Caution and Reference Effects*

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Abstract

We introduce the Cautious Utility model and show that it provides a new approach to three phenomena at the core of behavioral economics: the endowment effect, loss aversion, and violations of Expected Utility due to the certainty effect. In our model, all three phenomena stem from uncertainty about which utility to use and caution. We show how this model can help organize empirical evidence, some of which is incompatible with leading alternatives, and is both conceptually and behaviorally distinct from other popular approaches.

Keywords: Cautious Utility, Endowment Effect, Loss Aversion, Certainty Effect, Non-Expected Utility, Cumulative Prospect Theory.

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1 Introduction

Three phenomena hold a prominent place in behavioral economics: (i) the *endowment effect*—the minimum selling price is higher than the maximum buying price for the same item (Kahneman et al., 1991); (ii) *loss aversion*—subjects reject bets that return identical gains and losses with equal probability (Kahneman and Tversky, 1979); and (iii) the *certainty effect*—violations of Expected Utility in favor of certainty (Allais, 1953; Kahneman and Tversky, 1979). All three are widely documented and used in countless studies to explain numerous real-world observations. Several models have been developed to study them, most prominently Cumulative Prospect Theory (Tversky and Kahneman, 1991), where the endowment effect and loss aversion are linked to an asymmetry in the treatment of gains and losses ('losses loom larger than gains'), while the certainty effect is due to probability weighting.

This paper introduces a new approach to study all three phenomena: uncertainty about trade-offs and caution. In our model, individuals have a *set* of utility functions—as if unsure of how to precisely evaluate each option—and adopt the most pessimistic one. In previous work (Cerreia-Vioglio et al., 2015), we showed how this captures the certainty effect. This paper shows how it (i) captures the endowment effect and loss aversion in novel ways, even *without asymmetries between gains and losses*; (ii) provides new empirical predictions; and (iii) offers a tool to organize existing evidence at odds with leading models.

We call our model Cautious Utility. It works as follows. Individuals face lotteries over bundles in $\mathbb{R}^k$, where the first dimension is money and the others are goods (e.g., mugs). The standard approach assumes a utility over bundles and Expected Utility to evaluate lotteries. Instead, we have a *set* of utilities $\mathcal{W}$ and individuals use the most pessimistic one to evaluate each option. Specifically, if v is a utility function and p a lottery, call $c(p, v)$ the monetary certainty equivalent of that lottery using v—the amount of money indifferent to p for utility v. Cautious Utility assigns to the lottery p the value

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v).$$

The key ideas are (i) individuals may be unsure of how to evaluate bundles—they may entertain multiple utility functions as plausible; and (ii) facing this multiplicity, they choose with *caution*—using the utility that returns the lowest certainty equivalent.

For example, an individual who evaluates bundles of money (dimension 1) and mugs (dimension 2) may contemplate two utilities: $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. It is as if the individual were unsure of the trade-off: a mug could be worth \$1 (under v_1) or \$2 (under v_2). Cautious Utility stipulates that, in the face of this subjective uncertainty, the individual applies caution and uses the utility that returns the lowest monetary certainty equivalent.

Cautious Utility generalizes Cautious Expected Utility of Cerreia-Vioglio et al. (2015) and, like that model, captures the certainty effect and Allais' paradoxes. Intuitively, degenerate lotteries that return a given amount of money have the same certainty equivalent with any utility, making caution irrelevant. But caution does matter for general lotteries, lowering their value and generating an advantage for sure amounts.

After presenting our model in Section 2, in Section 3 we show that the same forces that generate the certainty effect also generate the endowment effect and loss aversion, providing a unified explanation. To best illustrate the role of caution, we focus on *symmetric sets* of utilities—either all utilities are symmetric for gains and losses, or, if one is not, the set also includes its specular one.

Our main result is that Cautious Utility implies the endowment effect and loss aversion, in addition to the certainty effect, even under symmetry. Without symmetry, reference effects emerge if at least one utility overweights losses: then caution and overweighting of losses jointly contribute to reference effects. To make explicit the role of caution, we show that individuals who use the opposite (sup instead of inf) exhibit the opposite of the certainty effect, of the endowment effect, and of loss aversion. In other words, all three phenomena are determined by the shape of the aggregator—inf vs. sup.

For an intuition, consider our example above, where the individual entertains two utilities, $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. Note how both utilities are symmetric for gains and losses. When buying a mug, the worst-case scenario is when it is least valuable: thus, the willingness to pay (WTP) is calculated using v_1 and equals \$1. When selling the mug, the worst case is when it has the highest value: the willingness to accept (WTA) is calculated using v_2 and equals \$2. Thus, WTA>WTP. Despite its simplicity, this example captures a broader result: we show that, whenever there is uncertainty on the trade-off between a good and money, the WTA is strictly above the WTP, generating the endowment effect even without any asymmetry between gains and losses.

The remainder of Section 3 shows how our approach extends to exchange asymmetries, to stochastic reference points, and presents convenient functional forms for estimation. We also demonstrate that the only preferences compatible with both our model and Cumulative Prospect Theory are those featuring none of our phenomena of interest. That is, the two models are not only conceptually different, but also behaviorally fully distinct.

In Section 4 we discuss how caution can help organize empirical evidence. Several studies show that the strength and frequency of the endowment effect vary substantially across goods, decreasing with familiarity, and with information about market values. This is in line with Cautious Utility, where the endowment effect depends on the uncertainty about the value of the good. Other recent evidence shows that the endowment effect and loss aversion are empirically unrelated, and, in fact, the latter is often absent. While in Prospect Theory the two behaviors are driven by the same parameter and must be related, in Cautious Utility they are decoupled. We also discuss other patterns consistent with our model but not with popular alternatives, and vice-versa.

Lastly, in Section 5 we give an axiomatic foundation to Cautious Utility. Dillenberger (2010) introduced Negative Certainty Independence to capture the certainty effect over money. We show that Cautious Utility is characterized by an extension to a form of certainty effect over bundles, together with very basic postulates (e.g., monotonicity). Paired with our result that Cautious Utility returns the endowment effect and loss aversion under symmetry, this means that, assuming symmetry and basic postulates, a property that captures the certainty effect for bundles *formally implies* the endowment effect and loss aversion.

1.1 Related Theoretical Literature

Cumulative Prospect Theory. The most popular model to study our behaviors of interest is Cumulative Prospect Theory (Tversky and Kahneman, 1992), henceforth CPT, which extends the original Prospect Theory (Kahneman and Tversky, 1979); see also Kahneman et al. (1991); Tversky and Kahneman (1991) and, for a textbook treatment, Wakker (2010). Violations of Expected Utility are captured by probability weighting. Reference dependence is captured separately, by positing that individuals evaluate changes relative to a reference point and that 'losses loom larger than gains.' The latter is formalized by assuming that the utility is not symmetric for gains and losses and losses weigh more—a common approach is to take $\lambda > 1$ and $v(-x) = -\lambda v(x)$ for $x > 0$. This asymmetry reduces the value of even bets around zero and increases the gap between WTA and WTP.

Cautious Utility is different. Probabilities are taken at face value, not weighted, and instead of a single asymmetric utility, we have many—possibly all symmetric. All three biases come from the same source, uncertainty about the utility and caution. As mentioned above, the difference is also behavioral: not only some behaviors can be captured by one model and not the other, but the two models are entirely distinct—the only preference compatible with both is standard Expected Utility with no reference effects.

Cautious Expected Utility. Our approach builds on Cerreia-Vioglio et al. (2015), which studies preferences over monetary lotteries on a bounded interval that admit the following *Cautious Expected Utility* representation: there exists a set $\mathcal{W}$ of strictly increasing and continuous functions over money such that the value of a lottery p is given by $\inf_{v\in\mathcal{W}} c(p, v)$. As explained in Cerreia-Vioglio et al. (2015), Cautious Expected Utility can be understood as a cautious completion of an incomplete preference, paralleling Gilboa et al. (2010), and therefore can be thought of as a possible 'risk' counterpart of the MaxMin Expected Utility model of Gilboa and Schmeidler (1989).

We extend this model and its characterization to bundles of goods (explicitly discussing gains/losses and symmetry) and to unbounded spaces. More importantly, we show that, in this extension, the same forces that generate the certainty effect over money also generate the endowment effect and loss aversion, providing a new model for these phenomena. Cerreia-Vioglio (2009) characterizes preferences that satisfy convexity and shows that they can be represented with a set of utilities and pessimism, connecting convexity with a preference for hedging in the face of uncertainty about the value of outcomes, future tastes, or the degree of risk aversion. Our model is a special case, as our preferences are convex.

Incomplete Preferences. An alternative approach to studying reference effects is via incomplete preferences (Bewley, 1986; Masatlioglu and Ok, 2005, 2014; Ortoleva, 2010; Ok et al., 2015). (These papers are typically silent about loss aversion or the certainty effect as they do not study risk preferences.) Agents have an incomplete preference relation and deviate from their reference point (or status quo) only if an alternative is better according to that relation. This generates status quo bias and the endowment effect. As incomplete preferences can be represented using multiple utilities, here too the endowment effect is related to the inability to compare bundles.

Cerreia-Vioglio et al. (2015) show that Cautious Expected Utility can be derived as a completion of an incomplete relation, and the same is true here. Indeed, the literature on incomplete preferences and status quo bias was an inspiration for our work. However, there are three critical differences. First, our preferences are complete: our agent uses caution as a criterion to complete them, and it is this criterion that drives our results. Second, risk plays a central role in our paper: we *derive* reference effects from a form of certainty effect and connect the different phenomena. There is no similar link in the models above. Third, the models above only specify behavior when the status quo, or the reference point, is available; here, instead, the behavior is specified independently of the availability of the status quo.

Perception, Imprecision, and Other Explanations. Central to our approach is that individuals have sets of utilities, expressing difficulty in making comparisons. This relates to the literature on difficulties in forming preferences, including preference imprecision (Dubourg et al., 1994, 1997; Butler and Loomes, 2007, 2011; Cubitt et al., 2015), imprecise perception and rational inattention (Gabaix and Laibson, 2017; Woodford, 2020; Frydman and Jin, 2020; Khaw et al., 2020), or cognitive uncertainty (Enke and Graeber, 2019). Other accounts of the endowment effect are based on memory (Johnson et al., 2007) and reference prices (Weaver and Frederick, 2012), while versions of saliency can generate all our behaviors of interest (Bordalo et al., 2012, 2013). The critical difference between our approach and all of these is the presence of caution.

2 Cautious Utility

We begin by introducing Cautious Utility; we discuss its axiomatic foundation in Section 5. Given $k \in \mathbb{N}$, let $\mathbb{R}^k$ be the space of k-dimensional bundles (e.g., money, mugs, pens, *etc.*). For ease of reference, the first dimension denotes money, used as a numeraire. Denote by ae_i the bundle whose i-th coordinate takes value $a \in \mathbb{R}$ while all the others are 0. With a small abuse of notation, we denote by 0 both the number and the vector whose components are all zero. Let Δ be the set of all lotteries, that is, (Borel) probability measures over $\mathbb{R}^k$ with compact support. We study a preference relation $\succcurlyeq$ over Δ.

Given $x \in \mathbb{R}^k$, we interchangeably use x and δ_x to denote the degenerate lottery that pays x with certainty. If $p \in \Delta$ and $v : \mathbb{R}^k \to \mathbb{R}$ is strictly increasing and continuous, then $\mathbb{E}_p(v)$ denotes the expected utility using v, i.e., $\int v \mathrm{d}p$, while $c(p, v) \in \mathbb{R}$ indicates, if it exists, its monetary certainty equivalent, i.e., the unique monetary value such that $\mathbb{E}_p(v) = v(c(p, v)e_1)$.

Definition 1. A preference relation $\succcurlyeq$ admits a Cautious Utility representation if there exists a set $\mathcal{W}$ of strictly increasing continuous utility functions $v : \mathbb{R}^k \to \mathbb{R}$ with $v(0) = 0$ such that (i) for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ satisfying $v(y+me_1) \geq v(x) \geq v(y-me_1)$ for all $v \in \mathcal{W}$; and (ii) the function $V : \Delta \to \mathbb{R}$, defined as

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \tag{1}$$

is a continuous utility representation of $\succcurlyeq$.

Cautious Utility builds upon two key tenets. First, agents have not one but a *set* of utilities: they may be unsure of which utility to use. For example, agents may be unsure of the trade-off between goods or how risk averse they should be. Second, agents act with *caution*: to evaluate each alternative, they use the utility with the *lowest* monetary certainty equivalent. Using certainty equivalents guarantees that the comparison across utilities is made after bringing each dimension to the same unit of measure (monetary amounts), avoiding comparisons across utilities, for which normalizations matter. Condition (i) in the definition guarantees that monetary certainty equivalents are always well-defined.

Example 1. *Consider $k = 2$, money and mugs. Suppose $\mathcal{W} = \{v_1, v_2\}$ where $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. It is as if the agent is unsure about the trade-off between money and mugs: one mug is equivalent to \$1 according to v_1, and to \$2 according to v_2. Because of caution, the model assigns to one mug a value $V(0, 1) = \min\{1, 2\} = 1$.*

To illustrate the role of caution, we also define a model that takes the opposite approach, where the agent uses the utility with the *highest* monetary certainty equivalent. We say that a preference relation admits an *Incautious Utility* representation if it is represented by (1) where sup replaces the inf.

Remark 1. *Despite the use of the most pessimistic utility, under Cautious Utility agents can be risk averse, seeking, or have varying risk attitudes: as in Expected Utility, this depends on the curvature of the utilities in* $\mathcal{W}$*. Cerreia-Vioglio et al. (2015) show that agents are risk averse when all functions are concave and risk seeking when all are convex. Similarly, if utilities are all concave for gains and convex for losses, the individual is risk averse for gains and risk seeking for losses. Overall, Cautious Utility does not restrict risk attitudes.*

Remark 2. *The use of the* inf *in Cautious Utility may at first appear too pessimistic, and one may wish to consider milder formulations. For example, the agent may use some weighted average of the* inf *and of the* sup*. A few considerations are in order. First, the set of utilities is subjective: it reflects the agent's preferences and is not the set of all possible utilities. Thus, the* inf *is taken only over a restricted collection. Second, this representation is not necessarily very pessimistic. For example, take a finite set* $\mathcal{W}$ *of quasi-linear utilities and some* $u \in \mathcal{W}$*, and an individual who uses the most pessimistic utility in* $\mathcal{W}' = \{(1-\gamma)\, u + \gamma v : v \in \mathcal{W}\}$*. Here* γ *can be understood as a 'pessimism weight:' the larger is* γ*, the lower the evaluation. When it is small, the individual is only 'mildly' pessimistic, yet preferences admit a Cautious Utility representation with set* $\mathcal{W}'$*. Finally, in Section 5, we show how Cautious Utility emerges from a natural axiom on the certainty effect; to the extent that one accepts this requirement, the model* necessarily *follows.*

2.1 Symmetry

To study gains and losses it is helpful to introduce a notion of symmetry. Recall that $v : \mathbb{R}^k \to \mathbb{R}$ is *odd* if $v(x) = -v(-x)$ for all $x \in \mathbb{R}^k$, that is, when there is no asymmetry in the treatment of positive and negative values. Indeed, a function is odd when it is symmetric with respect to the origin (like any power utility over the real line with an odd exponent). In line with this, we say that a set of functions $\mathcal{W}$ is *odd* if for each $v \in \mathcal{W}$ there exists $v' \in \mathcal{W}$ such that $v(x) = -v'(-x)$ for all $x \in \mathbb{R}^k$. Obviously, a set is odd when all utilities in it are odd.

The set $\mathcal{W}$ in Example 1 consists of two odd functions and is thus odd. In the following example, the set is odd even though none of the functions are.

Example 2. *Consider again* $k = 2$ *and* $\mathcal{W}$ *comprised of* $v(x_1, x_2) = f(x_1) + f(x_2)$ *and* $v'(x_1, x_2) = -f(-x_1) - f(-x_2)$*, where* $f(a) = a^{.25}$ *for* $a > 0$ *and* $f(a) = -(-a)^{.5}$ *for* $a \leq 0$*. Since* f *is not odd, neither are* v *nor* v'*. But* $v(x_1, x_2) = -v'(-x_1, -x_2)$ *for all* $x \in \mathbb{R}^2$*, which means that* $\mathcal{W} = \{v, v'\}$ *is odd.*

We say that a preference relation admits a *Symmetric Cautious Utility representation* if it admits a Cautious Utility representation with an odd set $\mathcal{W}$. *Symmetric Incautious Utility* is defined similarly.

In some of our results below, we deliberately assume symmetry. This is not because symmetry is necessarily appealing—there are many cases in which one may want to use utilities, and sets of utilities, that are not odd. However, assuming symmetry for some of our results will be helpful to show that Cautious Utility can generate reference effects *even* in the presence of symmetry. As we discuss below (Remark 3), adding typical asymmetries will only strengthen our results.

3 Caution, Loss Aversion, and the Endowment Effect

To study reference effects such as loss aversion and the endowment effect, we need to incorporate the role of the reference point. We take the standard approach, as in Prospect Theory, and define Cautious Utility on *relative changes* with respect to a *given* reference point: if y is the final allocation and r the reference bundle, then each bundle is viewed as $x = y - r$. For example, if the reference point is the current endowment, a bundle that returns an extra \$3 and takes away 2 mugs is evaluated as $(3, -2)$. This allows for direct comparisons with other models and gives complete flexibility on the reference point—it could be the endowment, the allocation of others, the expectation, *etc*. For now, we assume that the reference point is deterministic; we extend to stochastic reference points in Section 3.4.

