Dynamic Opinion Aggregation: Long-Run Stability and Disagreement*

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Abstract

This paper proposes a model of non-Bayesian social learning in networks that accounts for heuristics and biases in opinion aggregation. The updating rules are represented by nonlinear opinion aggregators from which we extract two extreme networks capturing strong and weak links. We provide graph-theoretic conditions for these networks that characterize opinions' convergence, consensus formation, and efficient or biased information aggregation. Under these updating rules, agents may ignore some of their neighbors' opinions, reducing the number of effective connections and inducing long-run disagreement for finite populations. For the wisdom of the crowd in large populations, we highlight a trade-off between how connected the society is and the nonlinearity of the opinion aggregator. Our framework bridges several models and phenomena in the non-Bayesian social learning literature, thereby providing a unifying approach to the field.

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1 Introduction

In recent years, studying people's opinion dynamics and reciprocal influence has become of utmost importance for economic research. This is mainly due to the significant increase in social media usage and the formation of global social networks. Under the classical Bayesian approach, agents act as statisticians who try to estimate a fundamental parameter based on their neighbors' opinions. An alternative approach, which is more descriptive and tractable, considers agents who assign fixed weights to their neighbors and repeatedly take weighted averages of the opinions they observe.[1] This is commonly known as the DeGroot linear updating rule. However, even among stationary updating rules, the DeGroot model is still quite unrealistic. In real life, individuals are often attracted to extreme or intermediate stances, and the set of their influencers varies and is not given by a fixed network of connections. These and other relevant properties are incompatible with simple repeated averaging and have made generalizing the DeGroot model and its insights the primary theoretical challenge in this literature.

While the convergence and long-run consensus properties of particular nonlinear rules have been widely studied in applied mathematics, computer science, and economics, a general treatment of convergence and consensus with nonlinear updating rules that naturally extends the tools of the DeGroot model is still missing. Moreover, the central question of information aggregation in large networks, namely, the wisdom of the crowd hypothesis, has received much less attention for nonlinear updating rules, primarily due to technical challenges. To be closed, these gaps in the social learning literature require new methodologies and mathematical tools.

This paper addresses these gaps by analyzing a general and unifying class of stationary nonlinear updating rules. It answers convergence, consensus, and information-aggregation questions by developing new mathematical tools well suited to studying nonlinear (and often nondifferentiable) rules and generalizing the ones used for the DeGroot model. We show that most of the insights of the DeGroot model can be generalized to this class of updating rules. However, we also highlight qualitative insights that are peculiar to nonlinear rules. For example, due to nonlinearities, agents may sometimes disregard some of their neighbors' opinions, reducing the number of effective connections and inducing long-run disagreement for finite populations. Moreover, regarding the wisdom of the crowd in large populations, we point out a trade-off between how connected society is and the nonlinearity of the opinion aggregator.

Robust opinion aggregators We consider agents on a network whose initial opinions equal a common fundamental parameter plus some agent-specific noise. Agents observe their neighbors' opinions and repeatedly incorporate them to update their own through functions that we call *robust opinion aggregators*. These aggregators map the last-period opinions of the neighbors of each agent into her current stance and satisfy the following natural properties:

1. **Normalization:** If the agents have reached a consensus, then none of them further updates

[1] See the empirical evidence in Breza et al. [14] and Chandrasekhar et al. [22] and the references therein. In addition, when modeling Bayesian updating in a network, tractability is easily lost; see Breza et al. [13]. Notable exceptions are Mossel et al. [58] and Mueller-Frank [59].

their opinions.

2. **Monotonicity:** If two opinion profiles are such that the first coordinatewise dominates the second, then this relation is preserved after aggregation.

3. **Translation invariance:** If the same constant shifts each agent's opinion, then the updates are shifted accordingly.

The first two properties are straightforwardly interpreted as minimal trust in the neighbors' opinions. Translation invariance is equivalent to assuming that agents only care about the opinions' differences rather than their intrinsic levels and rules out explosive/chaotic dynamics. This property is a consequence of a robust loss-minimization procedure that provides a foundation and an interpretation of the updating rule proposed (cf. Section 5). Importantly, recent field studies that compare Bayesian to non-Bayesian social learning models have obtained evidence consistent with our properties. For instance, Chandrasekhar et al. [22] find that if the sampled subjects reach a consensus, they remain stuck in their beliefs even when such behavior is objectively suboptimal: this is consistent with normalization. Similarly, they also find that most subjects respond monotonically to changes in their neighbors' opinions.

The properties of robust opinion aggregators imply that the influence among agents depends on their current opinions. This simple feature makes our model the first unifying framework to capture the many documented descriptive phenomena that we discuss in Section 3.[2] In such a framework, our main results deal with the long-run stability of opinions across two complementary dimensions. We first provide graph-theoretic conditions on robust opinion aggregators for different forms of convergence of opinions in finite populations. We then derive structural properties of robust opinion aggregators that either guarantee or prevent the identification of the fundamental parameter as the population grows.

The dynamics of robust opinion aggregation We first show that the opinions' *time averages* induced by *any* robust opinion aggregator *uniformly* converge so that a profile of long-run opinions always exists. This first benchmark result implies that an external agent can test the long-run learning properties of the updating procedure by computing time averages, a feature that we exploit in our results for large networks.

Moreover, this is the stepping stone for deriving convergence and consensus formation from the properties of the network structures associated with robust opinion aggregators. We say that an agent is *strongly influenced* by another if the former *always* reacts to variations in the latter's opinion, regardless of the current opinion profile in society. We show that if each agent has at least one strong link and the induced *strong network* is aperiodic, then opinions converge. This result is powerful for two reasons. First, it guarantees that, in a comprehensive class of models, the sole iteration of the aggregation procedure always leads to a stable distribution of opinions in the population (i.e., a Nash equilibrium under a best-response-dynamics interpretation). Second, it highlights the critical role of

[2] We defer the comparison with the existing models to Section 6.

strong ties in society in stabilizing opinions in the long run.[3]

Alternatively, we say that an agent is *weakly influenced* by another if the former reacts to variations in the latter's opinion for *at least one opinion profile*, and we show that opinions always converge only if the *weak network* is aperiodic. Therefore, whenever these extreme networks coincide (e.g., in the DeGroot model), opinions' convergence is characterized by network aperiodicity. However, whenever behavioral biases or robustness concerns in the updating rules induce a wedge between the two extreme networks, we cannot dispense from studying both to get a complete picture of the opinions' long-run behavior.

Our contribution to convergence to *consensus* is more conceptual than technical. It illustrates how the strong and weak networks are the key objects for nonlinear opinion aggregation since imposing extra conditions on them buys extra convergence properties. We show that if the strong network has a unique, strongly connected, and closed group that is aperiodic, convergence to consensus always obtains. Moreover, a necessary condition for forming consensus, regardless of the initial opinions, is that the weak network has a unique, strongly connected, and closed group that is aperiodic. Whenever the two networks coincide, convergence to consensus is fully characterized by the previous property. However, if they differ, then even in societies in which every two agents share some form of connection, we might observe persistent disagreement in the long run due to the weakness of these connections. Compared to the existing literature on convergence to consensus, we are the first to link a network structure derived from a given normalized, monotone, and translation invariant aggregator to convergence to consensus. However, several important works, such as Moreau [56], provide sufficient and necessary conditions given a fixed vector of initial opinions that can be used as part of an alternative route to our result about consensus. We defer to Section 6 a detailed comparison with these works.

Vox populi, vox Dei? We next study the information-aggregation properties of robust opinion aggregators.[4] In particular, we study whether the *wisdom of the crowd* is achieved, that is, whether in large networks, the agents' opinions converge to a true fundamental parameter (cf. Golub and Jackson [39]).

Given a robust opinion aggregator, we define strong and a weak influence vectors. These objects respectively capture the minimal and maximal influences among agents in the long run and give us a tool to study the limit opinions' variability. If the long-run *weak influence* of every agent vanishes sufficiently fast as the population grows, then the variance of their opinions vanishes as well. Conversely, if the long-run *strong influence* of at least one agent remains positive, then the aggregation procedure does not wash out all the idiosyncratic variability. Vanishing variability and symmetry of the robust opinion aggregator and the errors guarantee that the long-run opinions coincide with the true parameter in the large population limit.

Notably, our analysis of the large-population limit does not presume convergence or consensus.

[3] We follow one of the two interpretations that Granovetter [43] assigned to the adjectives "strong" and "weak" for social ties. Indeed, as also argued by Centola and Macy [18], there is a dual meaning behind the "strong-weak" classification of ties: one is relational, and the other is structural. We adhere to the former, whereby "strong ties connect close friends or kin whose interactions are frequent, affectively charged, and highly salient to each other", [18, pp. 703].

[4] *Vox populi, vox Dei* is Latin for "the voice of the people is the voice of God." It is often shortened to *Vox populi* as in the original paper of Galton [37] on the wisdom of the crowd. In that paper, Galton "aggregated" opinions using the empirical median, a robust opinion aggregator.

Therefore, the previous finite-population conclusions determine how the concentration of opinions in the large-population limit should be understood. When only convergence of time averages obtains, these results should be interpreted in terms of wisdom *from* the crowd; an external observer can identify the parameter by computing time averages of opinions. If standard convergence obtains, we have the usual wisdom *of* the crowd interpretation. In particular, even if consensus does not obtain for finite population sizes, a typical outcome in our model, our results still yield a form of stochastic consensus for large populations.

Even if the conditions above are interpretable, they might be computationally challenging to verify since they are expressed in terms of long-run influence. Therefore, we combine graph-theoretic conditions on the weak networks and a nonlinearity index of the aggregators to devise more primitive sufficient conditions for the wisdom of the crowd under the maintained symmetry assumptions. First, the aggregators are wise when the nonlinearity index is bounded across population sizes and the degrees in the weak network are growing sufficiently fast. Second, even if the degrees are bounded, but their distribution is balanced, and the connectivity of the weak network (measured by its second-largest eigenvalue in modulus) is high relative to the nonlinearity index, wisdom still obtains. For example, the former condition is satisfied in an Erdős–Rényi model with (sufficiently) slowly decreasing linking probability. In turn, the latter condition is satisfied by expander graphs with a sufficiently high (finite) degree or by the islands model of Golub and Jackson [40] with a moderate level of homophily.

The foundation of robust opinion aggregators The properties of robust opinion aggregators arise from the natural generalization of two foundations for non-Bayesian opinions' dynamics: repeated estimation of the underlying parameter with naive agents (cf. DeMarzo et al. [29]) and best-response dynamics in coordination games (cf. Golub and Jackson [40]). In particular, an opinion aggregator is robust if and only if there is a profile of distance-based loss functions with positive complementaries whose unique solution map coincides with the aggregator itself. Moreover, natural convexity and smoothness properties of the loss functions yield robust opinion aggregators with the sufficient (and necessary) conditions for convergence and consensus obtained in our main results. Therefore, it is possible to reinterpret these results in terms of convergence to Nash equilibria and the consistency of iterated robust estimation.

2 The model

This section introduces our model of opinion aggregation in social networks. Let $N = \{1, ..., n\}$, with $n \in \mathbb{N}$, denote a finite set of agents, and let I be an *arbitrary* closed interval of $\mathbb{R}$ with nonempty interior denoting the set of possible opinions. Let $B = I^n \subseteq \mathbb{R}^n$ denote the set of opinion profiles $x = (x_i)_{i=1}^n$. For example, the opinion profile may be the agents' subjective probability assessments of an event, and in this case, $I = [0, 1]$. In this paper, we consider different (directed) networks. We identify them with an $n \times n$ adjacency matrix A', that is, $a'_{ij} = 1$ if there is a directed link from agent i to agent j, and $a'_{ij} = 0$ otherwise.

Time is discrete, $t \in \mathbb{N}$, and the initial opinion of agent $i \in N$ in period 0 is given by a signal $X_i^0 = \mu + \varepsilon_i$, where $\mu \in \mathbb{R}$ is an underlying fundamental parameter and each $\varepsilon_i : \Omega \to \mathbb{R}$ is a random variable defined over a common probability space $(\Omega, \mathcal{F}, P)$.[5] Let A denote the *observation network* and $N_i = \{j \in N : a_{ij} = 1\}$ denote the *neighborhood* of agent i. The interpretation is that agent i can only observe the current opinions of her neighbors $j \in N_i$.

Let x_i^0 denote the realization of the period-0 opinion of agent i. We model the evolution of opinions in the following periods through an *opinion aggregator* $T : B \to B$ that for each profile of period-t opinions $x^t \in B$ returns the profile of period-$(t+1)$ updates $x^{t+1} = T\left(x^t\right)$. We let $T_i : B \to I$ denote the i-th component of T, the updating rule of agent i.[6] Let $e \in \mathbb{R}^n$ denote the vector whose components are all 1's.

Definition 1 *Let T be an opinion aggregator. We say that:*

1. *T is* normalized *if and only if $T(ke) = ke$ for all $k \in I$.*
2. *T is* monotone *if and only if for each $x, y \in B$*

$$x \geq y \implies T(x) \geq T(y).$$

3. *T is* translation invariant *if and only if*

$$T(x + ke) = T(x) + ke \qquad \forall x \in B, \forall k \in \mathbb{R} \ s.t. \ x + ke \in B.$$

We say that T is robust *if and only if T is normalized, monotone, and translation invariant.*

Normalization requires that whenever all the agents share the same opinion, each of the next-period updates coincides with that opinion. Monotonicity embodies a form of trust of the agents in the opinions observed by others. Translation invariance naturally arises when agents only care about their opinions' differences, as we show in Section 5. In our related work [20], we provide a game-theoretic foundation that relaxes this property to translation subinvariance, that is, agents react less than proportionally to uniform shifts. All our main convergence results continue to hold.[7]

Robust opinion aggregators are rich enough to describe several behavioral phenomena that we illustrate below: aversion/attraction to extreme opinions, rank-dependent social influence, confirmatory bias, and pure right/left bias. Moreover, they nest the widely studied DeGroot model, where T is also linear: $T(x) = Wx$ for all $x \in B$. Here, $W \in \mathcal{W}$ is the matrix collecting the vectors of weights, and $\mathcal{W}$ denotes the collection of stochastic matrices. This simple aggregation rule arises from either

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[5] For completeness, we present the stochastic structure of initial opinions here. However, this does not have a relevant role in the analysis until Section 4 on the wisdom of the crowd.

[6] The network structure (N, A) can be reflected in the opinion aggregator T by assuming that for each $i \in N$ and for each $x, x' \in B$

$$x_j = x'_j \qquad \forall j \in N_i \implies T_i(x) = T_i(x').$$

It is a natural assumption satisfied by all our illustrations, but it can be dispensed with for the general analysis.

[7] A careful inspection of the proofs shows that our convergence result continues to hold for opinion aggregators that are normalized, monotone, and Lipschitz continuous of order 1. Under normalization and monotonicity, this last property is equivalent to translation subinvariance. A natural concern is that for some opinion domains, the shift from, for example, $\frac{1}{4}$ and $\frac{1}{2}$ is perceived as larger than the shift from $\frac{1}{2}$ and $\frac{3}{4}$. If all the agents share this perception, all our results continue to hold after rescaling I according to the perceived differences. We thank an anonymous referee for this observation.

best-response dynamics in coordination games with quadratic payoffs or naive repeated maximum-likelihood estimation of a location parameter under Gaussian signals.[8] In both cases, each $T_i(x)$ is the minimizer over $c \in \mathbb{R}$ of the loss function

$$\sum_{j=1}^{n} w_{ij} (x_j - c)^2 \tag{1}$$

where $w_i \in \Delta = \left\{ p \in \mathbb{R}^n_+ : \sum_{j=1}^n p_j = 1 \right\}$ is the i-th row of W. In Section 5.1, we derive robust opinion aggregators from a more general *robust* loss-minimization problem that removes the quadratic and Gaussian assumptions. For this reason and the unifying role of the properties in Definition 1, we call the aggregators we analyze robust. Although natural, these properties exclude some extremely discontinuous behavior patterns, such as agents listening to each other only when their opinions are closer than some threshold (cf. Krause [49]). They also exclude updating rules in which agents always give some weight to an exogenously fixed opinion, as in Friedkin and Johnsen [33].

Turning to the analysis of opinions' dynamics, we deal with two kinds of limit of $\left\{T^t(x)\right\}_{t \in \mathbb{N}}$, the standard one induced by the supnorm $\| \, \|_\infty$ and the one of Cesaro (i.e., time-average limit),

$$\text{C-}\lim_t T^t(x) = \lim_\tau \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x),$$

where the limit on the right-hand side of the definition is the standard one.

Definition 2 *Let T be an opinion aggregator. We say that T is* Cesaro convergent *if and only if* C-$\lim_t T^t(x)$ *exists for all $x \in B$. We say that T is* convergent *if and only if* $\lim_t T^t(x)$ *exists for all $x \in B$.*

Given the initial opinions x^0, if the updates converge, then it is well known that Cesaro convergence obtains, and the time-average and the standard limit coincide. When T is Cesaro convergent, we define the *long-run opinion aggregator* $\bar{T} : B \to \mathbb{R}^n$ as

$$\bar{T}(x) = \text{C-}\lim_t T^t(x) \qquad \forall x \in B. \tag{2}$$

If convergence obtains, we study whether the profile of long-run opinions is represented by a unique consensus across all agents or by several coexisting conventions, that is, long-run disagreement. We denote by $D \subseteq B$ the consensus subset, that is, $x \in D$ if and only if $x_i = x_j$ for all $i, j \in N$.

Definition 3 *Let T be an opinion aggregator. We say that* convergence to consensus always obtains under T *if and only if T is convergent and $\bar{T}(x) \in D$ for all $x \in B$.*

3 The dynamics of robust opinion aggregation

This section studies the long-run properties of opinions for a given population size.

[8] For the former, see, among others, Ballester et al. [7], Calvó-Armengol et al. [17], Elliott and Golub [32], Golub and Jackson [40], and Golub and Morris [41]. For the latter, see DeMarzo et al. [29] and Golub and Jackson [39].

3.1 Convergence of the time averages

Our first result shows that even if the updates of a robust opinion aggregator might not converge, their time averages always stabilize in the long run.

Theorem 1 *If T is a robust opinion aggregator, then T is Cesaro convergent. Moreover, the long-run opinion aggregator $\bar{T}$ is a robust opinion aggregator such that $\bar{T} \circ T = \bar{T}$, and if $\hat{B}$ is a bounded subset of B, then*

$$\lim_{\tau} \left(\sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t (x) - \bar{T} (x) \right\|_{\infty} \right) = 0. \tag{3}$$

The *Cesaro limit* is described by the long-run opinion aggregator $\bar{T}$ that, for each initial profile of stances $x \in B$, returns the long-run average opinion of each agent. In particular, $\bar{T}$ is robust and satisfies the fixed-point equation $\bar{T} \circ T = \bar{T}$, hence generalizing the well-known notion of *eigenvector centrality* of the DeGroot model. Finally, whenever the initial opinions of the agents are known to belong to a bounded set, the initial realizations of their signals do not affect the *rate of convergence* of the time averages.

Median aggregator We now illustrate the content of Theorem 1 with a natural alternative to opinion aggregation via weighted means: the median aggregator. Assume that the agents best respond to the previous opponents' opinions, but instead of minimizing a weighted quadratic loss function (1), they minimize the weighted absolute deviations:

$$\sum_{j=1}^{n} w_{ij} |x_j - c| \qquad \forall x \in B, \forall c \in I \tag{4}$$

where the values w_{ij} are the entries of a stochastic matrix W. It is well known that the solution correspondence admits as a selection the robust opinion aggregator T,

$$T_i (x) = \min \left\{ c \in \mathbb{R} : \sum_{j : x_j \leq c} w_{ij} \geq 0.5 \right\} \qquad \forall x \in B, \forall i \in N, \tag{5}$$

that is, $T_i (x)$ is the (weighted) median of x.

Example 1 A group of agents $N = \{1, 2, 3, 4\}$ share their opinions $x^0 \in B = [0, 1]^4$. The weights assigned to the other agents are represented by the matrix

$$W = \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.1 & 0.2 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}.$$

Aggregation through weighted *averages* would achieve consensus in the limit (see, e.g., [39, Proposition 1]). However, the dynamics induced by using the median are qualitatively different.

If $x^0 = \left(x_1^0, 1, 1, 1\right)$, then the block of agents agreeing on the higher opinion is sufficiently large to attract agent 1 to the same opinion, and the limit (consensus) opinion of $(1, 1, 1, 1)$ is reached in one round of updating. Note that the initial opinion of agent 1 is irrelevant, given the agreement of the other agents. Similarly, the same limit consensus obtains if agent 2 disagrees with the initial consensus, that is if $x^0 = \left(1, x_2^0, 1, 1\right)$.

Instead, convergence to consensus fails if the initial opinions of *both* agents 1 and 2 fall. If $x^0 = (0, 1/2, 1, 1)$, then the first round of updating is $x^1 = (1/2, 1/2, 1, 1)$, and this opinion segregation will be the limit outcome: a strongly connected society fails to reach a consensus without a sufficiently large block of initial agreement. This highlights how with median aggregation, a *joint* deviation from consensus by a group of agents might be necessary to destabilize an initial consensus.[9]

If $x^0 = (0, 1/2, 0, 1)$, then the agents' first update is $x^1 = (0, 0, 1, 0)$, and agents 1 and 2 never change their opinions again, whereas agents 3 and 4 keep reciprocally switching their opinions. This shows that even convergence may not be guaranteed. However, given Theorem 1, we obtain that $\bar{T}\left(x^0\right) = (0, 0, 1/2, 1/2)$. ▲

On the one hand, the robust opinion aggregator defined in equation (5), with $w_{ii} = 0$ for all $i \in N$, yields a natural process of best-response dynamics under the payoffs of equation (4). In this case, Theorem 1 always guarantees that actions are going to stabilize on average over time, even when they do not converge. On the other hand, there is no compelling reason to assume that each agent has the same attraction for relatively central opinions.

For example, assume that the agents best respond to the previous opponents' opinions by computing a convex linear combination of an optimistic and a pessimistic aggregation. Formally, for each $i \in N$, consider a convex and closed set of probability weights $C_i \subseteq \Delta$ and a weight $\alpha_i \in [0, 1]$ and let

$$T_i(x) = \alpha_i \min_{w_i \in C_i} \sum_{j=1}^{n} w_{ij} x_j + (1 - \alpha_i) \max_{w_i \in C_i} \sum_{j=1}^{n} w_{ij} x_j \qquad \forall x \in B. \tag{6}$$

In words, agent i is uncertain about the relative importance of the opinions of the other agents, and this subjective uncertainty is represented by the set of possible weights C_i, while α_i measures the relative attractiveness of lower stances. This opinion aggregator is robust. Thus, Theorem 1 still guarantees convergence of time averages. To obtain standard convergence, as with the linear case, we need extra graph-based conditions. But, unlike with the DeGroot model, given the nonlinearity of T, there is no obvious notion of graph associated with it. In the next section, we show that two natural graphs $\underline{A}$ and $\bar{A}$ associated with T determine the long-run behavior of the agents' opinions. Indeed, for the aggregator in (6), we could either say that i is influenced by j if $w_{ij} > 0$ for *all* $w_i \in C_i$ or if $w_{ij} > 0$ for *some* $w_i \in C_i$. Intuitively, the resulting graphs $\underline{A}$ and $\bar{A}$ collect the links relevant under *every* scenario and those relevant under *some* scenarios. In stark contrast with the linear case, T is not always convergent to consensus even if every two agents are directly connected under $\bar{A}$, that is, $\bar{a}_{ij} = 1$ for all $i, j \in N$. Nevertheless, Theorem 2 provides necessary and sufficient conditions for convergence in terms of $\bar{A}$ and $\underline{A}$.

[9] In the corresponding DeGroot model with matrix W, both an individual and a joint deviation would still lead to a consensus but on a different opinion.

3.2 Stable long-run opinions

In the standard DeGroot model, convergence is tied to the properties of an underlying network structure. The structure can either be implicit and given by the indicator matrix $A(W)$ of W (e.g., Golub and Jackson [39]) or be explicit and given by a primitive observation network (e.g., DeMarzo et al. [29]).[10] Here, we follow the first approach and derive different network structures from a robust opinion aggregator T. The generalization of the second approach is deferred to Section 5.2.

We recall some common terminology from the network literature first. Consider an arbitrary network A' and let $\emptyset \neq M \subseteq N$ denote an arbitrary group. The network A' is *nontrivial* if and only if for each $i \in N$ there exists $j \in N$ such that $a'_{ij} = 1$. A path in M is a finite sequence of agents $i_1, i_2, ..., i_K \in M$ with $K \geq 2$, not necessarily distinct, such that $a'_{i_k i_{k+1}} = 1$ for all $k \in \{1, ..., K-1\}$. In this case, the length of the path is $K-1$. A cycle in M is a path in M such that $i_1 = i_K$. A cycle is simple if and only if the only repeated index in the sequence is the starting (and ending) one.[11] We say that M is *strongly connected* if and only if for each $i, j \in M$ there exists a path in M such that $i_1 = i$ and $i_K = j$. We say that M is *closed* if and only if for each $i \in M$, $a'_{ij} = 1$ implies $j \in M$. We say that M is aperiodic if and only if the greatest common divisor of the lengths of its simple cycles is 1. Finally, we say that A' is aperiodic if and only if each closed group M is *aperiodic*.[12]

In principle, there are multiple networks corresponding to the same robust aggregator T. We now give two natural definitions that formalize two extreme networks among agents induced by T. A piece of notation: $e^j \in \mathbb{R}^n$ denotes the j-th vector of the canonical basis.

Definition 4 *Let T be an opinion aggregator. We say that j* strongly influences *i if and only if there exists $\varepsilon_{ij} \in (0,1)$ such that for each $x \in B$ and for each $h > 0$ with $x + he^j \in B$*

$$T_i\left(x + he^j\right) - T_i(x) \geq \varepsilon_{ij} h. \tag{7}$$

We say that $\underline{A}(T)$ is the network of strong ties *of T if and only if for each $i, j \in N$ the ij-th entry is such that*

$$\underline{a}_{ij} = \begin{cases} 1 & \text{if } j \text{ strongly influences } i \\ 0 & \text{otherwise} \end{cases}.$$

We say that j weakly influences *i if and only if there exist $x \in B$ and $h > 0$ such that $x + he^j \in B$ and*

$$T_i\left(x + he^j\right) - T_i(x) > 0.$$

We say that $\bar{A}(T)$ is the network of weak ties *of T if and only if for each $i, j \in N$ the ij-th entry is such that*

$$\bar{a}_{ij} = \begin{cases} 1 & \text{if } j \text{ weakly influences } i \\ 0 & \text{otherwise} \end{cases}.$$

[10] Formally, the indicator matrix $A(W)$ of an arbitrary $W \in \mathcal{W}$ is such that its ij-th entry is equal to 1 if w_{ij} is strictly positive and 0 otherwise.

[11] More formally, a cycle (of length $K-1$) is simple if and only if for each $k, k' \in \{1, ..., K-1\}$: $i_k = i_{k'} \implies k = k'$.

[12] Our definition of aperiodic network coincides with the definition of strongly aperiodic network proposed by Golub and Jackson [39, Definition 7].

Equation (7) reflects uniform responsiveness of i to j: no matter what the current opinion profile is, the update of i increases at least linearly in the opinion of j. In actual social networks, strong links characterize only a subset of all connections: close friends, own past opinions (anchoring effect), or an extremely reliable source (more generally, the relational "strong ties" as in Granovetter [43] and Centola and Macy [18]).

In principle, there might be additional links (i.e., relational "weak ties") not in $\underline{A}(T)$ that are active only under particular circumstances. For instance, a person can completely discard a distant friend's opinion when it is too extreme compared to those of the rest of her neighbors. In contrast, for topics involving potential high-stakes risks (e.g., vaccinations), a person may be influenced by the opinion of someone outside her personal network, especially when the latter reports a highly negative stance (e.g., isolated severe adverse reactions to vaccines). These examples motivate the second part of Definition 4. Intuitively, i is weakly influenced by j if there are circumstances under which a change in j's opinion affects her update.

It is plain that $\underline{A}(T) \leq \bar{A}(T)$, and if T is linear with matrix W, then $A(W) = \underline{A}(T) = \bar{A}(T)$. Therefore, it is impossible to separate these two extreme networks in the DeGroot model. For a general robust opinion aggregator T, the strong directed network $\underline{A}(T)$ is the *minimal* network underlying T, while the weak directed network $\bar{A}(T)$ is the *maximal* network. Accordingly, they are instrumental in providing respectively *sufficient* and *necessary* conditions for convergence.

Theorem 2 *Let T be a robust opinion aggregator. The following statements are true:*

1. *If the network of strong ties $\underline{A}(T)$ is aperiodic and nontrivial, then T is convergent.*
2. *If T is convergent, then the network of weak ties $\bar{A}(T)$ is aperiodic and nontrivial.*

Therefore, if $\underline{A}(T) = \bar{A}(T)$, then T is convergent if and only if $\underline{A}(T)$ is aperiodic and nontrivial.

The first part of the result builds on the uniform convergence of the time averages of T^t to obtain standard convergence. Specifically, we need to use a Tauberian condition for T that turns uniform Cesaro convergence into standard convergence. We show that such a condition can be expressed in terms of the network of strong ties, and in particular, it requires that it be aperiodic and nontrivial. We defer to Section 6 a more detailed sketch of the proof that also elaborates on the technical contributions of each step of the proof.

Even if an agent does not strongly influence another, this does not always prevent communication between the two. Coherently, the second part of Theorem 2 states that if there exists a periodic behavior in a group that is closed with respect to weak ties, then there exists a profile of initial opinions such that the updates of this group will not stabilize. Indeed, since the agents in this group are never affected by outsiders, the cycle cannot be broken.

