The Intertemporal Government Budget Constraint

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Contents

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1 Introduction

The government intertemporal budget constraint is a stock-flow relationship determing the debt dynamics. Consider the general case in which debt is issued at different maturities and so payments in each period reflect the average cost of financing the debt.There are two ways of writing the government budget constraint: face-value and market prices. The difference between the two expressions depends on how the debt instrument are priced: face-value sets prices of the bonds at their value at maturity(which are equal to 1), market prices sets prices for all bonds at their current levels (which are different from one). From the point of view of the debt issuer what matters is the face-value, while form the point of view of the debt holder what matters is the market price value, unless bonds are held to maturity.

1.1 The IGBC at face value and market prices

The dynamics of the debt at face value is:

$$B_{G,t} = B_{G,t-1} + R_t^{av} B_{G,t-1} + PDef_t + SFA_t \quad (1)$$

$$R_t^{av} = \frac{IP_t}{B_{G,t-1}} \quad (2)$$

IP_t are interest payments at time t, that depend on the entire term structure of interest rates. SFA_t is a stock-flow adjustment term, commonly ignored in the textbook but of some relevance in practice. This term emerges because figures on debt levels and interest payments track changes in cash accounting, whereas figures on tax revenues, expenditure and primary deficits are obtained from accrual based accounting reflecting the need to plan the budget ahead. Data on stocks (debt) and flows (deficits) are therefore not mutually consistent, and government account reconciliation is are typically obtained by creating an artificial accounting figure, stock-flow adjustments. SFA are produced by a number of different factors, such as: net acquisitions of financial assets; transactions in liabilities that are excluded from standard government debt definitions,like derivatives; valuation effects caused by debt issuance above/below par, or redemption of debt above/below par; appreciation/depreciation of foreign-currency debt.

1.2 Roll-over risk and the GFN

Note that the debt dynamics considered so far is general in that the government here issue several debt instruments, typically bond at different maturities. Given that the one of the main risk related to government debt is roll-over risk, Si.e. the risk that a government may be unable to refinance its existing debt obligations (roll them over into new debt) as they come due, sometimes a modified version of the government intertemporal budget constraint is used to have an immediate gauge of the relevance of roll-over risk. Define Gross Financing Needs at the sum of the primary deficit and the bond coming to maturity in period t, B_t^m then we can rewrite the Government intertemporal budget constraint as follows:

$$B_{G,t} = B_{G,t-1} + R_t^{av} B_{G,t-1} - B_{G,t}^m + GFN_t + SFA_t \tag{3}$$

$$GFN_t = B_{G,t}^m + PDef_t \tag{4}$$

This version allows to keep track of GFN_t which is a natural driver of roll-over risk.

1.3 IGBC at market prices

The dynamics of the debt at market prices is instead:

$$P_t B_{G,t} = \frac{P_t}{P_{t-1}} P_{t-1} B_{G,t-1} + IP_t + PDef_t + SFA_t \tag{5}$$

Note that the dynamics of the debt at market prices depend on a revaluation effect that it is not present when the debt is computed at face value.

The revaluation effect is expressed as follows:

$$\left(1 + \frac{P_t - P_{t-1}}{P_{t-1}}\right) P_{t-1} B_{G,t-1} = \frac{P_t}{P_{t-1}} P_{t-1} B_{G,t-1} \tag{6}$$

2 Debt/GDP Dynamics

The dynamics of the debt at face value at time t scaled by GDP at time t, Y_t, is

$$B_{G,t} = B_{G,t-1} + R_t^{av} B_{G,t-1} + PDef_t + SFA_t \tag{7}$$

$$Y_t = Y_{t-1}(1 + g_t) \tag{8}$$

$$b_t = \frac{B_{G,t}}{Y_t} \tag{9}$$

$$d_t = \frac{PDef_t}{Y_t} \tag{10}$$

$$sfa_t = \frac{SFA_t}{Y_t} \tag{11}$$

The dynamics of the debt to GDP ratio becomes:

$$b_t = \frac{(1 + R_t^{av})}{(1 + g_t)} b_{t-1} + d_t + sfa_t \tag{12}$$

by subtracting b_{t-1} from the left and right hand side we have:

$$b_t - b_{t-1} = \frac{R_t^{av} - g_t}{(1 + g_t)} b_{t-1} + d_t + sfa_t \tag{13}$$

which shows that the debt to GDP dynamics depends on three components, namely, the snowball-effect, $\frac{R_t^{av} - g_t}{(1+g_t)} b_{t-1}$, the primary surplus to GDP, d_t, and the Stock-Flow Adjustment to GDP, sfa_t. The following Table illustrates the practical importance of SFA:

