The Government Intertemporal Budget Constraint

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Introduction

The government intertemporal budget constraint is a stock-flow relationship determining the debt dynamics.

- Consider the general case in which debt is issued at different maturities and so payments in each period reflect the average cost of financing the debt.

- There are two ways of writing the government budget constraint: face-value and market prices.
    - face-value sets prices of the bonds at their value at maturity(which are equal to 1)
    - market prices sets prices for all bonds at their current levels (which are different from one)

- From the point of view of the debt issuer what matters is the face-value, while from the point of view of the debt holder what matters is the market price value, unless bonds are held to maturity.

The dynamics of the debt at face value

The dynamics of the debt at face value is:

$$B_{G,t} = B_{G,t-1} + R_t^{av} B_{G,t-1} + PDef_t + SFA_t \qquad (1)$$

$$R_t^{av} = \frac{IP_t}{B_{G,t-1}} \qquad (2)$$

- IP_t are interest payments at time t, that depend on the entire term structure of interest rates, $PDef_t$ is the primary deficit, i.e. total deficit minus interest payments, SFA_t is a stock-flow adjustment term, commonly ignored in the textbooks but of clear relevance in practice.
  - This term emerges because figures on debt levels and interest payments track changes in cash accounting, whereas figures on tax revenues, expenditure and primary deficits are obtained from accrual based accounting reflecting the need to plan the budget ahead.
  - SFA are produced by a number of different factors, such as: net acquisitions of financial assets; transactions in liabilities that are excluded from standard government debt definitions, like derivatives; valuation effects caused by debt issuance above/below par, or redemption of debt above/below par; appreciation/depreciation of foreign-currency debt.

Debt Dynamics with Gross Financing Needs

- The debt dynamics considered so far is general in that the government here issue several debt instruments, typically bond at different maturities.

- Given that the one of the main risk related to government debt is roll-over risk, i.e. the risk that a government may be unable to refinance its existing debt obligations (roll them over into new debt) as they come due, sometimes a modified version of the government intertemporal budget constraint is used to have an immediate gauge of the relevance of roll-over risk.

- Define Gross Financing Needs as the sum of the primary deficit and the bond coming to maturity in period t, B_t^m then we can rewrite the Government intertemporal budget constraint as follows:

$$B_{G,t} = B_{G,t-1} + R_t^{av} B_{G,t-1} - B_{G,t}^m + GFN_t + SFA_t$$
$$GFN_t = B_{G,t}^m + PDef_t$$

This version allows to keep track of GFN_t which is a natural driver of roll-over risk.

The dynamics of the debt at market prices is instead:

$$P_t B_{G,t} = P_{t-1} B_{G,t-1} + Rev_t + IP_t + PDef_t + SFA_t \qquad (3)$$

Note that the dynamics of the debt at market prices depend on a revaluation effect, Rev_t, that it is not present when the debt is computed at face value.
The revaluation effect, is expressed as follows:

$$Rev_t = \frac{P_t - P_{t-1}}{P_{t-1}} P_{t-1} B_{G,t-1} \qquad (4)$$

Debt/GDP Dynamics

To obtain the dynamics of the debt at face value at time t scaled by GDP at time t, Y_t, consider the following:

$$
\begin{aligned}
B_{G,t} &= B_{G,t-1} + R_t^{av} B_{G,t-1} + PDef_t + SFA_t \\
Y_t &= Y_{t-1}(1 + g_t), \ g_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}} \\
b_t &= \frac{B_{G,t}}{Y_t}, \ d_t = \frac{PDef_t}{Y_t}, \ sfa_t = \frac{SFA_t}{Y_t}
\end{aligned}
$$

The dynamics of the debt to GDP ratio then becomes:

$$
b_t = \frac{(1 + R_t^{av})}{(1 + g_t)} b_{t-1} + d_t + sfa_t \tag{5}
$$

by subtracting b_{t-1} from the left and right hand side we have:

$$
b_t - b_{t-1} = \frac{R_t^{av} - g_t}{(1 + g_t)} b_{t-1} + d_t + sfa_t \tag{6}
$$

which shows that the debt to GDP dynamics depends on three components, namely, the snowball-effect, $\frac{R_t^{av} - g_t}{(1 + g_t)} b_{t-1}$, the primary surplus to GDP , d_t, and the Stock-Flow Adjustment to GDP, sfa_t.

