Derivation of the Euler Equation for Optimal Consumption Path

1 Intertemporal Optimization and the Euler Equation

The optimal intertemporal path of consumption can be derived using the **Ramsey-Euler equation**, which results from solving an intertemporal utility maximization problem.

1.1 The Optimization Problem

We consider a representative agent who maximizes lifetime utility:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where:

- c_t is consumption at time t,
- $u(c_t)$ is the instantaneous utility function,
- $0 < \beta < 1$ is the discount factor.

The agent faces a budget constraint:

$$a_{t+1} = (1 + r)a_t + y_t - c_t, \tag{2}$$

where:

- a_t is the asset stock,
- r is the constant interest rate,
- y_t is income at time t.

2 The Bellman Equation

The Bellman equation for this problem is:

$$V(a_t) = \max_{c_t} \left[u(c_t) + \beta V(a_{t+1}) \right]. \tag{3}$$

The **intuition behind the Bellman equation** is that the value of being in state a_t today is the sum of two parts:

1. The **immediate utility** from consuming c_t.
2. The **discounted future value** of being in state a_{t+1}, where future decisions are also made optimally.

The agent chooses c_t optimally by considering the trade-off between present consumption and future utility.

3 Derivation of the Euler Equation

To derive the Euler equation, we take the first-order condition (FOC) from the Bellman equation. Differentiating both sides with respect to c_t gives:

$$\frac{\partial V(a_t)}{\partial c_t} = u'(c_t) + \beta V'(a_{t+1}) \cdot \frac{da_{t+1}}{dc_t}. \tag{4}$$

Using the budget constraint, we see that:

$$\frac{da_{t+1}}{dc_t} = -1. \tag{5}$$

Thus, the first-order condition simplifies to:

$$u'(c_t) = \beta V'(a_{t+1}). \tag{6}$$

3.1 Envelope Theorem Application

Using the **envelope theorem**, we differentiate the Bellman equation with respect to assets a_t:

$$V'(a_t) = \frac{\partial}{\partial a_t} \max_{c_t} \left[u(c_t) + \beta V(a_{t+1}) \right]. \tag{7}$$

Since a_t affects a_{t+1} via the budget constraint, we obtain:

$$V'(a_t) = \beta V'(a_{t+1})(1 + r). \tag{8}$$

From the first-order condition:

$$V'(a_t) = u'(c_t)(1 + r). \tag{9}$$

Substituting this into the envelope condition:

$$u'(c_t) = \beta(1 + r)u'(c_{t+1}). \tag{10}$$

3.2 Euler Equation and Consumption Growth

Rearranging:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r). \tag{11}$$

This is the **Euler equation**, which governs the optimal intertemporal consumption path.

4 Special Case: CRRA Utility Function

For a **CRRA utility function**:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad u'(c) = c^{-\sigma}, \tag{12}$$

we obtain:

$$\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} = \beta(1+r). \tag{13}$$

Solving for c_{t+1}/c_t:

$$\frac{c_{t+1}}{c_t} = (\beta(1+r))^{\frac{1}{\sigma}}. \tag{14}$$

This equation describes the optimal growth of consumption over time. The main insights are:

- If $r > \frac{1}{\beta} - 1$, consumption grows over time.
- If $r < \frac{1}{\beta} - 1$, consumption declines over time.
- If $r = \frac{1}{\beta} - 1$, consumption remains constant.

5 Conclusion

The Euler equation provides the fundamental condition for optimal consumption over time. It shows how an individual balances **current** and **future consumption** based on the **interest rate**, **time preference**, and **risk aversion**. By solving this equation in specific models, we can determine the entire path of consumption.