical Finance*, 21, 743–774, 2011.

[8] A. Chateauneuf and J. H. Faro, Ambiguity through confidence functions, *Journal of Mathematical Economics*, 45, 535–558, 2009.

[9] S. H. Chew and J. S. Sagi, Event exchangeability: Probabilistic sophistication without continuity or monotonicity, *Econometrica*, 74, 771–786, 2006.

[10] S. H. Chew and J. S. Sagi, Small worlds: Modeling attitudes toward sources of uncertainty, *Journal of Economic Theory*, 139, 1–24, 2008.

[11] K. M. Chong and N. M. Rice, Equimeasurable rearrangements of functions, *Queen's Papers in Pure and Applied Mathematics*, 28, 1971.

[12] G. Debreu, *Theory of Value*, Yale University Press, New Haven, 1959.

[13] B. de Finetti, Sul significato soggettivo della probabilità, *Fundamenta Mathematicae*, 17, 298–329, 1931.

[14] E. Dekel, Asset demands without the independence axiom, *Econometrica*, 57, 163–169, 1989.

[15] P. Ghirardato and M. Marinacci, Ambiguity made precise: A comparative foundation, *Journal of Economic Theory*, 102, 251–289, 2002.

[16] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.

[17] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 141–153, 1989.

[18] S. Grant, Subjective probability without monotonicity: Or how Machina's mom may also be probabilistically sophisticated, *Econometrica*, 63, 159–189, 1995.

[19] S. Grant and B. Polak, Bayesian beliefs with stochastic monotonicity: An extension of Machina and Schmeidler, *Journal of Economic Theory*, 130, 264–282, 2006.

[20] S. Grant, B. Polak, and T. Strzalecki, Second-order expected utility, mimeo, 2009.

[21] L. P. Hansen and T. Sargent, *Robustness*, Princeton University Press, Princeton, 2008.

[22] P. Klibanoff, M. Marinacci, and S. Mukerji, A smooth model of decision making under ambiguity, *Econometrica*, 73, 1849–1892, 2005.

[23] I. Kopylov, Subjective probabilities on "small" domains, *Journal of Economic Theory*, 133, 236–265, 2007.

[24] P. S. Laplace, *Essai philosophique sur les probabilités*, Courcier, Paris, 1814.

[25] W. A. J. Luxemburg, Rearrangement-invariant Banach function spaces, *Queens Papers in Pure and Applied Mathematics*, 10, 83–144, 1967.

[26] M. J. Machina, D. Schmeidler, A more robust definition of subjective probability, *Econometrica*, 60, 745–780, 1992.

[27] M. J. Machina, D. Schmeidler, Bayes without Bernoulli: Simple conditions for probabilistically sophisticated choice, *Journal of Economic Theory*, 67, 106–128, 1995.

[28] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.

[29] M. Marinacci, Probabilistic sophistication and multiple priors, *Econometrica*, 70, 755–764, 2002.

[30] A. W. Marshall and I. Olkin, *Inequalities: theory of majorization and its applications*, Academic Press, New York, 1979.

[31] W. Neilson, A simplified axiomatic approach to ambiguity aversion, *Journal of Risk and Uncertainty*, 41, 113–124, 2010.

[32] J. Quiggin, A theory of anticipated utility, *Journal of Economic Behavior & Organization*, 3, 323–343, 1982.

[33] M. Rothschild and J. E. Stiglitz, Increasing risk: I. A definition, *Journal of Economic Theory*, 2, 225–243, 1970.

[34] M. Rothschild and J. E. Stiglitz, Increasing risk II: Its economics consequences, *Journal of Economic Theory*, 3, 66–84, 1971.

[35] R. Sarin and P. P. Wakker, Cumulative dominance and probabilistic sophistication, *Mathematical Social Sciences*, 40, 191–196, 2000.

[36] L. J. Savage, *The foundations of statistics*, Wiley, New York, 1954.

[37] D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 571–587, 1989.

[38] T. Strzalecki, Axiomatic foundations of multiplier preferences, *Econometrica*, 79, 47–73, 2011.

[39] T. Strzalecki, Probabilistic sophistication and variational preferences, *Journal of Economic Theory*, 146, 2117–2125, 2011.

[40] P. P. Wakker, Clarification of some mathematical misunderstandings about Savage's foundations of statistics 1954, *Mathematical Social Sciences*, 25, 199–202, 1993.

[41] M. Yaari, The dual theory of choice under risk, *Econometrica*, 55, 95–115, 1987.