Endowment Effect. First, we define the Willingness to Pay (WTP) and the Willingness to Accept (WTA). $\mathrm{WTP}_i(m)$ is the maximum amount of money that the agent is willing to pay to purchase m units of good $i \in \{2, \ldots, k\}$. Thus, it satisfies

$$0 \sim me_i - \mathrm{WTP}_i(m)e_1.$$

In words, the individual is indifferent between not buying (that is, getting 0) and acquiring m units of good i while foregoing $\mathrm{WTP}_i(m)$ units of money. Similarly, $\mathrm{WTA}_i(m)$ is the minimum amount of money that the agent is willing to accept to sell m units of the i-th good, thus satisfying

$$0 \sim -me_i + \mathrm{WTA}_i\,(m)\,e_1.$$ [1]

A preference $\succcurlyeq$ *exhibits the endowment effect* for good i if $\mathrm{WTA}_i(m) \geq \mathrm{WTP}_i(m)$ for all $m \in \mathbb{R}_+$. It exhibits the endowment effect if this is the case for all $i \in \{2, \ldots, k\}$. It exhibits the *opposite of the endowment effect* when the inequality is reversed.

[1] Formally, for each $i \in \{2, \ldots, k\}$, $\mathrm{WTP}_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mathrm{WTA}_i : \mathbb{R}_+ \to \mathbb{R}_+$ are defined by $\mathrm{WTP}_i\,(m) = \max\left\{l \in \mathbb{R}_+ : \delta_{me_i - le_1} \succcurlyeq \delta_0\right\}$ and $\mathrm{WTA}_i(m) = \min\left\{l \in \mathbb{R}_+ : \delta_{-me_i + le_1} \succcurlyeq \delta_0\right\}$. In our model, they are always well-defined and satisfy the simpler conditions above.

Loss Aversion. Following Kahneman and Tversky (1979), we use loss aversion to indicate the rejection of even bets around zero (see also Markowitz, 1952).[2] Formally, a preference $\succcurlyeq$ is *loss averse* on dimension $i \in \{1, \ldots, k\}$, if for each $a \in \mathbb{R}_{++}$

$$\delta_0 \succcurlyeq \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}.$$

It is loss averse if this is the case for all $i \in \{2, \ldots, k\}$. Gain seeking and loss neutrality are defined analogously, with $\succcurlyeq$ replaced by $\preccurlyeq$ and $\sim$, respectively. Finally, $\succcurlyeq$ is strictly loss averse (resp. gain seeking) on dimension i if this also holds strictly for some $a \in \mathbb{R}_{++}$.

Caution and Reference Effects. We are now ready to state our results on Cautious Utility and reference effects. We begin focusing on *Symmetric* Cautious Utility: since it is well-known how asymmetry in the treatment of gains and losses can generate both the endowment effect and loss aversion, to isolate the effects of caution we rule out this possibility. (The following discussion, and Remark 3 below, show how key results generalize.)

Proposition 1. *The following statements are true:*

1. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation, then (i) it exhibits the endowment effect and (ii) it is loss averse.*
2. *If $\succcurlyeq$ admits a Symmetric Incautious Utility representation, then (i) it exhibits the opposite of the endowment effect and (ii) it is gain seeking.*

Proposition 1 links loss aversion and the endowment effect to caution: when there are multiple utilities, if individuals apply caution then we have both loss aversion and the endowment effect; if they are incautious, we get the opposite. This result gives a new interpretation to both reference effects: they may derive *not from an asymmetry* in the treatment of gains and losses but from *uncertainty about the utility* joint with *caution*.

Proposition 1 has a simple intuition, illustrated by the following two examples.

Example 1 (cont.). *Consider again Example 1, where $k = 2$ and $\mathcal{W}$ consists of $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. The WTP is the amount \$$z \geq 0$ such that $(0,0) \sim (-z, m)$ for $m \geq 0$. Then,*

$$V(0,0) = \min\{0,0\} = 0 \qquad V(-z,m) = \min\{-z+m, -z+2m\} = -z+m$$
$$\Rightarrow \quad 0 = -\mathrm{WTP}_2(m) + m \quad \Rightarrow \quad \mathrm{WTP}_2(m) = m.$$

[2] The term loss aversion is often used also to denote the asymmetry parameter of the CPT model (the coefficient λ; Section 1.1), like risk aversion is at times used to indicate both the behavioral notion and the concavity parameter in Expected Utility.

Similarly, WTA is the amount $\$r \geq 0$ *such that* $(0,0) \sim (r,-m)$. *Then*

$$V(r,-m) = \min\{r-m, r-2m\} = r-2m \quad \Rightarrow \quad 0 = \mathrm{WTA}_2(m) - 2m \quad \Rightarrow \mathrm{WTA}_2(m) = 2m.$$

Therefore, $\mathrm{WTA}_2(m) = 2m > m = \mathrm{WTP}_2(m)$ *for all* $m > 0$*, the endowment effect. It is easy to see that results are inverted with the Incautious model.*

In this example, utilities are not only odd (symmetric) but even linear. The crucial feature is that they entail a different trade-off, or 'exchange rate,' between money and mugs: a mug is worth either \$1 or \$2. Together with caution, this generates the endowment effect. When buying, a cautious agent is pessimistic about the value of mugs, and the WTP is computed with the utility that values mugs at \$1. When selling, the opposite happens, and the WTA is calculated with the utility that values mugs at \$2. (Below, we show how it is generally true that WTA and WTP correspond to the highest and lowest values.) This creates the endowment effect. The opposite holds with Incautious Utility.[3]

Despite the endowment effect, this agent is loss neutral: $V(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}) = \min\{0,0\} = 0 = V(\delta_0)$ for $i = 1,2$. This shows that in Cautious Utility, the endowment effect may emerge *even without loss aversion*. We have an endowment effect when different utilities entail different trade-offs between a good and money. But this is not the only way to obtain the endowment effect, as illustrated in the example below.

Example 2 (cont.). *Consider again Example 2, where* $k = 2$ *and* $\mathcal{W}$ *includes* $v(x_1, x_2) = f(x_1) + f(x_2)$ *and* $v'(x_1, x_2) = -f(-x_1) - f(-x_2)$*, with* $f(a) = a^{.25}$ *for* $a > 0$ *and* $f(a) = -(-a)^{.5}$ *for* $a \leq 0$*. As opposed to Example 1, here money and mugs are treated identically, but the agent considers different 'curvatures.' Consider the lottery* $\frac{1}{2}\delta_{ae_1} + \frac{1}{2}\delta_{-ae_1}$, $a > 0$, $a \neq 1$*, and note that at least one of the two utilities must return a negative expected utility, that is,*

$$\min\left\{\frac{1}{2}v(ae_1) + \frac{1}{2}v(-ae_1), \frac{1}{2}v'(ae_1) + \frac{1}{2}v'(-ae_1)\right\} = \min\left\{\frac{1}{2}a^{.25} - \frac{1}{2}a^{.5}, \frac{1}{2}a^{.5} - \frac{1}{2}a^{.25}\right\} < 0.$$

But then, since $v(0) = v'(0) = 0$*, also the minimum of the certainty equivalents must be negative, and so must be the value of the lottery, that is,*

$$V\left(\frac{1}{2}\delta_{ae_1} + \frac{1}{2}\delta_{-ae_1}\right) < 0 = V(\delta_0).$$

Therefore, we have strict loss aversion on money; the same is true for mugs, since utilities are

[3] This also generates an 'endowment-effect in mugs-terms.' The minimum number of mugs the agent is willing to accept to give up \$1 is 1, but to obtain \$1, the individual is willing to 'pay' only .5 mugs.

the same. For the endowment effect, for any $m \geq 0$[4]

$$\mathrm{WTP}_2(m) = \min\{\sqrt{m}, m^2\} \qquad \text{and} \qquad \mathrm{WTA}_2(m) = \max\{\sqrt{m}, m^2\}.$$

Hence $\mathrm{WTA}_2(m) > \mathrm{WTP}_2(m)$ *for all* $m \neq 1$.

In this example, the individual considers two utilities, neither of which is odd but that are specular to each other. As they are not odd, the utility of ae_1 is not minus the utility of $-ae_1$, creating an asymmetry. But since $v(x) = -v'(-x)$, if one utility has an asymmetry in favor of one direction, the other has the opposite. This means that the expected utility of $\frac{1}{2}\delta_{ae_1} + \frac{1}{2}\delta_{-ae_1}$ is negative for at least one utility, and so is the certainty equivalent. Cautious agents, who consider the most pessimistic certainty equivalent, must assign a negative value to this lottery, becoming loss averse. In general, we show below that we have strict loss aversion whenever at least one utility is not odd: insofar as the individual entertains the possibility of weighting gains and losses differently, the model generates loss aversion.

These two examples illustrate two 'forces' that generate the endowment effect under Cautious Utility. First, uncertainty about the trade-off between money and goods, as in Example 1, leads to an endowment effect even without loss aversion. Second, uncertainty about how to aggregate gains and losses, as in Example 2, gives both loss aversion and the endowment effect. (Below, we show that we can also have loss aversion without the endowment effect.) In Section 4, we discuss how these forces are linked to empirical evidence.

Certainty Effect. To see how loss aversion and the endowment effect relate to the certainty effect, following Kahneman and Tversky (1979) we say that $\succcurlyeq$ exhibits the certainty effect if for all $x, y \in \mathbb{R}$ and $\alpha, \beta \in (0,1)$, if $\alpha\delta_{ye_1} + (1-\alpha)\delta_0 \sim \delta_{xe_1}$, then $\alpha\beta\delta_{ye_1} + (1-\alpha\beta)\delta_0 \succcurlyeq \beta\delta_{xe_1} + (1-\beta)\delta_0$.

Cautious Utility exhibits the certainty effect while ruling out the opposite violation of Independence (the case where $\succcurlyeq$ above is reversed and holds strictly at least once), therefore offering a unified explanation of the three phenomena. This follows directly from the functional form. Intuitively, while the agent acts with caution when evaluating general lotteries, caution does not play any role when evaluating monetary amounts—the monetary certainty equivalent of a degenerate lottery that yields \$$m$ is m for any utility. In fact, the implication is deeper. Section 5 shows that Cautious Utility can be *derived* from positing a form of certainty effect on risk preferences.

We conclude this discussion with a remark on the role of symmetry.

Remark 3. *We focus on* Symmetric *Cautious Utility to highlight the role of caution even under symmetry. Yet, even if utilities are not odd and neither is the set, Cautious Utility gives*

[4] $V(-z, m) = \min\{c(me_2 - ze_1, v), c(me_2 - ze_1, v')\} = 0$ implies $\min\{m^{.25} - z^{.5}, m^{.5} - z^{.25}\} = 0$, hence $\mathrm{WTP}_2(m) = \min\{\sqrt{m}, m^2\}$. An identical reasoning works for WTA. Proposition 2 below provides convenient formulas for these computations.

loss aversion and the endowment effect if at least one *utility overweights losses. In general, asymmetries of this kind simply add on to the other forces highlighted above. (This follows immediately from Propositions 2 and 3 below.) If all utilities underweight losses, Cautious Utility gives gain seeking and may, but need not, exhibit the endowment effect—depending on the relative strength of the underweight of losses and multiplicity of utilities. To illustrate, take* $\mathcal{W} = \{v, v'\}$ *where* $v(x_1, x_2) = f(x_1) + f(x_2)$ *and* $v'(x_1, x_2) = f(x_1) + \gamma f(x_2)$*, with* $f(a) = a$ *for* $a > 0$ *and* $f(a) = \lambda a$ *for* $a \leq 0$*, with* $\lambda > 0$ *and* $\gamma > 1$*. Both utilities underweight losses if* $\lambda < 1$*, while the bigger* γ *is, the bigger the uncertainty about the trade-off. Note that for* $m \geq 0$, $\mathrm{WTA}_2(m) = m\gamma\lambda$ *and* $\mathrm{WTP}_2(m) = \frac{m}{\lambda}$*, which means* $\mathrm{WTA}_2(m) \geq \mathrm{WTP}_2(m)$ *if and only if* $\lambda \geq \frac{1}{\sqrt{\gamma}}$*. Therefore, even when* $\lambda < 1$ *and all the utilities underweight losses and would give the opposite of the endowment effect, the model may still return the endowment effect if there is enough uncertainty about trade-offs (*γ *high enough).*

3.1 Strict Behavior and Comparative Statics

We now provide tools to easily compute the WTA and WTP, characterize when the former is strictly larger, and generate comparative statics. We focus on Cautious Utility, but 'specular' results hold for Incautious Utility. For any strictly increasing and continuous utility v, let WTA_i^v denote the WTA under Expected Utility, that is, the amount such that $v(-me_i + \mathrm{WTA}_i^v(m)e_1) = v(0)$. Define WTP_i^v analogously.

Proposition 2. *If* $\succcurlyeq$ *admits a Cautious Utility representation, then for each* $m \in \mathbb{R}_+$ *and* $i \in \{2, \ldots, k\}$

$$\mathrm{WTA}_i(m) = \sup_{v \in \mathcal{W}} \mathrm{WTA}_i^v(m) \quad \text{and} \quad \mathrm{WTP}_i(m) = \inf_{v \in \mathcal{W}} \mathrm{WTP}_i^v(m).$$

Proposition 2, that *does not* assume Symmetry, shows that the WTA in Cautious Utility is simply the highest of the WTAs obtained by the utilities in $\mathcal{W}$, while the WTP is the lowest of the WTPs. Caution leads individuals to focus on opposite ends of the range of values. This result has a series of implications. To derive them, note that Symmetry implies that the 'span' of WTAs and WTPs is the same.

Observation 1. *If* $\mathcal{W}$ *is odd, then* $\{\mathrm{WTA}_i^v(m) : v \in \mathcal{W}\} = \{\mathrm{WTP}_i^v(m) : v \in \mathcal{W}\}$ *for all* $m \in \mathbb{R}_+$*.*[5]

This shows how Symmetry guarantees that there is no built-in unevenness between WTA and WTP. Then, Proposition 2 immediately implies that Symmetric Cautious Utility exhibits the endowment effect: the WTA and WTP take the opposite ends of the same range. It also readily provides conditions under which the endowment effect holds strictly.

[5] This immediately follows by noting that if v and v' in $\mathcal{W}$ are such that $v(x) = -v'(-x)$ for all x, then $\mathrm{WTA}_i^v = \mathrm{WTP}_i^{v'}$.

Corollary 1. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation. Given $i \in \{2, ..., k\}$ and $m > 0$, the following statements are equivalent:*

(i) $\mathrm{WTA}_i(m) > \mathrm{WTP}_i(m)$*;*

(ii) There exist $v, v' \in \mathcal{W}$ such that $\mathrm{WTA}_i^v(m) \neq \mathrm{WTA}_i^{v'}(m)$*;*

(iii) There exist $v, v' \in \mathcal{W}$ such that $\mathrm{WTP}_i^v(m) \neq \mathrm{WTP}_i^{v'}(m)$.

It is enough that two utilities in $\mathcal{W}$ differ either in their WTA or in their WTP to create a strict wedge between the WTA and WTP of the agent. This can also be expressed using the standard notion of Marginal Rate of Substitution (MRS). Recall that, given a differentiable utility $v : \mathbb{R}^k \to \mathbb{R}$, the MRS of good i with respect to money is $\mathrm{MRS}_i^v(x) = \frac{v_i(x)}{v_1(x)}$, where v_j denotes the partial derivative of v with respect to x_j. We now show that if two utilities have different MRSs between money and goods, we have an endowment effect.

Corollary 2. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation. Given $i \in \{2, ..., k\}$, if each $v \in \mathcal{W}$ is continuously differentiable and there exist $v, v' \in \mathcal{W}$ such that $\mathrm{MRS}_i^v(x) \neq \mathrm{MRS}_i^{v'}(x)$ for all $x \in \mathbb{R}^k$ with $x_1, x_i \neq 0$, then $\mathrm{WTA}_i(m) > \mathrm{WTP}_i(m)$ for all $m \in \mathbb{R}_{++}$.*

This result also provides simple comparative statics. As standard, we use the ratio between WTA and WTP to define the strength of the endowment effect. As opposed to other models, this strength can vary with the good or the quantity of each good.

Example 3. *Consider $v_1(x_1, x_2, x_3) = x_1 + x_2 + \alpha x_3$ and $v_2(x_1, x_2, x_3) = x_1 + x_2 + \beta x_3$, with $\alpha > \beta > 0$. If $\mathcal{W} = \{v_1, v_2\}$, there is an endowment effect for good 3 but not for good 2: $\frac{\mathrm{WTA}_3(m)}{\mathrm{WTP}_3(m)} = \frac{\alpha}{\beta} \neq 1 = \frac{\mathrm{WTA}_2(m)}{\mathrm{WTP}_2(m)}$ for all $m > 0$.*

Example 4. *Consider $v_1(x_1, x_2) = x_1 + \alpha x_2$ (for $\alpha > 0$) and $v_2(x_1, x_2) = x_1 + x_2^3$. If $\mathcal{W} = \{v_1, v_2\}$, the endowment effect varies with the quantity: for $m \neq m'$ with $mm' \neq \alpha$, we have $\frac{\mathrm{WTA}_2(m)}{\mathrm{WTP}_2(m)} \neq \frac{\mathrm{WTA}_2(m')}{\mathrm{WTP}_2(m')}$.*

In general, the strength of the endowment effect depends on the range of possible tradeoffs that the agent considers for each good: the endowment effect is more substantial when the range is larger. For any set A, denote by $\mathrm{co}(A)$ its convex hull.