Table 1: Determinants of Italian Debt/GDP Ratio (% of GDP)

	2023	2024	2025	2026	2027
Level (gross of support)	137.3	137.8	138.9	139.8	139.6
Change from previous year	-3.2	0.5	1.1	0.9	-0.2
Factors determining changes in public debt:					
Primary balance (economic competence)	3.4	0.4	-0.3	-1.1	-2.2
Snowball effect	-4.5	-1.0	-0.7	0.1	0.7
of which: Interest (economic competence)	3.8	3.9	4.0	4.1	4.4
Stock-flow adjustment	-2.1	1.1	2.1	2.0	1.3
of which: Difference between cash and accruals	-2.6	1.6	1.8	1.3	0.8
Net accumulation of financial assets	0.2	-0.6	0.2	0.5	0.3
of which: Revenues from privatizations	0.0	0.0	-0.2	-0.3	-0.2
Debt revaluation effects	0.3	0.0	0.1	0.2	0.2
Other	0.0	0.0	0.0	0.0	0.0
p.m.: Implicit interest rate on debt (%)	2.9	3.0	3.0	3.1	3.2
Source: Table III.10, DEF 2024, page 75					

2.1 Debt Sustainability

The future path of the debt to GDP ratio can be obtained by solving by solving forward the intertemporal budget contraint:

$$b_t = \sum_{j=0}^{m} \left(\prod_{j=0}^{m} \left(\frac{1 + g_{t+j}}{1 + R_{t+j}^p} \right) (-d_{t+j} - sfa_{t+j}) \right) + \prod_{j=1}^{m} \left(\frac{1 + g_{t+j}}{1 + R_{t+j}^p} \right) b_{t+m} \tag{14}$$

Debt is defined to be sustainable when the following condition , known as the transversality condition, is satisfied:

$$\prod_{j=1}^{\infty}\left(\frac{1+g_{t+j}}{1+R^{p}_{t+j}}\right)b_{\infty}=0 \tag{15}$$

In practice, the dynamics of the debt to GDP ratio can be found by solving the following system:

$$sfa_t = \epsilon_t \tag{16}$$
$$R^{av}_t = rr_t + \pi_t = \mu_1 \tag{17}$$
$$g_t = \Delta y_t + \pi_t = \mu_2 \tag{18}$$
$$d_t = -f(R^{av}, g_t, b_{t-1}) \tag{19}$$
$$b_t = \frac{1+R^{av}_t}{1+g_t}b_{t-1} + d_t + sfa_t \tag{20}$$

Where the first three equations specify the processes for the average cost of financing the debt, GDP growth, and the stock-flow adjustment term. The fourth equation captures the policy rule followed by the fiscal authority and the last equation solves the intertemporal budget contraint period-by-period for the debt to GDP ratio. Simulation of this system allows to evaluate the effect of different fiscal policies on the DEBT/GDP ratio. Consider for example the case in which the government adopts a debt stabilizing rule without taking in consideration the stock-flow adjustment term. In this case the fiscal reaction function would take the following specification:

$$d_t = -((R^{av}_t - g_t)/(1+g_t)) * b_{t-1}$$

The resulting debt dynamics, when the initial value of the debt/gdp ratio is 1.2, is reported in Figure 1.