The third part of the result significantly generalizes Golub and Jackson [39, Theorem 2], which states that the aperiodicity of $A(W)$ characterizes convergence for linear aggregators. The class of robust opinion aggregators such that $\underline{A}(T) = \bar{A}(T)$ is much larger (see Proposition 4), but, as we illustrate with rank-dependent aggregators right below, in general, there exists a wedge between the two extreme networks $\underline{A}(T)$ and $\bar{A}(T)$.

Theorem 2 has important implications for our game-theoretic interpretation. Even if multiple closed groups do not strongly influence each other, simple best-response dynamics converge to a Nash equilibrium, provided that these groups are aperiodic under $\underline{A}(T)$. Instead, when T captures a process of pure information aggregation, it is natural to assume that information gathered in the past is not entirely dismissed in light of new evidence. This translates into the property that each agent strongly influences herself, a condition that guarantees convergence. Notably, in the empirical social learning literature, Chandrasekhar et al. [22] find that most subjects' behavior is consistent with a form of own-history dependence, even when it is objectively suboptimal.

Corollary 1 *Let T be a robust opinion aggregator. If T is* self-influential*, that is $\underline{a}_{ii} = 1$ for all $i \in N$, then T is convergent.*

We next introduce a general class of robust opinion aggregators that illustrates both the flexibility of our model and our convergence results. Their distinctive feature is rank-dependent influence across agents: a property that we have already encountered with the median aggregator.

Rank-dependent influence Consider a stochastic matrix W whose positive entries implicitly define the observation network. Formally, we say that T^f is a *rank-dependent aggregator* if and only if for each $i \in N$

$$T_i^f(x) = \sum_{j=1}^{n} x_{\pi(j)} \left[f_i\left(\sum_{l=1}^{j} w_{i\pi(l)}\right) - f_i\left(\sum_{l=1}^{j-1} w_{i\pi(l)}\right)\right] \qquad \forall x \in B \tag{8}$$

where π is a permutation of N such that $x_{\pi(1)} \leq ... \leq x_{\pi(n)}$ and $f_i : [0,1] \to [0,1]$ is a weakly increasing *distortion function* such that $f_i(0) = 0$ and $f_i(1) = 1$.[13]

In Figure 1, we illustrate some natural distortions. The first graph shows two distortion functions in which the red and blue agents are respectively attracted by extreme and moderate stances. The second graph shows two distortions that truncate part of the observed sample. The third graph shows pure directional biases: convex (resp. concave) distortion functions capture overweighting of higher

[13] The map $T_i^f : B \to I$ is a Choquet integral against the capacity obtained by distorting the probability vector $w_i \in \Delta$ with respect to the conjugated distortion $\bar{f}_i(\cdot) = 1 - f_i(1 - \cdot)$ (see [53, Example 4.6]), hence, T^f is robust. In particular that the functional form of T_i^f is analogous to the decision criterion in rank-dependent utility theory.

(resp. lower) opinions.

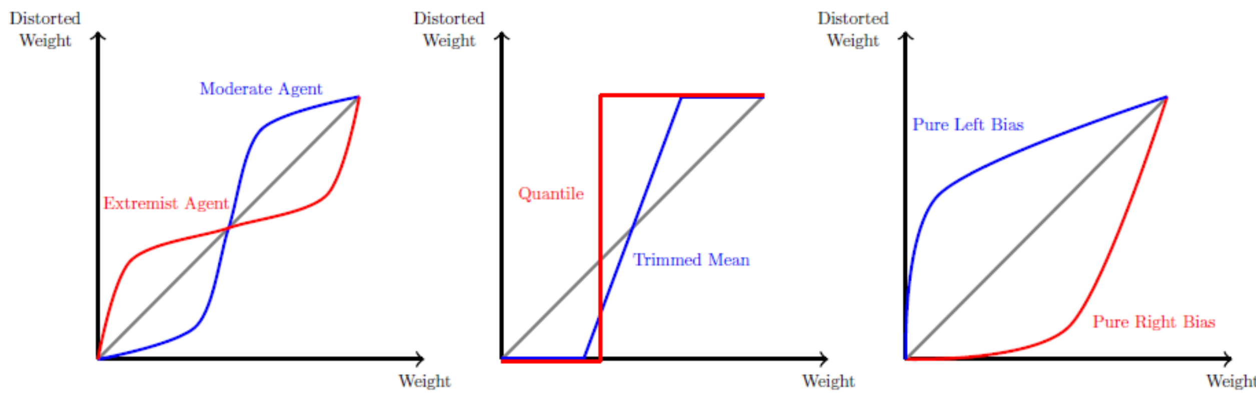

Figure 1

A flexible parametric distortion function is given by

$$f_i(s) = q_i^{\left(\frac{\ln s}{\ln q_i}\right)^{\alpha_i}} \qquad \forall s \in (0,1] \tag{9}$$

where $q_i \in (0,1)$ and $\alpha_i \in \mathbb{R}_{++}$.[14] The parameter α_i captures the attitudes of agent i with respect to extreme opinions: (relative to q_i) attraction ($\alpha_i \in (0,1)$) or aversion ($\alpha_i \in (1,\infty)$). The parameter q_i captures the relative concern of agent i for stating an opinion that is higher ($q_i \in (0,1/2)$) or lower ($q_i \in (1/2,1)$) than the opinions of her neighbors. To see why the parameter q_i captures the asymmetric concerns for disagreement of agent i, note that, as the aversion to extreme opinions increases ($\alpha_i \to \infty$), under a mild assumption, the corresponding rank-dependent aggregator converges pointwise to

$$T_i^{q_i}(x) = \min\left\{c \in \mathbb{R} : \sum_{j:x_j \le c} w_{ij} \ge q_i\right\} \qquad \forall x \in B, \tag{10}$$

that is, the weighted q_i-quantile.[15] In particular, we get back to the weighted median in (5) when $q_i = 0.5$. The q_i-quantiles capture the idea of an extreme truncation of the sample of opinions effectively taken into account. Indeed, the essential feature of these particular rank-dependent aggregators is the extreme flatness of the corresponding weight-distortion function $f_i(s) = 1_{[q_i,1]}(s)$ for all $s \in [0,1]$. With this, for each opinion profile $x \in B$, agent i is only influenced by the neighbor with the opinion corresponding to the q_i-quantile of the distribution of opinions induced by the profile x and the weights

[14] Clearly, f_i is defined only on $(0,1]$, but it also admits a unique continuous extension to $[0,1]$. The extension takes value 0 in 0. In particular, we obtain Prelec's probability-weighting function [62] when $q_i = 1\backslash e$. More generally, using an f_i different from the identity map is a way to introduce a *perception bias* a la Banerjee and Fudenberg [9] in a model of naive and nonequilibrium learning.

[15] It is well known that, given a probability vector $w_i \in \Delta$ and $x \in B$, the q_i-quantile of x is not uniquely defined, but can be any value in an interval $\left[q_i^-(x), q_i^+(x)\right]$. In this paper, we always consider $q_i^-(x)$, which corresponds to (10). As $\alpha_i \to \infty$, $T_i^f(x)$ converges to a value that belongs to $\left[q_i^-(x), q_i^+(x)\right]$. Finally, $\left[q_i^-(x), q_i^+(x)\right]$ collapses to a singleton whenever there does not exist a subset M of N such that $\sum_{l \in M} w_{il} = q_i$. A similar observation holds for (11).

$w_i \in \Delta$. In the case of continuous opinions, a less extreme form of truncation might be desirable. For example, agent i aggregates opinions with a trimmed mean with thresholds $\underline{q}_i, \bar{q}_i \in [0,1]$, $\underline{q}_i < \bar{q}_i$, if her distortion function is

$$f_i(s) = \begin{cases} 0 & \text{if} \quad s < \underline{q}_i \\ \frac{s-\underline{q}_i}{\bar{q}_i-\underline{q}_i} & \text{if} \quad \underline{q}_i \leq s \leq \bar{q}_i \\ 1 & \text{if} \quad s > \bar{q}_i \end{cases} \qquad \forall s \in [0,1]. \tag{11}$$

The q_i-quantile is the limit case in which both $\underline{q}_i$ and $\bar{q}_i$ converge to $q_i \in (0,1)$. Notice that flat regions of f_i imply that agent i disregards the opinions of some of her neighbors depending on the current ranking of opinions. For example, suppose that the opinion of j is currently the lowest among the opinions of the neighbors of agent i. If the weight that agent i puts on j's opinion is not too high, that is $w_{ij} < \underline{q}_i$, then i completely ignores j's opinion. Differently, whenever the weight on the opinion of j is high enough, that is $w_{ij} > \max\left\{\underline{q}_i, 1-\bar{q}_i\right\}$, agent i will always be influenced by j regardless of the current opinion profile. We illustrate this point with an example.

Example 2 (The islands model) Suppose that the agents are partitioned in m groups $\{M_p\}_{p=1}^m$, that is, $N = \cup_{p\in G} M_p$, where $M_p \cap M_{p'} = \emptyset$ for all $p, p' \in G = \{1,...,m\}$ such that $p \neq p'$. For example, these groups might capture the agents' similar cultural or social backgrounds. Also, consider a *strongly connected* observation network A with $a_{ii} = 1$ for all $i \in N$. So far, there is no relation between the neighborhood N_i of an agent i and the only group she belongs to, denoted M_{p_i}. In order to relate these two objects, let us define the *internal linking fraction* of $i \in N$ as

$$\ell_i = \frac{|\{j \in M_{p_i} : a_{ij} = 1\}| - 1}{|N_i|}.$$

According to our interpretation of the groups, the ℓ_i's capture the degree of homophily in the given network structure: agents with a high ℓ_i are connected with many neighbors belonging to their own group M_{p_i}. A stylized picture of real-world networks that has been fruitfully used in the literature (cf. Golub and Jackson [40]) is the islands structure with a large internal linking fraction for each agent.

Let each N_i be such that $|N_i| \geq 3$. Consider the stochastic matrix W such that $w_{ii} = \beta \in (1/|N_i|, 1/2)$, $w_{ij} = \frac{1-\beta}{|N_i|-1}$ if $j \in N_i \backslash \{i\}$, and $w_{ij} = 0$ otherwise, for all $i \in N$. Suppose that each agent $i \in N$ aggregates the opinions she observes in her neighborhood using a trimmed mean T_i with weights given by W and $\underline{q}_i = 1 - \bar{q}_i = \alpha/2$ where $\alpha \in [0, 2\beta)$. In words, every agent computes the weighted average of the opinions she observes, discarding both the $\alpha/2$ highest and lowest opinions and never fully discarding her own previous opinion, that is, $\underline{A}(T) \geq I$. Therefore, T is convergent by Corollary 1. The DeGroot model, obtained as a particular case by setting $\alpha = 0$, would still predict convergence to consensus in the long run. However, if there is sufficiently high homophily, that is, $\ell_i > 1 - \alpha/2$ for all $i \in N$, then disagreement is a typical outcome in the long-run. We next illustrate this point by studying the opinions' evolution in a society in which, starting from a consensus $ke \in B$, the stances of a nonempty subset $M \subseteq N$ of agents are shifted upward, that is,

$$x_i^0 = \begin{cases} k+\delta & \text{if} \quad i \in M \\ k & \text{otherwise}, \end{cases} \quad \forall i \in N$$

with $\delta > 0$ such that $k+\delta \in I$. For example, we can interpret this shock as follows: a subset of agents M is targeted by a marketing campaign and persuaded to increase the use of a certain technology (as in Sadler [63]). Crucially, the extent of opinion segregation in the new long-run dynamics will depend on the agents' identities in the subgroup in relation to the islands structure. If the shock is local, that is, $M = M_p$ for some $p \in G$, then the long-run limit will be such that $\lim_t T_i^t\left(x^0\right) > k$ if $i \in M$, and $\lim_t T_i^t\left(x^0\right) = k$ if instead $i \notin M$. But if the shock is dispersed, that is $|M \cap M_p| \leq 1$ for all $p \in G$, and the self-influentiality β is low enough, then the long-run limit will be such that $\lim_t T_i^t\left(x^0\right) = k$ for all $i \in N$.

If the number of islands m is much greater than the size of each island $|M_p|$, then the dispersed shock involves a much larger subgroup of agents. Nevertheless, the deviation of each subgroup member is washed out within each island, and the original consensus is restored. Instead, the original consensus is broken if the targeted set of agents M is smaller but more inward-looking, as in the first case. This phenomenon resembles the "complex contagion" theory of Centola and Macy [18], whereby a few "long ties" are not sufficient to spread an increased opinion globally. It is supported by the evidence on technology adoption in developing countries (see Beaman et al. [10]). In contrast, in the DeGroot model, both shocks lead to the formation of a new, higher consensus. ▲

Even if the observation network is strongly connected, there is no global convergence to consensus due to the wedge between the observation and the strong network. It is easy to see that whenever $\ell_i \geq 1 - \alpha/2$ for each $i \in N$, no agent strongly influences any agent, apart from herself. In general, the strong and weak networks for rank-dependent aggregators are completely characterized by the distortion functions $(f_i)_{i=1}^n$ and the matrix of weights W. Agent j strongly influences i if and only if her incremental weight, $f_i\left(\sum_{l \in M \cup \{j\}} w_{il}\right) - f_i\left(\sum_{l \in M} w_{il}\right)$, with respect to *any* baseline group $M \subseteq N \backslash \{j\}$ of agents is strictly positive. Similarly, agent j weakly influences i if and only if her incremental weight with respect to *some* baseline group of agents is strictly positive. This shows that the convergence of opinions to disagreement is a much more natural outcome for robust opinion aggregators, even in completely connected societies.

Remark 1 Suppose that the agents use a rank-dependent aggregator T^f with matrix of weights $W \in W$. Consider two disjoint groups $\overline{N}, \underline{N} \subseteq N$. If the members of both groups distort sufficiently toward zero the total weights of the outsiders, that is,

$$f_i\left(\sum_{j \in N \backslash \overline{N}} w_{ij}\right) = 0 \quad \forall i \in \overline{N} \quad \text{and} \quad f_l\left(\sum_{j \in \underline{N}} w_{lj}\right) = 1 \quad \forall l \in \underline{N}, \tag{12}$$

then convergence to consensus *does not* always obtain under T^f. For example, long-run disagreement arises whenever there is initial agreement within $\bar{N}$ on $b \in I$ and initial agreement within $\underline{N}$ on $a < b$ and all the other agents have intermediate opinions $x_i \in [a, b]$. In particular, equation (12) is compatible with an observation and a weak network, $A(W)$ and $\bar{A}(T^f)$, that are both *strongly connected.* ▲

This remark shows that it is not possible to resort to known results on convergence to consensus for nonlinear opinion-aggregation models to analyze this kind of long-run behavior (e.g., [56]). In turn, Theorem 2 gives easy-to-check sufficient conditions, in terms of strong links, to assess convergence of opinions. Finally, as we can easily see in Example 2, the exact composition of these groups is flexible and might change depending on their initial stances.

3.3 Long-run consensus

The following result shows that if we cannot partition the strong network into multiple strongly connected and closed groups, then convergence to consensus always obtains. Conversely, convergence to consensus implies that the weak network does not admit such a partition.

Proposition 1 *Let T be a robust opinion aggregator. The following statements are true:*

1. *If the network of strong ties $\underline{A}(T)$ is nontrivial, has a unique strongly connected and closed group M, and M is aperiodic under $\underline{A}(T)$, then convergence to consensus always obtains.*
2. *If convergence to consensus always obtains, then the network of weak ties $\bar{A}(T)$ is nontrivial, has a unique strongly connected and closed group M, and M is aperiodic under $\bar{A}(T)$.*

Therefore, if $\underline{A}(T) = \bar{A}(T)$, then convergence to consensus always obtains if and only if $\underline{A}(T)$ is nontrivial, has a unique strongly connected and closed group M, and M is aperiodic.

Point 1 states that if there exists a unique strongly connected set of agents in the society that do not have strong connections with the outsiders, then all the agents will eventually conform to this group. But if even the weak ties are not sufficient to connect two disjoint subgroups, then *long-run disagreement* can occur. It is then critical to identify strong and weak ties in the society to understand whether an intervention might generate a global consensus or just a localized one. However, the last part of the result confirms a general principle for robust opinion aggregators: if weak and strong ties coincide, then the results for convergence and consensus of the DeGroot model extend plainly. We next completely characterize the long-run opinion aggregator for a case with this property.

Quasi-arithmetic biased aggregation and opinion dispersion Consider agents that best respond to the previous opinions of the opponents in each period. In this interpretation of our dynamics, a restriction imposed by the quadratic loss in (1) is that upward and downward discrepancies are felt as equally harming by every agent. It might be the case that (some) agents are more concerned with one or the other. A smooth and tractable robust opinion aggregator that takes into account these asymmetries is obtained by minimizing

$$\phi_i^\theta(x - ce) = \sum_{j=1}^{n} w_{ij}\left[\exp\left(\theta\left(x_j - c\right)\right) - \theta\left(x_j - c\right)\right] \qquad \forall x \in \mathbb{R}^n, \forall c \in \mathbb{R} \tag{13}$$

where $\theta \neq 0$ and the values w_{ij} are the entries of a stochastic matrix W. In particular, whenever $\theta > 0$, upward deviations from i's current opinion are more penalized than downward deviations and vice versa whenever $\theta < 0$.

We next show that there exists a unique solution function T_i^θ for each minimization problem induced by ϕ_i^θ. In particular, for this parametric class, we derive an explicit formula for the induced robust long-run opinion aggregator.

Proposition 2 *Let I be bounded and let ϕ be the profile of loss functions $\left(\phi_i^\theta : \mathbb{R}^n \to \mathbb{R}_+\right)_{i=1}^n$ as in (13) with $W \in \mathcal{W}$ and $\theta \in \mathbb{R}\backslash\{0\}$. The following statements are true:*

1. *For each $i \in N$ we have that*

$$T_i^\theta(x) = \operatorname{argmin}_{c\in\mathbb{R}} \phi_i^\theta(x - ce) = \frac{1}{\theta}\ln\left(\sum_{j=1}^n w_{ij}\exp(\theta x_j)\right) \qquad \forall x \in B \tag{14}$$

and T^θ is a robust opinion aggregator with $\underline{A}\left(T^\theta\right) = \bar{A}\left(T^\theta\right) = A(W)$.

2. *For each $i \in N$ we have that*

$$\lim_{\theta\to\hat{\theta}} T_i^\theta(x) = \begin{cases} \max_{j:w_{ij}>0} x_j & \text{if } \hat{\theta} = \infty \\ \sum_{j=1}^n w_{ij}x_j & \text{if } \hat{\theta} = 0 \\ \min_{j:w_{ij}>0} x_j & \text{if } \hat{\theta} = -\infty \end{cases} \qquad \forall x \in B.$$

3. *If there exists a vector $s \in \Delta$ such that*

$$\lim_t W^t x = \left(\sum_{i=1}^n s_i x_i\right) e \qquad \forall x \in \mathbb{R}^n, \tag{15}$$

then convergence to consensus always obtains under T^θ and

$$\bar{T}^\theta(x) = \frac{1}{\theta}\ln\left(\sum_{i=1}^n s_i\exp(\theta x_i)\right) e \qquad \forall x \in B.$$

Point 1 gives an explicit functional form for the opinion aggregator, proving that the time-invariant version of the Log-Sum-Exp model of Tahbaz-Salehi and Jadbabaie [67] is also a robust opinion aggregator.[16] Point 2 shows that this functional form encompasses the linear case as a limit and allows for nonneutral behaviors toward the direction of disagreement. Equation (15) in point 3 is satisfied if and only if $A(W)$ has a unique strongly connected and closed group M and M is aperiodic under $A(W)$. In this case, we see how not just the network structure determines the limit influence of each agent, but the initial opinion also plays a key role. Indeed, the marginal contribution to the limit of agent i's initial opinion is proportional to $s_i\exp(\theta x_i)$. Therefore, when $\theta > 0$, the higher the initial signal realization of an individual, the higher her marginal contribution to the limit is. This

[16] Unlike us, Tahbaz-Salehi and Jadbabaie [67] allow for time-changing connections, but they assume uniform weights for all neighbors.

fact has relevant consequences. For example, consider one of the classical applications of non-Bayesian learning, technology adoption in a village of a developing country, with an opinion vector representing how much the agents have invested in the new technology (e.g., the share of land cultivated with the new technology). There, $\theta > 0$ captures the idea that the most innovative members of the society have a disproportionate influence on the others, maybe because their performance attracts relatively more attention. If resources are limited, that is, if the external actor can only increase adoption for an agent directly, relying on the network aggregation for the rest, the policy prescription is qualitatively different. Indeed, she should choose the agent j for which $s_j \exp(\theta x_j)$ is maximized, combining the standard eigenvector centrality s_j with a distortion increasing in the initial opinion x_j of agent j.[17]

4 Vox populi, vox Dei?

In the previous section, we considered a given deterministic profile of initial opinions and studied their evolution. However, for any given population size, the stochastic nature of the vector of initial opinions $X = \mu + \varepsilon$ implies that the long-run outcome $\bar{T}(X)$ will be stochastic as well. This section considers large networks to study the aggregate variability and the accuracy of long-run opinions under robust opinion aggregation, following the approach pioneered by Golub and Jackson [39]. Their question is whether the long-run opinions approach the true mean in large networks, that is, if a "law of large numbers" holds under DeGroot opinion aggregation. We take up that question for robust opinion aggregators. Remarkably, even seemingly very basic questions about this are unresolved. For example, take a large Erdős–Rényi network and assume that everyone uses a nonlinear rule such as rank-dependent influence. On the one hand, it seems that in a large network where everyone's neighborhood is small and influence locally looks symmetric, there is no channel for anyone's idiosyncratic noise to become influential enough to disrupt a law of large numbers. On the other hand, existing techniques seem basically powerless against this question. We next provide sufficient and necessary conditions for this concentration around the true parameters to hold.

Formally, we keep the same setup of Sections 2 and 3, with the caveat that here everything is parameterized by the size n of the population.

Assumptions In this section, we maintain the following assumptions:

1. $I = \mathbb{R}$.

2. For each $n \in \mathbb{N}$ we assume that $X_i(n) = \mu + \varepsilon_i(n)$ for all $i \in N$, where $\{\varepsilon_i(n)\}_{i \in N, n \in \mathbb{N}}$ is an array of uniformly bounded and independent random variables such that $\inf_{i \in N, n \in \mathbb{N}} \text{Var}(\varepsilon_i(n)) \geq \sigma^2 > 0$.

Some additional notation is helpful for the following analysis.

[17] More generally, in our model the influence among agents depends on their current opinions. This feature has immediate and relevant implications for designing network intervention policies. These policy interventions can assume different forms such as incentive distortions (Galeotti et al. [34]) or information design (Galperti and Perego [36]). We leave this important aspect for future research.

Notation By $\hat{I}$, we denote a bounded open interval such that $X_i(n)(\omega) \in \hat{I}$ for all $\omega \in \Omega$, $i \in N$, and $n \in \mathbb{N}$. We denote by $\ell \overset{\text{def}}{=} \sup \hat{I} - \inf \hat{I}$ the *signal range*. Moreover, we denote the collection of probability vectors in $\mathbb{R}^n$ by Δ_n.

We are interested in whether a growing society becomes wise (cf. Golub and Jackson [39]), that is, whether there is efficient aggregation of the information available in the network in the limit.

Definition 5 *Let $\{T(n)\}_{n\in\mathbb{N}}$ be a sequence of robust opinion aggregators. The sequence $\{T(n)\}_{n\in\mathbb{N}}$ has* vanishing variance *if and only if, for each $\iota \in \mathbb{N}$,*[18]

$$\operatorname{Var}\left(\bar{T}_\iota(n)(X_1(n), ..., X_n(n))\right) \to 0. \tag{16}$$

The sequence $\{T(n)\}_{n\in\mathbb{N}}$ is wise *if and only if, for each $\iota \in \mathbb{N}$,*

$$\bar{T}_\iota(n)(X_1(n), ..., X_n(n)) \xrightarrow{P} \mu. \tag{17}$$

When equation (16) holds, the aggregation procedure neutralizes the idiosyncratic variability of the agents' opinions. If, in addition, the agents' limit opinions are unbiased, then they concentrate around μ, and equation (17) holds. If $T(n)$ is *linear* with *strongly connected* matrix $W(n)$, then $\bar{T}(n)$ is linear and represented by a matrix $\bar{W}(n)$ whose rows all coincide with the left Perron-Frobenius eigenvector $s(T(n)) \in \Delta_n$ of $W(n)$, a standard measure of network centrality. DeMarzo et al. [29] and Golub and Jackson [39] call $s(T(n))$ the *influence vector,* and the latter authors show that $\{T(n)\}_{n\in\mathbb{N}}$ is wise if and only if $\lim_n \max_{k\in N} s_k(T(n)) = 0$, provided the errors $\varepsilon_i(n)$ have 0 mean. In this case, the vector $s(n)$ coincides with the gradient of $\bar{T}_i(n)$, thereby capturing the idea of the marginal contributions of the agents to the limit opinion of i.

As suggested by Theorem 1, for robust opinion aggregators, the marginal contributions to the limit opinion are captured by the partial derivatives of $\bar{T}_i(n)$. Even if our opinion aggregators might not be (Frechet) differentiable, they are Lipschitz continuous and hence almost everywhere differentiable by Rademacher's Theorem.[19] Let $\mathcal{D}\left(\bar{T}(n)\right)$ be the subset of $\hat{I}^n$ in which $\bar{T}(n)$ is differentiable.

Definition 6 *Let $T(n): \mathbb{R}^n \to \mathbb{R}^n$ be a robust opinion aggregator and $i \in N$. We say that $\underline{s}_i(T(n)) \in \mathbb{R}^n$ is the* strong influence vector *for i given $T(n)$ if and only if*

$$\underline{s}_{ij}(T(n)) = \inf_{x\in\mathcal{D}(\bar{T}(n))} \frac{\partial \bar{T}_i(n)}{\partial x_j}(x) \qquad \forall j \in N.$$

We say that $\bar{s}_i(T(n)) \in \mathbb{R}^n$ is the weak influence vector *for i given $T(n)$ if and only if*

$$\bar{s}_{ij}(T(n)) = \sup_{x\in\mathcal{D}(\bar{T}(n))} \frac{\partial \bar{T}_i(n)}{\partial x_j}(x) \qquad \forall j \in N.$$

[18] Note the following innocuous abuse of notation (given our interest in limit results): for each $\iota \in \mathbb{N}$, the sequences in equations (16) and (17) are well defined only starting from $n \geq \iota$. In fact, an agent with position ι can only belong to a society with size n greater than or equal to ι. A similar observation applies throughout the section, in particular, in Theorem 3.

[19] See Lemma 2 in Appendix A.

As for the notions of networks associated with a robust opinion aggregator, there are two natural definitions of influence vector. The values $\underline{s}_{ij}(T(n)), \bar{s}_{ij}(T(n)) \in [0,1]$ are respectively the minimal and maximal influence that, under the opinion aggregator $T(n)$, the initial opinion of j exerts on the limit opinion of i. Observe that whenever $T(n)$ is a robust opinion aggregator that satisfies point 1 of Proposition 1, for each $i, l \in N$, we have $\underline{s}_i(T(n)) = \underline{s}_l(T(n))$ and $\bar{s}_i(T(n)) = \bar{s}_l(T(n))$ since $\bar{T}_i = \bar{T}_l$. Moreover, both definitions of influence vector above coincide with the one of Golub and Jackson whenever $T(n)$ is linear and strongly connected since $\underline{s}_i(T(n)) = \bar{s}_i(T(n)) = s(T(n))$ for all $i \in N$.[20]

These objects are crucial for providing sufficient and necessary conditions for vanishing variance. To obtain also the wisdom of the crowd, the following additional symmetry assumptions are needed. We say that the array $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ is *symmetric* if and only if for each $i \in N$ and for each $n \in \mathbb{N}$, $\varepsilon_i(n)$ and $-\varepsilon_i(n)$ have the same distribution under P. Moreover, we say that the sequence $\{T(n)\}_{n\in\mathbb{N}}$ is *odd* if and only if $T(n)(-x) = -T(n)(x)$ for all $x \in \mathbb{R}^n$ and for all $n \in \mathbb{N}$.[21]

Theorem 3 *Let $\{T(n)\}_{n\in\mathbb{N}}$ be a sequence of robust opinion aggregators. The following statements are true:*

1. *If $\lim_n \sum_{j=1}^n (\bar{s}_{\iota j}(T(n)))^2 = 0$ for all $\iota \in \mathbb{N}$, then $\{T(n)\}_{n\in\mathbb{N}}$ has vanishing variance. If in addition $\{T(n)\}_{n\in\mathbb{N}}$ is odd and $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ is symmetric, then $\{T(n)\}_{n\in\mathbb{N}}$ is wise.*

2. *If $\limsup_n \max_{j\in N} \underline{s}_{\iota j}(T(n)) > 0$ for some $\iota \in \mathbb{N}$, then $\{T(n)\}_{n\in\mathbb{N}}$ does not have vanishing variance. In particular, $\{T(n)\}_{n\in\mathbb{N}}$ is not wise.*

Given $\iota \in \mathbb{N}$, the quantity $\sum_{j=1}^n (\bar{s}_{\iota j}(T(n)))^2$ is an upper bound for the sensitivity of $\bar{T}_\iota(n)$ to changes in the initial opinions of small subsets of agents. As long as this measure vanishes, the variance of the limit opinion of ι is going to 0. It is easy to show that this condition is implied by $\max_{j\in N} \bar{s}_{\iota j}(T(n)) = o\left(\frac{1}{\sqrt{n}}\right)$, that is, the *maximum weak influence* on ι is vanishing fast enough. Conversely, if the *maximum strong influence* $\max_{j\in N} \underline{s}_{\iota j}(T(n))$ on some agent ι is not vanishing, then the variability of her limit opinion does not disappear, preventing agent ι from learning μ.