Corollary 3. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation with finite set $\mathcal{W}$. For each $i, j \in \{2, \ldots, k\}$ and $m, m' \in \mathbb{R}_{++}$,*

$$\mathrm{co}\left(\left\{\mathrm{WTA}_i^v(m) : v \in \mathcal{W}\right\}\right) \supset \mathrm{co}\left(\left\{\mathrm{WTA}_j^v(m') : v \in \mathcal{W}\right\}\right) \implies \frac{\mathrm{WTA}_i(m)}{\mathrm{WTP}_i(m)} > \frac{\mathrm{WTA}_j(m')}{\mathrm{WTP}_j(m')}.$$

Turning to loss aversion, strict loss aversion holds when at least one utility is not odd.

Proposition 3. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation. Given $a \in \mathbb{R}_{++}$ and $i \in \{1, \ldots, k\}$, $\succcurlyeq$ is strictly loss averse on dimension i at a, that is, $\delta_0 \succ \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$, if and only if $-v(-ae_i) \neq v(ae_i)$ for some $v \in \mathcal{W}$.*

3.2 Loss Aversion, Endowment Effect, and Non-Expected Utility

Under Cautious Utility, loss aversion, the endowment effect, and violations of Expected Utility are conceptually related and stem from the same source. However, this *does not* mean that they should manifest themselves together or even be correlated.

Observation 2. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation, then:*

(i) The agent may be loss neutral on all dimensions yet exhibit the endowment effect strictly ($\mathrm{WTA}_i(m) > \mathrm{WTP}_i(m)$ for all $m > 0$ and for all $i \in \{2, ..., k\}$).

(ii) The agent may be strictly loss averse on all dimensions yet exhibit no endowment effect ($\mathrm{WTA}_i(m) = \mathrm{WTP}_i(m)$ for all $m > 0$ and for all $i \in \{2, ..., k\}$).

(iii) The agent may exhibit the certainty effect for monetary lotteries yet be loss neutral or exhibit no endowment effect.

(iv) The agent may follow Expected Utility for monetary lotteries yet exhibit the endowment effect. The agent may follow Expected Utility for monetary lotteries with only gains or only losses yet exhibit loss aversion.

Points (i) and (ii) show that the endowment effect and loss aversion are not necessarily related: we may have one without the other, and they need not be correlated. Recall that loss aversion is due to the uncertainty on how to aggregate gains and losses, while the endowment effect also relates to the uncertainty on the trade-off between goods and money. Example 1 provides a simple instance of (i), endowment effect without loss aversion.

For (ii), loss aversion without endowment effect, consider $\mathcal{W} = \{v, v'\}$, where $v(x_1, x_2) = (x_1 + x_2)^{.5}$ if $x_1 + x_2 \geq 0$ and $v(x_1, x_2) = -(-(x_1 + x_2))^{.25}$ otherwise; and $v'(x_1, x_2) = -v(-x_1, -x_2)$. Similar calculations as in Example 2 show strict loss aversion on each dimension, but no endowment effect.

Points (iii) and (iv) show the relation with the certainty effect. We can have violations of Expected Utility without loss aversion or the endowment effect, and the endowment effect independently of the certainty effect on monetary lotteries. This is intuitive: we have seen how the endowment effect can emerge from uncertainty about the trade-off between different goods, while violations of Expected Utility for monetary lotteries are due to uncertainty about how to evaluate monetary amounts. For (iii), take v and v' such that $v(x_1, x_2) = f(x_1 + x_2)$ and $v'(x_1, x_2) = g(x_1 + x_2)$ for some strictly increasing, continuous, and odd f and g. For the first part of (iv), use the set $\mathcal{W}$ of Example 1. For the second part, use the same set used for (ii). In most other models, like CPT, loss aversion and the endowment effect are linked by one parameter, while non-Expected Utility is separate, conceptually and empirically. In Cautious Utility, instead, non-Expected Utility is at the core, but it can manifest itself in ways that can generate loss aversion, the endowment effect, both, or neither. We will revisit Observation 2 when discussing the empirical evidence in Section 4.

3.3 Cautious Utility and Prospect Theory are Fully Distinct

We now show that Cautious Utility is not only conceptually different from CPT, but also fully behaviorally distinct, in the sense that the only preferences compatible with both models are those featuring none of the phenomena we are interested in.

To define CPT, we begin with the case in which only monetary lotteries are involved ($k = 1$). Consider a strictly increasing and continuous utility function $v : \mathbb{R} \to \mathbb{R}$ such that $v(0) = 0$, and two probability distortion functions $w^+, w^- : [0,1] \to [0,1]$, that are strictly increasing, continuous, and take value 0 at 0 and 1 at 1. For each lottery p over $\mathbb{R}$ with compact support, denote by F_p its corresponding CDF. Define

$$\mathrm{CPT}_{v,w^+,w^-}(p) = \int_{[0,\infty)} v(x)\, dw^+\left(F_p(x)\right) + \int_{(-\infty,0]} v(x)\, dw^-\left(F_p(x)\right).$$

This is similar to Expected Utility with utility v, except that probabilities are distorted (in their cumulative distribution). A widely used special case assumes $v(-x) = -\lambda v(x)$ for $x > 0$, where λ denotes the coefficient of loss aversion and regulates the asymmetry in the treatment of gains and losses, with $\lambda > 1$ capturing loss aversion.

There are two ways to extend CPT to bundles: probability distortions and reference-dependence can be applied to each dimension separately, or the agent can first compute the utility of each bundle, and then compare it to a ‘global’ reference point with utility zero. We consider both cases.

The first approach, widespread in the applied literature and studied by Bleichrodt et al. (2009), considers for each $i \in \{1, \ldots, k\}$ a strictly increasing and continuous utility $u_i : \mathbb{R} \to \mathbb{R}$ with $u_i(0) = 0$. For each lottery p over $\mathbb{R}^k$, let p_i be the marginal distribution over dimension i. Preferences admit an *Additive CPT representation* if they are represented by

$$V(p) = \sum_{i=1}^{k} \mathrm{CPT}_{u_i,w^+,w^-}(p_i).$$

The second approach was proposed by Tversky and Kahneman (1981, p. 456) and formally derived by Wakker and Tversky (1993). The agent has a strictly increasing and continuous utility over bundles $u : \mathbb{R}^k \to \mathbb{R}$, with $u(0) = 0$. For each lottery p, denote by p_u the distribution it induces over *utility* levels.[6] Preferences admit a *u-CPT representation* if

$$V(p) = \mathrm{CPT}_{v,w^+,w^-}(p_u).$$

Before stating our result, we need an extra property. A finite Cautious Utility representation

[6] That is, for all Borel subsets B of $\mathbb{R}$ and for all $p \in \Delta$, $p_u(B) = p(\{x \in \mathbb{R}^k : u(x) \in B\})$.

is *essential* if for each $\tilde{v} \in \mathcal{W}$ there exists $p \in \Delta$ such that

$$\min_{v \in \mathcal{W}} c(p, v) < \min_{v \in \mathcal{W} \setminus \{\tilde{v}\}} c(p, v).$$

This guarantees that no utility is redundant and that the set includes only the genuinely relevant elements. In all our examples above, the set is essential.

Proposition 4. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation as well as either an Additive CPT or a u-CPT representation, then $\succcurlyeq$ admits an Expected Utility representation. Moreover, if the representation is also finite and essential, then $\succcurlyeq$ is loss neutral and exhibits no endowment effect.*

Proposition 4 extends a similar result in Cerreia-Vioglio et al. (2015) that shows how Cautious Expected Utility and Rank Dependent Expected Utility are fully distinct.[7]

3.4 Stochastic Reference Points

Cautious Utility is applied to changes relative to a reference point: for a final allocation y and reference bundle r, Cautious Utility is applied to $x = y - r$. What if the reference point is stochastic? For example, the reference point may be the current portfolio of financial assets or a distribution of payoffs that the individual is expecting to receive.

Like we defined changes relative to a fixed reference point by 'subtracting' it, we can do the same when the reference point is a lottery. To illustrate, we proceed in steps. Given a reference lottery r that pays x_i with probability $r(x_i)$, the (degenerate) final allocation y is evaluated as the lottery that pays $y - x_i$ with probability $r(x_i)$: for example, if $k = 1$ and the reference point is $r = \frac{1}{2}\$10 + \frac{1}{2}\$0$, the final allocation \$7 is evaluated by Cautious Utility as the lottery $\frac{1}{2}(-\$3) + \frac{1}{2}\$7$. Intuitively, it is as if the agent were 'issuing' the reference lottery and paying its prizes in every contingency. Denote the subtraction of a lottery $r \in \Delta$ from $y \in \mathbb{R}^k$ as $y - r$; $r - y$ is defined similarly.

To extend this approach to stochastic final allocations, we need to take into account the correlation with the reference lottery. Consider a final allocation q and reference lottery r, suppose both are simple lotteries, and denote by $P_{q,r}(x, y)$ the joint probability that q returns x and r returns y. Then, define $q - r \in \Delta$ simply as $\sum_{x,y} \delta_{x-y} P_{q,r}(x, y)$.[8] For example, if the final allocation is $\frac{1}{2}\delta_x + \frac{1}{2}\delta_y$ and the reference lottery is $\frac{1}{2}\delta_z + \frac{1}{2}\delta_w$ then: if the two lotteries are independent, this is evaluated as $\frac{1}{4}\delta_{x-z} + \frac{1}{4}\delta_{x-w} + \frac{1}{4}\delta_{y-z} + \frac{1}{4}\delta_{y-w}$; if the two lotteries are perfectly correlated so that q returns x if and only if r returns z, this is evaluated as

[7] To see why essentiality is important, suppose $k = 1$ and $\mathcal{W} = \{v, v'\}$, where v is strictly increasing and concave and v' is such that $v(x) = -v'(-x)$ for all $x \in \mathbb{R}$. The set $\mathcal{W}$ is odd. But since v' is convex, it will never be used. Therefore, preferences are Expected Utility with utility v, which, by concavity, is loss averse.

[8] In general, define the map $T : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ by $T(x, y) = x - y$ for all $x, y \in \mathbb{R}^k$. Given $q, r \in \Delta$ with joint probability $P_{q,r}$, denote $q - r \in \Delta$ by $(q - r)(B) = P_{q,r}(T^{-1}(B))$ for all Borel sets B of $\mathbb{R}^k$.

$\frac{1}{2}\delta_{x-z} + \frac{1}{2}\delta_{y-w}$. Importantly, the value of the final allocation p when the reference point is p itself is 0: this is relevant, for example, for calculating the WTA of a lottery tickets, as it implies that keeping the lottery corresponds to 0.[9]

Endowment Effect for Lottery Tickets. Our results extend to the endowment effect for lotteries, widely documented empirically (see Section 4). Similarly to the deterministic case, we define WTA and WTP for a lottery p as

$$\mathrm{WTP}(p) = \max\{l \in \mathbb{R} : p - le_1 \succcurlyeq 0\} \text{ and } \mathrm{WTA}(p) = \min\{l \in \mathbb{R} : le_1 - p \succcurlyeq 0\}.$$

Like above, it can be shown that $\mathrm{WTP}(p)$ and $\mathrm{WTA}(p)$ are well defined and that

$$p - \mathrm{WTP}(p)\, e_1 \sim 0 \qquad \text{and} \qquad \mathrm{WTA}(p)\, e_1 - p \sim 0.$$

Given a strictly increasing and continuous utility v, let $\mathrm{WTA}^v(p)$ denote the WTA for p of an Expected Utility maximizer with utility v; define $\mathrm{WTP}^v(p)$ analogously. It is routine to check that they are well-defined under Cautious or Incautious Utility. Our results on WTA and WTP readily extend to this case of lotteries. (The proof follows from arguments which are identical to those used to prove Propositions 1 and 2 and is therefore omitted.)

Proposition 5. *If $\succcurlyeq$ admits a Cautious Utility representation $\mathcal{W}$ and $p \in \Delta$, then*

1. *If $\mathcal{W}$ is odd, then* $\mathrm{WTA}(p) \geq \mathrm{WTP}(p)$*;*
2. $\mathrm{WTA}(p) = \sup_{v \in \mathcal{W}} \mathrm{WTA}^v(p)$ *and* $\mathrm{WTP}(p) = \inf_{v \in \mathcal{W}} \mathrm{WTP}^v(p)$.

3.5 Discussion

We now turn to discuss convenient functional forms, the role of symmetry, and other implications of caution.

Convenient Functional Forms for Estimation. To bring a model to data, one needs a functional form with few parameters that can be easily estimated. We now discuss two convenient forms for Cautious Utility, obtained by generalizing Examples 1 and 2.

First, assume $\mathcal{W} = \{v_1, v_2\}$, where $v_1(x_1, x_2) = x_1 + \alpha x_2$ and $v_2(x_1, x_2) = x_1 + \beta x_2$ for some $\alpha, \beta \in \mathbb{R}_{++}$. Despite involving only two parameters, this special case can capture any range of trade-offs between money and the other good. As it generalizes Example 1, it also

[9] In accounting for the correlation between the reference lottery and the final allocation, our approach departs from the formulation of Köszegi and Rabin (2006, 2007) and adopts an approach closer to Schmidt et al. (2008). This is evident when the final allocation is the reference lottery itself, evaluated as 0 in our model, while in Köszegi and Rabin (2006, 2007) it is treated as a non-degenerate lottery.

captures the endowment effect, since $\text{WTA}_2(1) = \max\{\alpha, \beta\}$ and $\text{WTP}_2(1) = \min\{\alpha, \beta\}$, making the estimation immediate.

Alternatively, take $\mathcal{W} = \{v_1, v_2\}$ where $v_1(x_1, x_2) = g(x_1) + g(x_2)$ and $v'(x_1, x_2) = -g(-x_1) - g(-x_2)$, where $g(a) = a^\alpha$ for $a > 0$ and $g(a) = -(-a)^\beta$ for $a \leq 0$ with $\alpha, \beta \in \mathbb{R}_{++}$. As a generalization of Example 2, this two-parameters specification allows for the endowment effect, loss aversion, as well as different risk attitudes for gains and losses.

On Asymmetry of Utility Function(s). Our results mainly focus on *Symmetric* representations. This allows us to demonstrate in the starkest way how caution alone generates the endowment effect and loss aversion even without any asymmetry. We emphasize that our goal is not to argue that utilities are genuinely symmetric, but to propose a different force—caution—that may be at play possibly *jointly* with asymmetry. As noted in Remark 3, assuming that utilities overweight losses only strengthen our implications, giving us two forces pushing for the endowment effect and loss aversion. Because these forces do not coincide, they may together contribute to improving the fit with data. Which of the two is most important in each context, and whether we should expect one to be particularly relevant, is an empirical question we do not aim at addressing here.

Choice. Since Kahneman et al. (1990), some experiments measure not only WTA and WTP but also "Choice": the amount of money that makes the agent indifferent between receiving it or receiving one unit of the object, that is, \$$z$ such that $(z, 0) \sim (0, 1)$. Choice typically falls between WTA and WTP, though often very close to WTP. This is easy to obtain in Cautious Utility. With the utilities in Example 1, Choice coincides with WTP; with those in Example 2, it is strictly between the WTA and WTP (except for $m = 1$).

Exchange Asymmetries and Status Quo Bias. It is widely documented that individuals are status quo biased and often reject exchanges favoring to keep their current endowment (status quo) even when no money is involved (e.g., Knetsch, 1989). For example, an individual may be given a mug and asked to exchange it for a chocolate bar, or given a chocolate bar and asked to exchange it for a mug, and reject both. Under Cautious Utility, this happens whenever the individual considers a utility for which the mug is better, and another for which the chocolate bar is better. As a simple example, if mugs and chocolate bars are dimension 2 and 3, extend Example 1 and suppose $\mathcal{W} = \{v_1, v_2\}$ with $v_1(x_1, x_2, x_3) = x_1 + 2x_2 + x_3$ and $v_2(x_1, x_2, x_3) = x_1 + x_2 + 2x_3$. Then, $V(0, -1, 1) = \min\{-2 + 1, -1 + 2\} = -1 < 0 = V(0, 0, 0)$ and $V(0, 1, -1) = \min\{2 - 1, -2 + 1\} = -1 < 0 = V(0, 0, 0)$.

Randomization. Preferences under Cautious Utility are convex in probabilities, allowing for strict preference for randomization while ruling out the opposite (see also Cerreia-Vioglio et al. 2019). To illustrate, consider the same example above of an individual unsure about the trade-offs between mugs and chocolate bars. The individual is indifferent between one mug and one chocolate bar, since $V(0,1,0) = \min\{2,1\} = 1$ and $V(0,0,1) = \min\{1,2\} = 1$, but strictly prefers a 50/50 lottery p between the two, since $V(p) = \min\{0.5 \cdot 1 + 0.5 \cdot 2, 0.5 \cdot 2 + 0.5 \cdot 1\} = 1.5$. Unsure which is best, our individual prefers to 'hedge'.

