30490 Fiscal macro - General Exam

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Time: 1 hour and 15 minutes

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1 Question 1

Consider the version of the Diamond Model discussed in class.

Individuals live for two periods: they work when young and retire when old. They earn wages when young, save by investing in capital, and consume capital and returns when old. There is no depreciation, so the young's saving determines next period's capital stock.

The economy grows at rate $g = n + x$, where n is population growth and x productivity growth. The saving of the young determines capital per worker along the growth path.

When the young save one unit, they get $(1 + r)$ units when old. Suppose that the government puts in place a transfer scheme taking D_t from each of the young, and giving $(1 + n)D_t$ to each of the old within the same period. There are $(1 + n)$ young for each old. D_t increases at rate x over time.

1. Describe the population dynamics and the age structure of population in the model. What is the relationship between the Total Fertility Rate (number of children born to young couples), the rate of growth of the young, and the rate of growth of population?
2. Show that such a scheme is equivalent to a pay-as-you-go retirement system, in which the contributions from the young finance the benefits for the old, and per capita retirement contributions and benefits increase with productivity over time.
3. What are the conditions for this transfer scheme to increase welfare of each generation?

Answer to Question 1

1. In the Diamond model individuals live for two periods. At time t there are two generations alive: the young, N_t, and the old, N_{t-1}. The young work and save, while the old consume the return on their previous savings.

   Population evolves according to

   $$N_{t+1} = (1 + n)N_t,$$

   where n is the population growth rate. Therefore, in period t there are

   $$\frac{N_t}{N_{t-1}} = 1 + n$$

   young individuals for each old individual.

If the Total Fertility Rate is defined as the number of children born to each young couple, then, assuming two parents per couple, population replacement requires a fertility rate equal to 2. A fertility rate above 2 implies positive population growth, while a fertility rate below 2 implies negative population growth. Thus, the growth rate of the young population is directly related to fertility, ϕ:

$$1 + n = \frac{\phi}{2}.$$

Since every individual is young in one period and old in the next, the growth rate of the young population is also the growth rate of total population along a balanced demographic path.

2. The government takes a contribution D_t from each young individual. Since there are N_t young, total contributions are

$$D_t N_t.$$

These resources are transferred to the old. Since the number of old individuals is N_{t-1}, the benefit received by each old individual is

$$\frac{D_t N_t}{N_{t-1}}.$$

Using

$$N_t = (1 + n)N_{t-1},$$

the benefit per old individual is

$$\frac{D_t N_t}{N_{t-1}} = D_t(1 + n).$$

Hence each young pays D_t, and each old receives $(1 + n)D_t$. This is exactly a pay-as-you-go pension system: current contributions from the young finance current benefits for the old.

Moreover, since D_t grows at the rate of productivity growth x,

$$D_{t+1} = (1 + x)D_t,$$

both per capita contributions and per capita benefits grow with productivity over time.

3. Without the transfer scheme, one unit saved by a young individual yields

$$1 + r$$

units of consumption when old.

Under the pay-as-you-go scheme, one unit transferred to the old through the system yields an implicit gross return equal to

$$(1 + n)(1 + x).$$

Therefore, the scheme increases welfare if the implicit return of the PAYG system is larger than the return on capital:

$$(1 + n)(1 + x) > 1 + r.$$

Equivalently, for small growth rates,

$$n + x > r.$$

Since aggregate output grows at rate

$$g = n + x,$$

the condition can be written as

$$g > r.$$

Thus, the transfer scheme improves welfare when the economy is dynamically inefficient, that is, when the growth rate of the economy exceeds the interest rate. In that case, reducing capital accumulation through a PAYG transfer raises consumption possibilities for all generations.

2 Question 2

2.1 Question 2.1

Consider a 10-year zero-coupon bond with annual yield to maturity

$$Y_{t,T} = 5\%.$$

i. Compute its price $P_{t,T}$.

ii. Compute the corresponding continuously compounded yield $y_{t,T}$.

iii. Suppose that after one year the yield on the remaining 9-year bond increases to 5.5%. Compute the one-period holding-period return using log prices.

iv. Explain why the holding-period return differs from the initial yield to maturity.

