Debt Instruments

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Introduction

Government debt instruments play a pivotal role in fiscal macroeconomics, serving as tools for financing public expenditure and managing liquidity in the economy. This chapter introduces the variety of bonds that governments can issue, each with unique features and pricing mechanisms.
Specifically, we distinguish between the following types of bonds:

- **Nominal Bonds:** The value and payments of these bonds are fixed in nominal terms, without adjustment for inflation.
  - **Zero-Coupon Bonds:** These bonds do not pay periodic interest; instead, they are issued at a discount to their face value and redeemed at par upon maturity.
  - **Coupon Bonds:** These bonds pay periodic interest, known as coupons, until maturity, when the principal is repaid.
- **Inflation-Indexed Bonds:** These bonds adjust their principal and/or interest payments based on inflation, providing a hedge against price level changes.

The primary focus of these lectures is to illustrate how these different types of bonds are priced in financial markets. We will explore the theoretical frameworks and practical methods used to determine their value and compute yields. Additionally, we will examine how yields across different maturities can be identified and analyzed, providing insights into the term structure of interest rates.

Asset Pricing with Time-Varying Expected Returns

Consider a situation in which in each period k state of nature can occur and each state has a probability $\pi(k)$, in the absence of arbitrage opportunities the price of an asset i at time t can be written as follows:

$$P_{i,t} = \sum_{s=1}^{k} \pi_{t+1}(s) m_{t+1}(s) X_{i,t+1}(s)$$

where $m_{t+1}(s)$ is the discounting weight attributed to future pay-offs, which (as the probability π) is independent from the asset i, $X_{i,t+1}(s)$ are the payoffs of the assets (in case of stocks we have $X_{i,t+1} = P_{t+1} + D_{t+1}$, in case of zero coupon bonds, $X_{i,t+1} = P_{t+1}$), and therefore returns on assets are defined as $1 + R_{s,t+1} = \frac{X_{i,t+1}}{P_{i,t}}$.

Asset Pricing with Time-Varying Expected Returns

For the safe asset, whose payoffs do not depend on the state of nature, we have:

$$
\begin{aligned}
P_{s,t} &= X_{i,t+1} \sum_{s=1}^{k} \pi_{t+1}(s) m_{t+1}(s) \\
1 + R_{s,t+1} &= \frac{1}{\sum\limits_{j=1}^{m} \pi_{t+1}(s) m_{t+1}(s)} \\
1 + R_{s,t+1} &= \frac{1}{E_t(m_{t+1})}
\end{aligned}
$$

Asset Pricing with Time-Varying Expected Returns

consider now a risky asset :

$$
\begin{aligned}
E_t\left(m_{t+1}\left(1+R_{i,t+1}\right)\right) &= 1 \\
Cov\left(m_{t+1}R_{i,t+1}\right) &= 1-E_t\left(m_{t+1}\right)E_t\left(1+R_{i,t+1}\right) \\
E_t\left(1+R_{i,t+1}\right) &= -\frac{Cov\left(m_{t+1}R_{i,t+1}\right)}{E_t\left(m_{t+1}\right)}+\left(1+R_{s,t+1}\right)
\end{aligned}
$$

Turning now to excess returns we can write:

$$E_t\left(R_{i,t+1}-R_{s,t+1}\right)=-\left(1+R_{s,t+1}\right)cov\left(m_{t+1}R_{i,t+1}\right)$$

Assets whose returns are low when the stochastic discount factor is high (i.e. when agents values payoffs more) require an higher risk premium, i.e. an higher excess return on the risk-free rate.

Bond Returns: Yields-to-Maturity and Holding Period Returns

We distinguish between two types of bonds: those paying a coupon each given period and those that do not pay a coupon but just reimburse the entire capital upon maturity (zero-coupon bonds). Cash-flows from different types of bonds:

	$t+1$	$t+2$	$t+3$	$\ldots$	T
General	CF_{t+1}	CF_{t+2}	CF_{t+3}	$\ldots$	CF_T
Coupon bond	C	C	C	$\ldots$	$1+C$
1-period zero	1	0	0	$\ldots$	0
2-period zero	0	1	0	$\ldots$	0
$\vdots$				$\ldots$	
$(T-t)$-period zero	0	0	0	$\ldots$	1

Zero-Coupon Bonds:Yields

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{\left(1 + Y_{t,T}\right)^{T-t}}, \tag{1}$$

where $P_{t,T}$ is the price at time t of a bond maturing at time T, and $Y_{t,T}$ is yield to maturity. Taking logs of the left and the right-hand sides of the expression for $P_{t,T}$, and defining the continuously compounded *yield*, $y_{t,T}$, as $\log(1 + Y_{t,T})$, we have the following relationship:

$$p_{t,T} = -\left(T - t\right) y_{t,T}, \tag{2}$$

which clearly illustrates that the elasticity of the yield to maturity to the price of a zero-coupon bond is the maturity of the security. Therefore, the duration of the bond equals maturity as no coupons are paid.