4 Cautious Utility and Empirical Evidence

We now relate Cautious Utility to empirical evidence. We show how it can accommodate data incompatible with leading alternative models and may be used to organize widely documented patterns, although it fails to capture other regularities. Given its prominence, this section will contrast Cautious Utility mostly with CPT, yet we note that other models are also compatible with some of the patterns below. Our aim is not to run a competition between models based on theoretical elegance and descriptive validity, but rather to demonstrate the merit of adding caution as a potential source of reference effects. Naturally, we cannot discuss in full the immense evidence regarding our phenomena of interest, so we focus on differentiating aspects; we refer to DellaVigna (2009) and O'Donoghue and Sprenger (2018) for recent surveys of reference effects.

The Endowment Effect: Varying Strength and Information. A key aspect of the evidence of the endowment effect is how its strength and frequency vary *substantially* across goods and environments. It appears strongest with less common goods, it is reduced for ordinary market goods, and disappears for objects of known value, like monetary tokens.[10]

In addition, the endowment effect is severely affected by information. For example, Weaver and Frederick (2012) show that when subjects are given information on market values that suggest a high price, above typical WTAs, the endowment effect is very pronounced; when the information points to an intermediate price, between typical WTAs and WTPs, the endowment effect *shrinks or disappears*; when the information points to a very low price, below typical WTPs, the endowment effect increases again. Shogren et al. (1994) and List (2004a) find that the endowment effect is reduced by showing continuous trading in a public auction or by providing trading experience. List (2003, 2004b) shows how experienced traders exhibit much less endowment effect for goods they frequently trade.

Accommodating these patterns in CPT requires that the 'pain of losing' encoded by the

[10] In their widely-cited metastudy of the empirical literature, Horowitz and McConnell (2002, p. 427) describe the heterogeneity of ratios between WTAs and WTPs across forty-five studies and note: "With regard to patterns in the observed ratios, we find that, on average, the less the good is like an 'ordinary market good,' the higher is the ratio. The ratio is highest for public and non-market goods, next highest for ordinary private goods, and lowest for experiments involving forms of money. A generalization of this pattern holds even when we account for differences in survey design: ordinary goods have lower ratios than non-ordinary ones. This pattern is the major result we discover."

overweighting of losses not only varies across dimensions, but (i) it is higher for unfamiliar goods than for familiar ones; and (ii) it varies substantially and non-monotonically with information—it disappears when subjects observe trading or are informed of intermediate prices, while it increases if told very high or very low prices. These assumptions, particularly the second one, seem to us a less plausible way of modeling how the 'pain of losing' should vary.

Cautious Utility is not only compatible with these patterns but predicts them under very natural assumptions. Here, the endowment effect varies in strength depending on the uncertainty about trade-offs. We can expect more uncertainty, thus a bigger endowment effect, for less common goods, for inexperienced traders, and in the absence of information. We can expect the opposite for familiar goods, for professional traders, or after receiving information that brings utilities to converge, such as with market prices between the WTA and the WTP. When information increases doubts about trade-offs, such as with very high or very low market prices, the endowment effect may instead increase. Finally, we should see no effect when there is *no* uncertainty about the trade-off—e.g., monetary tokens.

Loss Aversion and its Relation to the Endowment Effect. As CPT ascribes both loss aversion and the endowment effect to the same parameter, it implies that (i) we cannot observe one without the other and (ii) that they must be empirically correlated.

However, the empirical evidence of rejection of fair bets around zero—that is, the behavioral definition of loss aversion—is much less robust than that of the endowment effect. Although many papers document it (Camerer, 1995; Starmer, 2000), several studies find it to be fragile (Ert and Erev, 2008, 2013), while others find loss neutrality. For example, Chapman et al. (2019, 2021) measure loss aversion using different techniques in large representative samples, and find average loss neutrality, even though they document (robustly) the endowment effect. L'Haridon et al. (2021) find loss neutrality also in the lab after carefully accounting for other features of CTP.

In terms of the correlation between loss aversion and the endowment effect, surprisingly few studies analyze it empirically. Gächter et al. (2007) and Dean and Ortoleva (2019) test this on students and find a (mild) positive correlation. However, Chapman et al. (2021) test it in large representative samples and find loss aversion to be *unrelated* to the endowment effect for lottery tickets. This is incompatible with CPT or any model that ascribes the endowment effect to loss aversion.

Instead, we have seen that Cautious Utility does not entail a relationship between loss aversion and the endowment effect; each may exist without the other, or they may coexist and be unrelated.[11]

[11] It is also easy to construct examples of Cautious Utility in which loss aversion for money is independent of the endowment effect for monetary lottery tickets (in the sense that we can identify a parametric family and a distribution of parameters in the population such that the endowment effect is distributed independently of

Gain-Loss Separability. A central feature of most models of reference-dependence under risk, including CPT, is *Gain-Loss Separability*: the overall utility of a lottery can be expressed as the sum of the utilities of its positive and negative components. Behaviorally, this means that if individuals prefer both the 'gain' part of p to that of q and the 'loss' part of p to that of q, they must prefer p to q. Formally, focusing on the simpler case of $k = 1$, for any p with finite support define its 'gain' part, p^+, and its 'loss' part, p^-, by

$$p^+(x) = \begin{cases} p(x) & x > 0 \\ p(\{y \in \text{supp}(p) | y \leq 0\}) & x = 0 \\ 0 & x < 0 \end{cases} \qquad p^-(x) = \begin{cases} p(x) & x < 0 \\ p(\{y \in \text{supp}(p) | y \geq 0\}) & x = 0 \\ 0 & x > 0 \end{cases}$$

That is, p^+ (resp. p^-) agrees with p on the strictly positive (resp. negative) prizes and assigns all the remaining probability to the prize zero. We say that $\succcurlyeq$ *satisfies Gain-Loss Separability* if for any p and q, if both $p^+ \succcurlyeq q^+$ and $p^- \succcurlyeq q^-$, then $p \succcurlyeq q$.

Gain-Loss Separability is satisfied not only by Expected Utility but also by CPT and RDU. However, experimental tests of this property find that it often fails. For example, Wu and Markle (2008) consider, among others, $p = \frac{1}{2}\delta_{4200} + \frac{1}{2}\delta_{-3000}$ and $q = \frac{3}{4}\delta_{3000} + \frac{1}{4}\delta_{-4500}$, and find (within-subjects) that a majority ranks $q^+ \succcurlyeq p^+$, $q^- \succcurlyeq p^-$, but $p \succ q$. See also Birnbaum and Bahra (2007); Por and Budescu (2013).

Cautious Utility can accommodate violations of Gain-Loss Separability. As different utilities can be relevant for different lotteries, it is easy to construct a set $\mathcal{W}$ where the utility used for mixed lotteries is different from the ones used with only gains or losses.[12]

Evidence of Randomization. Several papers document *preferences for randomization* for many types of objects; see Agranov and Ortoleva (2017, 2022) for a review. As we have seen, Cautious Utility allows for preferences for randomization. However, the model rules out strict preference for randomization with degenerate monetary amounts, documented in Feldman and Rehbeck (2020) and Agranov and Ortoleva (2021). Under RDU and CPT with pessimistic probability weighting, subjects should be averse to randomization, while with S-shaped probability weighting, preferences for randomization are expected in specific regions and not in others.

Evidence of the Certainty Effect. We briefly review the relationship with the evidence of the certainty effect and refer to Cerreia-Vioglio et al. (2015) for an in-depth discussion.

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the distribution of loss aversion).

[12]For example, let $p = \frac{1}{2}\delta_4 + \frac{1}{4}\delta_0 + \frac{1}{4}\delta_{-2}$ and $q = \frac{1}{3}\delta_9 + \frac{2}{3}\delta_{-\frac{3}{4}}$. Let $\mathcal{W} = \{v, v'\}$, where $v(x) = x$, and $v'(x) = \sqrt{x}$ if $x \geq 0$ and $v'(x) = -\sqrt{-x}$ if $x < 0$. Clearly $\mathcal{W}$ is odd. We have $p^+ \sim q^+$ (with $V(p^+) = 1$), $p^- \sim q^-$ (with $V(p^-) = -0.5$), but $p \nsim q$, since $V(p) = (1 - \frac{1}{4}\sqrt{2})^2 > (1 - \frac{2}{3}\sqrt{0.75})^2 = V(q)$.

An extensive literature documented Allais-type behavior when one option is risk-free.[13] Other stylized facts are that violations are less frequent when no option is risk-free and *mixed fanning*—indifference curves becoming flatter as we move towards better prizes in the Marschak–Machina triangle. All are compatible with Cautious Utility, RDU, and CPT.[14]

Another robustly documented pattern is that the strength of the certainty effect appears to vary significantly with stake sizes: Allais-type behavior is considerably less frequent for small stakes.[15] This is incompatible with RDU and CPT, as probability weighting does not change if all stakes are modified without changing their rank. Cautious Utility allows for it: for example, if one utility is most risk averse for all prizes close enough to zero, then behavior in this region is indistinguishable from Expected Utility, implying no certainty effect.

To our knowledge, less attention has been given to the certainty effect for losses since Kahneman and Tversky (1979), which document the opposite.[16] Such reversals are incompatible with Cautious Utility.

The 4-fold Pattern. An often-cited regularity of risk preferences is the *4-fold pattern*: risk aversion for gains of large probabilities and losses of small probability; risk seeking for gains of small probability and losses of large probability. This is easily captured by RDU and CPT, assuming probability weighting of inverted-S shape. Cautious Utility is compatible with risk aversion for gains and risk seeking for losses, but it cannot capture these two patterns globally together with risk seeking for all lotteries that give gains with small probabilities.[17]

5 Foundation

We conclude with the axiomatic foundation of Cautious Utility. Endow $\mathbb{R}^k$ with the usual Euclidean topology and Δ with a version of the weak topology.[18] Consider a binary relation $\succcurlyeq$ on Δ, on which we impose the following axioms.

[13] See Camerer (1995) for a review. Some papers document the opposite (Blavatskyy et al., 2022; Jain and Nielsen, 2020); this is compatible with Incautious Utility and less-often used versions of CPT and RDU.

[14] Bernheim and Sprenger (2020) provide evidence against rank dependency, the basic principle underlying probability weighting in CPT. For its design, their experiment cannot be used to neither refute nor support Cautious Utility.

[15] Conlisk (1989); Camerer (1989); Burke et al. (1996); Fan (2002); Huck and Müller (2012).

[16] Ruggeri et al. (2020) is a large-scale, multi-country replication of the experiments in Kahneman and Tversky (1979). While most effects replicate, the evidence of the opposite of the certainty effect for losses is weaker: few subjects exhibit the certainty effect in that range, but there is no evidence of its opposite, with the majority of subjects exhibiting a behavior compatible with Expected Utility (a pattern compatible with Cautious Utility and with CPT with no probability weighting for losses.)

[17] A few recent papers argue that risk seeking for gains with small probabilities may be due to a misunderstanding of lottery descriptions and not to innate preferences. Hertwig et al. (2004) and Abdellaoui et al. (2011) show that risk seeking for small probabilities disappears and is replaced by risk aversion if subjects are allowed to learn probabilities through sampling, instead of being simply given a description of a lottery.

[18] A generalized sequence $\{p_\alpha\}_{\alpha \in A}$ in Δ converges to p if and only if $\mathbb{E}_{p_\alpha}(v) \to \mathbb{E}_p(v)$ for all $v \in C\left(\mathbb{R}^k\right)$.

Axiom 1 (Weak Order). *The relation $\succcurlyeq$ is complete and transitive.*

Axiom 2 (Continuity). *For each $q \in \Delta$ the sets $\{p \in \Delta : p \succcurlyeq q\}$ and $\{p \in \Delta : q \succcurlyeq p\}$ are closed.*

Axiom 3 (Monotonicity). *For each $x, y \in \mathbb{R}^k$*

$$x > y \implies \lambda\delta_x + (1-\lambda)\, r \succcurlyeq \lambda\delta_y + (1-\lambda)\, r \qquad \forall\lambda \in (0,1]\,, \forall r \in \Delta$$

and $\lambda\delta_x + (1-\lambda)\, r \succ \lambda\delta_y + (1-\lambda)\, r$ for some $\lambda \in (0,1]$ and for some $r \in \Delta$.[19]

Axiom 4 (Monetary equivalent). *For each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$\lambda\delta_{y+me_1} + (1-\lambda)\, r \succcurlyeq \lambda\delta_x + (1-\lambda)\, r \succcurlyeq \lambda\delta_{y-me_1} + (1-\lambda)\, r \qquad \forall\lambda \in (0,1]\,, \forall r \in \Delta.$$

The first three postulates are standard. Monetary equivalent simply stipulates that for any two bundles $x, y \in \mathbb{R}^k$, there is a monetary amount m large enough that receiving that amount on top of y is better than x and losing that amount is worse than x, and this remains true even if we mix with some other lottery r. This axiom will guarantee that monetary certainty equivalents, WTAs, and WTPs are well defined.

The following axiom is our key assumption. It extends the Negative Certainty Independence (NCI) axiom of Dillenberger (2010) and Cerreia-Vioglio et al. (2015) to multi-dimensional bundles and generalizes the definition of certainty effect of Kahneman and Tversky (1979).

Axiom 5 (Multi-Dimensional Negative Certainty Independence (M-NCI)). *For each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$p \succcurlyeq \delta_{me_1} \implies \lambda p + (1-\lambda)\, r \succcurlyeq \lambda\delta_{me_1} + (1-\lambda)\, r \qquad \forall\lambda \in (0,1]\,, \forall r \in \Delta.$$

Like the original NCI, this property states that if a sure amount of money m is not preferred to a lottery p, then this ranking does not change if we mix both with another lottery. M-NCI is a weakening of standard Independence that captures the certainty effect. Intuitively, mixing m with a lottery eliminates its certainty appeal. Therefore, if m is worse than p when certain, it will remain so after the mixture.

For ease of comparison, it is also helpful to consider the inverse postulate that rules out the certainty effect while allowing for the opposite.

Axiom 6 (Multi-Dimensional Positive Certainty Independence (M-PCI)). *For each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$\delta_{me_1} \succcurlyeq p \implies \lambda\delta_{me_1} + (1-\lambda)\, r \succcurlyeq \lambda p + (1-\lambda)\, r \qquad \forall\lambda \in (0,1]\,, \forall r \in \Delta.$$

[19] $x > y$ means that $x_i \geq y_i$ for all i, where at least one of the inequalities is strict.

Our characterization theorem below focuses on *canonical* representations. To define them, we first introduce the following subrelation $\succcurlyeq'$:

$$p \succcurlyeq' q \stackrel{\text{def}}{\iff} \lambda p + (1-\lambda)\, r \succcurlyeq \lambda q + (1-\lambda)\, r \qquad \forall \lambda \in (0,1]\,, \forall r \in \Delta.$$

Intuitively, $\succcurlyeq'$ captures the rankings of which the agent is sure: $p \succcurlyeq' q$ when not only $p \succcurlyeq q$, but also any mixture featuring p is better than the corresponding mixture with q. It is easy to verify that $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies the Independence axiom of Expected Utility, and that it is incomplete (yet still transitive) whenever preferences are not Expected Utility. We say that a Cautious Utility representation $\mathcal{W}$ (see Definition 1) is *canonical* if it also represents $\succcurlyeq'$, in the sense that

$$p \succcurlyeq' q \iff \mathbb{E}_p\,(v) \geq \mathbb{E}_q\,(v) \qquad \forall v \in \mathcal{W}.$$

Theorem 1. *A binary relation $\succcurlyeq$ on Δ satisfies Axioms 1-5 if and only if it admits a canonical Cautious Utility representation.*

This theorem shows that Cautious Utility can be derived from an axiom that postulates the certainty effect, M-NCI, together with basic properties. It is routine to show that Incautious Utility is characterized by the same axioms with M-NCI replaced by M-PCI.

Technically, this result extends the main representation theorem of Cerreia-Vioglio et al. (2015) to a setup of lotteries over multi-commodity bundles and to an unbounded domain, necessary to define monetary certainty equivalents.

In the main text, we discussed Cautious Utility representations which are not necessarily canonical. When $\mathcal{W}$ is finite, as in most applications, preferences represented in this way satisfy all the above axioms (Axioms 1-5; see Remark 5 in the Appendix). Without additional structure, Monotonicity is guaranteed only in a weaker form, that is,

$$x \geq y \implies \lambda \delta_x + (1-\lambda)\, r \succcurlyeq \lambda \delta_y + (1-\lambda)\, r \qquad \forall \lambda \in (0,1]\,, \forall r \in \Delta.$$

Foundations of Symmetry. We now give a simple foundation to our symmetry assumption. For each $p \in \Delta$, denote by $\sigma(p) \in \Delta$ the lottery that, compared to p, swaps gains with losses, that is, $\sigma(p)\,(B) = p(-B)$ for all Borel subsets B of $\mathbb{R}^k$.

A natural form of symmetry posits that if p is better than q, then $\sigma(q)$ is better than $\sigma(p)$ (the two must be swapped as we are inverting gains and losses). But this would be too strong, as it rules out loss aversion and the endowment effect.[20] A weaker version posits that if not only $p \succcurlyeq q$, but also each mixture of p is better than the corresponding mixture of q, that is $p \succcurlyeq' q$, then we obtain $\sigma(q) \succcurlyeq \sigma(p)$. This is exactly the form of symmetry corresponding to the Symmetric Cautious (or Incautious) Utility model.

[20] Since $\delta_0 = \sigma(\delta_0)$ and $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} = \sigma\left(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}\right)$, we get $\delta_0 \sim \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$ (loss neutrality).