Answer to Question 2.1

2.2 Question 2.1

For a zero-coupon bond, the price is given by

$$P_{t,T} = \frac{1}{(1 + Y_{t,T})^{T-t}}.$$

Here $Y_{t,T} = 5\%$ and $T - t = 10$.

i. The price is

$$P_{t,T} = \frac{1}{(1.05)^{10}} = 0.6139.$$

ii. The continuously compounded yield $y_{t,T}$ satisfies

$$P_{t,T} = e^{-y_{t,T}(T-t)}.$$

Hence

$$y_{t,T} = -\frac{1}{10}\log(P_{t,T}) = \log(1.05) = 0.04879.$$

Therefore,

$$y_{t,T} = 4.879\%.$$

iii. After one year, the bond has 9 years remaining and the new yield is 5.5%. Its new price is

$$P_{t+1,T} = \frac{1}{(1.055)^{9}} = 0.6178.$$

The one-period holding-period return using log prices is

$$h_{t+1} = \log(P_{t+1,T}) - \log(P_{t,T}).$$

Thus,

$$h_{t+1} = \log(0.6178) - \log(0.6139) = 0.0063.$$

Hence,

$$h_{t+1} = 0.63\%.$$

iv. The holding-period return differs from the initial yield to maturity because the yield to maturity is the return earned only if the bond is held until maturity and the yield curve remains unchanged in the relevant sense. By contrast, the one-period holding-period return depends on the price at which the bond can be sold after one year. Since the yield on the remaining 9-year bond rises from 5% to 5.5%, the bond price falls relative to what it would have been under unchanged yields, reducing the one-period return.

2.3 Question 2.2

Assume that the expected holding-period return on a long-term bond satisfies

$$E_t(r^T_{t,t+1}) = y_{t,t+1} + \phi^T_{t,t+1}.$$

Explain why the yield on a long-term bond can be interpreted as depending on two components:

long-term yield = expected future short rates + term premium.

In your answer, discuss the role of:

i. expectations about future monetary policy;

ii. risk premia;

iii. the difference between the expectations hypothesis and a model with time-varying risk premia.

iv. Suppose that the observed yield curve is upward sloping. Discuss two possible interpretations:

- markets expect future short-term interest rates to rise;
- investors require a positive term premium to hold long-term bonds.

Explain why observing the yield curve alone is not sufficient to distinguish between these two explanations.

Answer to Question 2.2

2.4 Question 2.2

The expected one-period holding-period return on a long-term bond is

$$E_t(r^T_{t,t+1}) = y_{t,t+1} + \phi^T_{t,t+1},$$

where $y_{t,t+1}$ is the one-period short rate and $\phi^T_{t,t+1}$ is the risk, or term, premium required to hold a long-term bond.

A long-term yield can therefore be decomposed as

long-term yield = average expected future short rates + term premium.

More formally, the yield on a bond maturing at T can be interpreted as

$$y_{t,T} \simeq \frac{1}{T-t} \sum_{j=0}^{T-t-1} E_t y_{t+j,t+j+1} + TP_{t,T},$$

where $TP_{t,T}$ is the term premium.

i. Expectations about future monetary policy matter because short-term interest rates are largely controlled by the central bank. If investors expect the central bank to raise policy rates in the future, expected future short rates increase. This raises long-term yields today. Conversely, expectations of future monetary easing lower expected future short rates and therefore reduce long-term yields.

ii. Risk premia matter because long-term bonds are risky when sold before maturity. Their prices are sensitive to changes in interest rates. Investors may therefore require compensation for bearing duration risk, inflation risk, and uncertainty about future monetary policy. This compensation is the term premium. A higher term premium raises long-term yields even if expected future short rates do not change.

iii. Under the expectations hypothesis, long-term yields are determined only by current and expected future short-term interest rates. In its pure form,

$$TP_{t,T} = 0,$$

or, in a weaker version, the term premium is constant. Hence movements in long-term yields mainly reflect changes in expected future short rates.

In a model with time-varying risk premia, instead,

$$TP_{t,T}$$

changes over time. Therefore, long-term yields can move either because expectations of future monetary policy change or because the compensation required for holding long-term bonds changes.

iv. If the observed yield curve is upward sloping, there are two possible interpretations.

First, markets may expect future short-term interest rates to rise. In this case, long-term yields are higher because they incorporate higher expected future short rates:

$$E_t y_{t+j,t+j+1} > y_{t,t+1}.$$

Second, investors may require a positive term premium to hold long-term bonds. In this case, long-term yields are higher not necessarily because short rates are expected to rise, but because investors demand compensation for bearing the risks associated with long maturities:

$$TP_{t,T} > 0.$$

Observing the yield curve alone is not sufficient to distinguish between these explanations because a long-term yield combines both expected future short rates and the term premium. The same upward-sloping yield curve can be generated either by expectations of higher future monetary policy rates, by a positive term premium, or by a combination of both. Additional information, such as survey expectations, macroeconomic forecasts, or an explicit term-structure model, is needed to separate the two components.