Table: Determinants of Italian Debt/GDP Ratio (% of GDP)

	2023	2024	2025	2026	2027
Level (gross of support)	137.3	137.8	138.9	139.8	139.6
Change from previous year	-3.2	0.5	1.1	0.9	-0.2
Factors determining changes in public debt:					
Primary balance	3.4	0.4	-0.3	-1.1	-2.2
Snowball-effect	-4.5	-1.0	-0.7	0.1	0.7
of which: Interest	3.8	3.9	4.0	4.1	4.4
Stock-flow adjustment	-2.1	1.1	2.1	2.0	1.3
of which: Difference between cash and accruals	-2.6	1.6	1.8	1.3	0.8
Net accumulation of financial assets	0.2	-0.6	0.2	0.5	0.3
of which: Revenues from privatizations	0.0	0.0	-0.2	-0.3	-0.2
Debt revaluation effects	0.3	0.0	0.1	0.2	0.2
Other	0.0	0.0	0.0	0.0	0.0
p.m.: Implicit interest rate on debt (%)	2.9	3.0	3.0	3.1	3.2
				Source: Table III.10, DEF 2024, page 75	

Debt Sustainability

The future path of the debt to GDP ratio can be obtained by solving forward the intertemporal budget constraint:

$$b_t = \sum_{j=0}^{m} \left(\prod_{j=1}^{m} \left(\frac{1 + g_{t+j}}{1 + R^P_{t+j}} \right) (-d_{t+j} - sfa_{t+j}) \right) + \prod_{j=1}^{m} \left(\frac{1 + g_{t+j}}{1 + R^P_{t+j}} \right) b_{t+m} \qquad (7)$$

- Debt is defined to be sustainable when the following condition, known as the transversality condition, is satisfied:

$$\prod_{j=1}^{\infty} \left(\frac{1 + g_{t+j}}{1 + R^P_{t+j}} \right) b_\infty = 0 \qquad (8)$$

- Therefore, debt is sustainable when the current debt to GDP ratio is compensated by the stream of future surplus to GDP ratios corrected to take in account the effect of SFA, interest rates and growth.

Assessing Debt Sustainability

Assessing Debt sustainability requires to pin-down the dynamics of future surpluses, SFA, average cost of financing the debt and growth. In the simplest case, the dynamics of the debt to GDP ratio can be found by solving the following system:

$$
\begin{aligned}
sfa_t &= \epsilon_t \\
R_t^{av} &= rr_t + \pi_t = \mu_1 \\
g_t &= \Delta y_t + \pi_t = \mu_2 \\
d_t &= -f(R^{av}, g_t, b_{t-1}) \\
b_t &= \frac{1 + R_t^{av}}{1 + g_t} b_{t-1} + d_t + sfa_t
\end{aligned}
$$

- The first three equations specify the processes for the average cost of financing the debt, GDP growth, and the stock-flow adjustment term.

- The fourth equation captures the policy rule followed by the fiscal authority

- the last equation solves the intertemporal budget contraint period-by-period for the debt to GDP ratio.

Simulation of this system allows to evaluate the effect of different fiscal policies on the debt to GDP ratio.

Assessing Debt Sustainability

Consider for example the case in which the government adopts a debt stabilizing rule without taking in consideration the stock-flow adjustment term. In this case the fiscal reaction function would take the following specification:

$$d_t = -((R_t^{av} - g_t)/(1 + g_t)) * b_{t-1}$$

The resulting debt dynamics, when the initial value of the debt/gdp ratio is 1.2, is reported in Figure 1.