Axiom 7 (Weak Symmetry)**.** *For each $p, q \in \Delta$*

$$p \succcurlyeq' q \quad \Longrightarrow \quad \sigma(q) \succcurlyeq \sigma(p).$$

Proposition 6. *A binary relation $\succcurlyeq$ on Δ satisfies Axioms 1-5 and 7 if and only if it admits a canonical Symmetric Cautious Utility representation.*

6 Conclusion

This paper introduces a new way of modeling the endowment effect and loss aversion together with the certainty effect: caution.

Conceptually, caution can be viewed as a heuristic adopted to make decisions when agents are unsure of what to do. Called upon choosing in a situation of indecisiveness, our individuals adopt a conservative criterion. As such, caution can be understood as introducing a form of 'uncertainty aversion' even to choices with no objective risk—like choosing the price to pay for a mug. Agents unsure of the trade-offs have subjective uncertainty, and caution gives them a criterion of how to address it. Applied to resolve uncertainty about trade-offs, how to aggregate gains and losses, or risk aversion over money, caution generates the endowment effect, loss aversion, and the certainty effect.

Our results connect the three behaviors of interest to caution but also to each other. Our last set of results showed that, under Weak Symmetry and basic postulate, Symmetric Cautious Utility is characterized by M-NCI, a property that rules out the opposite of the certainty effect. Proposition 1 showed how this model gives us loss aversion and the endowment effect. Together, these results imply that, under Weak Symmetry and basic axioms, ruling out the opposite of the certainty effect over bundles (as encoded by M-NCI) *formally implies* loss aversion and the endowment effect: reference effects can be derived from a form of the certainty effect over bundles.

Appendix A: Preliminary Results

We begin by proving some ancillary results. Recall the definition of $\succcurlyeq'$ in Section 5. The goal of this section is to provide a Multi-Expected Utility representation for $\succcurlyeq'$.

Lemma 1. *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order. The following statements are true:*

1. *The relation $\succcurlyeq$ satisfies M-NCI if and only if for each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$p \succcurlyeq \delta_{me_1} \implies p \succcurlyeq' \delta_{me_1}. \qquad \text{(Equivalently } p \not\succcurlyeq' \delta_{me_1} \implies \delta_{me_1} \succ p.\text{)}$$

2. *If $\succcurlyeq$ satisfies Monotonicity, then for each $x, y \in \mathbb{R}^k$*

$$x > y \implies \delta_x \succ' \delta_y. \tag{2}$$

3. *If $\succcurlyeq$ satisfies Monetary equivalent, then for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$\delta_{y+me_1} \succcurlyeq' \delta_x \succcurlyeq' \delta_{y-me_1}. \tag{3}$$

Proof. All three points follow from the definition of $\succcurlyeq'$ and M-NCI, Monotonicity, and Monetary equivalent, respectively. ■

A.1 Aumann Utilities and Multi-Expected Utility Representations

In this section, in our formal results, we consider a binary relation $\succcurlyeq^*$ over Δ such that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \tag{4}$$

where $\mathcal{W} \subseteq C(\mathbb{R}^k)$. Recall that a function $v \in C(\mathbb{R}^k)$ is an Aumann utility if and only if

$$p \succ^* q \implies \mathbb{E}_p(v) > \mathbb{E}_q(v) \text{ and } p \sim^* q \implies \mathbb{E}_p(v) = \mathbb{E}_q(v).$$

We denote by e the vector whose components are all 1s. We endow $C(\mathbb{R}^k)$ with the distance $d : C(\mathbb{R}^k) \times C(\mathbb{R}^k) \to [0, \infty)$ defined by

$$d(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \min\left\{\max_{x \in [-ne, ne]} |f(x) - g(x)|, 1\right\} \quad \forall f, g \in C\left(\mathbb{R}^k\right).$$

It is routine to show that $\left(C\left(\mathbb{R}^k\right), d\right)$ is separable.[21] Moreover, if $\{f_m\}_{m\in\mathbb{N}} \subseteq C\left(\mathbb{R}^k\right)$ is such that $f_m \overset{d}{\to} f$, then $\{f_m\}_{m\in\mathbb{N}}$ converges uniformly on each compact subset of $\mathbb{R}^k$.

Proposition 7. *If $\succcurlyeq^*$ is as in (4) and such that*

$$x > y \implies \delta_x \succ^* \delta_y, \tag{5}$$

then $\succcurlyeq^*$ admits a strictly increasing Aumann utility.

Proof. By (4), observe that $x > y$ implies $v(x) \geq v(y)$ for all $v \in \mathcal{W}$. This implies that each $v \in \mathcal{W}$ is increasing. By Aliprantis and Border (2006, Corollary 3.5), there exists a countable d-dense subset D of $\mathcal{W}$. Clearly, we have that

$$p \succcurlyeq^* q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{6}$$

Vice-versa, consider $p, q \in \Delta$ such that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in D$. Since p and q have compact support, there exists $\bar{n} \in \mathbb{N}$ such that $[-\bar{n}e, \bar{n}e]$ contains both supports. Consider $v \in \mathcal{W}$. Since D is d-dense in $\mathcal{W}$, there exists a sequence $\{v_l\}_{l\in\mathbb{N}} \subseteq D$ such that $v_l \overset{d}{\to} v$. It follows that v_l converges uniformly on $[-\bar{n}e, \bar{n}e]$. This implies that

$$\begin{aligned}
\mathbb{E}_p(v) &= \int_{[-\bar{n}e,\bar{n}e]} v \mathrm{d}p = \lim_l \int_{[-\bar{n}e,\bar{n}e]} v_l \mathrm{d}p = \lim_l \mathbb{E}_p(v_l) \\
&\geq \lim_l \mathbb{E}_q(v_l) = \lim_l \int_{[-\bar{n}e,\bar{n}e]} v_l \mathrm{d}q = \int_{[-\bar{n}e,\bar{n}e]} v \mathrm{d}q = \mathbb{E}_q(v).
\end{aligned}$$

By (4) and (6) and since v was arbitrarily chosen, we can conclude that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{7}$$

Since D is countable, we can list its elements: $D = \{v_m\}_{m\in\mathbb{N}}$. Set $b_l = l + \max\{|v_l(-le)|, |v_l(le)|\}$ for all $l \in \mathbb{N}$ and $a_m = \Pi_{l=1}^m b_l \geq b_m$ for all $m \in \mathbb{N}$. Finally, define $v : \mathbb{R}^k \to \mathbb{R}$ by

$$v(x) = \sum_{m=1}^{\infty} \frac{v_m(x)}{a_m} \qquad \forall x \in \mathbb{R}^k. \tag{8}$$

We first prove that v is a well defined continuous function. Fix $x \in \mathbb{R}^k$. It follows that there exists $\bar{m} \in \mathbb{N}$ such that $x \in [-me, me]$ for all $m \geq \bar{m}$. Since each v_m is increasing, we have that $|v_m(x)| \leq \max\{|v_m(-me)|, |v_m(me)|\} \leq b_m \leq a_m$ for all $m \geq \bar{m}$. Since $a_m \geq m!$ for

[21] A proof is available upon request.

all $m \in \mathbb{N}$, it follows that

$$\frac{|v_m(x)|}{a_m} = \frac{|v_m(x)|}{b_m a_{m-1}} \leq \frac{1}{a_{m-1}} \leq \frac{1}{(m-1)!} \qquad \forall m \geq \bar{m}+1.$$

This implies that the right-hand side of (8) converges. Since x was arbitrarily chosen, v is well defined. Next, consider $n \in \mathbb{N}$. From the same argument above, we have that

$$\frac{|v_m(x)|}{a_m} \leq \frac{1}{(m-1)!} \quad \forall x \in [-ne, ne], \forall m \geq n+1.$$

By Weierstrass' M-test and since $\{v_m/a_m\}_{m\in\mathbb{N}}$ is a sequence of continuous functions, we can conclude that $v = \sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on $[-ne, ne]$, yielding that v is continuous on $[-ne, ne]$. Since n was arbitrarily chosen, it follows that v is continuous.

Finally, assume that $p \succ^* q$ (resp. $p \sim^* q$). By (7), we have that $\mathbb{E}_p(v_m) \geq \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m) = \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$). In particular, we have that $\mathbb{E}_p(v_m/a_m) \geq \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}/a_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}}/a_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m/a_m) = \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$). Since $\sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on compacta and the supports of p and q are compact, we can conclude that

$$\begin{aligned}
\mathbb{E}_p(v) - \mathbb{E}_q(v) &= \mathbb{E}_p\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) = \lim_l \sum_{m=1}^{l} \mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \lim_l \sum_{m=1}^{l} \mathbb{E}_q\left(\frac{v_m}{a_m}\right) \\
&= \lim_l \left[\sum_{m=1}^{l} \left(\mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\frac{v_m}{a_m}\right)\right)\right].
\end{aligned}$$

This implies that if $p \succ^* q$ (resp. $p \sim^* q$), then $\mathbb{E}_p(v) > \mathbb{E}_q(v)$ (resp. $\mathbb{E}_p(v) = \mathbb{E}_q(v)$), proving that v is an Aumann utility. In particular, by (5), v is strictly increasing. ∎

Consider a binary relation $\succcurlyeq^*$ on Δ. Define $\mathcal{W}_{\max}(\succcurlyeq^*)$ as the set of all strictly increasing functions $v \in C(\mathbb{R}^k)$ such that $v(0) = 0$ and $p \succcurlyeq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. We say that a set $\mathcal{W}$ in $C(\mathbb{R}^k)$ has full image if and only if

$$\forall x, y \in \mathbb{R}^k, \exists m \in \mathbb{R}_+ \text{ s.t. } v(y + me_1) \geq v(x) \geq v(y - me_1) \qquad \forall v \in \mathcal{W}.$$

Proposition 8. *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in (4). If $\succcurlyeq^*$ satisfies (2) and (3), then $\mathcal{W}_{\max}(\succcurlyeq^*)$ is a nonempty convex set with full image that satisfies (4).*

Proof. Consider $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and $\lambda \in (0,1)$. Since both functions are strictly increasing and continuous and such that $v_1(0) = 0 = v_2(0)$, it follows that $\lambda v_1 + (1-\lambda) v_2$ is strictly increasing, continuous, and takes value 0 in 0. Since $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$ and $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$. This implies that

$$\begin{aligned}\mathbb{E}_p(\lambda v_1 + (1-\lambda) v_2) &= \lambda \mathbb{E}_p(v_1) + (1-\lambda)\mathbb{E}_p(v_2) \\ &\geq \lambda \mathbb{E}_q(v_1) + (1-\lambda)\mathbb{E}_q(v_2) = \mathbb{E}_q(\lambda v_1 + (1-\lambda) v_2),\end{aligned}$$

proving that $\lambda v_1 + (1-\lambda) v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, $\mathcal{W}_{\max}(\succcurlyeq^*)$ is convex. By Proposition 7, there exists a strictly increasing $\hat{v} \in C(\mathbb{R}^k)$ such that

$$p \succ^* q \Longrightarrow \mathbb{E}_p(\hat{v}) > \mathbb{E}_q(\hat{v}) \text{ and } p \sim^* q \Longrightarrow \mathbb{E}_p(\hat{v}) = \mathbb{E}_q(\hat{v}).$$

Without loss of generality, we can assume that $\hat{v}(0) = 0$ (given $\hat{v}$, set $v = \hat{v} - \hat{v}(0)$) and, in particular, we have that $\hat{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is nonempty. Since $\succcurlyeq^*$ satisfies (3), it follows that $\mathcal{W}_{\max}(\succcurlyeq^*)$ has full image. Since $\succcurlyeq^*$ satisfies (2), v is increasing for all $v \in \mathcal{W}$. This implies that for each $v \in \mathcal{W}$ and for each $n \in \mathbb{N}$ the function $v_n = \left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v} - \left[\left(1-\frac{1}{n}\right)v(0) + \frac{1}{n}\hat{v}(0)\right] \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. Vice-versa, we have that

$$\begin{aligned}&\mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*) \\ &\quad \Longrightarrow \mathbb{E}_p\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \geq \mathbb{E}_q\left(\left(1-\frac{1}{n}\right)v + \frac{1}{n}\hat{v}\right) \quad \forall v \in \mathcal{W}, \forall n \in \mathbb{N} \\ &\quad\quad \Longrightarrow \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \Longrightarrow p \succcurlyeq^* q,\end{aligned}$$

proving that (4) holds with $\mathcal{W}_{\max}(\succcurlyeq^*)$ in place of $\mathcal{W}$. ■

We conclude by discussing Multi-Expected Utility representations which feature odd sets. To do this, we make two simple observations. First, recall the map $\sigma : \Delta \to \Delta$, which swaps gains with losses, defined by

$$\sigma(p)(B) = p(-B) \text{ for all Borel subsets of } \mathbb{R}^k \text{ and for all } p \in \Delta.$$

It is immediate to see that σ is affine and $\sigma(\sigma(p)) = p$ for all $p \in \Delta$. Second, by the Change of Variable Theorem (see, e.g., Aliprantis and Border 2006, Theorem 13.46), we have that

$$\mathbb{E}_{\sigma(r)}(v) = \int_{\mathbb{R}^k} v \mathrm{d}\sigma(r) = -\int_{\mathbb{R}^k} \bar{v} \mathrm{d}r = -\mathbb{E}_r(\bar{v}) \quad \forall r \in \Delta, \forall v \in C\left(\mathbb{R}^k\right) \tag{9}$$

where $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$ and for all $v \in C(\mathbb{R}^k)$.

Proposition 9. *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in (4) which satisfies (2) and (3). The following statements are equivalent:*

(i) For each $p, q \in \Delta$

$$p \succcurlyeq^* q \iff \sigma(q) \succcurlyeq^* \sigma(p).$$

(ii) For each $p, q \in \Delta$

$$p \succ^* q \implies \sigma(q) \succ^* \sigma(p).$$

(iii) $\mathcal{W}_{\max}(\succcurlyeq^*)$ is odd.

Moreover, if $\mathcal{W}$ in (4) is odd, then (i) and (ii) hold.

For the last part of the statement, that is proving that if $\mathcal{W}$ is odd, then (i) and (ii) hold, we can dispense with the assumption that $\succcurlyeq^*$ satisfies (2) and (3). The proof will clarify.

Proof. By Proposition 8, we have that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*).$$

In other words, for the first part of the statement, we can replace $\mathcal{W}$ in (4) with $\mathcal{W}_{\max}(\succcurlyeq^*)$.

(i) implies (ii). It is obvious.

(ii) implies (iii). Fix $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition of $\bar{v}$ and since each v in $\mathcal{W}_{\max}(\succcurlyeq^*)$ is strictly increasing, continuous, and such that $v(0) = 0$, we have that $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. By assumption and (9), we have that

$$p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq^* \sigma(p) \implies \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \implies -\mathbb{E}_q(\bar{v}) \geq -\mathbb{E}_p(\bar{v}) \implies \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}).$$

By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, we can conclude that $\bar{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is odd.

(iii) implies (i). By (9) and since $\mathcal{W}$ is odd and represents $\succcurlyeq^*$, we have that

$$\begin{aligned} p \succcurlyeq^* q &\iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \iff \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}) \quad \forall v \in \mathcal{W} \\ &\iff \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \quad \forall v \in \mathcal{W} \iff \sigma(q) \succcurlyeq^* \sigma(p), \end{aligned}$$

proving the implication (since $\mathcal{W}_{\max}(\succcurlyeq^*)$ represents $\succcurlyeq^*$) and also the second part of the statement. ■

A.2 Representing $\succcurlyeq'$

We can finally provide a Multi-Expected Utility representation for $\succcurlyeq'$.

Proposition 10. *If $\succcurlyeq$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent, then*

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq').$$

Moreover, $\mathcal{W}_{\max}(\succcurlyeq')$ is a nonempty convex set with full image.

Proof. By the same techniques of Cerreia-Vioglio (2009, Proposition 22) (see also Cerreia-Vioglio et al. 2017, Lemma 1 and Footnote 10), $\succcurlyeq'$ is a preorder that satisfies Sequential Continuity and Independence.[22] By Evren (2008, Theorem 2), there exists a set $\mathcal{W} \subseteq C\left(\mathbb{R}^k\right)$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Lemma 1 and since $\succcurlyeq$ is a Weak Order which satisfies Monotonicity and Monetary equivalent, we have that $\succcurlyeq'$ satisfies (2) and (3). By Proposition 8 and considering $\succcurlyeq'$ in place of $\succcurlyeq^*$, $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max}(\succcurlyeq')$, proving the statement. ■

Appendix B: Proof of the Main Results

We begin by proving the axiomatic foundation of our model (Theorem 1). We then proceed to the foundation of its symmetric version (Proposition 6). We then make two observations about the foundation of Incautious Utility and the necessity of the axioms (Remarks 4 and 5). We conclude by proving the other results in the paper.

Proof of Theorem 1. "Only if." We proceed by steps.

Step 1. There exists a continuous utility function $u : \Delta \to \mathbb{R}$ for $\succcurlyeq$ such that $u\left(\delta_{me_1}\right) = m$ for all $m \in \mathbb{R}$.