3 Question 3:

The probability with which a given government enters the state of default in each future period is computed by i) assigning each government to one of the three *credit risk classes* from the safest (A), to the risky (B) to the default class (labelled D) ii) assuming that a country defaults only when it reaches state D, and modelling the transition from one state to the other via a transition matrix that takes the following specification:

$$A_1 = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{pmatrix}$$

The risk free rate r is =0.02,

1. Describe the properties of the transition matrix
2. How many years will take to governments in class A and class B to default ?
3. How would a risk neutral agent price a perpetuity issued by a government in class A and in class B in the case where the recovery rate is zero ?
4. If Governments in class A and B were to issue perpetual loans, would different interest rates be charged on these loans ?

Answer to Question 3

4 Question 3

The transition matrix is

$$A_1 = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \end{pmatrix},$$

where the states are A, B, and D. State D is the default state.

1. The matrix A_1 is a Markov transition matrix. Each row sums to one:

   $$0.8 + 0.1 + 0.1 = 1, \qquad 0 + 0.7 + 0.3 = 1, \qquad 0 + 0 + 1 = 1.$$

   The entries are probabilities of moving from one credit-risk class to another. State D is absorbing because

   $$P(D \to D) = 1.$$

   Hence, once a government defaults, it remains in default. Moreover, the matrix is upper triangular, implying that credit quality can deteriorate from A to B or D, and from B to D, but cannot improve.

2. The transition matrix gives the probabilities of moving across credit states from one year to the next. Therefore, the time to default is a random variable: a government may default next year, or may remain in a non-default state and default later. Since state D is absorbing, default occurs the first time the Markov chain reaches D.

   To compute the number of years to default, we compute the expected hitting time of state D. Let T_A and T_B denote the expected number of years before reaching state D, starting respectively from class A and class B. The expected hitting time is obtained recursively: one year passes for sure, and then the government may remain in the same class, move to another non-default class, or enter default.

   Starting from B,

   $$T_B = 1 + 0.7T_B.$$

   Therefore,

   $$0.3T_B = 1, \qquad T_B = \frac{1}{0.3} = 3.33.$$

Thus, a government in class B defaults after 3.33 years on average.

Starting from A,

$$T_A = 1 + 0.8T_A + 0.1T_B.$$

Using $T_B = 3.33$,

$$T_A = 1 + 0.8T_A + 0.1(3.33).$$

Hence,

$$0.2T_A = 1.33, \qquad T_A = 6.67.$$

Thus, a government in class A defaults after 6.67 years on average.

3. Assume that the perpetuity pays one unit each year as long as the government has not defaulted, and that the recovery rate is zero. The risk-free rate is

$$r = 0.02,$$

so the discount factor is

$$\beta = \frac{1}{1+r} = \frac{1}{1.02}.$$

Let P_A and P_B be the prices of the perpetuity issued by governments in classes A and B.

For a government in class B,

$$P_B = \beta\left[0.7(1 + P_B) + 0.3(0)\right].$$

Therefore,

$$P_B = \frac{0.7\beta}{1 - 0.7\beta} = 2.1875.$$

For a government in class A,

$$P_A = \beta\left[0.8(1 + P_A) + 0.1(1 + P_B) + 0.1(0)\right].$$

Substituting $P_B = 2.1875$,

$$P_A = \frac{\beta\left[0.9 + 0.1P_B\right]}{1 - 0.8\beta}.$$

Hence,

$$P_A = 5.0852.$$

Therefore, the class A perpetuity has a higher price than the class B perpetuity because default is less likely and occurs later on average.

4. Yes. Different interest rates would be charged on the two perpetual loans.

Since the government in class B is riskier, its perpetuity has a lower price:

$$P_B < P_A.$$

A lower bond price corresponds to a higher yield or interest rate. Therefore, the interest rate charged to a government in class B would be higher than the interest rate charged to a government in class A.

Intuitively, investors require compensation for the higher probability of default. Hence,

$$i_B > i_A.$$

More explicitly, if each perpetuity promises to pay one unit per year, its yield is given by

$$i_j = \frac{1}{P_j},$$

where $j = A, B$.

Using the prices computed above,

$$P_A = 5.0852, \qquad P_B = 2.1875.$$

Therefore, the implied interest rate on the class A perpetuity is

$$i_A = \frac{1}{P_A} = \frac{1}{5.0852} = 0.1966.$$

Hence,

$$i_A = 19.66\%.$$

For the class B perpetuity,

$$i_B = \frac{1}{P_B} = \frac{1}{2.1875} = 0.4571.$$

Hence,

$$i_B = 45.71\%.$$

Therefore,

$$i_B > i_A,$$

because the government in class B has a higher probability of default. The riskier government must offer a higher promised interest rate in order to make investors willing to hold its debt.