Observe that whenever each $T(n)$ is linear and strongly connected, the sufficient and necessary conditions for the wisdom of the crowd in points 1 and 2 are equivalent to $\lim_n \max_{j\in N} s_j(T(n)) = 0$, the condition of Golub and Jackson [39] that characterizes the wisdom of the crowd for the DeGroot model.[22] Thus, we obtain their characterization as a particular case of our result. In general, there are two other conceptual differences between the previous results (e.g., Golub and Jackson [39] and Levy and Razin [51]) about the wisdom of the crowd and ours. First, we neither impose any parametric structure on the opinion aggregators nor assume that agents aggregate opinions according to functionals belonging to the same subclass (e.g., the median, quantiles, rank-dependent, quasi-arithmetic).

[20] Compared to [39], when our robust opinion aggregators do not induce consensus, the limit influence cannot be described by a unique, agent-independent, vector of $\mathbb{R}^n$. Therefore, we need to keep track of two influence vectors $\underline{s}_i(T(n))$ and $\bar{s}_i(T(n))$ for each agent $i \in \{1, ..., n\}$. Also, our definition of wise requires that for each fixed agent ι, the long-run opinion is converging in probability to μ. This definition collapses to the one of [39] under the additional assumption that $\bar{T}_\iota(n) = \bar{T}_{\iota'}(n)$ for all $\iota, \iota' \in \{1, ..., n\}$ and $n \in \mathbb{N}$.

[21] In the foundation of robust opinion aggregators that we propose in Section 5.1, loss functions that are symmetric with respect to opinions' deviations (i.e., even) induce odd opinion aggregators.

[22] Indeed, given $n \in \mathbb{N}$ and $i \in N$, if $\bar{s}_i(T(n)) \in \Delta_n$ (as in [39]), then $\sum_{j=1}^n \bar{s}_{ij}(T(n))^2 \leq \max_{j\in N} \bar{s}_{ij}(T(n))$.

Second, our results encompass the case of nonconvergent robust opinion aggregators. In such a case, $\bar{T}(n)$ is the limit of the updates' time averages. This extra layer of generality is helpful for the following question: can an external observer learn μ by observing only part of the updating dynamics of a subset of the agents? That is, can she achieve the *wisdom from the crowd*? We have a positive answer under the conditions of point 1: the external observer can use $\bar{T}_\iota(n)$ as a consistent estimator of the underlying parameter, even if the agents' opinions are not converging. In addition, when $T(n)$ is also convergent for all $n \in \mathbb{N}$, we have the *wisdom of the crowd*: all agents learn the true parameter. Finally, as the proof of Theorem 3 clarifies, our results are not only qualitative but also *quantitative*. For example, in point 1, not only do we prove that there is vanishing variance, but we provide an estimate of the variance, given a fixed population of size n.

The proof of Theorem 3 takes the following steps. For point 1, we treat each $\bar{T}_i(n)$ as an estimator of μ and borrow techniques from large-deviation theory. In particular, we observe that McDiarmid's concentration inequality can be used to bound the variance of $\bar{T}_i(n)$ whenever its variations with respect to the signal realizations can be bounded. Intuitively, these variations are proportional to the partial derivatives of $\bar{T}_i(n)$ with respect to the initial opinions of the other agents when these derivatives are defined. We formalize this idea by using a version of the Mean Value Theorem for Lipschitz functions to show that each $\bar{s}_{ij}(T(n))$ bounds the changes of $\bar{T}_i(n)$ as X_j varies. With this, we obtain a bound on the variance of $\bar{T}_i(n)$ that vanishes as $\sum_{j=1}^{n}(\bar{s}_{ij}(T(n)))^2$ does, yielding the first part of point 1. Next, we show that if both the errors and the opinion aggregator are symmetric, then $\bar{T}_i(n)$ is an unbiased estimator, so it converges in probability to μ.

For point 2, we show that the assumption on the strong influence vector implies that the variance of the long-run opinion of agent ι remains bounded away from zero for every n. This happens because (up to selecting a subsequence) for every n, there exists an agent j_n with a strong influence of at least $\alpha \in (0,1)$ on ι. In turn, this implies that we can decompose the long-run opinion of ι as the convex linear combination of $X_{j_n}(n)$ (with weight α) and a monotone function of the opinions of all agents. By Harris's inequality, the covariance between $X_{j_n}(n)$ and this monotone function is nonnegative. Therefore the overall variance of agent ι's long-run opinion is at least $\alpha^2 \operatorname{Var}(X_{j_n}(n)) \geq \alpha^2\sigma^2 > 0$.

4.1 Weak networks and the wisdom of the crowd

Point 1 of Theorem 3 provides an easy-to-interpret sufficient condition regarding the sequence of long-run opinion aggregators for both absence of aggregate variability and wisdom. However, it is important to have properties of the primitive sequence of robust opinion aggregators that induce long-run wisdom. To address this point via Theorem 3, we need to control the derivatives of the sequence of robust opinion aggregators $\{T(n)\}_{n\in\mathbb{N}}$ with their weak networks $\{\bar{A}(n)\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ and $i \in N$, we denote the *degree* of i in $\bar{A}(n)$ by $\bar{d}_i(n) = \sum_{j\in N} \bar{a}_{ij}(n)$. We define the maximum and minimum degrees by $\bar{d}_{\max}(n) = \max_{i\in N} \bar{d}_i(n)$ and $\bar{d}_{\min}(n) = \min_{i\in N} \bar{d}_i(n)$, respectively. Recall that $\hat{I}^n$ is the set of possible initial opinion vectors. Similarly to before, we denote by $\mathcal{D}(T(n)) \subseteq \hat{I}^n$ the subset of $\hat{I}^n$ where $T(n)$ is differentiable.

Definition 7 *Let $\{T(n)\}_{n\in\mathbb{N}}$ be a sequence of robust opinion aggregators and $\kappa \geq 1$. The sequence*

$\{T(n)\}_{n\in\mathbb{N}}$ *is* κ*-dominated if and only if*

$$\frac{\partial T_i(n)}{\partial x_j}(x) \leq \frac{\kappa}{\bar{d}_i(n)} \qquad \forall x \in \mathcal{D}(T(n)) \tag{18}$$

for all $i, j \in N$ *and for all* $n \in \mathbb{N}$.

For a *fixed* $n \in \mathbb{N}$, since each $T(n)$ is Lipschitz continuous, we can always satisfy the inequality in (18) by choosing $\kappa(n) = \bar{d}_{\max}(n)$.[23] Therefore, a sufficient condition for the sequence $\{T(n)\}_{n\in\mathbb{N}}$ to be κ-dominated for some $\kappa \geq 1$ is that $\sup_{n\in\mathbb{N}} \bar{d}_{\max}(n) < \infty$. Here, κ measures the deviation of $T(n)$ from the uniform linear aggregation of the opinions of the weak neighbors. This deviation can take two forms: i) some neighbors may be more important than others; ii) the relative weights may depend on the current opinion. The first form is already present in the linear model with nonuniform weights, while the second one is specific to robust opinion aggregators, as we next illustrate.

Example 3 Let $\{T^f(n)\}_{n\in\mathbb{N}}$ denote the sequence of rank-dependent aggregators with matrices of weights $\{W(n)\}_{n\in\mathbb{N}}$ and distortions $\{f_\iota\}_{\iota\in\mathbb{N}}$, with each f_ι continuous and locally Lipschitz on $(0,1)$.[24] This implies that there exists a set $F \subseteq (0,1)$ of measure 1 where each f_ι is differentiable. We assume that the weights are uniform over the (nontrivial) observation network, that is, for each $n \in \mathbb{N}$ and $i, j \in N$, it holds $w_{ij}(n) \in \{0, 1/|N_i(n)|\}$. In this case, we have that the inequality in (18) holds with $\kappa = \sup_{\iota\in\mathbb{N}} \sup_{x\in F} f_\iota'(x)$. If $\kappa < \infty$, then the sequence $\{T^f(n)\}_{n\in\mathbb{N}}$ is κ-dominated. For example, if all agents use the same distortion $f_\iota = \hat{f}$ that belongs to any of the cases in Figure 1 except for quantiles, then κ is finite. Alternatively, if all agents are using trimmed means with symmetric but potentially heterogenous trimming cutoffs $\left(\underline{q}_\iota, 1-\underline{q}_\iota\right)_{\iota\in\mathbb{N}}$ such that $\sup_{\iota\in\mathbb{N}} \underline{q}_\iota < 1/2$, then $\{T^f(n)\}_{n\in\mathbb{N}}$ is κ-dominated with $\kappa = 1/\left(1 - 2\sup_{\iota\in\mathbb{N}} \underline{q}_\iota\right)$ and each $T^f(n)$ is odd. ▲

We now give two alternative conditions under which a κ-dominated sequence of odd robust opinion aggregators is wise. For each $n \in \mathbb{N}$, if $\bar{A}(n)$ is strongly connected and undirected, the stochastic matrix of uniform weights associated with $\bar{A}(n)$ (i.e., the matrix whose ij-th entry is $\bar{a}_{ij}(n)/\bar{d}_i(n)$) has n real eigenvalues. We denote by $\lambda_2(n)$ the second largest eigenvalue in modulus (henceforth, SLEM) of this matrix: a standard (inverse) measure of connectivity.

Proposition 3 *Let* $\{T(n)\}_{n\in\mathbb{N}}$ *be a* κ*-dominated sequence of odd robust opinion aggregators and* $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ *be symmetric. The following statements are true:*

1. *If* $\lim_n \frac{\sqrt{n}}{\bar{d}_{\min}(n)} = 0$*, then* $\{T(n)\}_{n\in\mathbb{N}}$ *is wise.*
2. *If the weak networks* $\{\bar{A}(n)\}_{n\in\mathbb{N}}$ *are undirected and strongly connected,* $\sup_{n\in\mathbb{N}} \frac{\bar{d}_{\max}(n)}{\bar{d}_{\min}(n)} < \infty$*, and* $\sup_{n\in\mathbb{N}} \lambda_2(n) < \frac{1}{\kappa^2}$*, then* $\{T(n)\}_{n\in\mathbb{N}}$ *is wise.*

[23] In general, we can choose a much smaller $\kappa(n)$ (cf. Example 3). That said, since $T(n)$ is monotone and translation invariant, the gradient $\nabla T_i(n)(x)$ is a probability vector for all $i \in N$ and for all $x \in \mathcal{D}(T(n))$. This implies that $\kappa(n)$ can never be chosen to be smaller than 1. Moreover, it can be chosen to be 1 if and only if $T(n)(x) = W(n)x$ for all $x \in \mathbb{R}^n$, where $W(n)$ is the stochastic matrix of uniform weights associated with $\bar{A}(n)$. Intuitively, the less the derivative of T can change, the closer T is to be linear and the smaller κ can be chosen. For these reasons, we interpret κ as an index of nonlinearity.

[24] For example, this is the case if each f_ι is continuous on $[0,1]$ and either convex or concave.

The first part of the proposition shows that a sequence of odd robust opinion aggregators, which is κ-dominated, is wise, provided that the weak degree of each agent is increasing fast enough. On the one hand, the degree-growth condition in this statement is satisfied with high probability in standard random-graph models such as the Erdős–Rényi model with (sufficiently) slowly decreasing linking probability.

On the other hand, many real-world networks exhibit bounded degrees, even when the population size grows. In these cases, we can still obtain the wisdom of the crowd at the cost of requiring a high level of connectivity in the weak networks compared to the nonlinearity index κ. We now observe that this joint condition is satisfied by multiple graph models. For example, within the class of the $\bar{d}(n)$-regular graphs, in which each agent has exactly $\bar{d}(n)$ links, *Ramanujan graphs* have particularly high connectivity, with $\lambda_2(n) \leq 2/\sqrt{\bar{d}(n)}$. Importantly, for fixed $\bar{d} \in \mathbb{N}$, random graphs that are uniformly distributed over $\bar{d}$-regular graphs are "almost Ramanujan", in the sense that, with probability converging to 1, their SLEM will be lower than $2/\sqrt{\bar{d}}$ as n grows. Therefore, under this graph model, the connectivity condition reduces to $\bar{d} > 4\kappa^4$. In the context of Example 3 with agents using trimmed means with symmetric cutoffs, this condition amounts to $\bar{d} > 4\left(\frac{1}{1-2\sup_{\iota\in\mathbb{N}} \underline{q}_\iota}\right)^4$, which is satisfied with reasonable parameters such as $\sup_{\iota\in\mathbb{N}} \underline{q}_\iota \leq 1/8$ and $\bar{d} \geq 13$.

Even if regular graphs constitute a benchmark structure, given their balancedness properties, they still fail to capture the clustering of many real-world networks. The multi type random-graph model of Golub and Jackson [40, Definition 3] is an example that overcomes this limitation, allowing for homophily between agents of the same type. Notably, the realized degree distribution is balanced, and the SLEM of the realized network is close to the SLEM of the associated deterministic network of types.[25] Therefore, in order to guarantee the wisdom of the crowd, we need to show that the SLEM of the type network generating the weak networks of $\{T(n)\}_{n\in\mathbb{N}}$ is small enough compared to their coefficient of nonlinearity $1/\kappa^2$. Moreover, in their leading case of an islands model, this condition is always satisfied when the homophily index is low enough.

In Example 4 in Section 5, we illustrate how to use the sufficient conditions of Proposition 3 to obtain the wisdom of the crowd in a model in which agents repeatedly solve an estimation problem for the fundamental parameter μ.

Point 2 of Theorem 3 establishes that the persistent limit influence, of at least an individual, is sufficient to preserve the opinions' variability, even for large populations. It is not difficult to show that a more structural sufficient condition for persistent influence in terms of prominent families, as in Golub and Jackson [39, Definitions 5 and 6 and Proposition 3], can be given also for robust opinion aggregators.

5 Foundation of robust opinion aggregators

In this section, we give a microfoundation to robust opinion aggregators and their convergence and information-aggregation properties.

[25] The second statement is the content of their Theorem 2, while the balance condition is implied by their Lemma A.4. Golub and Jackson [40] also point out that a small SLEM guarantees that convergence speed to μ does not explode as the population size increases.

5.1 A characterization of robust opinion aggregators

Here, we characterize robust opinion aggregators as the solution to a distance-minimization problem. Formally, we endow each agent i with a loss function $\phi_i : \mathbb{R}^n \to \mathbb{R}_+$ and we assume that in each period, the agent solves

$$\min_{c\in\mathbb{R}} \phi_i (x - ce) \tag{19}$$

where $x \in B$ is the opinion profile of the previous period. Intuitively, in choosing her current opinion c, agent i minimizes a loss function that penalizes the disagreement (i.e., differences of opinions) with the last-period opinions of her neighbors. We next impose two minimal restrictions on the profile of loss functions $\phi = (\phi_i)_{i=1}^n$.

Definition 8 *The profile of loss functions ϕ is* sensitive *if and only if $\phi_i (he) > \phi_i (0)$ for all $i \in N$ and for all $h \in \mathbb{R}\setminus \{0\}$.*

If agent i observes a unanimous opinion (including herself), then her loss is minimized by declaring that same opinion. In particular, under a best-response-dynamics interpretation, sensitivity implies that all the constant profiles of actions are Nash equilibria of the induced game.

Definition 9 *The profile of loss functions ϕ has* increasing shifts *if and only if for each $i \in N$, $z, v \in \mathbb{R}^n$, and $h \in \mathbb{R}_{++}$*

$$z \geq v \implies \phi_i (z + he) - \phi_i (z) \geq \phi_i (v + he) - \phi_i (v) .$$

It has strictly increasing shifts *if and only if the above inequality is strict whenever $z \gg v$.*

The property of increasing shifts is a form of complementarity in disagreeing with two or more agents from the same side. It is implied by stronger properties usually required on supermodular games played on networks, such as degree complementarity (see, e.g., Galeotti et al. [35]).

We call *robust* a profile of loss functions that is sensitive and has increasing shifts. The collection of all these profiles is denoted by Φ_R. Given a robust profile of loss functions ϕ, we denote by $T^\phi : B \to B$ an arbitrary selection of the argmin correspondence

$$T^\phi (x) \in \prod_{i=1}^n \operatorname{argmin}_{c\in\mathbb{R}} \phi_i (x - ce) \qquad \forall x \in B. \tag{20}$$

The selfmap T^ϕ is an opinion aggregator and describes one possible updating rule induced by ϕ. The next theorem shows that our loss-function-based updating procedure naturally generalizes that of the DeGroot model (cf. Golub and Sadler [42]) without committing to any specific functional form (e.g., quadratic) of the loss function.[26]

Theorem 4 *Let T be an opinion aggregator. The following statements are equivalent:*

[26] In particular, it is always possible to derive a DeGroot aggregator via the loss function (1).

(i) There exists $\phi \in \Phi_R$ that has strictly increasing shifts and is such that $T = T^{\phi}$, that is, for each $i \in N$

$$T_i(x) = \text{argmin}_{c\in\mathbb{R}} \phi_i(x - ce) \qquad \forall x \in B; \tag{21}$$

(ii) T is a robust opinion aggregator.

The property of strictly increasing shifts guarantees that $\text{argmin}_{c\in\mathbb{R}} \phi_i(x - ce)$ is a singleton. However, it is violated in some interesting specifications of ϕ (see, e.g., equation (4)). In Proposition 9 in Appendix C, we show that the solution correspondence of problem (19) always admits a selection that is a robust opinion aggregator.

This theorem also suggests that as in DeMarzo et al. [29], we can interpret the induced opinion dynamics as repeated estimation of μ given the last-period neighbors' opinions. In particular, [29] only studied the case of maximum likelihood updating with Gaussian initial signals. Instead, we follow the general robust statistics approach: the agents minimize a loss function (see, e.g., the seminal contribution by Huber [45]) such as the absolute loss, the p-loss where the quadratic function in (1) is replaced by a general power $p \geq 1$ function, and the Huber loss. This approach is natural when the complexity of the network structure does not allow the agents to attach probabilistic beliefs to the data-generating process (see Breza et al. [14]).[27]

5.2 Loss functions and long-run dynamics

Next, we illustrate how our foundation is linked to the convergence and wisdom results for robust opinion aggregators. We focus on the familiar and particularly tractable class of loss functions given by

$$\phi_i(z) = \sum_{j=1}^{n} w_{ij}\rho_i(z_j) \qquad \forall z \in \mathbb{R}^n, \forall i \in N$$

where $W \in \mathcal{W}$ is a stochastic matrix whose positive entries implicitly define the observation network, and $\rho = (\rho_i : \mathbb{R} \to \mathbb{R}_+)_{i=1}^n$ is a profile of positive functions. The weight w_{ij} captures the relative importance of the opinion of j as perceived by i. We call such a profile *additively separable* and write $\phi = (W, \rho)$. We denote the set of *robust* and *additively separable* profiles of loss functions by Φ_A. Easy computations show that $(W, \rho) \in \Phi_A$ if and only if each ρ_i is convex, strictly decreasing on $\mathbb{R}_-$, and strictly increasing on $\mathbb{R}_+$. Additionally, if each ρ_i is strictly convex, then there exists a unique robust opinion aggregator T^{ϕ} that satisfies (20). Three relevant examples of robust opinion aggregators stemming from additively separable loss functions are the DeGroot aggregators, the quantile aggregators, and the opinion aggregator of Proposition 2.

Natural conditions on the profile of loss functions $\phi = (W, \rho)$ yield that both the strong network $\underline{A}(T^{\phi})$ and the weak network $\bar{A}(T^{\phi})$ coincide with the observation network given by W.[28]

[27] More recently, Dasaratha et al. [27] provide a foundation for stationary linear updating rules by analyzing an overlapping-generations model of agents that minimize a quadratic loss function after observing finitely many past actions of their neighbors and private Gaussian signals. While their setting is significantly different, with the main distinction that signals arrive in each period, we conjecture that our foundation for robust opinion aggregators can be used to analyze more general versions of their model in which the assumptions of Gaussian signals and quadratic losses are relaxed.

[28] In general, we can prove a similar result for profiles of loss functions that are not additively separable. In this case,

Proposition 4 *Let $\phi = (W, \rho) \in \Phi_A$. If I is compact and ρ_i is twice continuously differentiable and strongly convex for all $i \in N$, then there exists a unique T^ϕ that satisfies (20) and $\underline{A}\left(T^\phi\right) = \bar{A}\left(T^\phi\right) = A(W)$.*

Note that Proposition 4, paired with Theorem 2 and Proposition 1, characterizes convergence and convergence to consensus in terms of the observation network $A(W)$, provided that each ρ_i is sufficiently smooth and convex.

Finally, we illustrate how Proposition 3 can be applied to check the wisdom of the crowd in terms of the profile of loss functions. As a by-product, we obtain that, under Assumptions 1 and 2 of Section 4, the wisdom of the crowd can be achieved as long as the minimum degree of connections gets larger as the population size increases.

Example 4 Consider a sequence $\{T(n)\}_{n\in\mathbb{N}}$ of odd robust opinion aggregators as in Section 4 such that

$$T_i(n)(x) \in \operatorname{argmin}_{c\in\mathbb{R}} \sum_{j\in N_i(n)} \frac{\rho_i(n)(x_j - c)}{|N_i(n)|} \qquad \forall x \in \mathbb{R}^n$$

where the profile of loss functions $\phi(n) = (W(n), \rho(n)) \in \Phi_A$ used by the agents satisfies the assumptions in Proposition 4 and is such that $\rho_i(n)(-z) = \rho_i(n)(z)$ for all $z \in \mathbb{R}$, for all $i \in N$, and for all $n \in \mathbb{N}$. In this case, the weights $w_{ij}(n)$ of each $W(n)$ are uniform over their (nonempty) neighborhoods $N_i(n)$. Moreover, let $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ be symmetric and assume that there exists $\kappa \in \mathbb{R}$ such that

$$\frac{\rho_i''(n)(z)}{\rho_i''(n)(z')} \leq \kappa \quad \forall i \in N, \forall n \in \mathbb{N}, \forall z, z' \in [-\ell, \ell].$$

In particular, this condition is satisfied if $\rho_i(n) = \bar{\rho}$ for all $i \in N$ and for all $n \in \mathbb{N}$. By the Implicit Function Theorem, we have that $T(n)$ is differentiable and

$$\frac{\partial T_i(n)}{\partial x_j}(x) \leq \frac{\kappa}{|N_i(n)|} \leq \frac{\kappa}{\min_{k\in N}|N_k(n)|} \qquad \forall i, j \in N, \forall x \in \hat{I}^n, \forall n \in \mathbb{N}.$$

In words, the uniform bound on the sensitivity of the loss functions implies that the reciprocal weak influence among the agents can be bounded using the size of the minimal neighborhood in the growing network. By Proposition 4, we have that $\{T(n)\}_{n\in\mathbb{N}}$ is κ-dominated.[29]

By Proposition 3, wisdom is reached if the minimal degree in the society is growing sufficiently fast, that is,

$$\frac{1}{\min_{k\in N}|N_k(n)|} = o\left(\frac{1}{\sqrt{n}}\right). \tag{22}$$

Alternatively, if each $A(W(n))$ is undirected and strongly connected, $\sup_{n\in\mathbb{N}} \frac{\max_{k\in N}|N_k(n)|}{\min_{k\in N}|N_k(n)|} < \infty$, and $\sup_{n\in\mathbb{N}} \lambda_2(n) < \frac{1}{\kappa^2}$, then $\{T(n)\}_{n\in\mathbb{N}}$ is wise. For example, when the signal range ℓ is 1 and $\bar{\rho}(z) = \alpha z^4 + (1-\alpha) z^2$ for some $\alpha \in (0,1)$, the SLEM condition becomes $\sup_{n\in\mathbb{N}} \lambda_2(n) < \left(\frac{1-\alpha}{5\alpha+1}\right)^2$.▲

the differentiability and strong-convexity assumptions can also be weakened and replaced with a coercivity condition and a Lipschitz property of the difference quotients.

[29] In fact, by Proposition 4, we can prove that the weak network of $T(n)$ coincides with $A(W(n))$ on each set I^n where I is a compact interval with nonempty interior.

6 Related literature

The linear model This paper belongs to the literature on non-Bayesian opinion aggregation and nests the benchmark DeGroot model [28].[30] Within this model, Golub and Jackson [39] fully characterize convergence, convergence to consensus, and the wisdom of the crowd in terms of the network structure. For convergence, we significantly extend the scope of the conditions of [39, Theorem 2]. We show that in our nonlinear model, they are still sufficient for convergence and convergence to consensus when imposed on the strong network, while they are necessary when imposed on the weak network. For the wisdom of the crowd, we derive a general law of large numbers for robust opinion aggregators specializing in the one of [39] for the linear case. Here the main novelty is that the necessary and sufficient conditions for the wisdom of the crowd must be expressed respectively in terms of the strong and weak influence vectors, possibly creating a wedge that is not present in the linear model.

Convergence and the mathematics literature Our most novel contribution in terms of convergence is Theorem 2. Compared to the opinion-aggregation literature in computer science and economics, our techniques are completely functional analytic. This is natural since our aggregators are nonlinear. Formally, this creates an immediate overlap with the literature of maps iteration and fixed-point theory in which the iterates $\left\{T^t\left(x\right)\right\}_{t\in\mathbb{N}}$ and their convergence are studied in order to find the fixed points of T. Using functional analysis in place of linear algebra comes at a cost. On the one hand, it is a richer language but not immediately amenable to graph-theoretic notions, which are better expressed in terms of matrices. On the other hand, graph-theoretic properties are primitive within our framework. Thus, as a general contribution, our notions of networks of weak and strong ties build a useful link between nonlinear analysis and graph theory.

In more detail, the proof of point 1 of Theorem 2 relies on five major steps. We now comment on each step in relation to the literature. Given uniform Cesaro convergence of Theorem 1 and using Lorentz's Theorem, the first step (Lemma 4) observes that convergence of T is equivalent to asymptotic regularity. This technique seems to have first appeared in Bruck [16], who applied it to the case of nonexpansive maps in Hilbert spaces.[31] Because of this observation, showing that T is asymptotically regular is important. Conceptually, it poses the issue of what asymptotic regularity might mean at a graph-theoretic level. The second step moves to address these points. Proposition 5 is a quite simple yet new observation: if $\underline{A}\left(T\right)$ is nontrivial, then T admits a decomposition $T\left(x\right)=\varepsilon Wx+\left(1-\varepsilon\right)S\left(x\right)$ where $\varepsilon\in\left(0,1\right)$, W is a stochastic matrix such that $A\left(W\right)=\underline{A}\left(T\right)$, and S is a robust opinion aggregator. This grain of linearity is what allows us to bridge graph notions to the convergence properties of the operator T. Indeed, the third step (Lemma 5 and Proposition 6) shows that when W is a $\{0,1\}$-valued stochastic matrix that partitions the agents in m classes of agents that share the only individual in the class they observe (see Definition 10), then T is asymptotically regular. The third step thus offers an example of a graph-theoretic property encoded by W, which yields asymptotic

[30] For comprehensive surveys of this literature, see Acemoglu and Ozdaglar [3] and Golub and Sadler [42]. Banerjee et al. [8] consider an asynchronous departure from the DeGroot model.

[31] Recall that we endow $\mathbb{R}^n$ with the supnorm. Hence T might fail to be nonexpansive with respect to the Euclidean norm (see also Remark 2). Moreover, proving that asymptotic regularity is equivalent to convergence can also be obtained using the techniques of Browder and Petryshyn [15, Theorem 2].

regularity. In proving this step, we generalize the techniques of Edelstein and O'Brien [31, Lemma 1].[32] The decomposition used in the third step yields convergence, but it is a very special case. This concern is partially tamed by the fourth step (Lemma 6): if $\underline{A}(T)$ is aperiodic and nontrivial, then there exists $t \in \mathbb{N}$ such that T^t and T^{t+1} possess such a special decomposition, making T^t and T^{t+1} convergent. In the final step (proof of point 1 of Theorem 2), we prove that if T^t and T^{t+1} are convergent, so is T. To our knowledge, the second point of Theorem 2 does not have a counterpart in the literature.[33]

Convergence to consensus and the computer science literature The multidisciplinary literature on repeated averaging procedures mostly focuses on convergence to consensus: a relevant question we study in Section 3.3.[34] We now discuss the most important contributions to this issue. The closest paper to our functional approach is Moreau [56], who considers the iteration of a nonlinear and time-varying operator on a Euclidean space. Neither our results nor the ones in [56] nest the others. We restrict ourselves to time-homogeneous operators on a one-dimensional space and impose the additional condition of translation invariance (both papers assume normalization and monotonicity). The first two restrictions are substantial and make our approach less useful for some engineering applications considered in [56]. Instead, the requirement of translation invariance only boils down to different continuity assumptions between the two papers. Indeed, as we mentioned, the only implication of translation invariance used in our convergence result is Lipschitz continuity of order 1. Assumption 1.4 of [56] imposes a different continuity condition on an ancillary function that controls the shrinking rate of the operator. More generally, [56] (as well as a similar result by Krause [49, Theorem 8.3.4]) can only be used, after some additional steps, to derive point 1 of Proposition 1, which we obtain from Theorem 2.[35] However, [56] does not address issues that are relevant to us, such as convergence without consensus and the wisdom of the crowd. These questions significantly complicate the analysis, and we need to resort to completely different techniques coming from functional analysis as discussed above.[36] In addition, since our opinion aggregators are microfounded, under mild conditions they inherit the primitive-observation-network structure of the foundation (see Proposition 4). This imposes a strong discipline on the averaging process that allows us to provide bounds on the rate of convergence to consensus that are a function of the underlying network.[37]

Wisdom of the crowd and asymptotic learning Among the recent papers, the one closest to our wisdom of the crowd results is Molavi et al. [55]. However, both the questions and the methodology

[32] Their case is more general in terms of the domain of T in that B can be any convex subset of a normed vector space. However, their generality comes at a cost. In our jargon, they are only studying the case in which T is self-influential, which in our case would only yield the intermediate step needed to derive Corollary 1.