Zero-Coupon Bonds:Price and YTM

Price and YTM of zero-coupon bonds							
Mat	1	2	3	5	7	10	20
$P_{t,T}$	0.9524	0.9070	0.8638	0.7835	0.7106	0.6139	0.3769
$Y_{t,T}$	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
$p_{t,T}$	-0.0487	-0.0976	-0.1464	-0.2439	-0.3416	-0.4879	-0.9757
$y_{t,T}$	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488

Zero-Coupon Bonds: Holding Period Returns

The one-period uncertain holding-period return on a bond maturing at time T, $r_{t,t+1}^{T}$, is then defined as follows:

$$r_{t,t+1}^{T} \equiv p_{t+1,T} - p_{t,T} = -\left(T-t-1\right) y_{t+1,T} + \left(T-t\right) y_{t,T} \tag{3}$$

$$= y_{t,T} - \left(T-t-1\right)\left(y_{t+1,T} - y_{t,T}\right),$$

$$= \left(T-t\right) y_{t,T} - \left(T-t-1\right) y_{t+1,T}, \tag{4}$$

which means that yields and returns differ by a scaled measure of the change between the yield at time $t+1$, $y_{t+1,T}$, and the yield at time t, $y_{t,T}$.
Think of a situation in which the one-year YTM stands at 4.1 per cent while the 30-year YTM stands at 7 per cent. If the YTM of the thirty year bonds goes up to 7.1 per cent in the following period, then the period returns from the two bonds is the same.

A model of the term structure

Apply the no arbitrage condition to a one-period bond (the safe asset) and a T-period bond:

$$
\begin{aligned}
E_t\left(r_{t,t+1}^{T}-r_{t,t+1}^{1}\right) &= E_t\left(r_{t,t+1}^{T}-y_{t,t+1}\right)=\phi_{t,t+1}^{T} \\
E_t\left(r_{t,t+1}^{T}\right) &= y_{t,t+1}+\phi_{t,t+1}^{T}
\end{aligned}
$$

Solving forward the difference equation $p_{t,T}=p_{t+1,T}-r_{t,t+1}^{T}$, we have :

$$
\begin{aligned}
y_{t,T} &= \frac{1}{(T-t)}\sum_{i=0}^{n-1}E_t\left(r_{t+i,t+i+1}^{T}\right) \\
&= \frac{1}{(T-t)}\sum_{i=0}^{n-1}E_t\left(y_{t+i,t+i+1}+\phi_{t+i,t+i+1}^{T}\right)
\end{aligned}
$$

The model clearly shows that two unobservable factors drive bond yields

- Expectations of future monetary policy (risk free) rates over the residual life of the bonds
- Compensation for risk (risk premia)

Zero-Coupon Bonds:Yields

The relationship between price and yield to maturity of a constant coupon (C) bond is given by:

$$P^c_{t,T} = \frac{C}{\left(1+Y^c_{t,T}\right)} + \frac{C}{\left(1+Y^c_{t,T}\right)^2} + ... + \frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}.$$

When the bond sells at par, the yield to maturity equals the coupon rate. To measure the length of time that a bondholder has invested money for we need to introduce the concept of duration:

$$\begin{aligned} D^c_{t,T} &= \frac{\frac{C}{\left(1+Y^c_{t,T}\right)} + 2\frac{C}{\left(1+Y^c_{t,T}\right)^2} + \cdots + (T-t)\frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}}{P^c_{t,T}} \\ &= \frac{C\sum_{i=1}^{T-t}\frac{i}{\left(1+Y^c_{t,T}\right)^i} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}}}{P^c_{t,T}}. \end{aligned}$$

Coupon Bonds:HPR

It can be shown that in case of a coupon bond the period holding return can be approximated by extending the formula for zero-coupon bonds(in which case duration is equal to maturity) as follows:

$$r^c_{t+1} = D^c_{t,T} y^c_{t,T} - \left(D^c_{t,T} - 1\right) y^c_{t+1,T},$$

The formula can be made operational, given the information available in yields to maturity only, by approximating duration as follows:

$$D^c_{t,T} = \frac{1 - \left(1 + Y^c_{t,T}\right)^{-(T-t)}}{1 - \left(1 + Y^c_{t,T}\right)^{-1}}$$