Debt Dynamics under a debt-stabilizing rule

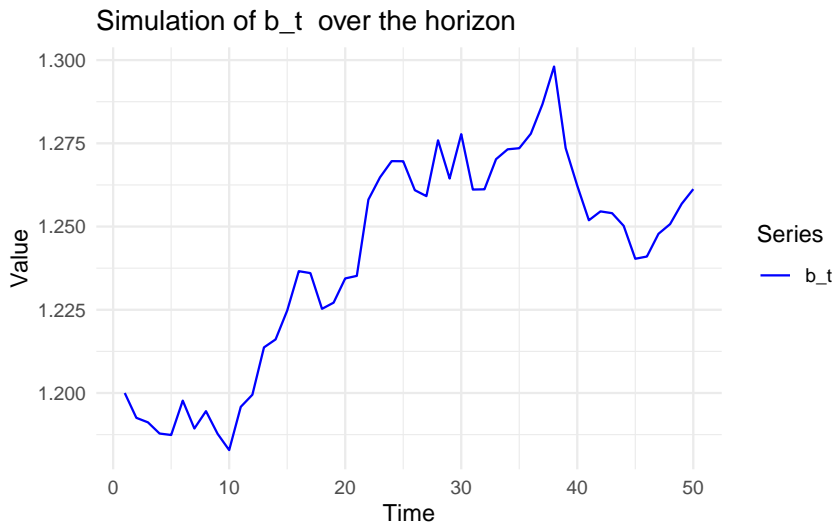## Simulation of b_t over the horizon

Figure: Debt to GDP Dynamics for 50 periods with $d_0=1.2$

Simulating Debt Dynamics with R

An R code is provided to experiment with different macroeconomic scenarios and different fiscal rules.

Debt and Inflation: The Traditional Approach

The traditional approach to analyze the interaction between government debt and inflation has two building blocks:

- the intertemporal government budget constraint, in which seignorage is explicitly allowed for and a portion of the debt is financed by money growth,
- a relationship determining how economic agents demand money.

$$
\begin{aligned}
B_{G,t} &= B_{G,t-1} - (M_t - M_{t-1}) + R_t^{av} B_{G,t-1} + PDef_t + SFA_t \\
\frac{M_t}{P_t y_t} &= e^{-\beta(rr_t + \pi_t)} \\
y_t &= \frac{Y_t}{P_t}
\end{aligned}
$$

Money Demand, Inflation and Seignorage

Taking logs of the money demand equation we have:

$$m_t - p_t - y_t = -\beta(rr_t + \pi_t) \tag{9}$$

This equation allows to pin down the "equilibrium" drivers of inflation. In fact,as in equilibrium the real interest rate and inflation are constant, we have:

$$\pi_t = \Delta m_t - \Delta y_t \tag{10}$$

Consider now the dynamics of the debt to GDP ratio :

$$b_t = \frac{(1 + R_t^{av})}{(1 + g_t)} b_{t-1} - \frac{M_t - M_{t-1}}{M_t} \frac{M_t}{P_t y_t} + d_t + sfa_t \tag{11}$$

Seignorage is the part of the debt to GDP ratio financed by printing money $\frac{M_t - M_{t-1}}{M_t} \frac{M_t}{P_t y_t}$
Note that seignorage results in an inflation tax, which is defined as follows:

$$\pi_t \frac{M_t}{P_t y_t} = \pi_t e^{-\beta(rr_t + \pi_t)} \tag{12}$$

The Laffer Curve for Inflation Tax, Inflation and Seignorage

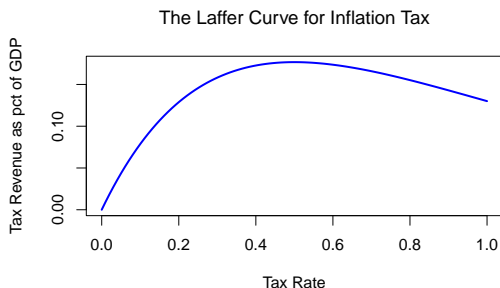For our chosen parameterization, the revenue from the inflation tax reflects a curve called the "Laffer Curve".