[33] Of course, there is also a vast literature on monotone dynamical systems in continuous time. This literature analyzes the limit set of the corresponding differential equation's solution by using completely different methods. We refer to [66] for a textbook treatment.

[34] See Chatterjee and Seneta [23] for a seminal paper exploring this question for time-inhomogeneous Markov chains.

[35] A similar internality condition is linked to the underlying network structure by Mueller-Frank [60].

[36] Other important papers that give different but related conditions for convergence to consensus include Angeli and Bliman [4], Cortes [26], that also shares with us the use of Clarke's differential (although in a completely different way), and Chen et al. [24], that generalize Moreau's model by allowing for delays in the transmission of opinions between neighbors.

[37] See the working-paper version of this paper [19].

are rather different. First, they follow Jadbabaie et al. [46] in considering social learning when agents both repeatedly receive external signals about an underlying state of the world and naively combine the beliefs of their neighbors. Instead, we follow the wisdom of the crowd approach of [39], and we study the long-run opinions as the size of the society grows to infinity. Therefore, we single out the role of the network structure and the opinion aggregator in efficiently combining the agents' *initial* information as the network's size increases. For the questions we explore, log-linear aggregators a la [55] can be studied in an equivalent linear system, thus making use of the results developed for the DeGroot model and its time-varying versions. So, our results cover their aggregators too after a suitable transformation.

Other related contributions Both Mueller-Frank [60] and Arieli et al. [5] address different robustness concerns in a social learning setting: in [60] it is with respect to external manipulation of the initial opinions, while in [5] it is with respect to the initial information structure of the agents. Further afield, Holme and Newman [44] and the subsequent literature study a model of opinion dynamics in which some of the links of the underlying network are broken and obtain polarization, as in our trimmed-means example. Unlike us, they consider the case in which the broken links are random and independent of the current opinion (while the ones that replace them must share the same opinion) and only provide numerical results.

Finally, our results also make use of some techniques coming from decision theory, and in particular Ghirardato et al. [38], Maccheroni et al. [52], and Schmeidler [65]. The papers [38] and [52] are the first to study functionals that satisfy normalization, monotonicity, and translation invariance, using nonstandard differential techniques. These techniques turn out to be particularly useful when we discuss the wisdom of the crowd. The third paper introduces the class of comonotonic additive functionals which includes rank-dependent aggregators. We instead consider (iterations of) operators as opposed to functionals. However, even under the usual decision-theoretic interpretation, our machinery and convergence results turn out to be useful, as shown in Cerreia-Vioglio et al. [21].

7 Conclusion

We see our results on the wisdom of the crowd as a natural starting point for further work. In Section 4.1, we considered a sequence of robust opinion aggregators $\{T(n)\}_{n\in\mathbb{N}}$ and a derived sequence of (uniform) DeGroot aggregators $\{W(n)\}_{n\in\mathbb{N}}$. Each $W(n)$ was constructed from the networks of weak ties $\bar{A}(n)$, which we assumed to be undirected. In a nutshell, we showed that if the Jacobian of each $T(n)$, whenever defined, is uniformly dominated by the corresponding $W(n)$, then the wisdom of the crowd holds, provided the dominating graphs exhibit enough connectivity. A careful inspection of the proof shows that $W(n)$ does not need to be induced by the network of weak ties. For example, it could be induced by any undirected multigraph, and the result would still hold. In both cases, connectivity is measured by the second largest eigenvalue in modulus, which can be computed because the graphs are undirected. It remains an open question whether the same type of result holds true when the graph is not assumed to be undirected, for example, by replacing the eigenvalue measure with another coefficient of ergodicity. Moreover, we conjecture that the condition on the second eigenvalue in part

2 of Proposition 3 can be relaxed in certain situations. Suppose that we start from two (sequences of) networks that are very symmetric and balanced, that would satisfy the conditions in part 2 of Proposition 3, and that thus achieve the wisdom of the crowd. Mashing them together and slightly connecting them would decrease connectivity as measured by the second largest eigenvalue, but it seems unrealistic that this change would disrupt the wisdom of the crowd.

On the more applied side, our results can be important tools for studying the transmission of idiosyncratic shocks to aggregate fluctuations in large economies. Even if we derived $\bar{T}$ as the operator mapping initial opinions to long-run opinions, Theorem 3 would apply to any nonlinear operator with the same properties. For example, we might consider a standard macroeconomic model of production networks and derive the equilibrium output and prices as functions of the idiosyncratic shocks of the firms. In their seminal paper, Acemoglu et al. [2] obtain *linear* equilibrium maps and provide sufficient conditions for the persistence of aggregate fluctuations in large economies. In our language, this means nonzero asymptotic variance as $n \to \infty$. Under more general specifications of the production functions or, perhaps more interestingly, under endogenous network formation (see, e.g., Acemoglu and Azar [1]), the equilibrium maps might well be nonlinear but still satisfy our properties. Therefore, our results would be the first step to extend and test the results of [2] in these more general and realistic settings. In all these cases, it would be interesting to derive the sufficient and necessary conditions for persistent aggregate fluctuations on the equilibrium operators from properties of the primitives, in the spirit of Proposition 3. This is the subject of current investigation.

Another avenue for future work would be to explore the role of robust opinion aggregators as a bridge between DeGroot-style continuous opinion aggregators and diffusion/contagion of a binary behavior such as adopting new technology. Indeed, (generalizations of) the discrete-opinion models of Morris [57], Kempe et al. [48], Centola and Macy [18], and Muller-Frank and Neri [61] can be obtained by considering a subclass of robust opinion aggregators with the property that each agent's updated opinion exactly coincides with one of the neighbors' opinions observed in the last period, a property that linear aggregators rule out. In the working paper [19] version of this paper, we show how our framework can deal with discrete (e.g., binary) opinions and obtain a result about convergence in that case. Obtaining sharper results on the wisdom of the crowd for such aggregators is an interesting open question.

A Appendix: convergence

All the missing proofs are in the Online Appendix (see Section D.1). The next three ancillary lemmas highlight the properties of T and the limiting operator $\bar{T}$, whenever it exists. Their proofs are based on routine arguments.

Lemma 1 *Let T be an opinion aggregator. The following statements are true:*

1. *If T is robust, then it admits an extension $S : \mathbb{R}^n \to \mathbb{R}^n$ which is also robust.*
2. *If T is normalized and monotone, then $\left\|T^t(x)\right\|_\infty \leq \|x\|_\infty$ for all $x \in B$ and for all $t \in \mathbb{N}$.*

Lemma 2 *If T is a robust opinion aggregator, then T^t is nonexpansive (i.e., Lipschitz continuous of order 1) for all $t \in \mathbb{N}$. In particular, T is nonexpansive.*

Despite being easy to derive, the property of nonexpansivity plays an important role in what follows and it also rules out the presence of chaotic behavior. The proof of next lemma instead relies on the property of "being a limit". It thus shows that the properties of T are often inherited by $\bar{T}$, provided the latter exists.

Lemma 3 *Let T be an opinion aggregator. If T is Cesaro convergent, then $\bar{T} : B \to B$, as defined in equation (2), is well defined and $\bar{T} \circ T = \bar{T}$. Moreover,*

1. *If T is nonexpansive, so is $\bar{T}$. In particular, $\bar{T}$ is continuous.*
2. *If T is normalized and monotone, so is $\bar{T}$.*
3. *If T is robust, so is $\bar{T}$.*
4. *If T is odd, so is $\bar{T}$, provided I is a symmetric interval, that is, $k \in I$ if and only if $-k \in I$.*

We can now prove that any sequence of updates of a robust opinion aggregator converges a la Cesaro and this convergence is uniform on bounded subsets of B.

Proof of Theorem 1. Consider $x \in B$. By point 2 of Lemma 1, we have that $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ is a bounded sequence and, in particular, relatively compact. By Lemma 2, T is nonexpansive. By Baillon et al. [6, Theorem 3.2 and Corollary 3.1], we can conclude that C-$\lim_t T^t(x)$ exists for all $x \in B$. By Lemma 3, $\bar{T}$ is a robust opinion aggregator such that $\bar{T} \circ T = \bar{T}$. Next, consider a bounded subset $\hat{B}$ of B. Define by $\tilde{B}$ the closed convex hull of $\hat{B}$. Since $\hat{B}$ is bounded and B is closed and convex, $\tilde{B}$ is a closed and bounded subset of B and, in particular, compact. For each $\tau \in \mathbb{N}$ define $S_\tau : \tilde{B} \to \mathbb{R}^n$ by

$$S_\tau(x) = \frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) \qquad \forall x \in \tilde{B}.$$

By Lemma 2, S_τ is well defined and nonexpansive for all $\tau \in \mathbb{N}$. The collection $\{S_\tau\}_{\tau\in\mathbb{N}}$ belongs to the space $C\left(\tilde{B}, \mathbb{R}^n\right)$ of continuous functions from $\tilde{B}$ to $\mathbb{R}^n$. This space is a Banach space once endowed with the supnorm: $\|f\|_* = \sup_{x\in\tilde{B}} \|f(x)\|_\infty$ for all $f \in C\left(\tilde{B}, \mathbb{R}^n\right)$. By [30, pp. 135–136] and since $\{S_\tau\}_{\tau\in\mathbb{N}}$ is a collection of nonexpansive maps, this implies that the sequence $\{S_\tau\}_{\tau\in\mathbb{N}} \subseteq C\left(\tilde{B}, \mathbb{R}^n\right)$ is equicontinuous. By contradiction, assume that $S_\tau \overset{\| \|_*}{\not\to} \bar{T}_{|\tilde{B}}$. This would imply that there exist $\varepsilon > 0$ and a subsequence $\{S_{\tau_m}\}_{m\in\mathbb{N}} \subseteq \{S_\tau\}_{\tau\in\mathbb{N}}$ such that $\left\|S_{\tau_m} - \bar{T}_{|\tilde{B}}\right\|_* \geq \varepsilon$ for all $m \in \mathbb{N}$. By the Arzela-Ascoli Theorem (see, e.g., [30, Theorem 7.5.7]) and since $\{S_{\tau_m}\}_{m\in\mathbb{N}}$ is equicontinuous and $\{S_{\tau_m}(x)\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^n$ is bounded for all $x \in \tilde{B}$, this would imply that there exist a subsequence $\left\{S_{\tau_{m(l)}}\right\}_{l\in\mathbb{N}}$ and a function $\hat{S} \in C\left(\tilde{B}, \mathbb{R}^n\right)$ such that $\lim_l \left\|S_{\tau_{m(l)}} - \hat{S}\right\|_* = 0$. By the previous part of the proof, recall that $\lim_\tau S_\tau(x) = \bar{T}(x)$ for all $x \in \tilde{B}$. By definition of $\| \|_*$, it would follow that $\bar{T}(x) = \lim_l S_{\tau_{m(l)}}(x) = \hat{S}(x)$ for all $x \in \tilde{B}$, that is, $\bar{T} = \hat{S}$ on $\tilde{B}$. This would imply that $0 < \varepsilon \leq \lim_l \left\| S_{\tau_{m(l)}} - \bar{T}_{|\tilde{B}} \right\|_* = 0$, a contradiction. We can conclude that

$$0 \leq \lim_\tau \sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_\infty \leq \lim_\tau \sup_{x \in \tilde{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_\infty = \lim_\tau \left\| S_\tau - \bar{T}_{|\tilde{B}} \right\|_* = 0,$$

proving the last part of the statement. ■

Remark 2 Theorem 1 could be seen as a version of the classic nonlinear ergodic theorem of Baillon (see, e.g., Krengel [50, Section 9.3]). The generalization we are relying upon is the one contained in Baillon et al. [6, Theorem 3.2 and Corollary 3.1]. Compared to our version, the part that would be missing is the one contained in (3). Observe that (3), not only guarantees uniform Cesaro convergence of $\left\{T^t(x)\right\}_{t \in \mathbb{N}}$, but also the independence from the initial condition of the rate of such convergence. This latter property might play an important role in applications and is missing in the aforementioned works. Finally, in the working-paper version of this paper, exploiting the finite dimensionality of our framework, we provide a self-contained proof. ▲

We next prove our first result on standard convergence: Theorem 2. We begin by presenting few facts which are useful for proving point 1. First, we identify a technical property, termed asymptotic regularity, which characterizes convergence. Second, we show how $\underline{A}(T)$ being nontrivial is equivalent to T having a useful decomposition. Finally, via this decomposition, we show that aperiodicity of $\underline{A}(T)$ yields asymptotic regularity, hence convergence. We then prove point 2 of Theorem 2 for an important special case: N strongly connected under $\bar{A}(T)$. The general case then follows by observing that a robust opinion aggregator can be restricted to any strongly connected component of $\bar{A}(T)$ and retain its properties, including convergence.

Lemma 4 *Let T be a robust opinion aggregator. The following statements are equivalent:*

(i) T is asymptotically regular, that is, $\lim_t \left\| T^{t+1}(x) - T^t(x) \right\|_\infty = 0$ for all $x \in B$;

(ii) T is convergent.

Proposition 5 *Let T be a robust opinion aggregator. The following statements are equivalent:*

(i) $\underline{A}(T)$ is nontrivial;

(ii) There exist $W \in \mathcal{W}$ and $\varepsilon \in (0,1)$ such that

$$T(x) = \varepsilon W x + (1 - \varepsilon) S(x) \qquad \forall x \in B \tag{23}$$

where S is a robust opinion aggregator.

Moreover, we have that W in (ii) can be chosen to be such that $A(W) = \underline{A}(T)$.

Proof. (i) implies (ii). For each $i,j \in N$ if j strongly influences i, consider $\varepsilon_{ij} \in (0,1)$ as in (7) otherwise let $\varepsilon_{ij} = 1/2$. Define $\tilde{W}$ to be such that $\tilde{w}_{ij} = \underline{a}_{ij}\varepsilon_{ij}$ for all $i,j \in N$ where $\underline{a}_{ij}$ is the ij-th entry of $\underline{A}(T)$. Since each row of $\underline{A}(T)$ is not null, for each $i \in N$ there exists $j \in N$ such that $\underline{a}_{ij} = 1$ and, in particular, $\tilde{w}_{ij} > 0$. This implies that $\sum_{l=1}^{n} \tilde{w}_{il} > 0$ for all $i \in N$. Define also $\varepsilon = \min\{\min_{i\in N} \sum_{l=1}^{n} \tilde{w}_{il}, 1/2\} \in (0,1)$. Define $W \in \mathcal{W}$ to be such that $w_{ij} = \tilde{w}_{ij}/\sum_{l=1}^{n} \tilde{w}_{il}$ for all $i,j \in N$. Clearly, we have that for each $i,j \in N$

$$w_{ij} > 0 \iff \tilde{w}_{ij} > 0 \iff \underline{a}_{ij} = 1. \tag{24}$$

This yields that $A(W) = \underline{A}(T)$. Next, consider $x,y \in B$ such that $x \geq y$. Define $y^0 = y$. For each $t \in \{1,...,n-1\}$ define $y^t \in B$ to be such that $y_i^t = x_i$ for all $i \leq t$ and $y_i^t = y_i$ for all $i \geq t+1$. Define $y^n = x$. Note that $x = y^n \geq ... \geq y^1 \geq y^0 = y$. It follows that for each $i \in N$

$$\begin{aligned}T_i(x) - T_i(y) &= \sum_{j=1}^{n} \left[T_i\left(y^j\right) - T_i\left(y^{j-1}\right)\right] \geq \sum_{j=1}^{n} \underline{a}_{ij}\varepsilon_{ij}\left(y_j^j - y_j^{j-1}\right) = \sum_{j=1}^{n} \tilde{w}_{ij}(x_j - y_j) \\ &= \left(\sum_{l=1}^{n} \tilde{w}_{il}\right)\left(\sum_{j=1}^{n} \frac{\tilde{w}_{ij}}{\sum_{l=1}^{n} \tilde{w}_{il}}(x_j - y_j)\right) = \left(\sum_{l=1}^{n} \tilde{w}_{il}\right)\left(\sum_{j=1}^{n} w_{ij}(x_j - y_j)\right) \geq \varepsilon \sum_{j=1}^{n} w_{ij}(x_j - y_j).\end{aligned}$$

It follows that

$$x \geq y \implies T(x) - T(y) \geq \varepsilon W(x-y) = \varepsilon(Wx - Wy). \tag{25}$$

Define $S: B \to \mathbb{R}^n$ by

$$S(x) = \frac{T(x) - \varepsilon W x}{1-\varepsilon} \qquad \forall x \in B. \tag{26}$$

By definition of S and since $W \in \mathcal{W}$ and T is normalized and translation invariant, it is immediate to see that $S(ke) = ke$ for all $k \in I$ and that S is translation invariant. Since (25) holds and $\varepsilon \in (0,1)$, routine computations yield that S is monotone. Since S is normalized and monotone, then $S(B) \subseteq B$, that is, S is a selfmap and, in particular, S is a robust opinion aggregator. By rearranging (26), (23) follows.

(ii) implies (i). Consider $i \in N$. Since W is a stochastic matrix, there exists $j \in N$ such that $w_{ij} > 0$. Let $x \in B$ and $h > 0$ be such that $x + he^j \in B$. By (23) and since S is monotone, we have that $T_i\left(x + he^j\right) - T_i(x) = \varepsilon w_{ij} h + (1-\varepsilon) S_i\left(x + he^j\right) - (1-\varepsilon) S_i(x) \geq \varepsilon w_{ij} h$, proving that j strongly influences i and $\underline{a}_{ij} = 1$. It follows that the i-th row of $\underline{A}(T)$ is not null. Since i was arbitrarily chosen, the statement follows.

Finally, by (24), note that W in (ii) can be chosen to be such that $A(W) = \underline{A}(T)$. ■

Point 1 of Theorem 2 builds on two assumptions: i) the matrix of strong ties $\underline{A}(T)$ has no null row; ii) each closed group of $\underline{A}(T)$ is aperiodic. The first assumption allows for a decomposition of T into a convex linear combination of a linear opinion aggregator with matrix W and a robust opinion aggregator S (cf. Proposition 5). We next show that if W takes a very particular form, which we dub partition matrix, then T is asymptotically regular and, in particular, convergent (Lemma 5 and Proposition 6 below). The second assumption yields that W can be always chosen such that W^t eventually "contains" a partition matrix. This will prove point 1 of Theorem 2.

Definition 10 *Let $J : B \to B$ be an opinion aggregator. We say that J is a* partition operator/matrix *if and only if there exists a family of disjoint nonempty subsets $\left\{\hat{N}_l\right\}_{l=1}^{m}$ of N such that $\cup_{l=1}^{m}\hat{N}_l = N$ and for each $l \in \{1, ..., m\}$ there exists $k_l \in \hat{N}_l$ such that $J_i(x) = x_{k_l}$ for all $i \in \hat{N}_l$.*

Note that a partition operator is linear. With a small abuse of notation, we will denote the matrix and the operator by the same symbol.

Lemma 5 *Let T be a robust opinion aggregator such that $T = \varepsilon J + (1-\varepsilon) S$ where $\varepsilon \in (0,1)$, J is a partition operator, and $S : B \to B$ is a robust opinion aggregator. Let C be a nonempty subset of B such that there exists $k > 0$ satisfying*

$$\|T(x) - x\|_\infty < k \qquad \forall x \in C. \tag{27}$$

If there exists $\delta > 0$ such that for each $t \in \mathbb{N}_0$ there exists $x \in C$ satisfying

$$\left\|T^{t+1}(x) - T^t(x)\right\|_\infty \geq \delta, \tag{28}$$

then $\left\{T^t(x) : x \in C \text{ and } t \in \mathbb{N}_0\right\}$ is unbounded.

Proposition 6 *Let T be a robust opinion aggregator. If T is such that $T = \varepsilon J + (1-\varepsilon) S$ where $\varepsilon \in (0,1)$, J is a partition operator, and S is a robust opinion aggregator, then T is asymptotically regular and, in particular, convergent.*

Proof. Fix $x \in B$. In Lemma 5, set $C = \{x\}$. Clearly, there exists $k > 0$ that satisfies $\|T(x) - x\|_\infty < k$. By point 2 of Lemma 1 and since T is a robust opinion aggregator, it follows that $\left\{T^t(x)\right\}_{t\in\mathbb{N}_0}$ is bounded. By Lemma 5, we have that for each $\delta > 0$ there exists $\bar{t} \in \mathbb{N}_0$ such that

$$\left\|T^{\bar{t}+1}(x) - T^{\bar{t}}(x)\right\|_\infty < \delta. \tag{29}$$

Since T is nonexpansive, $\left\{\left\|T^{t+1}(x) - T^t(x)\right\|_\infty\right\}_{t\in\mathbb{N}_0}$ is a decreasing sequence. By (29) and since $\left\{\left\|T^{t+1}(x) - T^t(x)\right\|_\infty\right\}_{t\in\mathbb{N}_0}$ is a decreasing sequence, we have that for each $\delta > 0$ there exists $\bar{t} \in \mathbb{N}$ such that $\left\|T^{t+1}(x) - T^t(x)\right\|_\infty < \delta$ for all $t \geq \bar{t}$, that is, $\lim_t \left\|T^{t+1}(x) - T^t(x)\right\|_\infty = 0$. Since x was arbitrarily chosen, it follows that T is asymptotically regular. By Lemma 4, this implies that T is convergent. ■

Lemma 6 below shows that if $\underline{A}(T)$ is aperiodic and nontrivial, then there exists $\bar{t} \in \mathbb{N}$ such that $T^{\bar{t}} = \gamma J + (1-\gamma) S$ (resp. $T^{\bar{t}+1} = \gamma J + (1-\gamma) S$) where J is a partition operator, $\gamma \in (0,1)$, and S is a robust opinion aggregator. The operator J only depends on $\underline{A}(T)$ while γ and S both depend on $\bar{t}$ (resp. $\bar{t}+1$). In turn, Proposition 6 yields that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent. This will be sufficient to imply the convergence of T.

Lemma 6 *Let T be a robust opinion aggregator. If $\underline{A}(T)$ is aperiodic and nontrivial, then there exists $\bar{t} \in \mathbb{N}$ such that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent.*

Proof. By Proposition 5 and since $\underline{A}(T)$ is nontrivial, we have that there exist $W \in \mathcal{W}$, $\varepsilon \in (0,1)$, and a robust opinion aggregator $S : B \to B$ such that

$$T(x) = \varepsilon W x + (1-\varepsilon) S(x) \quad \forall x \in B. \tag{30}$$

Moreover, W can be chosen to be such that $A(W) = \underline{A}(T)$. By [39, Theorems 2 and 3] and since $\underline{A}(T)$ is aperiodic, this implies that there exist $\bar{t} \in \mathbb{N}$ and a partition $\left\{\hat{N}_l\right\}_{l=1}^m$ of N such that for each $l \in \{1,...,m\}$ there exists $k_l \in \hat{N}_l$ satisfying $w_{ik_l}^{(\bar{t})}, w_{ik_l}^{(\bar{t}+1)} > 0$ for all $i \in \hat{N}_l$.[38] It follows that

$$W^{\bar{t}} = \delta_{\bar{t}} J + (1-\delta_{\bar{t}}) \tilde{W}_{\bar{t}} \text{ and } W^{\bar{t}+1} = \delta_{\bar{t}+1} J + (1-\delta_{\bar{t}+1}) \tilde{W}_{\bar{t}+1} \tag{31}$$

where $\delta_{\bar{t}}, \delta_{\bar{t}+1} \in (0,1)$, J is a partition operator/matrix,[39] and $\tilde{W}_{\bar{t}}$ as well as $\tilde{W}_{\bar{t}+1}$ are stochastic matrices. By (30) and induction, we also have that $T^{\bar{t}}(x) = \varepsilon^{\bar{t}} W^{\bar{t}} x + \left(1-\varepsilon^{\bar{t}}\right) \tilde{S}_{\bar{t}}(x)$ and $T^{\bar{t}+1}(x) = \varepsilon^{\bar{t}+1} W^{\bar{t}+1} x + \left(1-\varepsilon^{\bar{t}+1}\right) \tilde{S}_{\bar{t}+1}(x)$ for all $x \in B$, where $\tilde{S}_{\bar{t}}$ and $\tilde{S}_{\bar{t}+1}$ are robust opinion aggregators. By (31), it follows that $T^{\bar{t}} = \gamma_{\bar{t}} J + (1-\gamma_{\bar{t}}) \hat{S}_{\bar{t}}$ and $T^{\bar{t}+1} = \gamma_{\bar{t}+1} J + (1-\gamma_{\bar{t}+1}) \hat{S}_{\bar{t}+1}$ where $\gamma_{\bar{t}} = \varepsilon^{\bar{t}} \delta_{\bar{t}}$ (resp. $\gamma_{\bar{t}+1} = \varepsilon^{\bar{t}+1} \delta_{\bar{t}+1}$) and $\hat{S}_{\bar{t}}(x) = \frac{\varepsilon^{\bar{t}}(1-\delta_{\bar{t}})}{1-\varepsilon^{\bar{t}}\delta_{\bar{t}}} \tilde{W}_{\bar{t}} x + \frac{1-\varepsilon^{\bar{t}}}{1-\varepsilon^{\bar{t}}\delta_{\bar{t}}} \tilde{S}_{\bar{t}}(x)$ (resp. $\hat{S}_{\bar{t}+1}(x) = \frac{\varepsilon^{\bar{t}+1}\left(1-\delta_{\bar{t}+1}\right)}{1-\varepsilon^{\bar{t}+1}\delta_{\bar{t}+1}} \tilde{W}_{\bar{t}+1} x + \frac{1-\varepsilon^{\bar{t}+1}}{1-\varepsilon^{\bar{t}+1}\delta_{\bar{t}+1}} \tilde{S}_{\bar{t}+1}(x)$) for all $x \in B$. It follows that $\gamma_{\bar{t}}, \gamma_{\bar{t}+1} \in (0,1)$ and $\hat{S}_{\bar{t}}$ as well as $\hat{S}_{\bar{t}+1}$ are robust opinion aggregators. By Proposition 6, this implies that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent. ■

The reader only interested in the proof of point 1 can skip the next two results which are instrumental to prove point 2 of Theorem 2. To this end, we focus on the network of weak ties $\bar{A}(T)$. Assume that $\left\{C_{[r]}\right\}_{r \in \{0,...,d-1\}}$ is a family of disjoint nonempty subsets of N such that $\cup_{r=0}^{d-1} C_{[r]} = N$ with $d \geq 1$. Given $\left\{x^{[r]}\right\}_{r \in \{0,...,d-1\}} \subseteq B$, we denote by $x = \sum_{r=0}^{d-1} x^{[r]} 1_{C_{[r]}} \in B$ the vector whose i-th generic component is such that $x_i = x_i^{[r']}$ when $i \in C_{[r']}$ and $C_{[r']}$ is the only element in $\left\{C_{[r]}\right\}_{r \in \{0,...,d-1\}}$ containing i.

Lemma 7 *Let T be an opinion aggregator and $\left\{C_{[r]}\right\}_{r \in \{0,...,d-1\}}$ a family of disjoint nonempty subsets of N such that $\cup_{r=0}^{d-1} C_{[r]} = N$ with $d \geq 1$. If T is normalized and monotone, then $\bar{A}(T)$ is nontrivial. Moreover, if $\bar{\imath} \in N$ and $\{j \in N : \bar{a}_{\bar{\imath}j} = 1\} \subseteq C_{[r_{\bar{\imath}}]}$ for some $r_{\bar{\imath}} \in \{0,...,d-1\}$, then*

$$x = \sum_{r=0}^{d-1} x^{[r]} 1_{C_{[r]}} \implies T_{\bar{\imath}}(x) = T_{\bar{\imath}}\left(x^{[r_{\bar{\imath}}]}\right). \tag{32}$$

Proposition 7 *Let T be a robust opinion aggregator such that N is strongly connected under $\bar{A}(T)$. If T is convergent, then the network of weak ties $\bar{A}(T)$ is aperiodic and nontrivial.*

Proof. By Lemma 7 and since T is normalized and monotone, $\bar{A}(T)$ is nontrivial. By contradiction, assume that $\bar{A}(T)$ is not aperiodic, that is, there exists a closed group M which is not aperiodic under $\bar{A}(T)$. Since N is strongly connected under $\bar{A}(T)$, we have that N is the only closed group, yielding that the greatest common divisor of the lengths of the simple cycles in N is $d \geq 2$. For each

[38] As usual, we denote by $w_{ik_l}^{(\bar{t})}$ (resp. $w_{ik_l}^{(\bar{t}+1)}$) the entry in the i-th row and k_l-th column of the matrix $W^{\bar{t}}$ (resp. $W^{\bar{t}+1}$).

[39] That is, $J_i(x) = x_{k_l}$ for all $i \in \hat{N}_l$ and for all $l \in \{1,...,m\}$ where $\left\{\hat{N}_l\right\}_{l=1}^m$ and $\{k_l\}_{l=1}^m$ have been defined above.