Coupon Bonds: Model of the TS

In the case of long-dated coupon bonds the model for the term structure becomes:

$$
\begin{aligned}
y_{t,T} &= y^{*}_{t,T} + E[\Phi_T \mid I_t] = \frac{1-\gamma}{1-\gamma^{T-t}} \sum_{j=0}^{T-t-1} \gamma^{j} E[r_{t+j} \mid I_t] + E[\Phi_T \mid I_t] \\
\gamma &= 1/(1+\bar{y})
\end{aligned}
$$

Nominal and Real Bonds

The most common form of government bonds are nominal bonds that pay fixed coupons and principal.
Inflation-indexed bonds, which in the U.S. are known as Treasury Inflation Protected Securities (TIPS), are bonds whose coupons and principal adjust automatically with the evolution of a consumer price index.
They aim to pay investors a fixed inflation-adjusted coupon and principal, in other words they are real bonds and their yields are typically considered the best proxy for the term structure of real interest rates in the economy.

- Investors holding either inflation-indexed or nominal government bonds are exposed to the risk of changing real interest rates.
- In addition to real interest rate risk, nominal government bonds expose investors to inflation risk while real bonds do not. When future inflation is uncertain, the coupons and principal of nominal bonds can suffer from the eroding effects of inflationary surprises.
- Finally, both the nominal and real bond are theoretically affected by a premium for liquidity risk. Liquidity risk, is, the risk of having to sell (or buy) a bond in a thin market and, thus, at an unfair price and with higher transaction costs.

Break-even inflation

At time t the yields to maturity of nominal and real bonds maturing at T can be written as follows:

$$\begin{aligned} Y_{t,T}^{n} &= rr_{t,T} + E_t \pi_{t,T} + RP_t^{rr} + RP_t^{\pi} \\ Y_{t,T}^{r} &= rr_{t,T} + RP_t^{rr} + RP_t^{liq} \end{aligned}$$

the difference in the yield to maturity, usually referred to as the breakeven inflation rate $B_{t,T}$, can be written as:

$$B_{t,T} = E_t \pi_{t,T} + RP_t^{\pi} - RP_t^{liq}$$

The Case of 10-year BTP and BTP Italia

- BTP is a constant coupon bond with a standard relationship between price and yield to maturity. In the case of a 10-year bond we have

$$P_{t,T}^{BTP} = \frac{C}{\left(1 + Y_{t,t+10}^{BTP}\right)} + \frac{C}{\left(1 + Y_{t,t+10}^{BTP}\right)^2} + ... + \frac{1 + C}{\left(1 + Y_{t,t+10}^{BTP}\right)^{10}}.$$

- BTP Italia is an indexed bond in which the coupon paid is made of two components: the coupon, constant in each period, and the inflation adjustment for the coupon and the value of the principal. So the stream of payments for a BTP Italia goes as follows:

$$\begin{aligned}
C_{t+1} &= C(1 + \pi_{t+1}) + (1 + \pi_{t+1}) - 1, \\
C_{t+2} &= C(1 + \pi_{t+2}) + (1 + \pi_{t+1})(1 + \pi_{t+2}) - (1 + \pi_{t+1}), \\
... &= ... \\
C_{t+10} &= 1 + C(1 + \pi_{t+10}) + \prod_{i=1}^{10}(1 + \pi_{t+i}) - \prod_{i=1}^{9}(1 + \pi_{t+i}).
\end{aligned}$$

The Case of 10-year BTP and BTP Italia

- The price can be computed as:

$$P_{t,T}^{BTP,i} = \frac{C(1+\pi_{t,t+10})}{\left(1+Y_{t,t+10}^{BTP,i,n}\right)} + \frac{C(1+\pi_{t,t+10})^2}{\left(1+Y_{t,t+10}^{BTP,i,n}\right)^2} + \ldots + \frac{(1+C)(1+\pi_{t,t+10})^{10}}{\left(1+Y_{t,t+10}^{BTP,i,n}\right)^{10}}.$$

where $Y_{t,t+10}^{BTP,i,n}$ is the nominal yield to maturity of the BTP Italia and $\pi_{t,t+10}$ is the average expected inflation rate over the next 10-year.

- Price and Yields can then be computed as the value of a constant stream of payments discounted with a yield in real terms $Y_{t,t+10}^{BTPi}$:

$$P_{t,T}^{BTPi} = \frac{C}{\left(1+Y_{t,t+10}^{BTPi}\right)} + \frac{C}{\left(1+Y_{t,t+10}^{BTPi}\right)^2} + \ldots + \frac{1+C}{\left(1+Y_{t,t+10}^{BTPi}\right)^{10}}.$$

- So assuming that

$$Y_{t,t+10}^{BTP,i,n} = Y_{t,t+10}^{BTPi}$$

then, the breakeven inflation rate can be computed as

$$Y_{t,t+10}^{BTP} - Y_{t,t+10}^{BTPi}.$$