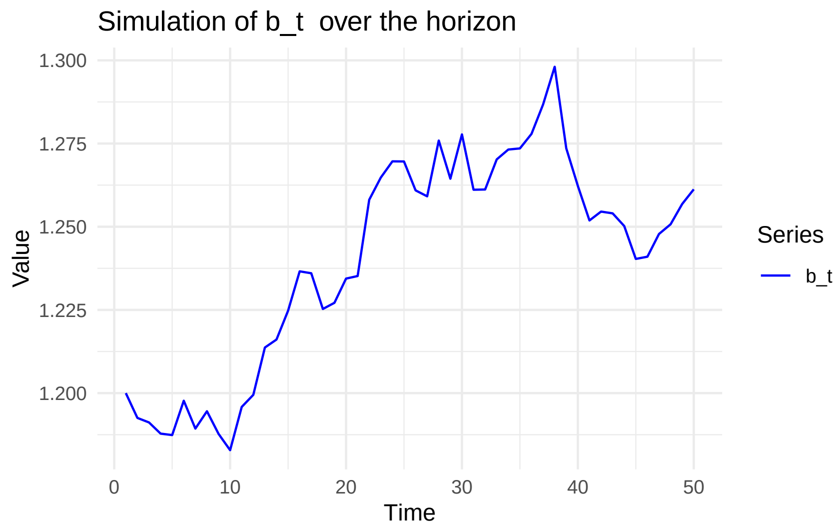

Figure 1: Debt to GDP Dynamics for 50 periods with d_0=1.2

To experiment with different scenarios and different rules you can use the following R code.

The first useful exercise would be to check that when sfa_t is set to zero in every period, perfect stabilization of the debt to GDP ratio at the initial level is achieved by the rule.

The next exercise is to consider what happens when the government adopts the following rule in a world in which $R_t^{av} > g_t$

$$d_t = -((R_t^{av} - g_t)/(1 + R_t^{av})) * b_{t-1}$$

As $R_t^{av} > g_t$, the rule leads to a sequence of primary surpluses that are smaller than the sequence that achieves debt stabilization (up to the effect of sfa_t).

The interesting question is if the rule leads to debt sustainability. Start by plotting the debt to GDP ratio delivered by this rule. Then, before rushing to conclusions, check for the validity of the transversality condition. Would you conclude that the validity of the transversality condition is a loose criterion for sustainability ? A further interesting question is to define a rule which will ensure nominal debt stability

$d_t = -(R_t^{av}/(1 + g_t)) * b_{t-1}$

In fact writing the debt dynamics in nominal term we have:

$$D_t = (1 + R_t^{av})D_{t-1} + DEF_t$$

so the nominal debt stabilizing surplus is

$$DEF_t = -(R_t^{av}) * D_{t-1}$$

and rescaling for GDP at time t we have

$$d_t = -(R_t^{av}/(1 + g_t)) * b_{t-1}$$

2.2 A code in R for simulating the Debt Dynamics

```
        # Programme to simulate  the dynamics of debt/GDP  with the simplest model
# Traditional framework
# Carlo A. Favero November 2024

rm(list = ls())
setwd(dirname(rstudioapi::getActiveDocumentContext()$path))
listofpackages <- c("ggplot2")

for (j in listofpackages){
  if (sum(installed.packages()[, 1] == j) == 0) {
    install.packages(j)
  }
  library(j, character.only = TRUE)
}

# Load necessary libraries
library(ggplot2)

# Parameters
mu_1 <- 0.05  # Example value for R^{av}_t (mu_1)
mu_2 <- 0.03  # Example value for g_t (mu_2)

# Initialization
n_periods <- 100
b <- numeric(n_periods)
d <- numeric(n_periods)
sfa <- numeric(n_periods)
disc_b<- numeric(n_periods)
bstar <- numeric(n_periods)
R_av <- rep(mu_1, n_periods)
g <- rep(mu_2, n_periods)
bstar <- rep(0.6, n_periods)

# Initial values for b_t and d_t
b[1] <- 1.2  # Arbitrary starting value for b_1
disc_b[1]<-b[1] # discounted b set equal to b in the first period
d[1] <- -0.03    # Initial value for d_1 (could be different)
sfa[1] <- 0

# Simulate the series
for (t in 2:n_periods) {
  R_av[t] <- mu_1
  g[t] <- mu_2
  sfa[t] <- rnorm(1, mean = 0.00, sd = 0.01)
  d[t] <- -((R_av[t] - g[t]) / (1 + g[t])) * b[t-1]
```

```r
  b[t] <- ((1 + R_av[t]) / (1 + g[t])) * b[t-1] + d[t] + sfa[t]
  disc_b[t]<- b[t]*((1 +  g[t]) / (1 + R_av[t]))^t
}


# Create a date series starting from 2024
start_date <- as.Date("2024-01-01")
dates <- seq.Date(from = start_date, by = "year", length.out = n_periods)

# Create a data frame for plotting
data <- data.frame(
  Date = dates,
  b_t = b,
  d_t = d,
  disc_b_t=disc_b
)

# Plot the series using ggplot2
ggplot(data, aes(x = Date)) +
  geom_line(aes(y = b_t, color = "b_t"), linewidth = 1) +
  labs(title = "Simulation of b_t over the horizon",
       y = "Value", x = "Date") +
  scale_color_manual(name = "Series", values = c("b_t" = "blue")) +
  theme_minimal()

ggplot(data, aes(x = Date)) +
  geom_line(aes(y = d_t, color = "d_t"), linewidth = 1) +
  labs(title = "Simulation of d_t over the horizon",
       y = "Value", x = "Date") +
  scale_color_manual(name = "Series", values = c("d_t" = "red")) +
  theme_minimal()

ggplot(data, aes(x = Date)) +
  geom_line(aes(y = disc_b_t, color = "disc_b_t"), linewidth = 1) +
  labs(title = "discounted value of future d_t over the horizon",
       y = "Value", x = "Date") +
  scale_color_manual(name = "Series", values = c("disc_b_t" = "green")) +
  theme_minimal()
```