$i \in N$ define $\bar{N}_i = \{j \in N : \bar{a}_{ij} = 1\}$. It follows that there exists a partition of N in cyclic classes $\left\{C_{[r]}\right\}_{r\in\{0,...,d-1\}}$ such that $\cup_{i\in C_{[r]}} \bar{N}_i \subseteq C_{[r]\oplus[1]}$ for all $r \in \{0, ..., d-1\}$ where $[r]$ are the elements of $\mathbb{Z}_d$ and $\oplus$ is the standard sum in $\mathbb{Z}_d$.[40] Since I has nonempty interior, there exist $a, b \in I$ such that $a > b$. Define the vector $x \in B$ to be such that $x = \sum_{r=0}^{d-1} \left(k_{[r]}e\right) 1_{C_{[r]}}$, where $k_{[0]} = a$ and $k_{[r]} = b$ for all $r \in \{1, ..., d-1\}$. By Lemma 7 and induction and since $\cup_{i\in C_{[r]}} \bar{N}_i \subseteq C_{[r]\oplus[1]}$ for all $r \in \{0, ..., d-1\}$, we have that

$$T^t(x) = \sum_{r=0}^{d-1} \left(k_{[r]\oplus t[1]}e\right) 1_{C_{[r]}} \qquad \forall t \in \mathbb{N}.$$

This implies that $\left\|T^{t+1}(x) - T^t(x)\right\|_\infty \geq a - b > 0$ for all $t \in \mathbb{N}$, a contradiction with Lemma 4 and T being convergent. ■

Proof of Theorem 2. 1. We adopt the usual convention $T^0(x) = x$ for all $x \in B$. By Lemma 6 and since $\underline{A}(T)$ is aperiodic and nontrivial, there exists $\bar{t} \in \mathbb{N}$ such that $T^{\bar{t}}$ and $T^{\bar{t}+1}$ are convergent. We next show that this implies that T is convergent. Fix $x \in B$. Since $T^{\bar{t}}$ is convergent, we can conclude that $\lim_k T^{k\bar{t}}(x)$ exists. Denote $\bar{x} = \lim_k T^{k\bar{t}}(x)$. Since T is continuous and so is $T^{\bar{t}}$, it is plain that $T^{\bar{t}}(\bar{x}) = \bar{x}$. This implies that

$$T^{\bar{t}}\left(T^s(\bar{x})\right) = T^{\bar{t}+s}(\bar{x}) = T^{s+\bar{t}}(\bar{x}) = T^s\left(T^{\bar{t}}(\bar{x})\right) = T^s(\bar{x}) \qquad \forall s \in \mathbb{N}_0.$$

By induction on k, this yields that for each $s \in \mathbb{N}_0$

$$T^{(k+1)\bar{t}}\left(T^s(\bar{x})\right) = T^{k\bar{t}}\left(T^{\bar{t}}\left(T^s(\bar{x})\right)\right) = T^{k\bar{t}}\left(T^s(\bar{x})\right) = T^s(\bar{x}) \quad \forall k \in \mathbb{N}.$$

In particular, by setting $k = s$, we obtain that for each $s \in \mathbb{N}$

$$T^{s(\bar{t}+1)}(\bar{x}) = T^{s\bar{t}}\left(T^s(\bar{x})\right) = T^s(\bar{x}). \tag{33}$$

Since $T^{\bar{t}+1}$ is convergent, we have that $\lim_s T^{s(\bar{t}+1)}(\bar{x})$ exists. By (33), this implies that $\lim_s T^s(\bar{x})$ exists. Denote $\hat{x} = \lim_s T^s(\bar{x})$. Since T is continuous, it is plain that $T(\hat{x}) = \hat{x}$. Since $\left\{T^{k\bar{t}}(\bar{x})\right\}_{k\in\mathbb{N}} \subseteq \left\{T^s(\bar{x})\right\}_{s\in\mathbb{N}}$ and $T^{k\bar{t}}(\bar{x}) = \bar{x}$ for all $k \in \mathbb{N}$, we have that

$$\bar{x} = \lim_k T^{k\bar{t}}(\bar{x}) = \lim_s T^s(\bar{x}) = \hat{x} \text{ and } T(\hat{x}) = \hat{x}. \tag{34}$$

We can now prove that $\left\{T^t(x)\right\}_{t\in\mathbb{N}}$ converges too. By (34) and since T is nonexpansive, we have that

$$\left\|\bar{x} - T^{t+1}(x)\right\|_\infty = \left\|T(\bar{x}) - T\left(T^t(x)\right)\right\|_\infty \leq \left\|\bar{x} - T^t(x)\right\|_\infty \qquad \forall t \in \mathbb{N},$$

yielding that $\left\{\left\|\bar{x} - T^t(x)\right\|_\infty\right\}_{t\in\mathbb{N}}$ is a decreasing sequence. Moreover, since $\bar{x} = \lim_k T^{k\bar{t}}(x)$, we have that the subsequence $\left\{\left\|\bar{x} - T^{k\bar{t}}(x)\right\|_\infty\right\}_{k\in\mathbb{N}} \subseteq \left\{\left\|\bar{x} - T^t(x)\right\|_\infty\right\}_{t\in\mathbb{N}}$ converges to 0. This implies that

[40] See Seneta [64, Section 1.3] and Kemeni and Snell [47, Section 1.4]. There is a minor caveat. The definition of aperiodic used in these works, and more in general in the Markov chains literature, is formally different from the one of Golub and Jackson [39], which we also adopt. Yet, when N is strongly connected, they are equivalent. Formally, N is aperiodic according to our current formulation if and only if the greatest common divisor of the lengths of *all* cycles, starting and ending at any node $i \in N$, is 1 (cf. Seneta [64, Definition 1.6]).

$\lim_t T^t(x) = \bar{x}$. Since x was arbitrarily chosen, the statement follows.

2. By Lemma 7 and since T is normalized and monotone, $\bar{A}(T)$ is nontrivial. Next, we consider a family of disjoint subsets $\left\{\hat{N}_l\right\}_{l=1}^{m+1}$ of N such that $\cup_{l=1}^{m+1}\hat{N}_l = N$ where $m \geq 1$ and the first m sets are nonempty. Given [64, Section 1.2], we choose the first m elements of $\left\{\hat{N}_l\right\}_{l=1}^{m+1}$ to be the classes (the partition) of essential indexes of $\bar{A}(T)$ and we collect all the possible inessential indexes of $\bar{A}(T)$ in $\hat{N}_{m+1}$. If $l \in \{1,...,m\}$, then $\hat{N}_l$ is closed and strongly connected and $\bar{a}_{ij} = 0$ for all $i \in \hat{N}_l$ and for all $j \in \hat{N}_l^c$. The set $\hat{N}_{m+1}$ might be empty. If $m = 1$ and $\hat{N}_{m+1} = \emptyset$, then N is strongly connected under $\bar{A}(T)$. In this case, by Proposition 7, $\bar{A}(T)$ is aperiodic. Assume that either $m > 1$ or $m = 1$ and $\hat{N}_{m+1} \neq \emptyset$. By contradiction, assume that $\bar{A}(T)$ is not aperiodic. This implies that there exists a closed group M which is not aperiodic under $\bar{A}(T)$. It is immediate to see that there exists $l \in \{1,...,m\}$ such that $\hat{N}_l \subseteq M$. Since $\hat{N}_l$ has (simple) cycles and the simple cycles of $\hat{N}_l$ are simple cycles of M and M is not aperiodic, the greatest common divisor of the lengths of the cycles of $\hat{N}_l$ is greater than the one of the cycles of M and, in particular, ≥ 2. Set $\hat{N}_l = \{i_1,...,i_r\}$. Clearly, $r \geq 2$. We introduce two maps $P : \mathbb{R}^r \to \mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{R}^r$. The first is defined by $x = P(\tilde{x})$ where $x_i = \min_{h\in\{1,...,r\}} \tilde{x}_h$ if $i \notin \hat{N}_l$ and $x_{i_h} = \tilde{x}_h$ for all $h \in \{1,...,r\}$. The second one is defined by $\tilde{x} = \pi(x)$ where $\tilde{x}_h = x_{i_h}$ for all $h \in \{1,...,r\}$. It is immediate to check that $P(\pi(z)) = z1_{\hat{N}_l} + \left(\min_{h\in\{1,...,r\}} z_{i_h} e\right) 1_{\hat{N}_l^c}$ for all $z \in \mathbb{R}^n$. Note that $P\left(\tilde{B}\right) \subseteq B$ and $\pi(B) \subseteq \tilde{B}$ where $\tilde{B} = I^r$. Next, we define $S : \tilde{B} \to \tilde{B}$ by $S(\tilde{x}) = \pi(T(P(\tilde{x})))$ for all $\tilde{x} \in \tilde{B}$. It is routine to check that S is a robust opinion aggregator. Moreover, by construction and since $\hat{N}_l$ is strongly connected and not aperiodic, we also have that the restricted set of agents $\tilde{N} = \{1,...,r\}$ is strongly connected and not aperiodic under $\bar{A}(S)$. Note that $S^t(\tilde{x}) = \pi\left(T^t(P(\tilde{x}))\right)$ for all $\tilde{x} \in \tilde{B}$. Indeed, by Lemma 7 and induction and since $\bar{a}_{ij} = 0$ for all $i \in \hat{N}_l$ and for all $j \in \hat{N}_l^c$, we have that for each $t \in \mathbb{N}$ and for each $\tilde{x} \in \tilde{B}$

$$\begin{aligned}
S^{t+1}(\tilde{x}) &= \pi\left(T\left(P\left(\pi\left(T^t(P(\tilde{x}))\right)\right)\right)\right) \\
&= \pi\left(T\left(T^t(P(\tilde{x}))\, 1_{\hat{N}_l} + \left(\min_{h\in\{1,...,r\}} T^t_{i_h}(P(\tilde{x}))\, e\right) 1_{\hat{N}_l^c}\right)\right) \\
&= \pi\left(T\left(T^t(P(\tilde{x}))\right)\right) = \pi\left(T^{t+1}(P(\tilde{x}))\right).
\end{aligned}$$

Since T is convergent and π is continuous, this implies that S is convergent. By Proposition 7 and since S is a convergent robust opinion aggregator such that $\tilde{N}$ is strongly connected under $\bar{A}(S)$, this is a contradiction with $\tilde{N}$ not being aperiodic. ■

Proof of Corollary 1. Since T is self-influential, it follows that each row of $\underline{A}(T)$ is not null, yielding that $\underline{A}(T)$ is nontrivial. Moreover, since there is a simple cycle of length 1 from i to i for all $i \in N$, each closed group is aperiodic. By Theorem 2, the statement follows. ■

In order to prove Proposition 1, we begin by making two simple observations about convergence and fixed points of the opinion aggregator T: i) convergence is always toward a fixed point of T; ii) simple properties on the network $\underline{A}(T)$ yield that those fixed points are constant vectors. We denote by $E(T)$ the set of fixed points/equilibria of T. Recall that D is the consensus subset, that is, $x \in D \subseteq B$ if and only if $x_i = x_j$ for all $i, j \in N$.

Proposition 8 *Let T be a robust opinion aggregator. If $\underline{A}(T)$ is nontrivial, has a unique strongly connected and closed group M, and M is aperiodic under $\underline{A}(T)$, then $E(T) = D$.*

Proof of Proposition 1. 1. Since $\underline{A}(T)$ is nontrivial, has a unique strongly connected and closed group M, and M is aperiodic under $\underline{A}(T)$, we have that any other closed group M' is a superset of M, yielding that M' is aperiodic under $\underline{A}(T)$. By Theorem 2 and Proposition 8 and since standard convergence implies Cesaro convergence and T is continuous, it is immediate to see that T is convergent and $\bar{T}(x) = \lim_t T^t(x) \in E(T) = D$ for all $x \in B$, proving the statement.

2. Consider the same family of disjoint subsets $\left\{\hat{N}_l\right\}_{l=1}^{m+1}$ of N, as in the proof of point 2 of Theorem 2. Recall that if $l \in \{1, ..., m\}$, then $\hat{N}_l$ is closed and strongly connected and $\bar{a}_{ij} = 0$ for all $i \in \hat{N}_l$ and for all $j \in \hat{N}_l^c$. Recall also that $\hat{N}_{m+1}$ might be empty. By Theorem 2 and since T is convergent (to consensus), $\bar{A}(T)$ is aperiodic and nontrivial. By contradiction and since $\bar{A}(T)$ is nontrivial and each closed group is aperiodic under $\bar{A}(T)$, assume that T does not have a unique strongly connected and closed group. Since $\bar{A}(T)$ is nontrivial, this implies that there are at least two distinct strongly connected and closed groups and, in particular, $m \geq 2$. Since I has nonempty interior, consider $a, b \in I$ such that $a > b$. Consider a vector $x \in B$ such that $x_i = a$ for all $i \in \hat{N}_1$, $x_i = b$ for all $i \in \hat{N}_l$ and for all $l \in \{2, ..., m\}$. Since T is convergent, define $\bar{x} = \lim_t T^t(x)$. By Lemma 7 and induction and since $\bar{a}_{ij} = 0$ for all $i \in \hat{N}_l$, for all $j \in \hat{N}_l^c$, and for all $l \in \{1, ..., m\}$, we have that

$$T_i^t(x) = x_i \qquad \forall i \in \hat{N}_l, \forall l \in \{1, ..., m\}, \forall t \in \mathbb{N},$$

proving that $\bar{x}_i = x_i$ for all $i \in \hat{N}_l$ and for all $l \in \{1, ..., m\}$. Since $a \neq b$, we have that $\bar{x}$ is not a constant vector, a contradiction with convergence to consensus. ■

B Appendix: vox populi, vox Dei?

All the missing proofs are in the Online Appendix (see Section D.2).

Proof of Theorem 3. Given $n \in \mathbb{N}$, for notational convenience, we define $\hat{B} = \hat{I}^n$. We first make a few observations. Since the random variables $\{X_i(n)\}_{i \in N, n \in \mathbb{N}}$ are uniformly bounded and $\bar{T}_i(n)$ is continuous for all $i \in N$ and for all $n \in \mathbb{N}$, it follows that $\omega \mapsto \bar{T}_i(n)(X_1(n)(\omega), ..., X_n(n)(\omega))$ is integrable for all $i \in N$ and for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and $i \in N$. By Rademacher's Theorem and since $\bar{T}(n)$ is nonexpansive, this implies that $\bar{T}(n)$ is almost everywhere differentiable. Let $\mathcal{D}(\bar{T}(n)) \subseteq \hat{B}$ be the subset of $\hat{B}$ where $\bar{T}(n)$ is differentiable. Clearly, $\bar{T}_i(n)$ is differentiable on $\mathcal{D}(\bar{T}(n))$ and Clarke differentiable. Since $\bar{T}_i(n)$ is monotone and translation invariant, note that $\nabla \bar{T}_i(n)(x) \in \Delta_n$ for all $x \in \mathcal{D}(\bar{T}(n))$. Consider $\bar{x} \in \hat{B}$. Recall that Clarke's differential is the set (see, e.g., [25, Theorem 2.5.1]):

$$\partial \bar{T}_i(n)(\bar{x}) = \text{co}\left\{p \in \Delta_n : p = \lim_k \nabla \bar{T}_i(n)\left(x^k\right) \text{ s.t. } x^k \to \bar{x} \text{ and } x^k \in \mathcal{D}\left(\bar{T}(n)\right)\right\}. \tag{35}$$

By Definition 6 and (35) and since i and n were arbitrarily chosen, note that

$$0 \leq \underline{s}_{ij}(T(n)) \leq p_j \leq \bar{s}_{ij}(T(n)) \quad \forall i, j \in N, \forall p \in \partial \bar{T}_i(n)(x), \forall x \in \hat{B}, \forall n \in \mathbb{N}. \tag{36}$$

1. We start the proof of point 1 with an ancillary claim.

Claim. For each $i, j \in N$ *and for each* $n \in \mathbb{N}$

$$\sup_{\left\{(x,t)\in \hat{B}\times\mathbb{R}: x+te^j\in\hat{B}\right\}} \left|\bar{T}_i(n)\left(x+te^j\right)-\bar{T}_i(n)(x)\right| \le \ell \bar{s}_{ij}(T(n)).$$

Proof of the Claim. Fix $i \in N$ and $n \in \mathbb{N}$ and consider $j \in N$, $x \in \hat{B}$, and $t \in \mathbb{R}$ such that $x + te^j \in \hat{B}$. Define $y = x + te^j$. By Lebourg's Mean Value Theorem, we have that there exist $\lambda \in (0,1)$ and $\bar{p} \in \partial \bar{T}_i(n)(z)$ where $z = \lambda y + (1-\lambda)x \in \hat{B}$ such that

$$\bar{T}_i(n)\left(x+te^j\right)-\bar{T}_i(n)(x) = \bar{T}_i(n)(y)-\bar{T}_i(n)(x) = \sum_{l=1}^{n} \bar{p}_l(y_l - x_l).$$

By (36), this implies that

$$\left|\bar{T}_i(n)\left(x+te^j\right)-\bar{T}_i(n)(x)\right| = \left|\bar{p}_j(y_j-x_j)\right| = \bar{p}_j\left|y_j-x_j\right| \le \ell\bar{p}_j \le \ell\bar{s}_{ij}(T(n)).$$

Since x and t were arbitrarily chosen, it follows that

$$\sup_{\left\{(x,t)\in \hat{B}\times\mathbb{R}: x+te^j\in\hat{B}\right\}} \left|\bar{T}_i(n)\left(x+te^j\right)-\bar{T}_i(n)(x)\right| \le \ell \bar{s}_{ij}(T(n)).$$

Since i, j, and n were also arbitrarily chosen, the statement follows. □

Consider now $n \in \mathbb{N}$ and $i \in N$. By McDiarmid's inequality as well as the previous claim, we can conclude that for each $\delta > 0$

$$\begin{aligned}
&P\left(\left\{\omega\in\Omega: \left|\bar{T}_i(n)(X_1(n)(\omega),...,X_n(n)(\omega)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right|^2 \ge \delta\right\}\right)\\
&= P\left(\left\{\omega\in\Omega: \left|\bar{T}_i(n)(X_1(n)(\omega),...,X_n(n)(\omega)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right| \ge \sqrt{\delta}\right\}\right)\\
&\qquad\le 2\exp\left(-\frac{2\delta}{\sum_{j=1}^{n}\left(\ell\bar{s}_{ij}(T(n))\right)^2}\right) = 2\exp\left(-\frac{2\delta}{\ell^2\sum_{j=1}^{n}\bar{s}_{ij}(T(n))^2}\right).
\end{aligned}$$

Next, by [11, Equation 21.9] and since i and n were arbitrarily chosen, observe that

$$\begin{aligned}
&\mathrm{Var}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right) \\
&= \mathbb{E}\left(\left(\bar{T}_i(n)(X_1(n),...,X_n(n)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right)^2\right) \\
&= \int_0^\infty P\left(\left\{\omega\in\Omega : \left(\bar{T}_i(n)(X_1(n)(\omega),...,X_n(n)(\omega)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right)^2 \geq t\right\}\right) dt \\
&= \int_0^{\ell^2} P\left(\left\{\omega\in\Omega : \left|\bar{T}_i(n)(X_1(n)(\omega),...,X_n(n)(\omega)) - \mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right)\right|^2 \geq t\right\}\right) dt \\
&\leq \int_0^{\ell^2} 2\exp\left(-\frac{2t}{\ell^2\sum_{j=1}^n \bar{s}_{ij}(T(n))^2}\right) dt \\
&= \ell^2\left(\sum_{j=1}^n \left(\bar{s}_{ij}(T(n))\right)^2\right)\left[1-\exp\left(-\frac{2}{\sum_{j=1}^n\left(\bar{s}_{ij}(T(n))\right)^2}\right)\right] \qquad \forall i\in N, \forall n\in\mathbb{N}.
\end{aligned}$$

If we consider $\iota\in\mathbb{N}$ and $n\geq\iota$, this implies that $\mathrm{Var}\left(\bar{T}_\iota(n)(X_1(n),...,X_n(n))\right)\to 0$, proving (16).

For the second statement of point 1, assume that $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ is symmetric and that $\{T(n)\}_{n\in\mathbb{N}}$ is odd. It is enough to show that $\bar{T}_i(n)$ is an unbiased estimator of μ for all $i\in N$ and for all $n\in\mathbb{N}$. By Theorem 1 as well as points 3 and 4 of Lemma 3 and since $I=\mathbb{R}$ and $T(n)$ is an odd robust opinion aggregator for all $n\in\mathbb{N}$, we have that $\bar{T}(n)$ is a well-defined odd robust opinion aggregator for all $n\in\mathbb{N}$. Since $\bar{T}(n)$ is odd for all $n\in\mathbb{N}$ and $\{\varepsilon_i(n)\}_{i\in N,n\in\mathbb{N}}$ is symmetric, this implies that for each $i\in N$ and for each $n\in\mathbb{N}$

$$\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP = \int_\Omega \bar{T}_i(n)(-\varepsilon_1(n),...,-\varepsilon_n(n))\,dP = -\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP.$$

It follows that $2\int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP = 0$ for all $i\in N$ and for all $n\in\mathbb{N}$. Since $\bar{T}(n)$ is translation invariant, we can conclude that for each $i\in N$ and for each $n\in\mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left(\bar{T}_i(n)(X_1(n),...,X_n(n))\right) = \int_\Omega \bar{T}_i(n)(X_1(n),...,X_n(n))\,dP \\
&= \int_\Omega \bar{T}_i(n)(\mu+\varepsilon_1(n),...,\mu+\varepsilon_n(n))\,dP = \mu + \int_\Omega \bar{T}_i(n)(\varepsilon_1(n),...,\varepsilon_n(n))\,dP = \mu,
\end{aligned}$$

proving that $\bar{T}_i(n)$ is an unbiased estimator of μ and thus concluding the proof of point 1.

2. Fix $n\in\mathbb{N}$ and $i,j\in N$. Consider $x,y\in\hat{B}$ such that $x\geq y$. By Lebourg's Mean Value Theorem and (36), we have that there exist $\lambda\in(0,1)$ and $p\in\partial\bar{T}_i(n)(z)$ where $z=\lambda x+(1-\lambda)y\in\hat{B}$ such that $\bar{T}_i(n)(x)-\bar{T}_i(n)(y)=\sum_{l=1}^n p_l(x_l-y_l)\geq p_j(x_j-y_j)\geq \underline{s}_{ij}(T(n))(x_j-y_j)$. Since x and y were arbitrarily chosen, we have that

$$\bar{T}_i(n)(x)-\bar{T}_i(n)(y)\geq \underline{s}_{ij}(T(n))(x_j-y_j) \quad \forall x,y\in\hat{B} \text{ s.t. } x\geq y. \tag{37}$$

By definition and since $\bar{T}(n)$ is a robust opinion aggregator, we have that $\underline{s}_{ij}(T(n))\in[0,1]$. If $\underline{s}_{ij}(T(n))<1$, define $R_{ij}(n):\hat{B}\to\mathbb{R}$ by $R_{ij}(n)(x)=\left(\bar{T}_i(n)(x)-\underline{s}_{ij}(T(n))x_j\right)/\left(1-\underline{s}_{ij}(T(n))\right)$

for all $x \in \hat{B}$. By (37), it is immediate to see that $R_{ij}(n)$ is monotone and

$$\bar{T}_i(n)(x) = \underline{s}_{ij}(T(n))\, x_j + \left(1 - \underline{s}_{ij}(T(n))\right) R_{ij}(n)(x) \qquad \forall x \in \hat{B}. \tag{38}$$

If $\underline{s}_{ij}(T(n)) = 1$, then $\bar{T}_i(n)(x) = x_j$ for all $x \in \hat{B}$ and we can choose $R_{ij}(n) : \hat{B} \to \mathbb{R}$ to be any monotone functional and obtain (38). Since n, i, and j were arbitrarily chosen, it follows that (38) holds for all $i, j \in N$ and for all $n \in \mathbb{N}$.

By assumption, there exists $\iota \in \mathbb{N}$ such that $\alpha = \limsup_n \max_{j\in N} \underline{s}_{\iota j}(T(n))/2 > 0$. It follows that there exist a subsequence $\{T(n_m)\}_{m\in\mathbb{N}}$ and a sequence $\{j_m\}_{m\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\underline{s}_{\iota j_m}(T(n_m)) \geq \alpha$ and $j_m \leq n_m$ for all $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$. By (38) and Harris' inequality (see, e.g., [12, Theorem 2.15]) and since $\{X_i(n_m)\}_{i\in N}$ is a collection of independent random variables, we have that

$$\begin{aligned}
&\operatorname{Var}\left(\bar{T}_\iota(n_m)(X_1(n_m), ..., X_{n_m}(n_m))\right) \\
&= \left(1 - \underline{s}_{\iota j_m}(T(n_m))\right)^2 \operatorname{Var}\left(R_{\iota j_m}(n_m)(X_1(n_m), ..., X_{n_m}(n_m))\right) + \left(\underline{s}_{\iota j_m}(T(n_m))\right)^2 \operatorname{Var}\left(X_{j_m}(n_m)\right) \\
&+ 2\left(1 - \underline{s}_{\iota j_m}(T(n_m))\right) \underline{s}_{\iota j_m}(T(n_m)) \operatorname{Cov}\left(R_{\iota j_m}(n_m)(X_1(n_m), ..., X_{n_m}(n_m)), X_{j_m}(n_m)\right) \\
&\geq \alpha^2 \operatorname{Var}\left(X_{j_m}(n_m)\right) = \alpha^2 \operatorname{Var}\left(\varepsilon_{j_m}(n_m)\right) \geq \alpha^2 \sigma^2 > 0.
\end{aligned}$$

Since m was arbitrarily chosen, we can conclude that $\{T(n)\}_{n\in\mathbb{N}}$ does not have vanishing variance. Moreover, since $\{X_i(n)\}_{i\in N, n\in\mathbb{N}}$ is an array of uniformly bounded random variables, so is the array $\left\{\bar{T}_i(n)(X_1(n), ..., X_n(n))\right\}_{i\in N, n\in\mathbb{N}}$. This implies that $\bar{T}_\iota(n)(X_1(n), ..., X_n(n))$ cannot converge in probability to a constant (otherwise, $\{T(n)\}_{n\in\mathbb{N}}$ would have vanishing variance), proving that $\{T(n)\}_{n\in\mathbb{N}}$ is not wise. ■

C Appendix: discussion

All the missing proofs are in the Online Appendix (see Section D.3). Given the profile of loss functions $\phi = (\phi_i)_{i=1}^n$, define $\mathbf{T}^\phi : B \rightrightarrows B$ as

$$\mathbf{T}^\phi(x) = \prod_{i=1}^n \operatorname{argmin}_{c\in\mathbb{R}} \phi_i(x - ce) \qquad \forall x \in B. \tag{39}$$

The next two ancillary lemmas are instrumental in showing that $\mathbf{T}^\phi$ is well defined and behaved.

Lemma 8 *Let ϕ be a profile of loss functions. If $\phi \in \Phi_R$, then for each $i \in N$ and $\tilde{z} \in \mathbb{R}^n$*

$$\tilde{z} \gg 0 \implies \phi_i(\tilde{z}) > \phi_i\left(\tilde{z} - \min_{j\in N} \tilde{z}_j e\right),$$

and

$$0 \gg \tilde{z} \implies \phi_i(\tilde{z}) > \phi_i\left(\tilde{z} - \max_{j\in N} \tilde{z}_j e\right).$$

Lemma 9 *Let ϕ be a profile of loss functions. If $\phi \in \Phi_R$, then for each $i \in N$ and for each $x \in \mathbb{R}^n$ the function $f_{i,x} : \mathbb{R} \to \mathbb{R}_+$, defined by $f_{i,x}(c) = \phi_i(x - ce)$ for all $c \in \mathbb{R}$, is continuous and convex. Moreover, if ϕ has strictly increasing shifts, then $f_{i,x}$ is strictly convex for all $i \in N$ and for all $x \in \mathbb{R}^n$.*

To prove (i) implies (ii) of Theorem 4, we prove a more general result, namely, that the solution correspondence (39) of problem (19), always admits a selection which is a robust opinion aggregator.