Proof of the Step. Let $p \in \Delta$. Since p has compact support, there exists $n \in \mathbb{N}$ such that $[-ne, ne]$ contains the support of p. By Lemma 1 and since $\succcurlyeq$ satisfies Weak Order and Monetary equivalent, we have that there exist $m', m'' \in \mathbb{R}_+$ such that $\delta_{m'e_1} \succcurlyeq' \delta_{ne} \succcurlyeq' \delta_{-m'e_1}$ and $\delta_{m''e_1} \succcurlyeq' \delta_{-ne} \succcurlyeq' \delta_{-m''e_1}$. By Lemma 1 and since $\succcurlyeq$ satisfies Weak Order and Monotonicity, if we set $m = \max\{m', m''\}$, we obtain that $\delta_{me_1} \succcurlyeq' \delta_{ne}, \delta_{-ne} \succcurlyeq' \delta_{-me_1}$. By Proposition 10 and since $\succcurlyeq$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent and each element of $\mathcal{W}_{\max}(\succcurlyeq')$ is increasing, we have that

$$\delta_{me_1} \succcurlyeq' \delta_{ne} \succcurlyeq' p \succcurlyeq' \delta_{-ne} \succcurlyeq' \delta_{-me_1}.$$

Since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, we can conclude that $\delta_{me_1} \succcurlyeq p \succcurlyeq \delta_{-me_1}$. Consider the sets $U = \left\{m \in \mathbb{R} : \delta_{me_1} \succcurlyeq p\right\}$ and $L = \left\{m \in \mathbb{R} : p \succcurlyeq \delta_{me_1}\right\}$. It follows that U and L are nonempty. Since $\succcurlyeq$ satisfies Weak Order, we have that $U \cup L = \mathbb{R}$. By the same arguments of Aliprantis and Border (2006, Theorem 15.8) and invoking Aliprantis and Border (2006, Theorem 2.55), the map $x \mapsto \delta_x$ is a (continuous) embedding. Since $\succcurlyeq$ satisfies Continuity, this implies that both U and L are closed. Since $\mathbb{R}$ is connected and $U \cup L = \mathbb{R}$, we can conclude that $U \cap L$ is nonempty and, in particular, $p \sim \delta_{me_1}$ for all $m \in U \cap L$. By Lemma

[22]That is, for each two generalized sequences $\{p_\alpha\}_{\alpha \in A}$ and $\{q_\alpha\}_{\alpha \in A}$ in Δ

$$p_\alpha \succcurlyeq' q_\alpha \quad \forall \alpha \in A,\ p_\alpha \to p, \text{ and } q_\alpha \to q \implies p \succcurlyeq' q.$$

1 and since $\succcurlyeq$ satisfies Weak Order, Monotonicity, and M-NCI, we have that $m \geq m'$ if and only if $\delta_{me_1} \succcurlyeq' \delta_{m'e_1}$ if and only if $\delta_{me_1} \succcurlyeq \delta_{m'e_1}$. This implies that $U \cap L$ is a singleton. We denote by $m_p \in \mathbb{R}$ the unique element such that $p \sim \delta_{m_p e_1}$. Since p was arbitrarily chosen, we define $u : \Delta \to \mathbb{R}$ by $u(p) = m_p$ for all $p \in \Delta$. By construction, we have that $u\left(\delta_{me_1}\right) = m$ for all $m \in \mathbb{R}$. Moreover, since $\succcurlyeq$ satisfies Weak Order, we have that

$$p \succcurlyeq q \iff \delta_{m_p e_1} \succcurlyeq \delta_{m_q e_1} \iff m_p \geq m_q \iff u(p) \geq u(q),$$

proving that u is a utility function for $\succcurlyeq$. Finally, since $\succcurlyeq$ satisfies Continuity, this implies that for each $t \in \mathbb{R}$

$$\{p \in \Delta : u(p) \geq t\} = \left\{p \in \Delta : u(p) \geq u\left(\delta_{te_1}\right)\right\} = \left\{p \in \Delta : p \succcurlyeq \delta_{te_1}\right\}$$

is closed, proving that u is upper semicontinuous. A specular argument yields lower semicontinuity, proving that u is continuous. □

Step 2. $\succcurlyeq'$ is represented by $\mathcal{W}_{\max}(\succcurlyeq')$ which has full image, in particular,

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq') \iff c(p,v) \geq c(q,v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq'). \tag{10}$$

Proof of the Step. By Proposition 10, the first part of (10) follows. Since $\mathcal{W}_{\max}(\succcurlyeq')$ has full image and each element of $\mathcal{W}_{\max}(\succcurlyeq')$ is strictly increasing and continuous, $c(p,v)$ is well defined for all $p \in \Delta$ and for all $v \in \mathcal{W}_{\max}(\succcurlyeq')$, and also the second part of (10) follows. □

Step 3. For each $p \in \Delta$ we have that $\inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(p,v) \in \mathbb{R}$.

Proof of the Step. Fix $p \in \Delta$. By the same arguments of the first part of Step 1, there exists $m \in \mathbb{R}_+$ such that $\mathbb{E}_{\delta_{me_1}}(v) \geq \mathbb{E}_p(v) \geq \mathbb{E}_{\delta_{-me_1}}$ for all $v \in \mathcal{W}_{\max}(\succcurlyeq')$. It follows that $m \geq \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(p,v) \geq -m$.

Step 4. For each $p \in \Delta$ we have that

$$u(p) \leq \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(p,v).$$

Proof of the Step. Fix $p \in \Delta$. By Step 3, $\bar{m} = \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(p,v)$ is a real number. Pick $m \in \mathbb{R}$ such that $m > \bar{m}$. This implies that there exists $v \in \mathcal{W}_{\max}(\succcurlyeq')$ such that $c(p,v) < m = c\left(\delta_{me_1}, v\right)$. By Step 2, it follows that $p \not\succcurlyeq' \delta_{me_1}$. By Lemma 1 and Step 1 and since $\succcurlyeq$ satisfies Weak Order and M-NCI, we have that $\delta_{me_1} \succ p$, yielding that $m = u\left(\delta_{me_1}\right) > u(p)$. Since m was arbitrarily chosen to be just strictly greater than $\bar{m}$, we have that $u(p) \leq \bar{m}$, proving the statement. □

Step 5. For each $p \in \Delta$ we have that

$$u(p) \geq \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v).$$

Proof of the Step. Fix $p \in \Delta$. By Step 3, $\bar{m} = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v)$ is a real number. By Step 2, we have that

$$c(p,v) \geq \bar{m} = c\left(\delta_{\bar{m}e_1}, v\right) \qquad \forall v \in \mathcal{W}_{\max}(\succcurlyeq').$$

By Steps 1 and 2 and since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, this implies that $p \succcurlyeq' \delta_{\bar{m}e_1}$ and, in particular, $p \succcurlyeq \delta_{\bar{m}e_1}$, that is, $u(p) \geq u\left(\delta_{\bar{m}e_1}\right) = \bar{m}$, proving the statement. □

By imposing $\mathcal{W} = \mathcal{W}_{\max}(\succcurlyeq')$, the implication follows from Steps 1, 2, 4, and 5.

"If." It is routine (cf. Remark 5). ■

Proof of Proposition 6. "Only if." By the proof of Theorem 1 and since $\succcurlyeq$ satisfies Axioms 1-5, we have that $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Cautious Utility representation, in particular, $\mathcal{W}_{\max}(\succcurlyeq')$ represents $\succcurlyeq'$ and $\succcurlyeq$. By definition of $\succcurlyeq'$ and since $\succcurlyeq$ satisfies Weak Symmetry, $\succcurlyeq'$ satisfies Independence, and σ is affine and idempotent, this implies that

$$\begin{aligned}
p \succcurlyeq' q &\implies \lambda p + (1-\lambda)\sigma(r) \succcurlyeq' \lambda q + (1-\lambda)\sigma(r) \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \sigma(\lambda q + (1-\lambda)\sigma(r)) \succcurlyeq \sigma(\lambda p + (1-\lambda)\sigma(r)) \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \lambda\sigma(q) + (1-\lambda)\sigma(\sigma(r)) \succcurlyeq \lambda\sigma(p) + (1-\lambda)\sigma(\sigma(r)) \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \lambda\sigma(q) + (1-\lambda) r \succcurlyeq \lambda\sigma(p) + (1-\lambda) r \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \sigma(q) \succcurlyeq' \sigma(p).
\end{aligned}$$

By Lemma 1 and since $\succcurlyeq$ satisfies Weak Order, Monotonicity, and Monetary equivalent, we have that $\succcurlyeq'$ satisfies (2) and (3). By Proposition 9, we can conclude that $\mathcal{W}_{\max}(\succcurlyeq')$ is odd, proving that $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Symmetric Cautious Utility representation.

"If." By Theorem 1, Axioms 1-5 follow. Since $\mathcal{W}$ is a canonical Symmetric Cautious Utility representation, $\mathcal{W}$ represents $\succcurlyeq'$. By the second part of Proposition 9 and the discussion thereafter, we can conclude that for each $p, q \in \Delta$

$$p \succcurlyeq' q \iff \sigma(q) \succcurlyeq' \sigma(p).$$

Since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, this implies that for each $p, q \in \Delta$

$$p \succcurlyeq' q \implies \sigma(q) \succcurlyeq' \sigma(p) \implies \sigma(q) \succcurlyeq \sigma(p),$$

proving that $\succcurlyeq$ satisfies Weak Symmetry. ■

The two proofs above provide a foundation of the Cautious Utility model and its symmetric version. In the next remark, we discuss the foundation of the Incautious one.

Remark 4. *Recall that an Incautious Utility representation features the same exact objects of a Cautious one except that the* inf *is replaced by* sup. *It is then important to observe that the Multi-Expected Utility representation of $\succcurlyeq'$ in Appendix A and the symmetry property of its representation (Propositions 8–10) have been derived without ever using the M-NCI axiom. The same is true for Steps 1–3 in the proof of Theorem 1 where Step 3 could have been written with* sup *in place of* inf *using the same arguments.*[23] *Thus, substituting M-NCI with M-PCI allows for replacing in Steps 4 and 5 the* inf *with* sup. *Finally, Proposition 6 is a result just in terms of $\succcurlyeq'$ without ever relying on M-NCI.*

Next, we comment on the necessity of the axioms when the Cautious Utility representation chosen might not be canonical, as we always posit in the main text.

Remark 5. *Consider a Cautious Utility representation $\mathcal{W}$ for $\succcurlyeq$ (not necessarily canonical). By definition, we have that $\mathcal{W}$ is a set of strictly increasing and continuous utility functions $v : \mathbb{R}^k \longrightarrow \mathbb{R}$ such that for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$v\left(y + me_1\right) \geq v\left(x\right) \geq v\left(y - me_1\right) \quad \forall v \in \mathcal{W}, \tag{11}$$

$v(0) = 0$ for all $v \in \mathcal{W}$, and $V : \Delta \longrightarrow \mathbb{R}$, defined by

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \quad \forall p \in \Delta,$$

is a continuous utility function for $\succcurlyeq$. It is then immediate to observe that $\succcurlyeq$ satisfies Weak Order and Continuity. As for the other axioms, define the binary relation

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}.$$

Clearly, $\succcurlyeq^*$ is a preorder that satisfies Independence. Fact 1–4 below follow from immediate computations and the definition of $\succcurlyeq^*$. Fact 5 follows by the second part of Proposition 9 and the discussion thereafter, provided $\mathcal{W}$ is odd:

1. *For each $p, q \in \Delta$*
$$p \succcurlyeq^* q \implies p \succcurlyeq q.$$
2. *For each $p \in \Delta$ and for each $m \in \mathbb{R}$*
$$p \succcurlyeq \delta_{me_1} \implies p \succcurlyeq^* \delta_{me_1}.$$

[23]In Step 1, using M-PCI in place of M-NCI seamlessly yields the same result.

3. *For each* $x, y \in \mathbb{R}^k$
$$x > y \implies \delta_x \succ^* \delta_y.$$

4. *For each* $x, y \in \mathbb{R}^k$ *there exists* $m \in \mathbb{R}_+$
$$\delta_{y+me_1} \succcurlyeq^* \delta_x \succcurlyeq^* \delta_{y-me_1} \text{ and } \delta_{y+me_1} \succcurlyeq \delta_x \succcurlyeq \delta_{y-me_1}.$$

5. *For each* $p, q \in \Delta$
$$p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq^* \sigma(p).$$

By point 1 and since $\succcurlyeq'$ *is the largest subrelation of* $\succcurlyeq$ *that satisfies Independence, we have that* $\succcurlyeq^*$ *is a subrelation of* $\succcurlyeq'$, *that is,* $p \succcurlyeq^* q \implies p \succcurlyeq' q$. *In light of this and given the definition of* $\succcurlyeq'$, *point 2 (resp. point 4) implies that* $\succcurlyeq$ *satisfies M-NCI (resp. Monetary equivalent). Point 3 implies a weaker form of the Monotonicity axiom with strict inequalities replaced by weak ones.*[24] *If the set* $\mathcal{W}$ *is also finite (as in all our examples), then Monotonicity holds as stated: with strict inequalities. Finally, points 1 and 5 imply that* $\succcurlyeq$ *satisfies a weaker form of symmetry, that is* $p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq \sigma(p)$, *whenever* $\mathcal{W}$ *is odd. This form of symmetry is sufficient to obtain our results on the endowment effect and loss aversion.*

Proof of Proposition 2. Consider a Cautious Utility representation $\mathcal{W}$ for $\succcurlyeq$ (not necessarily canonical). For each $i \in \{2, ..., k\}$ recall that $\mathrm{WTA}_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mathrm{WTP}_i : \mathbb{R}_+ \to \mathbb{R}_+$ are the functions defined by

$$\mathrm{WTA}_i(m) = \min \left\{ l \in \mathbb{R}_+ : \delta_{-me_i + le_1} \succcurlyeq \delta_0 \right\} \qquad \forall m \in \mathbb{R}_+ \tag{12}$$

and

$$\mathrm{WTP}_i(m) = \max \left\{ l \in \mathbb{R}_+ : \delta_{me_i - le_1} \succcurlyeq \delta_0 \right\} \qquad \forall m \in \mathbb{R}_+. \tag{13}$$

By points 1, 3, and 4 of Remark 5 and since $\succcurlyeq$ is represented by a continuous utility, these functions are well defined and $\delta_{-me_i + \mathrm{WTA}_i(m)e_1} \sim \delta_0$ as well as $\delta_{me_i - \mathrm{WTP}_i(m)e_1} \sim \delta_0$ for all $m \in \mathbb{R}_+$ and for all $i \in \{2, ..., k\}$. Given $v \in \mathcal{W}$, recall that we define WTA_i^v and WTP_i^v according to definitions (12) and (13) for the corresponding Expected Utility preference with Bernoulli utility v. By (11) and since each $v \in \mathcal{W}$ is strictly increasing and continuous, it is immediate to see that $\mathrm{WTA}_i^v(m)$ and $\mathrm{WTP}_i^v(m)$ are the unique solutions of the equations

$$v(-me_i + le_1) = 0 \text{ and } v(me_i - le_1) = 0.$$

By (11) and since v is strictly increasing and continuous, this implies that both WTA_i^v and WTP_i^v are continuous functions. Fix $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$. By point 2 of Re-

[24] That is, given $x, y \in \mathbb{R}^k$, $x \geq y$ implies $\lambda\delta_x + (1-\lambda) r \succcurlyeq \lambda\delta_y + (1-\lambda) r$ for all $\lambda \in (0, 1]$ and for all $r \in \Delta$.

mark 5 and the definition of $\succcurlyeq^*$, and since each $\hat{v}$ in $\mathcal{W}$ satisfies $\hat{v}(0) = 0$, we have that $\delta_{-me_i+\mathrm{WTA}_i(m)e_1} \succcurlyeq^* \delta_0$ and $\delta_{me_i-\mathrm{WTP}_i(m)e_1} \succcurlyeq^* \delta_0$, that is,

$$\hat{v}\left(-me_i + \mathrm{WTA}_i(m)\, e_1\right) \geq 0 \text{ and } \hat{v}\left(me_i - \mathrm{WTP}_i(m)\, e_1\right) \geq 0 \quad \forall \hat{v} \in \mathcal{W}.$$

By the definitions of WTA_i^v and WTP_i^v and since each $\hat{v}$ in $\mathcal{W}$ is strictly increasing, this implies that $\mathrm{WTA}_i(m) \geq \mathrm{WTA}_i^{\hat{v}}(m)$ and $\mathrm{WTP}_i^{\hat{v}}(m) \geq \mathrm{WTP}_i(m)$ for all $\hat{v} \in \mathcal{W}$, yielding that

$$\mathrm{WTA}_i(m) \geq \sup_{\hat{v}\in\mathcal{W}} \mathrm{WTA}_i^{\hat{v}}(m) \text{ and } \inf_{\hat{v}\in\mathcal{W}} \mathrm{WTP}_i^{\hat{v}}(m) \geq \mathrm{WTP}_i(m). \tag{14}$$

Vice-versa, by the definitions of WTA_i^v and WTP_i^v and since each v in $\mathcal{W}$ is strictly increasing, we have that

$$v\left(-me_i + \sup_{\hat{v}\in\mathcal{W}} \mathrm{WTA}_i^{\hat{v}}(m)\, e_1\right) \geq 0 \text{ and } v\left(me_i - \inf_{\hat{v}\in\mathcal{W}} \mathrm{WTP}_i^{\hat{v}}(m)\, e_1\right) \geq 0 \quad \forall v \in \mathcal{W}.$$

By the definition of $\succcurlyeq^*$ and point 1 of Remark 5, we obtain that $\delta_{-me_i+\sup_{\hat{v}\in\mathcal{W}} \mathrm{WTA}_i^{\hat{v}}(m)e_1} \succcurlyeq^* \delta_0$ and $\delta_{me_i-\inf_{\hat{v}\in\mathcal{W}} \mathrm{WTP}_i^{\hat{v}}(m)e_1} \succcurlyeq^* \delta_0$, and, in particular, $\delta_{-me_i+\sup_{\hat{v}\in\mathcal{W}} \mathrm{WTA}_i^{\hat{v}}(m)e_1} \succcurlyeq \delta_0$ and $\delta_{me_i-\inf_{\hat{v}\in\mathcal{W}} \mathrm{WTP}_i^{\hat{v}}(m)e_1} \succcurlyeq \delta_0$. By the definitions of WTA_i and WTP_i, this implies that

$$\mathrm{WTA}_i(m) \leq \sup_{\hat{v}\in\mathcal{W}} \mathrm{WTA}_i^{\hat{v}}(m) \text{ and } \inf_{\hat{v}\in\mathcal{W}} \mathrm{WTP}_i^{\hat{v}}(m) \leq \mathrm{WTP}_i(m).$$

Since m and i were arbitrarily chosen, we can conclude that

$$\mathrm{WTA}_i(m) = \sup_{\hat{v}\in\mathcal{W}} \mathrm{WTA}_i^{\hat{v}}(m) \text{ and } \mathrm{WTP}_i(m) = \inf_{\hat{v}\in\mathcal{W}} \mathrm{WTP}_i^{\hat{v}}(m) \quad \forall m \in \mathbb{R}_+, \forall i \in \{2, ..., k\},$$

proving the statement. ■

Proof of Proposition 1. We begin with a part which is common to both models. Consider $i \in \{2, \ldots, k\}$, $m \in \mathbb{R}_+$, and $v : \mathbb{R}^k \to \mathbb{R}$ strictly increasing, continuous, and such that $v(0) = 0$. Recall that $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$. In particular, $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. It is then immediate to see that

$$v\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) = 0 \iff \bar{v}\left(me_i - \mathrm{WTA}_i^v(m)\, e_1\right) = 0 \iff \mathrm{WTA}_i^v(m) = \mathrm{WTP}_i^{\bar{v}}(m). \tag{15}$$

We can now prove points 1.i and 2.i.