Figure: Laffer Curve. $TAX = \pi_t e^{-\beta(rr_t + \pi_t)}, \beta = 2, rr = 0.02$

Seignorage in Italy

Our parameterization of the Laffer curve that allows to replicate the ratio of money (M3) to GDP in the case of Italy. We report herewith the historical trend of seignorage in Italy in 70s and 80s:

Italy
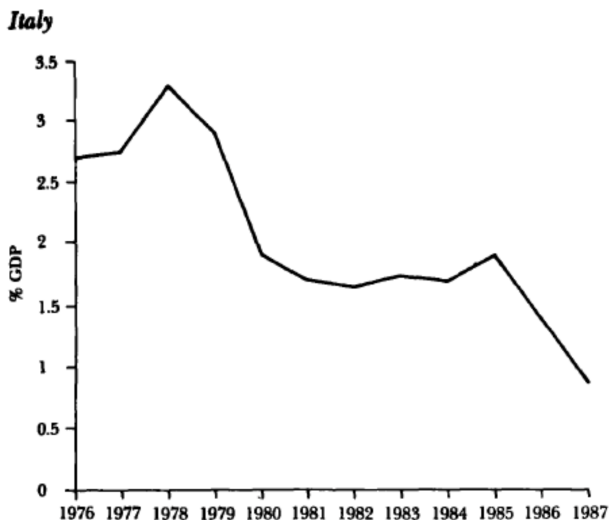
Debt dynamics with money financing

The full dynamic system to simulate the path of the debt to GDP ratio becomes now:

$$
\begin{aligned}
sfa_t &= \epsilon_t \\
\Delta y_t &= \mu_1 \\
\Delta m_t &= \mu_2 \\
rr_t &= \mu_3 \\
\pi_t &= \Delta m_t - \Delta y_t \\
g_t &= \Delta y_t + \pi_t \\
R_t^{av} &= = rr_t + + \pi_t \\
\frac{M_t}{P_t y_t} &= e^{-\beta(rr_t + \pi_t)} \\
d_t &= -f(R^{av}, g_t, b_{t-1}) \\
b_t &= \frac{(1 + R_t^{av})}{(1 + g_t)} b_{t-1} - \Delta m_t \frac{M_t}{P_t y_t} + d_t + sfa_t
\end{aligned}
$$

Simulating Debt Dynamics with Seignorage in R

An R code is provided to simulate of debt dynamics when seignorage is allowed for. Several interesting simulations can be conducted with the code,

- in particular one could evaluate first a baseline scenario of unsustainaible debt (obtained setting, for example, the primary surplus at 1.25 per cent of GDP and money growth at zero in a case where r¿g)

- to then check if it is possible to reach debt sustainability via seignorage without any intervention on the primary surplus.
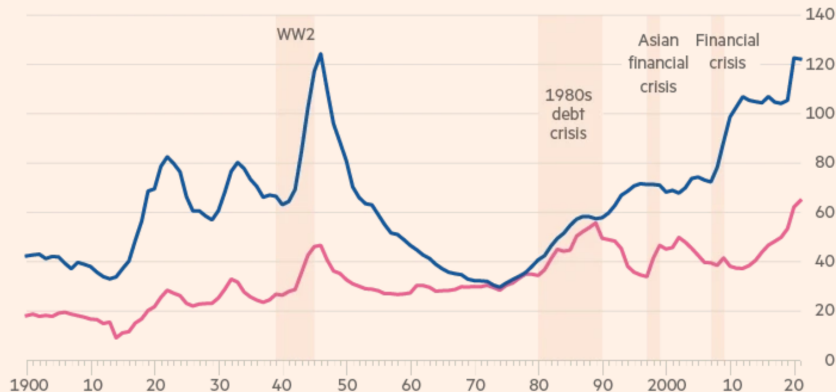
Trends in Debt to GDP

Global debt levels soar to highest since second world war

Median debt-to-GDP ratio (%)

— Developed economies
— Emerging economies

WW2

1980s debt crisis

Asian financial crisis

Financial crisis

Median debt-to-GDP ratio of country grouping based on G20 advanced and G20 emerging economies
Sources: IMF Historical Debt database; IMF WEO databse; Goldman Sachs GIR
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