Proposition 9 *Let ϕ be a profile of loss functions. If $\phi \in \Phi_R$, then the correspondence $\mathbf{T}^{\phi}$ is well defined and admits a selection T^{ϕ} which is a robust opinion aggregator. Moreover, if ϕ has strictly increasing shifts, then $\mathbf{T}^{\phi} = T^{\phi}$ is single-valued and, in particular, is a robust opinion aggregator.*

Proof. Fix $i \in N$. We begin by considering the correspondence $\mathbf{T}_i^{\phi} : B \rightrightarrows I$ defined by $\mathbf{T}_i^{\phi}(x) = \operatorname{argmin}_{c \in \mathbb{R}} \phi_i(x - ce)$ for all $x \in B$. We next show that $\mathbf{T}_i^{\phi}$ is well defined, nonempty-, convex-, and compact-valued, and such that for each $x, y \in B$

$$x \geq y \implies \mathbf{T}_i^{\phi}(x) \geq_{\text{SSO}} \mathbf{T}_i^{\phi}(y) \tag{40}$$

where $\geq_{\text{SSO}}$ is the strong set order. Fix $x \in B$. We next show that

$$\forall d \notin \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right], \exists c \in \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right] \text{ s.t. } \phi_i(x - ce) < \phi_i(x - de). \tag{41}$$

Consider d as above. We have two cases either $d < \min_{j \in N} x_j$ or $d > \max_{j \in N} x_j$. In the first case, we have that $x - de \gg 0$, in the second case, we have that $0 \gg x - de$. By Lemma 8 and since $\phi \in \Phi_R$, if we set $\tilde{c} = \min_{j \in N} x_j - d$ (resp. $\max_{j \in N} x_j - d$), we obtain that $\phi_i(x - de) > \phi_i(x - de - \tilde{c}e) = \phi_i(x - ce)$ where $c = \min_{j \in N} x_j \in [\min_{j \in N} x_j, \max_{j \in N} x_j]$ (resp. $c = \max_{j \in N} x_j \in [\min_{j \in N} x_j, \max_{j \in N} x_j]$), proving (41). By (41), we can conclude that

$$\min_{c \in \mathbb{R}} \phi_i(x - ce) = \min_{c \in I} \phi_i(x - ce) = \min_{c \in \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right]} \phi_i(x - ce) \tag{42}$$

as well as $\operatorname{argmin}_{c \in \mathbb{R}} \phi_i(x - ce) = \operatorname{argmin}_{c \in I} \phi_i(x - ce) = \operatorname{argmin}_{c \in \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right]} \phi_i(x - ce)$. By Weierstrass' Theorem and since, by Lemma 9, the map $c \mapsto \phi_i(x - ce)$ is continuous and convex, it follows that the above minimization problems admit solution and each argmin is a compact and convex set. Since x was arbitrarily chosen, this implies that $\mathbf{T}_i^{\phi}$ is well defined, nonempty-, convex-, and compact-valued and, in particular,

$$\emptyset \neq \mathbf{T}_i^{\phi}(x) \subseteq \left[\min_{j \in N} x_j, \max_{j \in N} x_j\right] \subseteq I \quad \forall x \in B. \tag{43}$$

We next prove (40). In order to do so, we rewrite explicitly (42) as a problem of parametric optimization/monotone comparative statics. Next, define $f : I \times B \to \mathbb{R}$ by $f(c, x) = -\phi_i(x - ce)$ for all $(c, x) \in I \times B$. It is immediate to see that $\mathbf{T}_i^{\phi}(x) = \operatorname{argmax}_{c \in I} f(c, x)$ for all $x \in B$. We next show that f has increasing differences in (c, x). Consider $x, y \in B$ as well as $c, d \in I$ such that $c \geq d$ and $x \geq y$. Define $z = x - ce$, $v = y - ce$, and $h = c - d$. Note that $z \geq v$ and $h \in \mathbb{R}_+$. Since $\phi \in \Phi_R$, it follows that

$$\begin{aligned} f(c, x) - f(d, x) &= \phi_i(x - de) - \phi_i(x - ce) = \phi_i(z + he) - \phi_i(z) \\ &\geq \phi_i(v + he) - \phi_i(v) = \phi_i(y - de) - \phi_i(y - ce) = f(c, y) - f(d, y). \end{aligned}$$

This shows that f satisfies the property of increasing differences in (c, x). By [54, Theorem 5], $\mathbf{T}_i^{\phi}$ satisfies (40). We finally show that $\mathbf{T}_i^{\phi}$ is such that for each $x \in B$ and for each $k \in \mathbb{R}$ such that $x + ke \in B$

$$c^\star \in \mathbf{T}_i^{\phi}(x) \iff c^\star + k \in \mathbf{T}_i^{\phi}(x + ke). \tag{44}$$

Fix $x \in B$. Consider $k \in \mathbb{R}$ such that $x + ke \in B$. Consider $c^\star \in \mathbf{T}_i^{\phi}(x)$. By definition, it follows that $\phi_i(x - c^\star e) \leq \phi_i(x - ce)$ for all $c \in \mathbb{R}$. This implies that $\phi_i(x + ke - (c^\star + k)e) = \phi_i(x - c^\star e) \leq \phi_i(x - (d - k)e) = \phi_i(x + ke - de)$ for all $d \in \mathbb{R}$. By definition of $\mathbf{T}_i^{\phi}$, this implies that $c^\star + k \in \mathbf{T}_i^{\phi}(x + ke)$. Vice versa, if $c^\star + k \in \mathbf{T}_i^{\phi}(x + ke)$, then $\phi_i(x + ke - (c^\star + k)e) \leq \phi_i(x + ke - de)$ for all $d \in \mathbb{R}$, yielding that $\phi_i(x - c^\star e) = \phi_i(x + ke - (c^\star + k)e) \leq \phi_i(x - ce)$ for all $c \in \mathbb{R}$, proving that $c^\star \in \mathbf{T}_i^{\phi}(x)$.

To sum up, since $i \in N$ was arbitrarily chosen, we proved that, for each $i \in N$, $\mathbf{T}_i^{\phi}$ is well defined, nonempty-, convex-, and compact-valued, and satisfies (40) as well as (44). Observe also that $\mathbf{T}^{\phi} : B \rightrightarrows B$ is the product correspondence $\mathbf{T}^{\phi} = \prod_{i=1}^{n} \mathbf{T}_i^{\phi}$. We are ready to show that $\mathbf{T}^{\phi}$ admits a selection T^{ϕ} which is a robust opinion aggregator. Define $T^{\phi} : B \to B$ to be such that $T_i^{\phi}(x) = \min \mathbf{T}_i^{\phi}(x)$ for all $x \in B$ and for all $i \in N$. Since $\mathbf{T}_i^{\phi}(x)$ is nonempty and compact for all $x \in B$ and for all $i \in N$, it follows that $T_i^{\phi}(x)$ is well defined and, in particular, $T_i^{\phi}(x) \in \mathbf{T}_i^{\phi}(x)$ for all $x \in B$ and for all $i \in N$, proving that T^{ϕ} is a selection of $\mathbf{T}^{\phi}$. By (43), it follows that $\mathbf{T}_i^{\phi}(ke) = \{k\}$ for all $k \in I$ and for all $i \in N$, proving that $T_i^{\phi}(ke) = k$ for all $k \in I$ and for all $i \in N$, that is, that T^{ϕ} is normalized. Next, consider $x, y \in B$ such that $x \geq y$. By (40), we have that $T_i^{\phi}(x) \geq T_i^{\phi}(y)$ for all $i \in N$, proving monotonicity of T_i^{ϕ} for all $i \in N$ and so of T^{ϕ}. Finally, consider $x \in B$ and $k \in \mathbb{R}$ such that $x + ke \in B$. By (44) and definition of $T_i^{\phi}(x)$ as well as $T_i^{\phi}(x + ke)$, we have that $T_i^{\phi}(x) \in \mathbf{T}_i^{\phi}(x)$ for all $i \in N$, yielding that $T_i^{\phi}(x) + k \in \mathbf{T}_i^{\phi}(x + ke)$ for all $i \in N$ and, in particular, $T_i^{\phi}(x) + k \geq T_i^{\phi}(x + ke)$ for all $i \in N$. This implies that $T_i^{\phi}(x + ke) = T_i^{\phi}(x) + k$ for all $i \in N$, proving translation invariance.[41]

Finally, by Lemma 9, if ϕ has strictly increasing shifts, then the map $c \mapsto \phi_i(x - ce)$ is strictly convex, yielding that each $\mathbf{T}_i^{\phi}$ is single-valued and so is $\mathbf{T}^{\phi}$. ■

Proof of Theorem 4. (i) implies (ii). By Proposition 9 and since $\phi \in \Phi_R$ and has strictly increasing shifts, the implication follows.

(ii) implies (i). Let $T : B \to B$ be a robust opinion aggregator. By point 1 of Lemma 1, there exists an extension from $\mathbb{R}^n$ to $\mathbb{R}^n$. With a small abuse of notation, we denote it by the same symbol T. Fix $i \in N$. Define $\phi_i^T : \mathbb{R}^n \to \mathbb{R}_+$ by $\phi_i^T(z) = (T_i(z))^2$ for all $z \in \mathbb{R}^n$. Next, consider $h \in \mathbb{R} \setminus \{0\}$. Since T is normalized, it follows that $\phi_i^T(he) = (T_i(he))^2 = h^2 > 0 = (T_i(0))^2 = \phi_i^T(0)$. Since i and h were arbitrarily chosen, this implies that $\phi = \left(\phi_i^T\right)_{i=1}^n$ is sensitive. Since T is translation invariant, we have that

$$\phi_i^T(z + he) = (T_i(z + he))^2 = (T_i(z) + h)^2 = (T_i(z))^2 + 2hT_i(z) + h^2 \quad \forall h \in \mathbb{R}, \forall z \in \mathbb{R}^n. \tag{45}$$

[41] Fix $i \in N$. By the previous part of the proof, for each $x \in B$ and for each $k \in \mathbb{R}$ such that $x + ke \in B$, we have that $T_i^{\phi}(x + ke) \leq T_i^{\phi}(x) + k$. Next, note that if $x \in B$ and $x + ke \in B$, then $(x + ke) - ke = x \in B$. It follows that $T_i^{\phi}(x) = T_i^{\phi}((x + ke) - ke) \leq T_i^{\phi}(x + ke) - k$, proving the opposite inequality.

Consider $z, v \in \mathbb{R}^n$ and $h \in \mathbb{R}_{++}$. By (45) and since T is monotone, we can conclude that

$$z \geq v \implies \phi_i^T (z + he) - \phi_i^T (z) = 2hT_i (z) + h^2 \geq 2hT_i (v) + h^2 = \phi_i^T (v + he) - \phi_i^T (v) .$$

Since i was arbitrarily chosen, it follows that $\phi = \left(\phi_i^T\right)_{i=1}^n$ has increasing shifts and, in particular, $\phi \in \Phi_R$. Next, consider $z, v \in \mathbb{R}^n$ such that $z \gg v$. Set $k = \min_{j \in N} (z_j - v_j)$. It follows that $k > 0$ and $z \geq v + ke$. Since T is monotone and translation invariant and $k > 0$, we can conclude that $T(z) \geq T(v + ke) = T(v) + ke \gg T(v)$. Since $z, v \in \mathbb{R}^n$ were arbitrarily chosen, it follows that $z \gg v \implies T(z) \gg T(v)$. By (45), this implies that if $z, v \in \mathbb{R}^n$ and $h \in \mathbb{R}_{++}$, then

$$z \gg v \implies \phi_i^T (z + he) - \phi_i^T (z) = 2hT_i (z) + h^2 > 2hT_i (v) + h^2 = \phi_i^T (v + he) - \phi_i^T (v) .$$

Since i was arbitrarily chosen, it follows that $\phi = \left(\phi_i^T\right)_{i=1}^n$ has strictly increasing shifts. We next prove (21). By Proposition 9 and since $\phi = \left(\phi_i^T\right)_{i=1}^n \in \Phi_R$ has strictly increasing shifts, we have that $\mathbf{T}_i^\phi (x) = \text{argmin}_{c \in \mathbb{R}} \phi_i^T (x - ce)$ is well defined and single-valued for all $x \in B$ and for all $i \in N$. Finally, fix $i \in N$ and $x \in B$. By (45), we have that $\phi_i^T (x - ce) = (T_i (x))^2 - 2cT_i (x) + c^2$ for all $c \in \mathbb{R}$, which, as a function of c, is quadratic and minimized at $c = T_i (x)$, proving the statement. ■

References

[1] D. Acemoglu and P. D. Azar, Endogenous production networks, *Econometrica*, 88, 33–82, 2020.

[2] D. Acemoglu, V. M. Carvalho, A. Ozdaglar, and A. Tahbaz-Salehi, The network origins of aggregate fluctuations, *Econometrica*, 80, 1977–2016, 2010.

[3] D. Acemoglu and A. Ozdaglar, Opinion dynamics and learning in social networks, *Dynamic Games and Applications*, 1, 3–49, 2011.

[4] D. Angeli and P. A. Bliman, Extension of a result by Moreau on stability of leaderless multi-agent systems, *Proceedings of the 44th IEEE Conference on Decision and Control*, 759–764, 2005.

[5] I. Arieli, Y. Babichenko, and S. Shlomov, Virtually additive learning, *Journal of Economic Theory*, 197, 105322, 2021.

[6] J. B. Baillon, R. E. Bruck, and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston Journal of Mathematics*, 4, 1–9, 1978.

[7] C. Ballester, A. Calvó-Armengol, and Y. Zenou, Who's who in networks. Wanted: The key player, *Econometrica*, 74, 1403–1417, 2006.

[8] A. Banerjee, E. Breza, A. G. Chandrasekhar, and M. Mobius, Naive learning with uninformed agents, *American Economic Review*, 111, 3540–3574, 2021.

[9] A. Banerjee and D. Fudenberg, Word-of-mouth learning, *Games and Economic Behavior*, 46, 1–22, 2004.

[10] L. Beaman, A. BenYishay, J. Magruder, and A. M. Mobarak, Can network theory-based targeting increase technology adoption?, *American Economic Review*, 111, 1918–1943, 2021.

[11] P. Billingsley, *Probability and Measure*, 3rd ed., John Wiley & Sons, New York, 1995.

[12] S. Boucheron, G. Lugosi, and P. Massart, *Concentration Inequalities: A Nonasymptotic Theory of Independence*, Oxford University Press, Oxford, 2013.

[13] E. Breza, A. G. Chandrasekhar, B. Golub, and A. Parvathaneni, Networks in economic development, *Oxford Review of Economic Policy*, 35, 678–721, 2019.

[14] E. Breza, A. G. Chandrasekhar, and A. Tahbaz-Salehi, Seeing the forest for the trees? An investigation of network knowledge, *National Bureau of Economic Research*, 24359, 2018.

[15] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, *Bulletin of the American Mathematical Society*, 72, 571–575, 1966.

[16] R. E. Bruck, On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set, *Israel Journal of Mathematics*, 29, 1–16, 1978.

[17] A. Calvó-Armengol, J. De Martí, and A. Prat, Communication and influence, *Theoretical Economics,* 10, 649–690, 2015.

[18] D. Centola and M. Macy, Complex contagions and the weakness of long ties, *American Journal of Sociology*, 113, 702–734, 2007.

[19] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Robust opinion aggregation and its dynamics, IGIER Working Paper, 662, 2020.

[20] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, Adaptation, coordination, and inertia in network games, mimeo, 2022.

[21] S. Cerreia-Vioglio, R. Corrao, and G. Lanzani, (Un-)Common preferences, ambiguity, and coordination, mimeo, 2023.

[22] A. G. Chandrasekhar, H. Larreguy, and J. P. Xandri, Testing models of social learning on networks: Evidence from two experiments, *Econometrica*, 88, 1–32, 2020.

[23] S. Chatterjee and E. Seneta, Towards consensus: Some convergence theorems on repeated averaging, *Journal of Applied Probability*, 14, 89–97, 1977.

[24] Y. Chen, J. Lü, and Z. Lin, Consensus of discrete-time multi-agent systems with transmission nonlinearity, *Automatica*, 49, 1768–1775, 2013.

[25] F. H. Clarke, *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, 1990.

[26] J. Cortés, Distributed algorithms for reaching consensus on general functions, *Automatica,* 44, 726–737, 2008.

[27] K. Dasaratha, B. Golub, and N. Hak, Learning from neighbors about a changing state, *The Review of Economic Studies*, forthcoming.

[28] M. H. DeGroot, Reaching a consensus, *Journal of the American Statistical Association*, 69, 118–121, 1974.

[29] P. M. DeMarzo, D. Vayanos, and J. Zwiebel, Persuasion bias, social influence, and unidimensional opinions, *Quarterly Journal of Economics*, 118, 909–968, 2003.

[30] J. Dieudonne, *Foundations of Modern Analysis*, Academic Press, New York, 1960.

[31] M. Edelstein and R. C. O'Brien, Nonexpansive mappings, asymptotic regularity and successive approximations, *Journal of the London Mathematical Society*, 17, 547–554, 1978.

[32] M. Elliott and B. Golub, A network approach to public goods, *Journal of Political Economy*, 127, 730–776, 2019.

[33] N. E. Friedkin and E. C. Johnsen, Social Influence and Opinions, *The Journal of Mathematical Sociology*, 15, 193–205, 1990.

[34] A. Galeotti, B. Golub, and S. Goyal, Targeting interventions in networks, *Econometrica*, 88, 2445–2471, 2020.

[35] A. Galeotti, S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv, Network games, *Review of Economic Studies*, 77, 218–244, 2010.

[36] S. Galperti and J. Perego, Information systems, mimeo, 2020.

[37] F. Galton, Vox populi, *Nature*, 75, 450–451, 1907.

[38] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.

[39] B. Golub and M. O. Jackson, Naïve learning in social networks and the wisdom of crowds, *American Economic Journal: Microeconomics*, 2, 112–149, 2010.

[40] B. Golub and M. O. Jackson, How homophily affects the speed of learning and best-response dynamics, *Quarterly Journal of Economics*, 127, 1287–1338, 2012.

[41] B. Golub and S. Morris, Expectations, networks, and conventions, mimeo, 2018.

[42] B. Golub and E. Sadler, Learning in social networks, in *The Oxford Handbook of the Economics of Networks* (Y. Bramoullé, A. Galeotti, and B. Rogers, eds.), Oxford University Press, New York, 2016.

[43] M. S. Granovetter, The strength of weak ties, *American Journal of Sociology*, 78, 1360–1380, 1973.

[44] P. Holme and M. E. Newman, Nonequilibrium phase transition in the coevolution of networks and opinions, *Physical Review E,* 74, 2006.

[45] P. J. Huber, Robust estimation of a location parameter, *Annals of Mathematical Statistics*, 35, 73–101, 1964.

[46] A. Jadbabaie, P. Molavi, A. Sandroni, and A. Tahbaz-Salehi, Non-Bayesian social learning, *Games and Economic Behavior*, 76, 210–225, 2012.

[47] J. G. Kemeny and J. L. Snell, *Finite Markov Chains*, 2nd ed., Springer, New York, 1976.

[48] D. Kempe, J. Kleinberg, and E. Tardos, Maximizing the spread of influence through a social network, *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, 137–146, 2003.

[49] U. Krause, *Positive Dynamical Systems in Discrete Time: Theory, Models, and Applications*, de Gruyter, Berlin, 2015.

[50] U. Krengel, *Ergodic Theorems*, De Gruyter, Berlin, 1986.

[51] G. Levy and R. Razin, Information diffusion in networks with the Bayesian peer influence heuristic, *Games and Economic Behavior*, 109, 262–270, 2018.

[52] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.

[53] M. Marinacci and L. Montrucchio, Introduction to the mathematics of ambiguity, *Uncertainty in Economic Theory*, (I. Gilboa, ed.), New York: Routledge, 2004.

[54] P. Milgrom and C. Shannon, Monotone comparative statics, *Econometrica*, 62, 157–180, 1994.

[55] P. Molavi, A. Tahbaz-Salehi, and A. Jadbabaie, A theory of non-Bayesian social learning, *Econometrica*, 86, 445–490, 2018.

[56] L. Moreau, Stability of multiagent systems with time-dependent communication links, *IEEE Transactions on automatic control,* 50, 169–182, 2005.

[57] S. Morris, Contagion, *The Review of Economic Studies*, 67, 57–78, 2000.

[58] E. Mossel, N. Olsman, and O. Tamuz, Efficient Bayesian learning in social networks with Gaussian estimators, *Proceedings of the 54th Annual Allerton Conference on Communication, Control, and Computing*, 425–432, 2016.

[59] M. Mueller-Frank, A general framework for rational learning in social networks, *Theoretical Economics*, 8, 1–40, 2013.

[60] M. Mueller-Frank, Manipulating opinions in social networks, mimeo, 2018.

[61] M. Mueller-Frank and C. Neri, A general analysis of boundedly rational learning in social networks, *Theoretical Economics*, 16, 317–357, 2021.

[62] D. Prelec, The probability weighting function, *Econometrica*, 66, 497–527, 1998.

[63] E. Sadler, Influence campaigns, *American Economic Journal: Microeconomics*, forthcoming.

[64] E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd ed., Springer, New York, 1981.

[65] D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 571–587, 1989.

[66] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, American Mathematical Society, 2008.

[67] A. Tahbaz-Salehi and A. Jadbabaie, A one-parameter family of distributed consensus algorithms with boundary: From shortest paths to mean hitting times, *Proceedings of the 45th IEEE Conference on Decision and Control*, 4664–4669, 2006.

D Online Appendix

In this section, we confine all the missing proofs. They appear in the order in which the corresponding statements appear in the text, unless they are new ancillary results.

D.1 Convergence

Proof of Lemma 1. 1. Since T is robust, we have that $T_i : B \to \mathbb{R}$ is monotone and translation invariant for all $i \in N$.[42] By [5, Theorem 4], T_i is a niveloid for all $i \in N$. By [5, Theorem 1], T_i admits an extension $S_i : \mathbb{R}^n \to \mathbb{R}$ which is a niveloid for all $i \in N$. By [5, Theorem 4], S_i is monotone and translation invariant for all $i \in N$. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ to be such that the i-th component of $S(x)$ is $S_i(x)$ for all $i \in N$ and for all $x \in \mathbb{R}^n$. It is immediate to see that S is monotone and translation invariant. Fix $k' \in I$. Since S is translation invariant and T is normalized, it follows that for each $k \in \mathbb{R}$

$$S(ke) = S\left(k'e + \left(k - k'\right)e\right) = S\left(k'e\right) + \left(k - k'\right)e = T\left(k'e\right) + \left(k - k'\right)e = k'e + \left(k - k'\right)e = ke,$$

proving that S is normalized and, in particular, that S is robust.

2. By induction, if T is normalized and monotone, then T^t is normalized and monotone for all $t \in \mathbb{N}$. Consider $x \in B$ and $t \in \mathbb{N}$. Define $k_\star = \min_{i \in N} x_i$ and $k^\star = \max_{i \in N} x_i$. Note that $\|x\|_\infty = \max\{|k_\star|, |k^\star|\}$, $k_\star, k^\star \in I$, and $k_\star e \le x \le k^\star e$. Since T^t is normalized and monotone, we have that

$$k_\star e = T^t(k_\star e) \le T^t(x) \le T^t(k^\star e) = k^\star e,$$

yielding that $\left|T^t(x)\right| \le \max\{|k_\star|, |k^\star|\} e$ and $\left\|T^t(x)\right\|_\infty \le \|x\|_\infty$. Since t and x were arbitrarily chosen, the statement follows. ■

Proof of Lemma 2. Since T is a robust opinion aggregator, T_i is normalized, monotone, and translation invariant for all $i \in N$. By [5, Theorem 4], it follows that T_i is a niveloid for all $i \in N$. By [5, p. 346], it follows that $|T_i(x) - T_i(y)| \le \|x - y\|_\infty$ for all $x, y \in B$ and for all $i \in N$. This implies that

$$\|T(x) - T(y)\|_\infty = \max_{i \in N} |T_i(x) - T_i(y)| \le \|x - y\|_\infty \qquad \forall x, y \in B,$$

proving that T is nonexpansive.

By induction, we next show that T^t is nonexpansive for all $t \in \mathbb{N}$. Since we have shown that T is nonexpansive, T^t is nonexpansive for $t = 1$, proving the initial step. By the induction hypothesis, assume that T^t is nonexpansive, we have that for each $x, y \in B$

$$\left\|T^{t+1}(x) - T^{t+1}(y)\right\|_\infty = \left\|T\left(T^t(x)\right) - T\left(T^t(y)\right)\right\|_\infty \le \left\|T^t(x) - T^t(y)\right\|_\infty \le \|x - y\|,$$

proving the inductive step. The statement follows by induction. ■

Proof of Lemma 3. Let $x \in B$. Since T is a selfmap, we have that $\left\{T^t(x)\right\}_{t \in \mathbb{N}} \subseteq B$. Since B is convex, we have that $\frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) \in B$ for all $\tau \in \mathbb{N}$. Since x was arbitrarily chosen, this implies

[42] With a small abuse of terminology, we use the same name for similar properties that pertain to functionals and operators.

that $A_\tau : B \to B$, defined by $A_\tau(x) = \sum_{t=1}^{\tau} T^t(x)/\tau$ for all $x \in B$, is well defined for all $\tau \in \mathbb{N}$. Since B is closed, we have that $\bar{T}(x) = \lim_\tau A_\tau(x) = \lim_\tau \frac{1}{\tau}\sum_{t=1}^{\tau} T^t(x) \in B$ for all $x \in B$, proving that $\bar{T}$ is well defined. By the same computations contained in [1, Lemma 20.12], despite T being nonlinear, one has

$$A_\tau(T(x)) = \frac{\tau+1}{\tau}A_{\tau+1}(x) - \frac{1}{\tau}T(x) \qquad \forall x \in B, \forall \tau \in \mathbb{N}.$$

This implies that

$$\bar{T}(T(x)) = \lim_\tau A_\tau(T(x)) = \lim_\tau \frac{\tau+1}{\tau}\lim_\tau A_{\tau+1}(x) - \lim_\tau \frac{1}{\tau}T(x) = \bar{T}(x) \qquad \forall x \in B,$$

proving that $\bar{T} \circ T = \bar{T}$.

1. By the same inductive argument contained in the proof of Lemma 2, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is nonexpansive. Since the convex linear combination of nonexpansive maps is nonexpansive, the map $A_\tau : B \to B$ is nonexpansive for all $\tau \in \mathbb{N}$. We can conclude that for each $x, y \in B$

$$\left\|\bar{T}(x) - \bar{T}(y)\right\|_\infty = \left\|\lim_\tau A_\tau(x) - \lim_\tau A_\tau(y)\right\|_\infty = \lim_\tau \|A_\tau(x) - A_\tau(y)\|_\infty \le \|x-y\|_\infty,$$

proving that $\bar{T}$ is nonexpansive. Continuity of $\bar{T}$ trivially follows.

2. By induction, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is normalized and monotone. Since the convex linear combination of normalized and monotone operators is normalized and monotone, the map $A_\tau : B \to B$ is normalized and monotone for all $\tau \in \mathbb{N}$. We can conclude that $\bar{T}(ke) = \lim_\tau A_\tau(ke) = ke$ for all $k \in I$ as well as

$$x \ge y \implies \bar{T}(x) = \lim_\tau A_\tau(x) \ge \lim_\tau A_\tau(y) = \bar{T}(y),$$

proving that $\bar{T}$ is normalized and monotone.

3. Since T is robust, T is normalized, monotone, and translation invariant. By the previous point, $\bar{T}$ is normalized and monotone. By induction, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is translation invariant. Since the convex linear combination of translation invariant operators is translation invariant, the map $A_\tau : B \to B$ is translation invariant for all $\tau \in \mathbb{N}$. We can conclude that for each $x \in B$ and for each $k \in \mathbb{R}$ such that $x + ke \in B$

$$\bar{T}(x+ke) = \lim_\tau A_\tau(x+ke) = \lim_\tau [A_\tau(x) + ke] = \bar{T}(x) + ke,$$

proving that $\bar{T}$ is translation invariant and, in particular, robust.

4. By induction, we have that for each $t \in \mathbb{N}$ the map $T^t : B \to B$ is odd. Since the convex linear combination of odd maps is odd, the map $A_\tau : B \to B$ is odd for all $\tau \in \mathbb{N}$. We can conclude that

$$\bar{T}(-x) = \lim_\tau A_\tau(-x) = \lim_\tau [-A_\tau(x)] = -\bar{T}(x) \quad \forall x \in B,$$

proving that $\bar{T}$ is odd. ■

In order to prove Lemma 4, we are going to rely upon Lorentz's Theorem.

Theorem 5 (Lorentz) *Let $\{x^t\}_{t\in\mathbb{N}} \subseteq \mathbb{R}^n$ be a bounded sequence. The following statements are equivalent:*

(i) There exists $\bar{x} \in \mathbb{R}^n$ *such that*

$$\forall \varepsilon > 0 \; \exists \bar{\tau} \in \mathbb{N} \; \forall m \in \mathbb{N} \textit{ s.t. } \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} < \varepsilon \qquad \forall \tau \geq \bar{\tau}$$

and $\lim_t \left\| x^{t+1} - x^t \right\|_\infty = 0$*;*

(ii) $\lim_t x^t = \bar{x}$*.*

Proof of Lemma 4. By Theorem 1 and since T is robust, we have that if $\hat{B}$ is a bounded subset of B, then

$$\lim_{\tau} \left(\sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty} \right) = 0 \tag{46}$$

where $\bar{T} : B \to B$ is a robust opinion aggregator such that $\bar{T} \circ T = \bar{T}$. Since $\bar{T}(T(x)) = \bar{T}(x)$ for all $x \in B$, by induction, we have that $\bar{T}(T^m(x)) = \bar{T}(x)$ for all $m \in \mathbb{N}$ and for all $x \in B$.

(i) implies (ii). Fix $x \in B$. Define the sequence $x^t = T^t(x)$ for all $t \in \mathbb{N}$. By point 2 of Lemma 1, we have that $\{x^t\}_{t \in \mathbb{N}}$ is bounded. Set $\hat{B} = \{x^t\}_{t \in \mathbb{N}}$. Note that for each $\tau \in \mathbb{N}$ and for each $m \in \mathbb{N}$

$$\frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} = \frac{1}{\tau} \sum_{t=1}^{\tau} T^{m+t}(x) = \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x)).$$

Since (46) holds, if we define $\bar{x} = \bar{T}(x)$, then we have that for each $m \in \mathbb{N}$

$$\lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} = \lim_{\tau} \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x)) = \bar{T}(T^m(x)) = \bar{T}(x) = \bar{x}.$$

It follows that

$$\sup_{m \in \mathbb{N}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} x^{m+t} - \bar{x} \right\|_{\infty} = \sup_{m \in \mathbb{N}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(T^m(x)) - \bar{T}(T^m(x)) \right\|_{\infty} \leq \sup_{x \in \hat{B}} \left\| \frac{1}{\tau} \sum_{t=1}^{\tau} T^t(x) - \bar{T}(x) \right\|_{\infty}.$$

Since (46) holds and T is asymptotically regular, we have that $\{x^t\}_{t \in \mathbb{N}}$ satisfies point (i) of Theorem 5. By Theorem 5, we have that $\lim_t T^t(x) = \lim_t x^t$ exists. Since x was arbitrarily chosen, the implication follows.