3 Debt Composition:some special cases

3.1 The Government Intertemporal Budget Constraint with a one-period discount bond

Consider the case in which all debt is financed with a one-period discount bond that it issued in period t with a price P_t and repaid in period t+1 at a price of one. No-arbitrage determines the price of the government discount bond P_t in presence of the price of a safe asset which is determined by the CB, i.e. the one-period monetary policy rate R_t, so if the government debt is also risk-free, by no arbitrage we have:

$$P_t = \frac{1}{1+R_t} \tag{21}$$

In this case, the dynamics of the debt at market prices is the following:

$$\begin{aligned} P_t B_{G,t} &= P_{t-1}B_{G,t-1} + (1-P_{t-1})B_{G,t-1} + PDef_t + SFA_t & (22) \\ &= B_{G,t-1} + PDef_t + SFA_t & (23) \\ PDef_t &= G_t - T_t & (24) \end{aligned}$$

where $1 - P_{t-1}$ represents the cost of financing each bond issued at time t-1 with a price P_{t-1} and repaid at the face value of 1 at time t. $PDef_t$ is the primary deficit obtained by subtracting government revenue to government non-interest expenditure, SFA_t is a stock-flow adjustment term ignored in the textbooks but very relevant empirically. by using the no-arb condition we can re-write the constraint as follows:

$$B_{G,t} = (1+R_t)(B_{G,t-1} + PDef_t + SFA_t) \tag{25}$$

which is different from the budget constraint at face value:

$$B_{G,t} = B_{G,t-1} + R_{t-1}B_{G,t-1} + PDef_t + SFA_t \tag{26}$$

3.2 The Government Intertemporal Budget Constraint with a perpetuity

Consider now the case in which all debt is financed with perpetuities issued in any period and repriced in every period so that the price of perpetuities is the same independently on the issue period. The price of the perpetuity is:

$$P_t = \frac{1}{R_t^p} \tag{27}$$

However, as the perpetuity is riskier than the safe asset, no-arbitrage implies:

$$R_t^p = R_t + RP_t \tag{28}$$

where again the price of a safe asset which is determined by the CB, that controls the one-period monetary policy rate R_t. When the debt is financed via a perpetuity repriced in every period (setting SFA=0) and paying a coupon of 1 in each period we have:

$$P_t B_{G,t} = P_{t-1} B_{G,t-1} + \pi_t P_{t-1} B_{G,t-1} + B_{G,t-1} + PDef_t \tag{29}$$

$$P_t = \frac{1}{R_t^p} \tag{30}$$

$$\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{31}$$

by dividing the intertemmporal budget constraint by P_t we have:

$$B_{G,t} = B_{G,t-1} + R_{t-1}^p B_{G,t-1} + PDef_t \tag{32}$$

4 Debt and Inflation: The Traditional Approach

The traditional approach to analyze the interaction between government debt and inflation has two building blocks: the intertemporal government budget contraint, in which seignorage is explicitly allowed for and a portion of the debt is financed by money growth, and a relationship determining how money is demanded by economic agents. In the intertemporal budget constraint seignorage is explicitly allowed for and a portion of the debt is financed by money growth. The money demand relationship pins down the mapping between money growth and inflation .