1.i. Consider $i \in \{2, \ldots, k\}$ and $m \in \mathbb{R}_+$. Let $v', v'' \in \mathcal{W}$. Without loss of generality, we can assume that $\mathrm{WTA}_i^{v'}(m) \geq \mathrm{WTA}_i^{v''}(m)$. By Proposition 2 and (15), and since $\bar{v}'' \in \mathcal{W}$, we have that

$$\mathrm{WTA}_i(m) = \sup_{v\in\mathcal{W}} \mathrm{WTA}_i^v(m) \geq \mathrm{WTA}_i^{v'}(m) \geq \mathrm{WTA}_i^{v''}(m) =$$

$$= \mathrm{WTP}_i^{\bar{v}''}(m) \geq \inf_{v\in\mathcal{W}} \mathrm{WTP}_i^v(m) = \mathrm{WTP}_i(m).$$

Since $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$ were arbitrarily chosen, the statement follows.

2.i. We first discuss how Proposition 2 becomes for Incatious Utility. If $\succcurlyeq$ admits an Incautious Utility representation, then for each $m \in \mathbb{R}_+$ and for each $i \in \{2, ..., k\}$

$$\mathrm{WTA}_i(m) \leq \inf_{v\in\mathcal{W}} \mathrm{WTA}_i^v(m) \quad \text{and} \quad \mathrm{WTP}_i(m) \geq \sup_{v\in\mathcal{W}} \mathrm{WTP}_i^v(m).$$

In order to derive these inequalities, we only need to observe that, for Incatious Utility, point 2 of Remark 5 becomes: for each $p \in \Delta$ and for each $m \in \mathbb{R}$

$$\delta_{me_1} \succcurlyeq p \implies \delta_{me_1} \succcurlyeq^* p.$$

The inequalities above then follow by a specular argument up to (14). We can now prove 2.i. Consider $i \in \{2, \ldots, k\}$ and $m \in \mathbb{R}_+$. Let $v', v'' \in \mathcal{W}$. Without loss of generality, we can assume that $\mathrm{WTA}_i^{v'}(m) \geq \mathrm{WTA}_i^{v''}(m)$. By the inequalities above and (15) and since $\bar{v}' \in \mathcal{W}$, we have that

$$\mathrm{WTA}_i(m) \leq \inf_{v\in\mathcal{W}} \mathrm{WTA}_i^v(m) \leq \mathrm{WTA}_i^{v''}(m) \leq \mathrm{WTA}_i^{v'}(m) =$$

$$= \mathrm{WTP}_i^{\bar{v}'}(m) \leq \sup_{v\in\mathcal{W}} \mathrm{WTP}_i^v(m) \leq \mathrm{WTP}_i(m).$$

Since $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$ were arbitrarily chosen, the statement follows.

We next prove points 1.ii and 2.ii.

1.ii. Consider $i \in \{1, \ldots, k\}$ and $a \in \mathbb{R}_{++}$. By contradiction, assume that $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} \succ \delta_0$. By point 2 of Remark 5, we have that $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} \succcurlyeq^* \delta_0$. By point 5 of Remark 5, $\delta_0 = \sigma(\delta_0) \succcurlyeq^* \sigma\left(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}\right) = \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. By point 1 of Remark 5, we can conclude that $\delta_0 \succcurlyeq \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$, a contradiction.

2.ii. We begin by recalling that for Incautious Utility points 1 as well as 3–5 of Remark 5 hold while point 2 becomes: for each $p \in \Delta$ and for each $m \in \mathbb{R}$

$$\delta_{me_1} \succcurlyeq p \implies \delta_{me_1} \succcurlyeq^* p. \tag{16}$$

Consider $i \in \{1, \ldots, k\}$ and $a \in \mathbb{R}_{++}$. By contradiction, assume that $\delta_0 \succ \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. By (16), we have that $\delta_0 \succcurlyeq^* \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. By point 5 of Remark 5, $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} =$

$\sigma\left(\frac{1}{2}\delta_{ae_i}+\frac{1}{2}\delta_{-ae_i}\right) \succcurlyeq^* \sigma(\delta_0)=\delta_0$. By point 1 of Remark 5, we can conclude that $\frac{1}{2}\delta_{ae_i}+\frac{1}{2}\delta_{-ae_i} \succcurlyeq \delta_0$, a contradiction. ■

Proof of Corollary 1. Fix $i \in \{2,...,k\}$ and $m>0$. Before proving the statement, we make few observations. Given $v \in \mathcal{W}$, recall that $\bar{v}:\mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x)=-v(-x)$ for all $x \in \mathbb{R}^k$. Since $\mathcal{W}$ is odd, $\bar{v} \in \mathcal{W}$. Moreover, it is immediate to check that $\bar{\bar{v}}=v$ for all $v \in \mathcal{W}$. By the first part of the proof of Proposition 1, in particular (15), we have that for each $v \in \mathcal{W}$

$$\mathrm{WTA}_i^v(m)=\mathrm{WTP}_i^{\bar{v}}(m). \tag{17}$$

Since $\bar{v} \in \mathcal{W}$ and $\bar{\bar{v}}=v$ for all $v \in \mathcal{W}$, we can conclude that for each $v \in \mathcal{W}$

$$\mathrm{WTA}_i^{\bar{v}}(m)=\mathrm{WTP}_i^{\bar{\bar{v}}}(m)=\mathrm{WTP}_i^{v}(m). \tag{18}$$

(i) implies (ii). By Proposition 2 and since $\mathrm{WTA}_i(m)>\mathrm{WTP}_i(m)$, we have that $\sup_{v\in\mathcal{W}}\mathrm{WTA}_i^v(m)=\mathrm{WTA}_i(m)>\mathrm{WTP}_i(m)=\inf_{v\in\mathcal{W}}\mathrm{WTP}_i^v(m)$. By (18), this implies that there exist $v,v' \in \mathcal{W}$ such that $\mathrm{WTA}_i^{v'}(m)>\mathrm{WTP}_i^v(m)=\mathrm{WTA}_i^{\bar{v}}(m)$. Since $\bar{v} \in \mathcal{W}$, this proves the implication.

(ii) implies (iii). By assumption, there exist $v,v' \in \mathcal{W}$ such that $\mathrm{WTA}_i^v(m) \neq \mathrm{WTA}_i^{v'}(m)$. By (17), we have that

$$\mathrm{WTP}_i^{\bar{v}}(m)=\mathrm{WTA}_i^v(m)\neq\mathrm{WTA}_i^{v'}(m)=\mathrm{WTP}_i^{\bar{v}'}(m).$$

Since $\bar{v},\bar{v}' \in \mathcal{W}$, this proves the implication.

(iii) implies (i). By assumption, there exist $v,v' \in \mathcal{W}$ such that $\mathrm{WTP}_i^v(m) \neq \mathrm{WTP}_i^{v'}(m)$. Without loss of generality, we can assume that $\mathrm{WTP}_i^v(m)>\mathrm{WTP}_i^{v'}(m)$. By Proposition 2 and (18) and since $\bar{v} \in \mathcal{W}$, we have that $\mathrm{WTA}_i(m)=\sup_{v\in\mathcal{W}}\mathrm{WTA}_i^v(m)\geq\mathrm{WTA}_i^{\bar{v}}(m)=$
$=\mathrm{WTP}_i^v(m)>\mathrm{WTP}_i^{v'}(m)\geq\inf_{v\in\mathcal{W}}\mathrm{WTP}_i^v(m)=\mathrm{WTP}_i(m)$, proving the implication. ■

Proof of Corollary 2. Fix $i \in \{2,...,k\}$. Consider $v,v' \in \mathcal{W}$ which are continuously differentiable and such that $\mathrm{MRS}_i^v(x)\neq\mathrm{MRS}_i^{v'}(x)$ for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$. By definition of WTA_i^v and $\mathrm{WTA}_i^{v'}$ and since v and v' are strictly increasing, $\mathrm{WTA}_i^v(m)>0$ and $\mathrm{WTA}_i^{v'}(m)>0$ for all $m>0$. In particular, given $m>0$, we have that if $x=-me_i+\mathrm{WTA}_i^v(m)\,e_1$, then $x_1 \neq 0$ and $x_i \neq 0$. Since MRS_i^v and $\mathrm{MRS}_i^{v'}$ are well defined for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$ and v and v' are strictly increasing, we have that the partial derivative with respect to the first component is strictly positive for both v and v' for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$. By the Implicit Function Theorem and the definition of WTA_i^v and since v is strictly increasing, we have that WTA_i^v is continuously differentiable on $(0,\infty)$ and the derivative at $m>0$ is $\mathrm{MRS}_i^v\left(-me_i+\mathrm{WTA}_i^v(m)\,e_1\right)$. For ease of notation, define $f_v,f_{v'}:(0,\infty)\to\mathbb{R}$ by $f_v(m)=\mathrm{MRS}_i^v\left(-me_i+\mathrm{WTA}_i^v(m)\,e_1\right)$ and $f_{v'}(m)=\mathrm{MRS}_i^{v'}\left(-me_i+\mathrm{WTA}_i^v(m)\,e_1\right)$ for all $m>0$. Since v and v' are continuously differentiable and $\mathrm{MRS}_i^v(x)\neq\mathrm{MRS}_i^{v'}(x)$ for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$, we can conclude that f_v and $f_{v'}$ are continuous on $(0, \infty)$ and such that $f_v(m) \neq f_{v'}(m)$ for all $m > 0$. By the Intermediate Value Theorem, this implies that either $f_v(m) < f_{v'}(m)$ for all $m > 0$ or $f_v(m) > f_{v'}(m)$ for all $m > 0$. Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(m) = v'\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)$ for all $m \geq 0$. Since v' and $m \mapsto \mathrm{WTA}_i^v(m)$ are continuous and $\mathrm{WTA}_i^v(0) = 0$, note that h is continuous and $h(0) = 0$. Since v' is continuously differentiable and so is $\mathrm{WTA}_i^v(m)$ on $(0, \infty)$, we have that h is continuously differentiable on $(0, \infty)$ and

$$\begin{aligned} h'(m) &= \frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) f_v(m) - \frac{\partial v'}{\partial x_i}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) \\ &= \frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)\left(f_v(m) - \frac{\frac{\partial v'}{\partial x_i}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)}{\frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)}\right) \\ &= \frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)\left(f_v(m) - f_{v'}(m)\right) \quad \forall m > 0. \end{aligned}$$

Since $\frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) > 0$ for all $m > 0$, we can conclude that either $h'(m) < 0$ or $h'(m) > 0$ for all $m > 0$. In the first (resp. second) case, since h' is continuous on $(0, \infty)$, we have that

$$h(m) - h(m/2n) = \int_{m/2n}^{m} h'(t)\, dt < 0 \text{ (resp. } > 0) \quad \forall m > 0, \forall n \in \mathbb{N}.$$

Since h is continuous, $h(0) = 0$, and the sequence is $\{h(m) - h(m/2n)\}_{n \in \mathbb{N}}$ is decreasing (resp. increasing), we have that

$$v'\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) = h(m) = \lim_n \left[h(m) - h(m/2n)\right] < 0 \text{ (resp. } > 0) \quad \forall m > 0.$$

In the first (resp. second) case, by definition of $\mathrm{WTA}_i^{v'}(m)$ and since v' is strictly increasing, we have that $\mathrm{WTA}_i^v(m) < \mathrm{WTA}_i^{v'}(m)$ (resp. $>$) for all $m > 0$. By Corollary 1 and since $v, v' \in \mathcal{W}$, this implies the statement. ■

Proof of Corollary 3. Consider $l \in \{2, \ldots, k\}$ and $m'' \in \mathbb{R}_{++}$. By (15) and since $\mathcal{W}$ is odd, we have that

$$\left\{\mathrm{WTA}_l^v(m'') : v \in \mathcal{W}\right\} = \left\{\mathrm{WTP}_l^{\bar{v}}(m'') : v \in \mathcal{W}\right\} = \left\{\mathrm{WTP}_l^v(m'') : v \in \mathcal{W}\right\}.$$

By Proposition 2 and since $\mathcal{W}$ is finite, this implies that

$$\mathrm{WTA}_l(m'') = \max_{v \in \mathcal{W}} \mathrm{WTA}_l^v(m'') = \max \operatorname{co}\left(\left\{\mathrm{WTA}_l^v(m'') : v \in \mathcal{W}\right\}\right)$$

and

$$\begin{aligned}\mathrm{WTP}_l\left(m''\right) &= \min_{v\in\mathcal{W}} \mathrm{WTP}_l^v\left(m''\right) = \min \mathrm{co}\left(\left\{\mathrm{WTP}_l^v\left(m''\right) : v \in \mathcal{W}\right\}\right) \\ &= \min \mathrm{co}\left(\left\{\mathrm{WTA}_l^v\left(m''\right) : v \in \mathcal{W}\right\}\right).\end{aligned}$$

We can conclude that if $\mathrm{co}\left(\left\{\mathrm{WTA}_i^v\left(m\right) : v \in \mathcal{W}\right\}\right) \supset \mathrm{co}\left(\left\{\mathrm{WTA}_j^v\left(m'\right) : v \in \mathcal{W}\right\}\right)$, then

$$\mathrm{WTA}_i\left(m\right) \geq \mathrm{WTA}_j\left(m'\right) \text{ and } \mathrm{WTP}_i\left(m\right) \leq \mathrm{WTP}_j\left(m'\right) \tag{19}$$

and one of the two inequalities is strict, since the inclusion is proper. Moreover, since $\mathcal{W}$ is finite and since each $v \in \mathcal{W}$ is strictly increasing, we have that $\mathrm{WTP}_i\left(m\right) = \mathrm{WTP}_i^v\left(m\right) > 0$ for some $v \in \mathcal{W}$. By (19) and since $\mathrm{WTP}_i\left(m\right) > 0$, we have that $\mathrm{WTP}_j\left(m'\right) > 0$, proving the statement. ■

Proof of Proposition 3. Set $p = \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. Since $\succcurlyeq$ admits a Symmetric Cautious Utility representation and, in particular, $\mathcal{W}$ is odd, we have that

$$\begin{aligned}\delta_0 \succ p &\iff 0 > \inf_{v\in\mathcal{W}} c\left(p, v\right) \iff \exists v \in \mathcal{W} \quad 0 > c\left(p, v\right) \\ &\iff \exists v \in \mathcal{W} \quad 0 = v\left(0\right) > \frac{1}{2}v\left(ae_i\right) + \frac{1}{2}v\left(-ae_i\right) \\ &\iff \exists v \in \mathcal{W} \quad -v\left(-ae_i\right) > v\left(ae_i\right), \\ &\iff \exists v \in \mathcal{W} \quad -v\left(-ae_i\right) \neq v\left(ae_i\right)\end{aligned}$$

proving the statement. ■

We next show that if $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical. This fact will be key in proving the second part of Proposition 4.