(ii) implies (i). Fix $x \in B$. Define $x^t = T^t(x)$ for all $t \in \mathbb{N}$. Since T is convergent, we have that $\{x^t\}_{t \in \mathbb{N}}$ converges and, in particular, is bounded. By Theorem 5, we have that $\lim_t \left\| T^{t+1}(x) - T^t(x) \right\|_\infty = \lim_t \left\| x^{t+1} - x^t \right\|_\infty = 0$. Since x was arbitrarily chosen, the implication follows. ■

Proof of Lemma 5. We first offer two definitions and make two observations. Define the diameter of $\{T^t(x) : x \in C \text{ and } t \in \mathbb{N}_0\}$ by $\bar{D}$.[43] Given $x \in B$, define $x^t = T^t(x)$ as well as $y^t = S(x^t)$ for all $t \in \mathbb{N}_0$. Since T is nonexpansive, recall that $\left\{ \left\| x^t - x^{t-1} \right\|_\infty \right\}_{t \in \mathbb{N}}$ is a decreasing sequence for all $x \in B$. Note that this implies that $\|T(x) - x\|_\infty \geq \left\| T^{t+1}(x) - T^t(x) \right\|_\infty$ for all $t \in \mathbb{N}_0$ and for all $x \in B$, yielding that $k > \delta$.

By contradiction, assume that $\{T^t(x) : x \in C \text{ and } t \in \mathbb{N}_0\}$ is bounded. This implies that $\bar{D} < \infty$. Consider $M \in \mathbb{N} \setminus \{1\}$ and $P \in \mathbb{N}$ to be such that $M\delta > \bar{D} + \delta + 1$ and $\left\lfloor \frac{P}{M} \right\rfloor > \max\left\{1, \frac{k}{(1-\varepsilon)\varepsilon^M}\right\}$.

[43] Recall that the diameter of a subset $\hat{A}$ of B is the quantity $\sup\left\{ \|x - y\|_\infty : x, y \in \hat{A} \right\}$.

By (28) and since $P \in \mathbb{N}$, there exists $x \in C$ such that $\left\|x^{P+1} - x^P\right\|_\infty = \left\|T^{P+1}(x) - T^P(x)\right\|_\infty \geq \delta$. Now, we list seven useful facts:

1. By (27) and since $\left\{\left\|x^t - x^{t-1}\right\|_\infty\right\}_{t\in\mathbb{N}}$ is a decreasing sequence, it follows that $k \geq \left\|x^{i+1} - x^i\right\|_\infty \geq \delta$ for all $i \in \{1, ..., P\}$.

2. By definition of $\left\{y^t\right\}_{t\in\mathbb{N}_0}$ and since S is nonexpansive, we have that $\left\|y^t - y^{t-1}\right\|_\infty \leq \left\|x^t - x^{t-1}\right\|_\infty$ for all $t \in \mathbb{N}$.

3. By definition of $\left\{x^t\right\}_{t\in\mathbb{N}_0}$ and since $T = \varepsilon J + (1-\varepsilon) S$, we have that $x^t = T\left(x^{t-1}\right) = \varepsilon J\left(x^{t-1}\right) + (1-\varepsilon) y^{t-1}$ for all $t \in \mathbb{N}$, that is,
$$y^{t-1} = \frac{1}{1-\varepsilon}x^t - \frac{\varepsilon}{1-\varepsilon}J\left(x^{t-1}\right) \qquad \forall t \in \mathbb{N}.$$
By point 2, this yields that $\left\|\frac{1}{1-\varepsilon}\left(x^{t+1} - x^t\right) - \frac{\varepsilon}{1-\varepsilon}\left(J\left(x^t\right) - J\left(x^{t-1}\right)\right)\right\|_\infty = \left\|y^t - y^{t-1}\right\|_\infty \leq \left\|x^t - x^{t-1}\right\|_\infty$ for all $t \in \mathbb{N}$.

4. Let L be an integer in $\mathbb{N}$ such that
$$L > \frac{k}{(1-\varepsilon)\varepsilon^M}. \tag{47}$$
Define $b_m = \delta + m(1-\varepsilon)\varepsilon^M$ for all $m \in \{0, ..., L\}$. It follows that the collection of intervals $\{[b_m, b_{m+1}]\}_{m=0}^{L-1}$ contains L elements whose union is a superset of $[\delta, k]$.

5. Note that $\varepsilon^{M-1}\frac{1-\varepsilon^i}{\varepsilon^i} = \varepsilon^{M-i-1} - \varepsilon^{M-1} \leq \varepsilon^{M-i-1}$ for all $i \in \{1, ..., M-1\}$. Since $\varepsilon \in (0,1)$, this implies that
$$(1-\varepsilon)\varepsilon^M \sum_{i=1}^{M-1} \frac{1-\varepsilon^i}{\varepsilon^i} \leq (1-\varepsilon)\varepsilon \sum_{i=1}^{M-1} \varepsilon^{M-i-1} = (1-\varepsilon)\varepsilon \sum_{i=0}^{M-2} \varepsilon^i \leq (1-\varepsilon)\varepsilon\frac{1}{1-\varepsilon} \leq \varepsilon < 1.$$

6. Let $t \in \mathbb{N}$, $j \in N$, and $b, \kappa, c \geq 0$. If $x_j^{t+1} - x_j^t \geq b - c$ and $\left\|x^t - x^{t-1}\right\|_\infty \leq b + \kappa$, then (by point 3): $\frac{b-c}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\left(x_{k_l}^t - x_{k_l}^{t-1}\right) = \frac{b-c}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\left(J_j\left(x^t\right) - J_j\left(x^{t-1}\right)\right) \leq b + \kappa$ where l is such that $j \in \hat{N}_l$. This yields that
$$x_{k_l}^t - x_{k_l}^{t-1} \geq b - \frac{c}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon}\kappa. \tag{48}$$

7. Let $t \in \mathbb{N}$, $j \in N$, and $b, \kappa, c \geq 0$. If $x_j^t - x_j^{t+1} \geq b - c$ and $\left\|x^t - x^{t-1}\right\|_\infty \leq b + \kappa$, then (by point 3): $\frac{b-c}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\left(x_{k_l}^{t-1} - x_{k_l}^t\right) = \frac{b-c}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon}\left(J_j\left(x^{t-1}\right) - J_j\left(x^t\right)\right) \leq b + \kappa$ where l is such that $j \in \hat{N}_l$. This yields that
$$x_{k_l}^{t-1} - x_{k_l}^t \geq b - \frac{c}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon}\kappa. \tag{49}$$

By definition of P, we have that $\lfloor P/M \rfloor$ satisfies (47). By point 4, there exists a collection of intervals $\{[b_m, b_{m+1}]\}_{m=0}^{\lfloor P/M \rfloor - 1}$ which covers $[\delta, k]$. By point 1, $[\delta, k]$ contains $\left\{\left\|x^{i+1} - x^i\right\|_\infty\right\}_{i=1}^P$. Since we have $\lfloor P/M \rfloor$ intervals and the first P elements (of the sequence $\left\{\left\|x^{t+1} - x^t\right\|_\infty\right\}_{t\in\mathbb{N}}$) belong to these intervals, we have that there exists one of them, $\hat{I} = [b_{\bar{m}}, b_{\bar{m}+1}]$, which contains at least M elements of $\left\{\left\|x^{i+1} - x^i\right\|_\infty\right\}_{i=1}^P$. Since $\left\{\left\|x^t - x^{t-1}\right\|_\infty\right\}_{t\in\mathbb{N}}$ is decreasing, we have that there exists $K \in \mathbb{N}_0$ such that $\left\|x^{K+i+1} - x^{K+i}\right\|_\infty \in \hat{I}$ for all $i \in \{1, ..., M\}$. This implies that there exists $j \in \{1, ..., n\}$ such that $\left|x_j^{K+M+1} - x_j^{K+M}\right| \geq b_{\bar{m}}$ and $||x^{K+M} - x^{K+M-1}||_\infty \leq b_{\bar{m}+1} = b_{\bar{m}} + (1-\varepsilon)\,\varepsilon^M$. We have two cases:

a. $x_j^{K+M+1} - x_j^{K+M} \geq b_{\bar{m}}$. Set $b = b_{\bar{m}}$, $c = 0$, and $\kappa = (1-\varepsilon)\,\varepsilon^M$. By (48), we can conclude that

$$x_{k_l}^{K+M} - x_{k_l}^{K+M-1} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon)}{\varepsilon}. \tag{50}$$

By (finite) induction, we next prove that

$$x_{k_l}^{K+M+1-i} - x_{k_l}^{K+M-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{\left(1-\varepsilon^i\right)}{\varepsilon^i} \qquad \forall i \in \{1, ..., M-1\}. \tag{51}$$

By (50), the statement is true for $i = 1$. Next, we assume it is true for $i \in \{1, ..., M-1\}$ and prove it is still true for $i+1$ when $i+1 \in \{1, ..., M-1\}$. This implies that $i \leq M-2$. Define $t = K + M - i$. By the induction hypothesis, we have that

$$x_{k_l}^{t+1} - x_{k_l}^{t} = x_{k_l}^{K+M+1-i} - x_{k_l}^{K+M-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{\left(1-\varepsilon^i\right)}{\varepsilon^i}.$$

Moreover, we also have that $\left\|x^t - x^{t-1}\right\|_\infty = \left\|x^{K+M-i} - x^{K+M-i-1}\right\|_\infty \leq b_{\bar{m}} + (1-\varepsilon)\,\varepsilon^M$. Set $b = b_{\bar{m}}$, $c = (1-\varepsilon)\,\varepsilon^M \frac{(1-\varepsilon^i)}{\varepsilon^i}$, and $\kappa = (1-\varepsilon)\,\varepsilon^M$. By (48), we can conclude that

$$\begin{aligned} x_{k_l}^{K+M+1-(i+1)} - x_{k_l}^{K+M-(i+1)} &= x_{k_l}^{K+M-i} - x_{k_l}^{K+M-i-1} = x_{k_l}^{t} - x_{k_l}^{t-1} \\ &\geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{\left(1-\varepsilon^i\right)}{\varepsilon^i}\frac{1}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon}(1-\varepsilon)\,\varepsilon^M \\ &= b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{\left(1-\varepsilon^{i+1}\right)}{\varepsilon^{i+1}}, \end{aligned}$$

proving (51). By (51) and summation as well as point 5, this implies that

$$x_{k_l}^{K+M} - x_{k_l}^{K+1} \geq (M-1)\,b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \sum_{i=1}^{M-1} \frac{1-\varepsilon^i}{\varepsilon^i} \geq (M-1)\,b_{\bar{m}} - 1,$$

that is, $\left\|x^{K+M} - x^{K+1}\right\|_\infty \geq x_{k_l}^{K+M} - x_{k_l}^{K+1} \geq (M-1)\,b_{\bar{m}} - 1$. Since $b_{\bar{m}} \geq \delta > 0$, we have that $(M-1)\,b_{\bar{m}} \geq (M-1)\,\delta > \bar{D} + 1$. We can conclude that $\bar{D} \geq \left\|x^{K+M} - x^{K+1}\right\|_\infty \geq (M-1)\,b_{\bar{m}} - 1 > \bar{D}$, a contradiction.

b. $x_j^{K+M} - x_j^{K+M+1} \geq b_{\bar{m}}$. Set $b = b_{\bar{m}}$, $c = 0$, and $\kappa = (1-\varepsilon)\,\varepsilon^M$. By (49), we can conclude that

$$x_{k_l}^{K+M-1} - x_{k_l}^{K+M} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{1-\varepsilon}{\varepsilon}. \tag{52}$$

By (finite) induction, we next prove that

$$x_{k_l}^{K+M-i} - x_{k_l}^{K+M+1-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^M \frac{\left(1-\varepsilon^i\right)}{\varepsilon^i} \qquad \forall i \in \{1, ..., M-1\}. \tag{53}$$

By (52), the statement is true for $i = 1$. Next, we assume it is true for $i \in \{1, ..., M-1\}$ and prove it is still true for $i+1$ when $i+1 \in \{1, ..., M-1\}$. This implies that $i \leq M-2$. Define

$t = K + M - i$. By the induction hypothesis, we have that

$$x_{k_l}^{t} - x_{k_l}^{t+1} = x_{k_l}^{K+M-i} - x_{k_l}^{K+M+1-i} \geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^{M}\frac{\left(1-\varepsilon^{i}\right)}{\varepsilon^{i}}.$$

Moreover, we also have that $\left\|x^{t} - x^{t-1}\right\|_{\infty} = \left\|x^{K+M-i} - x^{K+M-i-1}\right\|_{\infty} \leq b_{\bar{m}} + (1-\varepsilon)\,\varepsilon^{M}$. Set $b = b_{\bar{m}}$, $c = (1-\varepsilon)\,\varepsilon^{M}\frac{(1-\varepsilon^{i})}{\varepsilon^{i}}$, and $\kappa = (1-\varepsilon)\,\varepsilon^{M}$. By (49), we can conclude that

$$\begin{aligned} x_{k_l}^{K+M-(i+1)} - x_{k_l}^{K+M+1-(i+1)} &= x_{k_l}^{K+M-i-1} - x_{k_l}^{K+M-i} = x_{k_l}^{t-1} - x_{k_l}^{t} \\ &\geq b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^{M}\frac{\left(1-\varepsilon^{i}\right)}{\varepsilon^{i}}\frac{1}{\varepsilon} - \frac{1-\varepsilon}{\varepsilon}(1-\varepsilon)\,\varepsilon^{M} \\ &= b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^{M}\frac{\left(1-\varepsilon^{i+1}\right)}{\varepsilon^{i+1}}, \end{aligned}$$

proving (53). By (53) and summation as well as point 5, this implies that

$$x_{k_l}^{K+1} - x_{k_l}^{K+M} \geq (M-1)\,b_{\bar{m}} - (1-\varepsilon)\,\varepsilon^{M}\sum_{i=1}^{M-1}\frac{1-\varepsilon^{i}}{\varepsilon^{i}} \geq (M-1)\,b_{\bar{m}} - 1,$$

that is, $\left\|x^{K+1} - x^{K+M}\right\|_{\infty} \geq x_{k_l}^{K+1} - x_{k_l}^{K+M} \geq (M-1)\,b_{\bar{m}} - 1$. Since $b_{\bar{m}} \geq \delta > 0$, we have that $(M-1)\,b_{\bar{m}} \geq (M-1)\,\delta > \bar{D} + 1$. We can conclude that $\bar{D} \geq \left\|x^{K+1} - x^{K+M}\right\|_{\infty} \geq (M-1)\,b_{\bar{m}} - 1 > \bar{D}$, a contradiction.

Points a and b prove the statement. ■

Proof of Lemma 7. Consider generic $x, y \in B$ and $l \in N$. Define $y^{0} = y$. For each $t \in \{1, ..., n-1\}$ define $y^{t} \in B$ to be such that $y_i^{t} = x_i$ for all $i \leq t$ and $y_i^{t} = y_i$ for all $i \geq t+1$. Define $y^{n} = x$. Note that $y^{j} - y^{j-1} = (x_j - y_j)\,e^{j}$ for all $j \in \{1, ..., n\}$. We also have that

$$T_l(x) - T_l(y) = T_l(y^{n}) - T_l\left(y^{0}\right) = \sum_{j=1}^{n}\left[T_l\left(y^{j}\right) - T_l\left(y^{j-1}\right)\right]. \tag{54}$$

Since I has nonempty interior, we have that there exist $a, b \in I$ such that $a > b$. By contradiction, assume that $\bar{A}(T)$ is not nontrivial, that is, there exists $i \in N$ such that $\bar{a}_{ij} = 0$ for all $j \in N$, yielding that $T_i\left(z + he^{j}\right) = T_i(z)$ for all $h \in \mathbb{R}$ and for all $z \in B$ such that $z + he^{j} \in B$. Set $x = ae$ and $y = be$. By (54) and since T is normalized, it follows that $0 < a - b = T_i(ae) - T_i(be) = 0$, a contradiction, proving the first part of the statement. Next, consider $\bar{\imath} \in N$ and define $\bar{N}_{\bar{\imath}} = \{j \in N : \bar{a}_{\bar{\imath}j} = 1\}$. By assumption, we have that $\bar{N}_{\bar{\imath}} \subseteq C_{[r_{\bar{\imath}}]}$. Let x be as in (32) and $y = x^{[r_{\bar{\imath}}]}$. By definition of $\bar{A}(T)$, it is immediate to see that $\bar{a}_{\bar{\imath}j} = 0$ only if $T_{\bar{\imath}}\left(z + he^{j}\right) = T_{\bar{\imath}}(z)$ for all $h \in \mathbb{R}$ and for all $z \in B$ such that $z + he^{j} \in B$. Consider $j \in \{1, ..., n\}$. We have two cases: either $j \in \bar{N}_{\bar{\imath}}$ or $j \notin \bar{N}_{\bar{\imath}}$. In the first case, since $\bar{N}_{\bar{\imath}} \subseteq C_{[r_{\bar{\imath}}]}$, we have that $y^{j} - y^{j-1} = \left(x_j^{[r_{\bar{\imath}}]} - x_j^{[r_{\bar{\imath}}]}\right)e^{j} = 0$ and $T_{\bar{\imath}}\left(y^{j}\right) - T_{\bar{\imath}}\left(y^{j-1}\right) = 0$. In the second case, since $\bar{a}_{\bar{\imath}j} = 0$, we have that $T_{\bar{\imath}}\left(y^{j}\right) = T_{\bar{\imath}}\left(y^{j-1} + \left(x_j - x_j^{[r_{\bar{\imath}}]}\right)e^{j}\right) = T_{\bar{\imath}}\left(y^{j-1}\right)$, yielding that $T_{\bar{\imath}}\left(y^{j}\right) - T_{\bar{\imath}}\left(y^{j-1}\right) = 0$. By (54), it follows that $T_{\bar{\imath}}(x) - T_{\bar{\imath}}\left(x^{[r_{\bar{\imath}}]}\right) = 0$. ■

Proof of Proposition 8. By Proposition 5, since $\underline{A}(T)$ is nontrivial, there exist $W \in \mathcal{W}$ and $\varepsilon \in (0,1)$ such that

$$T(x) = \varepsilon Wx + (1-\varepsilon)\,S(x) \qquad \forall x \in B \tag{55}$$

where $S : B \to B$ is a robust opinion aggregator. Moreover, W can be chosen to be such that $A(W) = \underline{A}(T)$. By induction and (55), we have that if $t \in \mathbb{N}$, then there exist $\gamma \in (0,1)$ and a robust opinion aggregator $\tilde{S} : B \to B$ (which both depend on t) such that

$$T^t(x) = \gamma W^t x + (1-\gamma)\tilde{S}(x) \qquad \forall x \in B. \tag{56}$$

As usual, we denote the ij-th entry of W^t by $w_{ij}^{(t)}$. Since T is normalized, observe that $E(T) \supseteq D$. By induction, if $t \in \mathbb{N}$, then $D \subseteq E(T) \subseteq E(T^t)$. Since $A(W) = \underline{A}(T)$, it follows that $A(W)$ has a unique strongly connected and closed group M, and M is aperiodic under $A(W)$. By [6, Corollaries 8.1 and 8.2], W is such that there exist $\bar{t} \in \mathbb{N}$ and $k \in N$ such that $w_{ik}^{(\bar{t})} > 0$ for all $i \in N$. Let $\tilde{S}$ denote the robust opinion aggregator for $\bar{t}$ in equation (56). We next show that $E\left(T^{\bar{t}}\right) = D$. By contradiction, assume that there exists $x \in B \backslash D$ such that $T^{\bar{t}}(x) = x$. Define $x_i = \min_{l \in N} x_l$ and $x_j = \max_{l \in N} x_l$. It follows that $x_j > x_i$ and $i \neq j$. We have two cases:

1. $x_k < x_j$. It follows that

$$0 = \left\|T^{\bar{t}}(x) - x\right\|_\infty \geq \left|T_j^{\bar{t}}(x) - x_j\right| = \left|\gamma \sum_{l=1}^n w_{jl}^{(\bar{t})} x_l + (1-\gamma)\tilde{S}_j(x) - x_j\right|$$
$$= \gamma \sum_{l=1}^n w_{jl}^{(\bar{t})}(x_j - x_l) + (1-\gamma)\left(x_j - \tilde{S}_j(x)\right) \geq \gamma w_{jk}^{(\bar{t})}(x_j - x_k) > 0,$$

a contradiction.

2. $x_k > x_i$. It follows that

$$0 = \left\|T^{\bar{t}}(x) - x\right\|_\infty \geq \left|T_i^{\bar{t}}(x) - x_i\right| = \left|\gamma \sum_{l=1}^n w_{il}^{(\bar{t})} x_l + (1-\gamma)\tilde{S}_i(x) - x_i\right|$$
$$= \gamma \sum_{l=1}^n w_{il}^{(\bar{t})}(x_l - x_i) + (1-\gamma)\left(\tilde{S}_i(x) - x_i\right) \geq \gamma w_{ik}^{(\bar{t})}(x_k - x_i) > 0,$$

a contradiction.

Cases 1 and 2 prove that $E\left(T^{\bar{t}}\right) = D$, and hence that $E(T) = D$. ■

Proof of Proposition 2. We omit the proof of point 2 which follows from well-known facts.[44]

1. Consider $\theta \in \mathbb{R}\backslash\{0\}$ and $\rho : \mathbb{R} \to \mathbb{R}_+$ defined by $\rho(\tilde{s}) = e^{\theta\tilde{s}} - \theta\tilde{s}$ for all $\tilde{s} \in \mathbb{R}$. It is easy to see that ρ is strictly convex and differentiable. Given $x \in B$ and $i \in N$, consider also the function $c \mapsto \phi_i^\theta(x - ce) = \sum_{j=1}^n w_{ij}\rho(x_j - c)$. Since ρ is strictly convex and differentiable, so is $c \mapsto \phi_i^\theta(x - ce)$. Given $x \in B$ and $i \in N$, this implies that the minimizer of the function $c \mapsto \phi_i^\theta(x - ce)$ is then uniquely pinned down by the first order conditions. Moreover, as we will immediately see, minimizing $c \mapsto \phi_i^\theta(x - ce)$ over I is equivalent to minimize it over $\mathbb{R}$. We compute the first order conditions where $c^\star$ is the optimal value:

$$-\sum_{j=1}^n w_{ij}\left[\theta \exp(\theta(x_j - c^\star)) - \theta\right] = 0 \implies \sum_{j=1}^n w_{ij}\exp(\theta x_j) = \exp(\theta c^\star) \implies c^\star = \frac{1}{\theta}\ln\left(\sum_{j=1}^n w_{ij}\exp(\theta x_j)\right) \in I.$$

[44] The result for $\hat{\theta} = \infty$ is also known as Laplace's method (see, e.g., [4, Theorem 4.1]). The case for $\hat{\theta} = -\infty$ is instead obtained from the previous one and by observing that $\theta x_j = -\theta(-x_j)$ and that $\theta \to -\infty$ yields $-\theta \to \infty$. The case of $\hat{\theta} = 0$ is a standard result in risk theory.

Since i and x were arbitrarily chosen, equation (14) is satisfied. It is routine to show that T^{θ} is a robust opinion aggregator. As for the second part, fix $i, j \in N$. Observe that T_i^{θ} is continuously differentiable in the interior of B. Moreover, $\frac{\partial T_i^{\theta}}{\partial x_j}(x) > 0$ for some $x \in \operatorname{int} B$ if and only if there exists $\varepsilon \in (0,1)$ such that $\frac{\partial T_i^{\theta}}{\partial x_j}(x) \geq \varepsilon$ for all $x \in \operatorname{int} B$ if and only if $w_{ij} > 0$. By the Mean Value Theorem and since i and j were arbitrarily chosen, this implies that $\underline{A}\left(T^{\theta}\right) = \bar{A}\left(T^{\theta}\right) = A(W)$.

3. Let $S : \mathbb{R}^n \to \mathbb{R}^n_{++}$ be defined by $S_i(x) = \exp(\theta x_i)$ for all $i \in N$ and for all $x \in \mathbb{R}^n$. Define $\hat{T} : \mathbb{R}^n \to \mathbb{R}^n$ by $\hat{T}(x) = Wx$ for all $x \in \mathbb{R}^n$. We next show that

$$\left(T^{\theta}\right)^t = S^{-1}\hat{T}^t S \qquad \forall t \in \mathbb{N}. \tag{57}$$

By definition of T^{θ}, if $t = 1$, then $T^{\theta}(x) = S^{-1}(WS(x))$ for all $x \in B$, yielding (57). Next, assume that (57) holds for t. We have that $\left(T^{\theta}\right)^{t+1} = T^{\theta}\left(T^{\theta}\right)^t = S^{-1}\hat{T}SS^{-1}\hat{T}^tS = S^{-1}\hat{T}^{t+1}S$, proving that (57) holds for $t+1$. By induction, (57) follows. Consider $x \in B$. By (15), it follows that $\lim_t \hat{T}^t(S(x)) = \lim_t W^t S(x) = \left(\sum_{i=1}^n s_i \exp(\theta x_i)\right) e \in \mathbb{R}^n_{++}$. By (57) and since S^{-1} is continuous, we have that $\lim_t \left(T^{\theta}\right)^t(x) = \left(\frac{1}{\theta}\ln\left(\sum_{i=1}^n s_i \exp(\theta x_i)\right)\right) e = \bar{T}^{\theta}(x)$. Since x was arbitrarily chosen, the statement follows. ■

D.2 Vox populi, vox Dei?

In order to prove point 2 of Proposition 3, we need an ancillary fact whose intuition is simple. If $\{T(n)\}_{n\in\mathbb{N}}$ is κ-dominated, this means that the Jacobian of each $T(n)$, whenever defined, is dominated by a stochastic matrix $W(n)$. As argued in the main text, $W(n)$ summarizes the connections in the weak network while $\kappa \geq 1$ partially captures the nonlinearities of $T(n)$: in fact, the less the Jacobians change, the smaller κ can be chosen with $\kappa = 1$ if and only if each $T(n)$ is linear with matrix $W(n)$. In order to prove that $\{T(n)\}_{n\in\mathbb{N}}$ is wise, we necessarily need to show that the maximal weak influence vanishes for each agent, that is, $\lim_n \sum_{j=1}^n (\bar{s}_{\iota j}(T(n)))^2 = 0$ for all $\iota \in \mathbb{N}$. But, given $n \in \mathbb{N}$, the weak influence vectors $\bar{s}_i(T(n))$ are computed via the Jacobian of $\bar{T}(n)$, not the one of $T(n)$. So we need to control the former, via the latter. If we assume for this discussion that $T(n)$ is convergent, $\bar{T}(n)$ is the limit of the iterations of $T(n)$. Thus, by a suitable version of the chain rule, at each round t of iteration, we have two contrasting forces: the opinions in the network $W(n)$ converge to a value given by its eigenvector centrality $\bar{w}(n)$, exactly like in the linear model, while the nonlinearities of $T(n)$ compound exponentially fast (making wisdom difficult to attain). The next lemma, in (60), allows us to separate these two forces isolating also the role of the connectivity of the network $W(n)$. Intuitively, if connectivity is high enough, then the convergence in $W(n)$ is so fast that it cannot be offset by the nonlinearities of $T(n)$ and if each agent has little influence, that is, each component of $\bar{w}(n)$ goes to zero fast, then the maximal weak influence vanishes too.

To ease notation, we discuss the next ancillary result by dropping the n indexing. Let $\mathcal{W}_{\text{un}}$ denote the subset of $\mathcal{W}$ such that $W \in \mathcal{W}_{\text{un}}$ if and only if there exists an undirected and strongly connected graph with an $n \times n$ adjacency matrix A such that $w_{ij} = \frac{a_{ij}}{d_i}$ for all $i, j \in N$ where $d_i = \sum_{l=1}^n a_{il}$. It is well known that if $W \in \mathcal{W}_{\text{un}}$, then W is reversible and there exists a unique left Perron-Frobenius eigenvector $\bar{w} \in \Delta$, that is $\bar{w}^{\mathrm{T}} W = \bar{w}^{\mathrm{T}}$, and

$$\bar{w}_i = \frac{d_i}{\sum_{j=1}^n d_j} \qquad \forall i \in N.$$

In particular, note that

$$0 \leq \bar{w}_k \leq \frac{1}{n}\frac{\max_{i\in N} d_i}{\min_{i\in N} d_i} \qquad \forall k \in N. \tag{58}$$

Finally, recall that if $W \in \mathcal{W}_{\text{un}}$ and $n \geq 2$, then the eigenvalues of W are real and, accounting for multiplicity, such that $1 = \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq ... \geq \tilde{\lambda}_n \geq -1$. We denote by λ_2 ($= \max_{i=2,\ldots,n} \left|\tilde{\lambda}_i\right|$) the second largest eigenvalue in modulus (SLEM).

Lemma 10 *Let T be a robust opinion aggregator and $n \geq 2$. If there exist $\kappa \geq 1$ and $W \in \mathcal{W}_{\text{un}}$ such that*

$$\frac{\partial T_i}{\partial x_j}(x) \leq \kappa w_{ij} \qquad \forall x \in \mathcal{D}(T), \forall i, j \in N, \tag{59}$$

then

$$\bar{s}_{ij}(T) \leq \kappa^t \bar{w}_j + \sqrt{\frac{\max_{i \in N} d_i}{\min_{i \in N} d_i}} \kappa^t \lambda_2^t \quad \forall i, j \in N, \forall t \in \mathbb{N} \tag{60}$$

where $\lambda_2 \in \mathbb{R}_+$ is the SLEM of W.