$$B_{G,t} = B_{G,t-1} - (M_t - M_{t-1}) + R_t^{av} B_{G,t-1} + PDef_t + SFA_t \tag{33}$$

$$\frac{M_t}{P_t y_t} = e^{-\beta(rr_t + \pi_t)} \tag{34}$$

$$y_t = \frac{Y_t}{P_t} \tag{35}$$

4.1 Money,Seignorage and Inflation Tax

Taking logs of the money demand equation we have:

$$m_t - p_t - y_t = -\beta(rr_t + \pi_t) \tag{36}$$

This equation allows to pin down the "equilibrium" drivers of inflation. In fact,as in equilibrium the real interest rate and inflation are constant, we have:

$$\pi_t = \Delta m_t - \Delta y_t \tag{37}$$

Note that seignorage (i.e. financing debt with printing money) results in an inflation tax, which is defined as follows:

$$\pi_t \frac{M_t}{P_t y_t} = \pi_t e^{-\beta(rr_t + \pi_t)} \tag{38}$$

For our chosen parameterization, the revenue from the inflation tax reflects a curve called the "Laffer Curve". The Tax revenue initially grows but after the tax rate exceeds a treshold the reduction in the tax base more than compensate the increase in the tax rate and the tax revenue decreases. Note that for a parameterization of the Laffer curve that allows

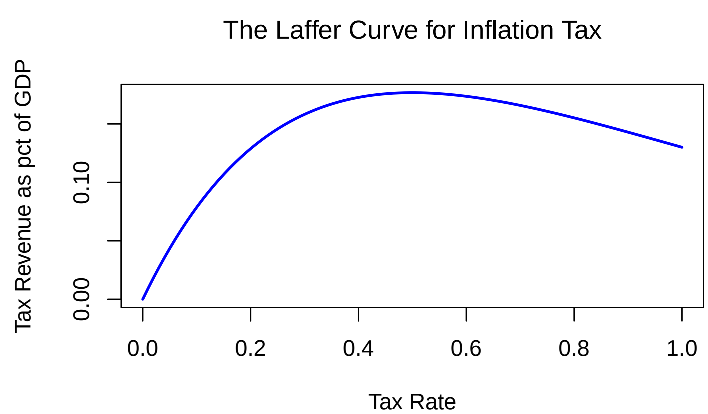

Figure 2: Laffer Curve. $TAX = \pi_t e^{-\beta(rr_t + \pi_t)}$,$\beta = 2, rr = 0.02$

to replicate the ratio of money (M3) to GDP in the case of Italy , the curve reaches its maximum at an inflation rate of 50 per cent, making the downard sloping part of the curve of very little practical relevance.

4.2 Debt Dynamics with Seignorage

Consider now the dynamics of the debt to GDP ratio :

$$b_t = \frac{(1+R_t^{av})}{(1+g_t)} b_{t-1} - \frac{M_t - M_{t-1}}{M_t} \frac{M_t}{P_t y_t} + d_t + sfa_t \tag{39}$$

The full dynamic system to simulate the path of the debt to GDP ratio becomes now:

$$\begin{aligned}
sfa_t &= \epsilon_t \\
\Delta y_t &= \mu_1 \\
\Delta m_t &= \mu_2 \\
rr_t &= \mu_3 \\
\pi_t &= \Delta m_t - \Delta y_t \\
g_t &= \Delta y_t + \pi_t \\
R_t^{av} &= = rr_t + + \pi_t \\
\frac{M_t}{P_t y_t} &= e^{-\beta(rr_t + \pi_t)} \\
d_t &= -f(R^{av}, g_t, b_{t-1}) \\
b_t &= \frac{(1+R_t^{av})}{(1+g_t)} b_{t-1} - \Delta m_t \frac{M_t}{P_t y_t} + d_t + sfa_t
\end{aligned}$$