Lemma 2. *If $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical.*

Proof. Define $\succcurlyeq^*$ to be such that $p \succcurlyeq^* q$ if and only if $\mathbb{E}_p\left(v\right) \geq \mathbb{E}_q\left(v\right)$ for all $v \in \mathcal{W}$ where $\mathcal{W}$ is a finite essential Cautious Utility representation of $\succcurlyeq$. Since $\mathcal{W}$ is finite, we have that the smallest convex cone containing $\mathcal{W}$, denoted by $\mathrm{cone}\left(\mathcal{W}\right)$, is closed with respect to the $\sigma\left(C\left(\mathbb{R}^k\right), \Delta\right)$-topology and so is the set $\mathrm{cone}\left(\mathcal{W}\right) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}}$. By definition of $\mathcal{W}_{\max}\left(\succcurlyeq^*\right)$, it follows that $\mathrm{cone}\left(\mathcal{W}\right) \setminus \{0\} \subseteq \mathcal{W}_{\max}\left(\succcurlyeq^*\right)$. By Proposition 8, Remark 5, and (Evren, 2008, Theorem 5) and since $\mathcal{W}$ is a Cautious Utility representation, we have that (where the closure is in the $\sigma\left(C\left(\mathbb{R}^k\right), \Delta\right)$-topology)

$$\begin{aligned}\operatorname{cone}(\mathcal{W}) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}} &= \operatorname{cl}\left(\operatorname{cone}\left(\mathcal{W}_{\max}\left(\succcurlyeq^*\right)\right) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}}\right) \\ &\supseteq \operatorname{cl}\left(\mathcal{W}_{\max}\left(\succcurlyeq^*\right) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}}\right) \\ &\supseteq \mathcal{W}_{\max}\left(\succcurlyeq^*\right) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}},\end{aligned}$$

yielding that $\operatorname{cone}(\mathcal{W}) \setminus \{0\} \supseteq \mathcal{W}_{\max}\left(\succcurlyeq^*\right)$ and, in particular, $\operatorname{cone}(\mathcal{W}) \setminus \{0\} = \mathcal{W}_{\max}\left(\succcurlyeq^*\right)$. Since the functional $v \mapsto c(p, v)$ is quasiconcave over $\operatorname{cone}(\mathcal{W}) \setminus \{0\}$ for all $p \in \Delta$, it is immediate to see that

$$V(p) = \min_{v\in\mathcal{W}} c(p, v) = \min_{v\in\operatorname{cone}(\mathcal{W})\setminus\{0\}} c(p, v) \quad \forall p \in \Delta.$$

By Remark 5 and since $\mathcal{W} = \{v_i\}_{i=1}^n$ is a finite Cautious Utility representation, we have that $\succcurlyeq$ satisfies Axioms 1- 5. By Theorem 1 and its proof, $\mathcal{W}_{\max}\left(\succcurlyeq'\right)$ is a canonical Cautious Utility representation for $\succcurlyeq$. In particular, we have that

$$V(p) = \min_{v\in\mathcal{W}} c(p, v) = \min_{v\in\operatorname{cone}(\mathcal{W})\setminus\{0\}} c(p, v) = \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(p, v) \quad \forall p \in \Delta.$$

Since $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies the Independence axiom and $p \succcurlyeq^* q$ implies $p \succcurlyeq q$, we have that $\succcurlyeq^*$ is a subrelation of $\succcurlyeq'$ and $\mathcal{W}_{\max}\left(\succcurlyeq'\right) \subseteq \mathcal{W}_{\max}\left(\succcurlyeq^*\right) = \operatorname{cone}(\mathcal{W}) \setminus \{0\}$. By contradiction, assume that $\mathcal{W}_{\max}\left(\succcurlyeq'\right) \neq \operatorname{cone}(\mathcal{W}) \setminus \{0\}$. Since $\mathcal{W}_{\max}\left(\succcurlyeq'\right)$ is a convex set closed with respect to strictly positive scalar multiplications, this implies that $\mathcal{W} \not\subseteq \mathcal{W}_{\max}\left(\succcurlyeq'\right)$. If $\mathcal{W}$ is a singleton, then $\succcurlyeq$ is Expected Utility and, in particular, $\succcurlyeq'$ is complete and coincides with $\succcurlyeq$. This implies that $\mathcal{W} = \{v_1\}$ and $\mathcal{W}_{\max}\left(\succcurlyeq'\right) = \{\lambda v_1\}_{\lambda>0} = \operatorname{cone}(\mathcal{W}) \setminus \{0\}$, a contradiction. Assume $\mathcal{W}$ is not a singleton. Consider $\breve{v} \in \mathcal{W}\setminus\mathcal{W}_{\max}\left(\succcurlyeq'\right)$. Since $\mathcal{W}$ is essential, there exists $\bar{p} \in \Delta$ such that $\min_{v\in\mathcal{W}} c(\bar{p}, v) < \min_{v\in\mathcal{W}\setminus\{\breve{v}\}} c(\bar{p}, v)$. Since $\mathcal{W} = \{v_i\}_{i=1}^n$ and $n \geq 2$, without loss of generality, we can set $\breve{v} = v_n \notin \mathcal{W}_{\max}\left(\succcurlyeq'\right)$. In particular, we have that

$$\inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = \min_{v\in\mathcal{W}} c(\bar{p}, v) = c(\bar{p}, v_n) < c(\bar{p}, v_i) \quad \forall i \in \{1, ..., n-1\}. \tag{20}$$

Consider a sequence $\{\hat{v}_m\}_{m\in\mathbb{N}} \subseteq \mathcal{W}_{\max}\left(\succcurlyeq'\right)$ such that $c(\bar{p}, \hat{v}_m) \downarrow \inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. By construction and since $\mathcal{W}_{\max}\left(\succcurlyeq'\right) \subseteq \operatorname{cone}(\mathcal{W}) \setminus \{0\}$, there exists a collection of scalars $\left\{\lambda_{m,i}\right\}_{m\in\mathbb{N}, i\in\{1,\dots,n\}} \subseteq [0, \infty)$ such that $\hat{v}_m = \sum_{i=1}^n \lambda_{m,i} v_i$ for all $m \in \mathbb{N}$. Since $\hat{v}_m$ is strictly increasing, we have that for each $m \in \mathbb{N}$ there exists $i \in \{1, ..., n\}$ such that $\lambda_{m,i} > 0$. Define $\lambda_{m,\sigma} = \sum_{i=1}^n \lambda_{m,i} > 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and for each $i \in \{1, ..., n\}$ define also $\bar{\lambda}_{m,i} = \lambda_{m,i}/\lambda_{m,\sigma}$ as well as $\tilde{v}_m = \sum_{i=1}^n \bar{\lambda}_{m,i} v_i = \hat{v}_m/\lambda_{m,\sigma}$. Since $\lambda_{m,\sigma} > 0$ for all $m \in \mathbb{N}$, it is immediate to see that $c(\bar{p}, \tilde{v}_m) = c(\bar{p}, \hat{v}_m)$ for all $m \in \mathbb{N}$ and, in particular, $c(\bar{p}, \tilde{v}_m) \downarrow$

$\inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p},v)$. For each $m\in\mathbb{N}$ denote by $\bar{\lambda}_m$ the $\mathbb{R}^n$ vector whose i-th component is $\bar{\lambda}_{m,i}$. Since $\left\{\bar{\lambda}_m\right\}_{m\in\mathbb{R}}$ is a sequence in the $\mathbb{R}^n$ simplex, there exists a subsequence $\left\{\bar{\lambda}_{m_l}\right\}_{l\in\mathbb{N}}$ such that $\bar{\lambda}_{m_l,i}\to\bar{\lambda}_i\in[0,1]$ for all $i\in\{1,...,n\}$ and $\sum_{i=1}^n\bar{\lambda}_i=1$. It is immediate to see that $\tilde{v}_{m_l}=\sum_{i=1}^n\bar{\lambda}_{m_l,i}v_i\overset{\sigma(C(\mathbb{R}^k),\Delta)}{\longrightarrow}\sum_{i=1}^n\bar{\lambda}_iv_i=\tilde{v}$ where $\tilde{v}$ is continuous, strictly increasing, and such that $\tilde{v}(0)=0$. Moreover, for each $p,q\in\Delta$ we have that $p\succcurlyeq' q$ implies $\mathbb{E}_p(\tilde{v})\geq\mathbb{E}_q(\tilde{v})$, proving that $\tilde{v}\in\mathcal{W}_{\max}(\succcurlyeq')$. Note that $\bar{\lambda}_n<1$, otherwise, we would have that $v_n=\tilde{v}\in\mathcal{W}_{\max}(\succcurlyeq')$, a contradiction. By (20) and since $\bar{\lambda}_n<1$ and the functional $v\mapsto c(p,v)$ is explicitly quasiconcave over $\operatorname{co}(\mathcal{W})$ for all $p\in\Delta$,[25] we have that

$$c(\bar{p},v_n)<c(\bar{p},\tilde{v})=\lim_l c\left(\bar{p},\tilde{v}_{m_l}\right)=\lim_m c(\bar{p},\tilde{v}_m)=\inf_{v\in\mathcal{W}_{\max}(\succcurlyeq')}c(\bar{p},v)=c(\bar{p},v_n),$$

a contradiction. It follows that $\mathcal{W}_{\max}(\succcurlyeq')=\operatorname{cone}(\mathcal{W})\setminus\{0\}$ and, in particular, $\mathcal{W}$ represents also $\succcurlyeq'$. This implies that $\mathcal{W}$ is canonical. ■

Proof of Proposition 4. We first prove the first part of the statement assuming $\succcurlyeq$ satisfies u-CPT, then we will move to the additive case. Since $u(0)=0$ and u is strictly increasing, it follows that there exists $\bar{t}>0$ such that $[-\bar{t},\bar{t}]\subseteq\operatorname{Im}u$. Let $\Delta_0([0,\bar{t}])$ be the set of finitely supported probabilities over $[0,\bar{t}]$. Consider $\tilde{p}\in\Delta_0([0,\bar{t}])$. By definition, we have that there exist two unique collections $\{t_i\}_{i=1}^n\subseteq[0,\bar{t}]$ and $\{\lambda_i\}_{i=1}^n\subseteq[0,1]$ such that $\operatorname{supp}p=\{t_i\}_{i=1}^n$, $\sum_{i=1}^n\lambda_i=1$, and $\tilde{p}=\sum_{i=1}^n\lambda_i\delta_{t_i}$. Without loss of generality, we can assume that $t_1<...<t_n$. We define $\tilde{V}:\Delta_0([0,\bar{t}])\to\mathbb{R}$ by

$$\tilde{V}(\tilde{p})=\sum_{j=1}^{n-1}\left(w^+\left(\sum_{i=j}^n\lambda_i\right)-w^+\left(\sum_{i=j+1}^n\lambda_i\right)\right)v\left(t_j\right)+w^+(\lambda_n)\,v(t_n)$$

for all $\tilde{p}\in\Delta_0([0,\bar{t}])$. We next show that for each $\tilde{p}\in\Delta_0([0,\bar{t}])$ and for each $\tilde{t}\in[0,\bar{t}]$, if $\tilde{V}(\tilde{p})=\tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda\tilde{p}+(1-\lambda)\delta_{\tilde{t}})=\tilde{V}(\delta_{\tilde{t}})$ for all $\lambda\in(0,1)$. Consider $\tilde{p}\in\Delta_0([0,\bar{t}])$ and $\tilde{t}\in[0,\bar{t}]$ such that $\tilde{V}(\tilde{p})=\tilde{V}(\delta_{\tilde{t}})$. Given $\tilde{p}\in\Delta_0([0,\bar{t}])$, since $\{t_i\}_{i=1}^n\subseteq[0,\bar{t}]\subseteq\operatorname{Im}u$, there exists $\{x_i\}_{i=1}^n\subseteq\mathbb{R}^k$ such that $u(x_i)=t_i$ for all $i\in\{1,...,n\}$. Consider $p=\sum_{i=1}^n\lambda_i\delta_{x_i}$. It is immediate to see that $\tilde{V}(\tilde{p})=V(p)$. Since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, there exists $c\in\mathbb{R}$ such that $p\sim\delta_{ce_1}$. This implies that $V(p)=V\left(\delta_{ce_1}\right)$ and, in particular, $u(ce_1)\in[0,\bar{t}]$. Moreover, since u and v are strictly increasing, we have that $u(ce_1)=\tilde{t}\in[0,\bar{t}]$ and $V\left(\delta_{ce_1}\right)=\tilde{V}(\delta_{\tilde{t}})$. By Remark 5 and since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, we have that $\succcurlyeq$ satisfies M-NCI. This yields that

[25] Formally, see e.g. (Aliprantis and Border, 2006, p. 300), given $p\in\Delta$, for each $h\in\mathbb{N}\setminus\{1\}$, for each $\{v_l\}_{l=1}^h\subseteq\operatorname{co}(\mathcal{W})$, and for each $\{\lambda_l\}_{l=1}^h\subseteq[0,1]$ such that $\sum_{l=1}^h\lambda_l=1$ and $\lambda_h<1$

$$c(p,v_i)>c(p,v_h)\quad\forall i\in\{1,...,h-1\}\implies c\left(p,\sum_{i=1}^h\lambda_iv_i\right)>c(p,v_h).$$

$\lambda p + (1-\lambda)\,\delta_{ce_1} \sim \delta_{ce_1}$ for all $\lambda \in (0,1)$. This implies that

$$\tilde{V}\,(\lambda\tilde{p} + (1-\lambda)\,\delta_{\tilde{t}}) = V\left(\lambda p + (1-\lambda)\,\delta_{ce_1}\right) = V\left(\delta_{ce_1}\right) = \tilde{V}\,(\delta_{\tilde{t}})\,.$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that w^+ is the identity. The same proof, performed with $[-\bar{t},0]$ in place of $[0,\bar{t}]$ and w^+ replaced by w^-, yields that w^- is the identity. These two facts together allows us to conclude that $V(\cdot) = \mathrm{CPT}\,(v,w^+,w^-)\left(F_{\cdot,u}\right)$ is an Expected Utility functional with utility $v \circ u : \mathbb{R}^k \to \mathbb{R}$. We next assume that $\succcurlyeq$ admits an Additive CPT representation. As before consider $\bar{t} > 0$. Define $\Delta_0\,([0,\bar{t}])$ and $\tilde{V}$ as before with v replaced by u_1. For each $\tilde{p} \in \Delta_0\,([0,\bar{t}])$ define p in Δ to be the product measure $\tilde{p} \otimes \delta_0 ... \otimes \delta_0$. It is immediate to see that $\tilde{V}\,(\tilde{p}) = V\,(p)$ for all $\tilde{p} \in \Delta_0\,([0,\bar{t}])$. As before, we can show that for each $\tilde{p} \in \Delta_0\,([0,\bar{t}])$ and for each $\tilde{t} \in [0,\bar{t}]$, if $\tilde{V}\,(\tilde{p}) = \tilde{V}\,(\delta_{\tilde{t}})$, then $\tilde{V}\,(\lambda\tilde{p} + (1-\lambda)\,\delta_{\tilde{t}}) = \tilde{V}\,(\delta_{\tilde{t}})$ for all $\lambda \in (0,1)$. Consider $\tilde{p} \in \Delta_0\,([0,\bar{t}])$ and $\tilde{t} \in [0,\bar{t}]$ such that $\tilde{V}\,(\tilde{p}) = \tilde{V}\,(\delta_{\tilde{t}})$. This implies that $V\,(p) = V\left(\tilde{t}e_1\right)$, that is, $p \sim \delta_{\tilde{t}e_1}$. By Remark 5 and since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, we have that $\succcurlyeq$ satisfies M-NCI. This yields that $\lambda p + (1-\lambda)\,\delta_{\tilde{t}e_1} \sim \delta_{\tilde{t}e_1}$ for all $\lambda \in (0,1)$. This implies that

$$\tilde{V}\,(\lambda\tilde{p} + (1-\lambda)\,\delta_{\tilde{t}}) = V\left(\lambda p + (1-\lambda)\,\delta_{\tilde{t}e_1}\right) = V\left(\delta_{\tilde{t}e_1}\right) = \tilde{V}\,(\delta_{\tilde{t}})\,.$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that w^+ is the identity. The same proof, performed with $[-\bar{t},0]$ in place of $[0,\bar{t}]$ and w^+ replaced by w^-, yields that w^- is the identity. This implies that $\succcurlyeq$ admits an Expected Utility representation with utility $u : \mathbb{R}^k \to \mathbb{R}$ defined by $u\,(x) = \sum_{i=1}^{k} u_i\,(x_i)$ for all $x \in \mathbb{R}^k$.

As for the second part of the statement, by Lemma 2 and since $\mathcal{W}$ is a finite essential Cautious Utility representation, we have that $\mathcal{W}$ is a canonical representation, that is, $\mathcal{W} = \{v_i\}_{i=1}^{n}$ represents also $\succcurlyeq'$. Since $\succcurlyeq$ is Expected Utility with utility $v \circ u$ (where in the additive case v is the identity and u is additively separable), we have that $\succcurlyeq'$ coincides with $\succcurlyeq$, yielding that for each $i \in \{1,...,n\}$ there exists $\lambda_i > 0$ such that $v_i = \lambda_i(v \circ u)$. This implies that $c\,(p,v_i) = c\,(p, v \circ u)$ for all $p \in \Delta$ and for all $i \in \{1,...,n\}$. Since $\mathcal{W}$ is essential, this implies that $\mathcal{W}$ is a singleton. Since $\mathcal{W} = \{v_1\}$ and $\mathcal{W}$ is odd, this implies that v_1 is odd and, in particular, $\succcurlyeq$ is loss neutral and exhibits no endowment effect. ■

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