Proof. Define $\hat{B} = \hat{I}^n$. Before starting, we introduce an useful object: the Clarke differential of T. By Rademacher's Theorem and since T is robust, T is Lipschitz continuous and, in particular, almost everywhere differentiable on $\mathbb{R}^n$. Recall that $\mathcal{D}(T)$ denotes the set of points of $\hat{B}$ where T is differentiable. We denote the Jacobian of T at $x \in \mathcal{D}(T)$ by $J_T(x)$. Since T is a robust opinion aggregator, we have that $J_T(x) \in \mathcal{W}$ for all $x \in \mathcal{D}(T)$. Finally, given $x \in \hat{B}$, we denote the Clarke differential of T at x by $\partial T(x)$ (see, e.g., [3, Definition 2.6.1]) where

$$\partial T(x) = \text{co}\left\{W \in \mathcal{W} : W = \lim_k J_T\left(x^k\right) \text{ s.t. } x^k \to x \text{ and } x^k \in \mathcal{D}(T)\right\}.$$

By Theorem 1, recall that $\bar{T} \circ T = \bar{T}$, yielding that $\bar{T}_i \circ T = \bar{T}_i$ for all $i \in N$. By the Chain rule (see, e.g., [3, Theorem 2.6.6]), we have that

$$\partial \bar{T}_i(x) \subseteq \text{co}\left\{\partial \bar{T}_i(T(x)) \partial T(x)\right\} \qquad \forall i \in N, \forall x \in \hat{B} \tag{61}$$

where $\partial \bar{T}_i(T(x)) \partial T(x)$ is the set of probability vectors $p \in \Delta$ such that $p^{\mathrm{T}} = q^{\mathrm{T}} \tilde{W}$ where $q \in \partial \bar{T}_i(T(x))$ and $\tilde{W} \in \partial T(x)$. By definition of $\partial T(x)$ and since T satisfies (59), we have that

$$\tilde{W} \leq \kappa W \qquad \forall \tilde{W} \in \partial T(x), \forall x \in \hat{B}. \tag{62}$$

We next prove by induction that for each $x \in \hat{B}$, for each $i \in N$, for each $p \in \partial \bar{T}_i(x)$, and for each $t \in \mathbb{N}$ there exists $q \in \Delta$ such that

$$p^{\mathrm{T}} \leq q^{\mathrm{T}}\left(\kappa^t W^t\right). \tag{63}$$

By (62), we have that $q^{\mathrm{T}} \tilde{W} \leq q^{\mathrm{T}}(\kappa W)$ for all $q \in \partial \bar{T}_i(T(x))$, for all $\tilde{W} \in \partial T(x)$, for all $x \in \hat{B}$, and for all $i \in N$. By (61) and since $\partial \bar{T}_i(T(x)) \subseteq \Delta$ for all $i \in N$, this implies that (63) holds for $t = 1$. Next, we assume that the statement holds for t and we show it holds for $t+1$. Consider $x \in \hat{B}$, $i \in N$, and $p \in \partial \bar{T}_i(x)$. By (61), we have that there exist $\left\{\tilde{q}^k\right\}_{k=1}^m \subseteq \partial \bar{T}_i(T(x))$, $\left\{\tilde{W}_k\right\}_{k=1}^m \subseteq \partial T(x)$, and $\{\alpha_k\}_{k=1}^m \subseteq [0,1]$ such that $\sum_{k=1}^m \alpha_k = 1$ and $p^{\mathrm{T}} = \sum_{k=1}^m \alpha_k \left(\tilde{q}^k\right)^{\mathrm{T}} \tilde{W}_k$. By inductive hypothesis and since $\left\{\tilde{q}^k\right\}_{k=1}^m \subseteq \partial \bar{T}_i(T(x))$ and $T(x) \in \hat{B}$, for each $k \in \{1, ..., m\}$ we have that $\left(\tilde{q}^k\right)^{\mathrm{T}} \kappa W \leq \left(\hat{q}^k\right)^{\mathrm{T}}\left(\kappa^t W^t\right) \kappa W = \left(\hat{q}^k\right)^{\mathrm{T}}\left(\kappa^{t+1} W^{t+1}\right)$ for some $\hat{q}^k \in \Delta$. By (62), this yields that

$$p^{\mathrm{T}} = \sum_{k=1}^m \alpha_k \left(\tilde{q}^k\right)^{\mathrm{T}} \tilde{W}_k \leq \sum_{k=1}^m \alpha_k \left(\tilde{q}^k\right)^{\mathrm{T}} (\kappa W) \leq \left(\sum_{k=1}^m \alpha_k \left(\hat{q}^k\right)^{\mathrm{T}}\right)\left(\kappa^{t+1} W^{t+1}\right).$$

Since $\sum_{k=1}^{m} \alpha_k \hat{q}^k \in \Delta$ and x, i, as well as p were arbitrarily chosen, the inductive step follows. By induction, (63) holds.

By [2, Theorem 20.1.5] and since $W \in \mathcal{W}_{\text{un}}$, we have that the rows of W^t converge exponentially fast to the left Perron-Frobenius eigenvector $\bar{w}$ and the speed is controlled by λ_2, that is,

$$\max_{i,j \in N} \left| w_{ij}^{(t)} - \bar{w}_j \right| \leq \sqrt{\frac{\max_{i \in N} d_i}{\min_{i \in N} d_i}} \lambda_2^t \quad \forall t \in \mathbb{N}.$$

Consider $\bar{x} \in \hat{B}$, $i \in N$, $p \in \partial \bar{T}_i(\bar{x})$, and $t \in \mathbb{N}$. By (63), this implies that $p^{\mathrm{T}} \leq q^{\mathrm{T}} \left(\kappa^t W^t \right) = \kappa^t q^{\mathrm{T}} W^t$ for some $q \in \Delta$, yielding that

$$\begin{aligned} p_j &\leq \kappa^t \sum_{i=1}^{n} q_i w_{ij}^{(t)} = \kappa^t \bar{w}_j + \kappa^t \sum_{i=1}^{n} q_i \left(w_{ij}^{(t)} - \bar{w}_j \right) \\ &\leq \kappa^t \bar{w}_j + \kappa^t \sum_{i=1}^{n} q_i \left| w_{ij}^{(t)} - \bar{w}_j \right| \leq \kappa^t \bar{w}_j + \kappa^t \sqrt{\frac{\max_{i \in N} d_i}{\min_{i \in N} d_i}} \lambda_2^t \qquad \forall j \in N. \end{aligned}$$

Since $\bar{x}$, p, and t were arbitrarily chosen, and $\nabla \bar{T}_i(x) \in \partial \bar{T}_i(x)$ for all $x \in \mathcal{D}\left(\bar{T}\right)$, we have that

$$\bar{s}_{ij}(T) = \sup_{x \in \mathcal{D}\left(\bar{T}\right)} \frac{\partial \bar{T}_i}{\partial x_j}(x) \leq \kappa^t \bar{w}_j + \kappa^t \sqrt{\frac{\max_{i \in N} d_i}{\min_{i \in N} d_i}} \lambda_2^t \quad \forall j \in N, \forall t \in \mathbb{N}.$$

Since i was arbitrarily chosen, the statement follows. ■

Proof of Proposition 3. 1. Fix $n \in \mathbb{N}$ and define $\hat{B} = \hat{I}^n$. Since $T(n)$ is a robust opinion aggregator, we have that $T(n)$ is Lipschitz continuous. By Rademacher's Theorem, this implies that $T(n)$ is almost everywhere differentiable on $\hat{B}$ and, in particular, Clarke differentiable. Since $T_j(n)$ is monotone and translation invariant for all $j \in N$, note that $\nabla T_j(n)(x) \in \Delta_n$ for all $x \in \mathcal{D}(T(n))$ and for all $j \in N$. Recall that the Clarke's differential is the set (see, e.g., [3, Theorem 2.5.1]):

$$\partial T_j(n)(\bar{x}) = \operatorname{co}\left\{ p \in \Delta_n : p = \lim_k \nabla T_j(n)\left(x^k\right) \text{ s.t. } x^k \to \bar{x} \text{ and } x^k \in \mathcal{D}(T(n)) \right\} \qquad \forall \bar{x} \in \hat{B}, \forall j \in N. \tag{64}$$

By Theorem 1, recall that $\bar{T}(n) \circ T(n) = \bar{T}(n)$. Fix $\bar{x} \in \hat{B}$. Define by $\Pi_{j=1}^n \partial T_j(n)(\bar{x})$ the collection of all $n \times n$ square matrices whose j-th row is an element of $\partial T_j(n)(\bar{x})$. From the previous part of the proof, we have that $\Pi_{j=1}^n \partial T_j(n)(\bar{x}) \subseteq \mathcal{W}$. For each $i \in N$, define

$$\begin{aligned} &\partial \bar{T}_i(n)(T(n)(\bar{x}))\, \Pi_{j=1}^n \partial T_j(n)(\bar{x}) \\ &= \left\{ \tilde{w} \in \Delta_n : \exists p \in \partial \bar{T}_i(n)(T(n)(\bar{x})), \exists W \in \Pi_{j=1}^n \partial T_j(n)(\bar{x}) \text{ s.t. } p^{\mathrm{T}} W = \tilde{w}^{\mathrm{T}} \right\}. \end{aligned}$$

By the Chain Rule (see, e.g., [3, Theorem 2.6.6 and point e of Proposition 2.6.2]), we have that for each $i \in N$

$$\partial \bar{T}_i(n)(\bar{x}) \subseteq \operatorname{co}\left\{ \partial \bar{T}_i(n)(T(n)(\bar{x}))\, \Pi_{j=1}^n \partial T_j(n)(\bar{x}) \right\}. \tag{65}$$

By assumption, we have that for each $i, j \in N$

$$\sup_{x \in \mathcal{D}(T(n))} \frac{\partial T_i(n)}{\partial x_j}(x) \leq \frac{\kappa}{\bar{d}_i(n)} \leq \frac{\kappa}{\bar{d}_{\min}(n)}. \tag{66}$$

By (64) and (66), we have that $0 \leq p_j \leq \frac{\kappa}{\bar{d}_{\min}(n)}$ for all $p \in \partial T_i(n)(\bar{x})$ and for all $i, j \in N$. By (65), $0 \leq p_j \leq \frac{\kappa}{\bar{d}_{\min}(n)}$ for all $p \in \partial \bar{T}_i(n)(\bar{x})$ and for all $i, j \in N$. Finally, observe that if $x \in \mathcal{D}(\bar{T}(n))$, we have that $\nabla \bar{T}_i(n)(x) \in \partial \bar{T}_i(n)(x)$ and, in particular, $\frac{\partial \bar{T}_i(n)}{\partial x_j}(x) \leq \frac{\kappa}{\bar{d}_{\min}(n)}$ for all $i, j \in N$. This yields that

$$\bar{s}_{ij}(T(n)) = \sup_{x \in \mathcal{D}(\bar{T}(n))} \frac{\partial \bar{T}_i(n)}{\partial x_j}(x) \leq \frac{\kappa}{\bar{d}_{\min}(n)} \quad \forall i, j \in N.$$

By point 1 of Theorem 3 and since $\lim_n \frac{\sqrt{n}}{\bar{d}_{\min}(n)} = 0$ and n was arbitrarily chosen, we have that for each $\iota \in \mathbb{N}$

$$\lim_n \sum_{j=1}^{n} (\bar{s}_{\iota j}(T(n)))^2 \leq \lim_n \sum_{j=1}^{n} \left(\frac{\kappa}{\bar{d}_{\min}(n)}\right)^2 = \lim_n \frac{n\kappa^2}{(\bar{d}_{\min}(n))^2} = 0,$$

yielding the statement.

2. For each $n \in \mathbb{N}$ denote by $W(n) \in \mathcal{W}$ the stochastic matrix whose ij-th entry is $\bar{a}_{ij}(n)/\bar{d}_i(n)$. By assumption, each $W(n)$ is in $\mathcal{W}_{\text{un}}$ and has a unique left Perron-Frobenius eigenvector that we denote $\bar{w}(n) \in \Delta_n$. By assumption, it follows that there exist $\bar{\kappa} > 1$ and $\varepsilon > 0$ such that $\{T(n)\}_{n \in \mathbb{N}}$ is $\bar{\kappa}$-dominated and $\sup_{n \in \mathbb{N}} \lambda_2(n) < \frac{1}{\bar{\kappa}^{2+\varepsilon}}$. Set $\bar{m} = \sup_{n \in \mathbb{N}} \sqrt{\frac{\bar{d}_{\max}(n)}{\bar{d}_{\min}(n)}} \in \mathbb{R}_+$ and $t_n = \max\{1, \lfloor \log_{\bar{\kappa}^2} (\max_{k \in N} \bar{w}_k(n))^{-\alpha} \rfloor\}$ for all $n \in \mathbb{N}$ where $\alpha = \frac{1+\delta}{1+\varepsilon}$ with $\delta \in (0, \varepsilon)$. Note that $\alpha \in (0,1)$ and $(1+\varepsilon)\alpha = 1+\delta$. By (58), we have that $0 \leq \max_{k \in N} \bar{w}_k(n) \leq \bar{m}^2/n$ for all $n \in \mathbb{N}$ and, in particular, $\lim_n \max_{k \in N} \bar{w}_k(n) = 0$. By Lemma 10, recall that

$$0 \leq \bar{s}_{ij}(T(n)) \leq \bar{\kappa}^{t_n} \bar{w}_j(n) + \bar{m}\bar{\kappa}^{t_n} \lambda_2^{t_n}(n) \quad \forall i, j \in N, \forall n \in \mathbb{N} \backslash \{1\}.$$

It follows that

$$(\bar{s}_{ij}(T(n)))^2 \leq \bar{\kappa}^{2t_n} \bar{w}_j(n)^2 + 2\bar{\kappa}^{t_n} \bar{w}_j(n) \bar{m}\bar{\kappa}^{t_n} \lambda_2^{t_n}(n) + \bar{m}^2 \bar{\kappa}^{2t_n} \lambda_2^{2t_n}(n) \quad \forall i, j \in N, \forall n \in \mathbb{N} \backslash \{1\}$$

and

$$\sum_{j=1}^{n} (\bar{s}_{ij}(T(n)))^2 \leq a_n + b_n + c_n \qquad \forall i \in N, \forall n \in \mathbb{N} \backslash \{1\} \tag{67}$$

where $a_n = \sum_{j=1}^n \bar{\kappa}^{2t_n} \bar{w}_j(n)^2$, $b_n = \sum_{j=1}^n 2\bar{m}\bar{\kappa}^{2t_n} \lambda_2^{t_n}(n) \bar{w}_j(n)$, and $c_n = \sum_{j=1}^n \bar{m}^2 \bar{\kappa}^{2t_n} \lambda_2^{2t_n}(n)$ for all $n \in \mathbb{N} \backslash \{1\}$. Note that these three sequences only depend on n and not on $i, j \in N$. We next show that $\lim_n a_n = \lim_n b_n = \lim_n c_n = 0$. Since $\lim_n \max_{k \in N} \bar{w}_k(n) = 0$ and $\bar{\kappa} > 1$, observe that $\lim_n (\max_{k \in N} \bar{w}_k(n))^{-\alpha} = \infty$ and $\lim_n \log_{\bar{\kappa}^2} (\max_{k \in N} \bar{w}_k(n))^{-\alpha} = \infty$. This implies that $\lim_n t_n = \infty$. Moreover, there exists $\bar{n} \in \mathbb{N} \backslash \{1\}$ such that $\log_{\bar{\kappa}^2} (\max_{k \in N} \bar{w}_k(n))^{-\alpha} - 1 \leq t_n = \lfloor \log_{\bar{\kappa}^2} (\max_{k \in N} \bar{w}_k(n))^{-\alpha} \rfloor \leq \log_{\bar{\kappa}^2} (\max_{k \in N} \bar{w}_k(n))^{-\alpha}$ for all $n \geq \bar{n}$.

- Since $1 - \alpha \in (0,1)$, $\bar{\kappa} > 1$, and $\lim_n \max_{k \in N} \bar{w}_k(n) = 0$, observe that for each $n \geq \bar{n}$

$$\begin{aligned} 0 \leq a_n &= \bar{\kappa}^{2t_n} \sum_{j=1}^{n} \bar{w}_j(n)^2 \leq \bar{\kappa}^{2t_n} \max_{k \in N} \bar{w}_k(n) \sum_{j=1}^{n} \bar{w}_j(n) \\ &= \bar{\kappa}^{2t_n} \max_{k \in N} \bar{w}_k(n) \leq (\bar{\kappa}^2)^{\log_{\bar{\kappa}^2} (\max_{k \in N} \bar{w}_k(n))^{-\alpha}} \max_{k \in N} \bar{w}_k(n) = \left(\max_{k \in N} \bar{w}_k(n)\right)^{1-\alpha} \to 0 \text{ as } n \to \infty. \end{aligned}$$

- Since $\bar{\kappa} > 1$, we have that $0 \leq \sup_{n \in \mathbb{N}} \bar{\kappa}^2 \lambda_2(n) \leq \frac{1}{\bar{\kappa}^{\varepsilon}} < 1$. Since $t_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ and

$\lim_n t_n = \infty$, this implies that

$$0 \leq b_n = 2\bar{m}\bar{\kappa}^{2t_n}\lambda_2^{t_n}(n)\sum_{j=1}^{n}\bar{w}_j(n) = 2\bar{m}\left(\bar{\kappa}^2\lambda_2(n)\right)^{t_n} \leq 2\bar{m}\left(\sup_{n\in\mathbb{N}}\bar{\kappa}^2\lambda_2(n)\right)^{t_n} \to 0 \text{ as } n \to \infty.$$

- Since $\sup_{n\in\mathbb{N}}\lambda_2(n) \leq \frac{1}{\bar{\kappa}^{2+\varepsilon}}$, we have that $\sup_{n\in\mathbb{N}}\lambda_2^2(n) \leq \frac{1}{\bar{\kappa}^{4+2\varepsilon}}$, that is, $0 \leq \sup_{n\in\mathbb{N}}\bar{\kappa}^2\lambda_2^2(n) \leq \frac{1}{\bar{\kappa}^{2+2\varepsilon}}$. Since $t_n \in \mathbb{N}$ for all $n \in \mathbb{N}$, this implies that $\left(\sup_{n\in\mathbb{N}}\bar{\kappa}^2\lambda_2^2(n)\right)^{t_n} \leq \left(\frac{1}{\bar{\kappa}^{2+2\varepsilon}}\right)^{t_n}$ for all $n \in \mathbb{N}$. Since $(1+\varepsilon)\alpha = 1+\delta$ and $\delta > 0$, we obtain that for each $n \geq \bar{n}$

$$\begin{aligned}
0 \leq c_n &= \bar{m}^2 n\bar{\kappa}^{2t_n}\lambda_2^{2t_n}(n) = \bar{m}^2 n\left(\bar{\kappa}^2\lambda_2^2(n)\right)^{t_n} \leq \bar{m}^2 n\left(\frac{1}{\bar{\kappa}^{2+2\varepsilon}}\right)^{t_n} = \bar{m}^2 n\left(\frac{1}{\bar{\kappa}^{2(1+\varepsilon)}}\right)^{t_n} \\
&\leq \bar{m}^2 n\left(\bar{\kappa}^2\right)^{-(1+\varepsilon)\left(\log_{\bar{\kappa}^2}(\max_{k\in N}\bar{w}_k(n))^{-\alpha}-1\right)} = \bar{m}^2 n\bar{\kappa}^{2(1+\varepsilon)}\left(\bar{\kappa}^2\right)^{-(1+\varepsilon)\log_{\bar{\kappa}^2}(\max_{k\in N}\bar{w}_k(n))^{-\alpha}} \\
&= \bar{m}^2\bar{\kappa}^{2(1+\varepsilon)}n\left(\max_{k\in N}\bar{w}_k(n)\right)^{(1+\varepsilon)\alpha} \leq \bar{m}^2\bar{\kappa}^{2(1+\varepsilon)}n\left(\frac{\bar{m}^2}{n}\right)^{(1+\varepsilon)\alpha} \\
&= \bar{m}^{4+2\delta}\bar{\kappa}^{2(1+\varepsilon)}n^{-\delta} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

By (67), we have $\lim_n \sum_{j=1}^{n}\left(\bar{s}_{\iota j}(T(n))\right)^2 = 0$ for all $\iota \in \mathbb{N}$. By point 1 of Theorem 3 and since $\{T(n)\}_{n\in\mathbb{N}}$ is a sequence of odd robust opinion aggregators and $\{\varepsilon_i(n)\}_{i\in N, n\in\mathbb{N}}$ is symmetric, point 2 of the statement follows. ■

D.3 Discussion

Proof of Lemma 8. Fix $i \in N$. Consider $\tilde{z} \in \mathbb{R}^n$ such that $\tilde{z} \gg 0$. Define $z = \tilde{z} - \min_{j\in N}\tilde{z}_j e$, $v = 0$, and $h = \min_{j\in N}\tilde{z}_j$. Note that $z \geq v$ as well as $h \in \mathbb{R}_{++}$. Since ϕ has increasing shifts and is sensitive, we obtain that

$$\phi_i(\tilde{z}) - \phi_i\left(\tilde{z} - \min_{j\in N}\tilde{z}_j e\right) = \phi_i(z+he) - \phi_i(z) \geq \phi_i(v+he) - \phi_i(v) = \phi_i\left(\min_{j\in N}\tilde{z}_j e\right) - \phi_i(0) > 0,$$

proving the first inequality. A similar argument yields the second inequality. ■

Proof of Lemma 9. Fix $i \in N$ and $x \in \mathbb{R}^n$. Define $g_{i,x} : \mathbb{R} \to \mathbb{R}_+$ by $g_{i,x}(c) = \phi_i(x+ce)$ for all $c \in \mathbb{R}$. Consider $c_1, c_2 \in \mathbb{R}$ such that $c_1 > c_2$ and $h > 0$. Since $\phi \in \Phi_R$ and $x + c_1 e \geq x + c_2 e$, it follows that

$$\begin{aligned}
g_{i,x}(c_1+h) - g_{i,x}(c_1) &= \phi_i((x+c_1 e)+he) - \phi_i(x+c_1 e) \\
&\geq \phi_i((x+c_2 e)+he) - \phi_i(x+c_2 e) = g_{i,x}(c_2+h) - g_{i,x}(c_2).
\end{aligned}$$

By [8, Problem N, pp. 223–224], it follows that $g_{i,x}$ is midconvex. Next, fix $c \in \mathbb{R}$ and $c' \in (c-1, c+1)$. Set $c_1 = 2c - c'$, $c_2 = c-1$, and $h = c' - (c-1)$. Since $c_1 > c_2$, $h > 0$, and $\phi_i \geq 0$, we have that

$$g_{i,x}(c') - g_{i,x}(c-1) \leq g_{i,x}(c+1) - g_{i,x}(2c-c') \implies 0 \leq g_{i,x}(c') \leq g_{i,x}(c-1) + g_{i,x}(c+1).$$

Since c' was arbitrarily chosen, we have that $g_{i,x}$ is bounded on $(c-1, c+1)$. By [8, Theorem C, p. 215], it follows that $g_{i,x}$ is continuous and convex. Finally, observe that $f_{i,x} = g_{i,x} \circ h$ where $h(c) = -c$ for all $c \in \mathbb{R}$, yielding that $f_{i,x}$ is convex and continuous being the composition of a convex and continuous function with an affine and continuous function. Next, assume that ϕ has also strictly increasing shifts and, in particular, has increasing shifts. By the previous part of the proof, $g_{i,x}$ is convex. By contradiction, assume that $g_{i,x}$ is not strictly convex. This implies that there exists an interval $[d_2, d_1]$, with $d_2 < d_1$, where $g_{i,x}$ is affine. Define $c_1 = \frac{1}{2}d_1 + \frac{1}{2}d_2$, $c_2 = d_2$, and $h = (d_1 - d_2)/2$. Note that $c_1 > c_2$ and $h > 0$. Since ϕ has strictly increasing shifts, by the same computations of the previous part of the proof, we have that

$$g_{i,x}(d_1) - g_{i,x}\left(\frac{1}{2}d_1 + \frac{1}{2}d_2\right) = g_{i,x}(c_1 + h) - g_{i,x}(c_1)$$
$$> g_{i,x}(c_2 + h) - g_{i,x}(c_2) = g_{i,x}\left(\frac{1}{2}d_1 + \frac{1}{2}d_2\right) - g_{i,x}(d_2),$$

yielding that $g_{i,x}\left(\frac{1}{2}d_1 + \frac{1}{2}d_2\right) < \frac{1}{2}g_{i,x}(d_1) + \frac{1}{2}g_{i,x}(d_2)$, a contradiction with affinity. Since $g_{i,x}$ is strictly convex, so is $f_{i,x} = g_{i,x} \circ h$. ■

Proof of Proposition 4. Before starting, we make few observations about strong convexity (see, e.g., [8, p. 268]). Since each ρ_i is strongly convex and twice continuously differentiable, we have that for each $i \in N$ there exists $\alpha_i > 0$ such that $\rho_i''(s) \geq \alpha_i$ for all $s \in \mathbb{R}$. Moreover, we have that for each $i \in N$

$$\left(\rho_i'(s_1) - \rho_i'(s_2)\right)(s_1 - s_2) \geq \alpha_i (s_1 - s_2)^2 \quad \forall s_1, s_2 \in \mathbb{R}. \tag{68}$$

Finally, since each ρ_i is twice continuously differentiable and I is compact, for each $i \in N$ we have that there exists $L_i > 0$ such that

$$\left|\rho_i'(s_1) - \rho_i'(s_2)\right| \leq L_i |s_1 - s_2| \quad \forall s_1, s_2 \in [\min I - \max I, \max I - \min I]. \tag{69}$$

Recall that $\phi_i : \mathbb{R}^n \to \mathbb{R}_+$ is defined by $\phi_i(z) = \sum_{j=1}^n w_{ij}\rho_i(z_j)$ for all $z \in \mathbb{R}^n$ and for all $i \in N$. By assumption, $\phi \in \Phi_A \subseteq \Phi_R$. Since $\rho_i'' \geq \alpha_i > 0$ for all $i \in N$, this implies that ρ_i is strictly convex for all $i \in N$. Standard computations yield that ϕ has strictly increasing shifts. By Proposition 9, we have that $\mathbf{T}^\phi = T^\phi$ is single-valued and a robust opinion aggregator from B to B. Moreover, $T_i^\phi(x)$ is the unique solution of

$$\min_{c \in \mathbb{R}} \phi_i(x - ce) = \min_{c \in I} \phi_i(x - ce) \qquad \forall i \in N, \forall x \in B. \tag{70}$$

Fix $i \in N$. Since ρ_i is differentiable and convex, so is the map $c \mapsto \phi_i(x - ce)$ for all $x \in B$. The solution of (70) is then given by the first order condition $\sum_{j=1}^n w_{ij}\rho_i'\left(x_j - T_i^\phi(x)\right) = 0$ for all $x \in B$. Consider $x \in B$, $h > 0$, and $l \in N$ such that $x + he^l \in B$. We have that

$$\sum_{j=1}^n w_{ij}\rho_i'\left(x_j - T_i^\phi(x)\right) = 0 \text{ and } \sum_{j=1}^n w_{ij}\rho_i'\left(x_j + he_j^l - T_i^\phi\left(x + he^l\right)\right) = 0. \tag{71}$$

Note that if $w_{il} = 0$, then $\sum_{j=1}^n w_{ij}\rho_i'\left(x_j + he_j^l - c\right) = \sum_{j=1}^n w_{ij}\rho_i'(x_j - c)$ for all $c \in \mathbb{R}$, proving that $T_i^\phi\left(x + he^l\right) = T_i^\phi(x)$. Since x and h were arbitrarily chosen, we have that $w_{il} = 0$ implies $\bar{a}_{il} = 0$. In particular, since i and l were arbitrarily chosen, we have that $A(W) \geq \bar{A}\left(T^\phi\right)$.

Next, assume that $w_{il} > 0$. By (69), (71), and (68) and since T^ϕ is monotone and $h > 0$, we can conclude that

$$
\begin{aligned}
L_i\left(T_i^{\phi}\left(x+he^l\right)-T_i^{\phi}(x)\right) &\geq \sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j+he_j^l-T_i^{\phi}(x)\right)-\sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j+he_j^l-T_i^{\phi}\left(x+he^l\right)\right)\\
&= \sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j+he_j^l-T_i^{\phi}(x)\right)-\sum_{j=1}^{n} w_{ij}\rho_i'\left(x_j-T_i^{\phi}(x)\right)\\
&= w_{il}\left[\rho_i'\left(x_l+h-T_i^{\phi}(x)\right)-\rho_i'\left(x_l-T_i^{\phi}(x)\right)\right] \geq w_{il}\alpha_i h,
\end{aligned}
$$

proving that $T_i^{\phi}\left(x+he^l\right)-T_i^{\phi}(x) \geq \varepsilon_{il}h$ where $\varepsilon_{il} = L_i^{-1}w_{il}\alpha_i/2 \in (0,1)$. Since x and h were arbitrarily chosen, we have that $w_{il}>0$ implies $\underline{a}_{il}=1$. In particular, since i and l were arbitrarily chosen, we have that $\underline{A}\left(T^{\phi}\right) \geq A(W)$. Since $\bar{A}\left(T^{\phi}\right) \geq \underline{A}\left(T^{\phi}\right)$, we can conclude that $A(W) = \bar{A}\left(T^{\phi}\right) = \underline{A}\left(T^{\phi}\right)$, proving the statement. ■

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis*, 3rd ed., Springer-Verlag, Berlin, 2006.

[2] P. Brémaud, *Discrete Probability Models and Methods*, Springer, New York, 2017.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, 1990.

[4] S. Cerreia-Vioglio, F. Maccheroni, and M. Marinacci, A characterization of probabilities with full support and the Laplace method, *Journal of Optimization Theory and Applications*, 181, 470–478, 2019.

[5] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and A. Rustichini, Niveloids and their extensions: Risk measures on small domains, *Journal of Mathematical Analysis and Applications*, 413, 343–360, 2014.

[6] M. O. Jackson, *Social and Economic Networks*, Princeton University Press, Princeton, 2008.

[7] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Mathematica*, 80, 167–190, 1948.

[8] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.