4.3 A code in R for simulating the Debt Dynamics with Seignorage

The Following R code allows the simulation of debt dynamics when seignorage is allowed for. Several interesting simulations can be conducted with the code, in particular one could evaluate first a baseline scenario of unsustanaible debt (obtained setting, for example, the primary surplus at 1.25 per cent of GDP and money growth at zero in a case where r¿g) and then checking if it is possible to reach debt sustainable via seignorage without any intervention on the primary surplus.

```r
# Programme to  simulate dynamics of debt with seignorage
# Traditional framework
# Carlo A. Favero November 2024

# set up directory and load packages
rm(list = ls())
setwd(dirname(rstudioapi::getActiveDocumentContext()$path))
listofpackages <- c("ggplot2")

for (j in listofpackages){
  if (sum(installed.packages()[, 1] == j) == 0) {
    install.packages(j)
  }
  library(j, character.only = TRUE)
}

############################
# The Laffer Curve
############################
# Define constants
rr <- 0.03
beta <- 2

# Define the range for pi_t
pi_t <- seq(0, 1, length.out = 100)

# Calculate y_t as a function of pi_t
y_t <- pi_t*exp(-beta * (rr + pi_t))

# Create the plot using ggplot2
plot <- ggplot(data = data.frame(pi_t, y_t), aes(x = pi_t, y = y_t)) +
  geom_line(color = "blue", linewidth = 1.2) +
  geom_hline(yintercept = 0, color = "black", linewidth = 0.8, linetype =
     "dashed") +  # Add zero line
  labs(x = expression(paste("Tax Rate")),
       y = expression(paste("Tax Revenue as % of GDP")),
       title = expression(paste("The Laffer Curve for Inflation Tax"))) +
  theme_minimal()

# Save the plot as a PDF file
ggsave("Laffer_Curve_Inflation_Tax.pdf", plot = plot, width = 8, height = 6)


############################
# The Debt Dynamics
############################
```

```
# Parameters
mu_1 <- 0.03  # real rate
mu_2 <- 0.01  # real gdp growth
mu_31 <- 0.03  # money growth first period
mu_32 <- 0.05  # money growth after policy switch

# Initialization
n_periods <- 100
R_r<- numeric(n_periods)
R_av<- numeric(n_periods)
g_r<- numeric(n_periods)
infl<- numeric(n_periods)
mg<- numeric(n_periods)
g<- numeric(n_periods)
sfa <- numeric(n_periods)
itb <- numeric(n_periods)
b <- numeric(n_periods)
d <- numeric(n_periods)
disc_b<- numeric(n_periods)
bstar <- numeric(n_periods)
bstar <- rep(0.6, n_periods)

# Initial values for b_t and d_t
b[1] <- 1.2  # Arbitrary starting value for b_1
disc_b[1]<-b[1] # discounted b set equal to b in the first period
d[1] <- -0.03    # Initial value for d_1 (could be different)
sfa[1] <- 0
itb[1] <- 1

# Simulate the series
for (t in 2:n_periods) {
  sfa[t] <- rnorm(1, mean = 0.00, sd = 0.01)
  R_r[t] <- mu_1
  g_r[t] <- mu_2
  # Set money growth conditionally
  if (t <= 20) {
    mg[t] <- mu_31
  } else {
    mg[t] <- mu_32
  }
  infl[t]<-mg[t]-g_r[t]
  R_av[t]<-R_r[t]+infl[t]
  g[t]<- g_r[t]+infl[t]
  itb[t]<-exp(-beta *  R_av[t])
  d[t] <- 0.0125 #-((R_av[t] - g[t]) / (1 + g[t])) * b[t-1]
  b[t] <- ((1 + R_av[t]) / (1 + g[t])) * b[t-1] -mg[t]*itb[t] + d[t] + sfa[t]
```

```
  disc_b[t]<- b[t]*((1 +  g[t]) / (1 + R_av[t]))^t
}


# Create a date series starting from 2024
start_date <- as.Date("2024-01-01")
dates <- seq.Date(from = start_date, by = "year", length.out = n_periods)

# Create a data frame for plotting
data <- data.frame(
  Date = dates,
  b_t = b,
  d_t = d,
  disc_b_t=disc_b
)

# Plot the series using ggplot2
ggplot(data, aes(x = Date)) +
  geom_line(aes(y = b_t, color = "b_t"), linewidth = 1) +
  labs(title = "Simulation of b_t over the horizon",
       y = "Value", x = "Date") +
  scale_color_manual(name = "Series", values = c("b_t" = "blue")) +
  theme_minimal()

ggplot(data, aes(x = Date)) +
  geom_line(aes(y = d_t, color = "d_t"), linewidth = 1) +
  labs(title = "Simulation of d_t over the horizon",
       y = "Value", x = "Date") +
  scale_color_manual(name = "Series", values = c("d_t" = "red")) +
  theme_minimal()

ggplot(data, aes(x = Date)) +
  geom_line(aes(y = disc_b_t, color = "disc_b_t"), linewidth = 1) +
  labs(title = "discounted value of future d_t over the horizon",
       y = "Value", x = "Date") +
  scale_color_manual(name = "Series", values = c("disc_b_t" = "green")) +
  theme_minimal()
```