# Signed Integral Representations of Comonotonic Additive Functionals*

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**Abstract**

We establish integral representation results for suitably pointwise continuous and comonotonic additive functionals of bounded variation defined on Stone lattices. As an application, we prove a comonotonic version of the Daniell-Stone Theorem.

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## 1 Introduction

In the past twenty years and after the seminal papers of Schmeidler [17] and Artzner, Delbaen, Eber and Heath [4], Choquet integrals played an important role in Mathematical Economics and Finance. Two different frameworks are typically used in these fields. The first, introduced by Schmeidler [16] and [17], adopts as function space the space $B(\Sigma)$ of bounded measurable functions where $\Sigma$ is an algebra. This approach is particularly well suited for Decision Theory. The second approach, studied by Zhou [18] and [19], relies on a *Stone vector lattice* $L$. A particular case of Stone vector lattice is the space $C(S)$ of continuous functions over a compact space $S$: a more familiar setting in the theory of integral representations. The purpose of this paper is threefold:

(i) to provide a unified treatment that encompasses these two different settings, $B(\Sigma)$ and $C(S)$. This is achieved by considering *Stone lattices* with suitable density properties in a vector lattice of bounded functions. This notion allows us to use techniques from both frameworks which we combine and extend;

(ii) to extend Choquet integral representations from monotone set functions, often called capacities, to general, not necessarily monotone, set functions. Besides the mathematical interest of our exercise, nonmonotone Choquet integrals are of interest in applications (see, e.g., [8]);

(iii) to provide a genuine version of the Daniell-Stone Theorem (see, e.g., [9, Chapter 4] and [15, Chapter 16]) for comonotonic additive functionals defined on a Stone vector lattice.

Our main results are Theorem 13 and Theorem 22. Theorem 13 shows that a functional $V : L \to \mathbb{R}$ defined on a comonotonic Stone lattice $L$ is comonotonic additive, of bounded variation, and pointwise outer

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continuous if and only if there exists a unique outer continuous set function $\nu : \Sigma_L \to \mathbb{R}$, defined on the collection $\Sigma_L = \{(f \geq t) : f \in L \text{ and } t \in \mathbb{R}\}$ of upper level sets, such that

$$V(f) = \int_0^{\infty} \nu(f \geq t)\, dt + \int_{-\infty}^{0} [\nu(f \geq t) - \nu(S)]\, dt \qquad \forall f \in L. \tag{1}$$

Here, the integrals in the right hand side are in the sense of Riemann. Theorem 13 extends to the non-monotone case the integral representation results of Schmeidler [16] and Zhou [18]. In doing so, it also extends to the nonadditive case some classic integral representation results with signed measures, as shown in Section 5. Moreover, it extends to comonotonic Stone lattices (the nonmonotone) related results of Murofushi, Sugeno, and Machida [13] derived for the case $B(\Sigma)$.[1]

Theorem 22 extends the Daniell-Stone Theorem to the comonotonic additive case. Specifically, if the functional $V$ is also superadditive and pointwise continuous then the integral representation (1) of $V$ displays a continuous and supermodular $\nu$ defined on the entire $\sigma$-algebra generated by $L$. Surprisingly, $\nu$ maintains its uniqueness features despite its larger domain. For this reason, the integral on the right hand side of (1) is a genuine Choquet integral. The classic Daniell-Stone Theorem corresponds to the particular case where $V$ is additive or, equivalently, modular and $\nu$ turns out to be $\sigma$-additive.

Finally in proving Theorem 13, we establish some new results on the decomposition of set functions of bounded variation. This allows to refine the representation in (1).

The paper is organized as follows. After some preliminaries in Section 2, we establish in Section 3 the decomposition results just mentioned. Sections 4 and 6 contain the main integral representation results while Section 5 shows that our results generalize some classic ones.

# 2 Preliminaries

## 2.1 Sets

A collection $\Sigma$ of subsets of a space $S$ is a *lattice (with zero and unit)* if given any two sets $A, B \in \Sigma$ both $A \cup B$ and $A \cap B$ belong to $\Sigma$ (and $\varnothing, S \in \Sigma$). We assume that all lattices $\Sigma$ considered in this paper contain $\varnothing$ and $S$; moreover, generic subsets $A$ and $B$ are understood to belong to $\Sigma$.

A function $\nu : \Sigma \to \mathbb{R}$ is a *set function* if $\nu(\varnothing) = 0$. In particular, a set function $\nu : \Sigma \to \mathbb{R}$ is:

(i) *positive* if $\nu(A) \geq 0$ for all $A$;

(ii) *monotone* if $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$;

(iii) *supermodular (convex)* if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all $A$ and $B$;

(iv) *submodular (concave)* if $\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B)$ for all $A$ and $B$;

(v) *additive* if $\nu(A \cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint $A$ and $B$;

(vi) *outer* (resp., *inner*) *continuous at* $A$ if $\lim_{n\to\infty} \nu(A_n) = \nu(A)$ whenever $A_n \downarrow A$ (resp., $A_n \uparrow A$);

(vii) *continuous at* $A$ if it is both inner and outer continuous at $A$;

(viii) *outer* (resp., *inner*) *continuous* if it is outer (resp., inner) continuous at each $A$;

(ix) *continuous* if it is continuous at each $A$;

[1] These earlier results were rediscovered by Marinacci and Montrucchio [12].

(x) *countably additive* if $\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$ for all countable collections of pairwise disjoint sets $\{A_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.

Monotone set functions are called *capacities*. Notice that capacities are always positive. If $\Sigma$ is an algebra, additive set functions are called *charges* and the countably additive ones are called *measures*. Finally, observe that a set function which is modular – that is, both supermodular and submodular – is additive.

When $\Sigma$ is an algebra, each set function $\nu : \Sigma \to \mathbb{R}$ has a *dual set function* $\bar{\nu} : \Sigma \to \mathbb{R}$ given by $\bar{\nu}(A) = \nu(S) - \nu(A^c)$. It is easy to see that a set function $\nu$ is outer (resp., inner) continuous if and only if its dual $\bar{\nu}$ is inner (resp., outer) continuous.

## 2.2 Functions

Throughout the paper, $L$ is a nonempty collection of bounded functions $f : S \to \mathbb{R}$ where $S$ is a nonempty set. We consider $L$ endowed with the metric induced by the supnorm $\|\cdot\|$. The collection $L$ is

(i) a *lattice* if $f \vee g, f \wedge g \in L$ whenever $f, g \in L$;

(ii) a *Stone lattice* if it is a lattice and $\alpha f + \beta \in L$ for all $f \in L$ and all $\alpha, \beta \in \mathbb{R}$;[2]

(iii) a *Stone vector lattice* if it is both a Stone lattice and a vector space. That is, if it is a vector lattice that contains $1_S$.

Two functions $f, g \in L$ are comonotonic (i.e., commonly monotonic) if

$$(f(s) - f(s'))(g(s) - g(s')) \geq 0 \qquad \forall s, s' \in S,$$

see Denneberg [7] for alternative characterizations of comonotonicity. Next, we introduce a key notion for our analysis.

**Definition 1** *A Stone lattice $L$ is* comonotonic *if there is a Stone vector lattice $E$ such that $L \subseteq E$ and, given any two comonotonic $f, g \in E$ and given any $\varepsilon > 0$, there exist two comonotonic $f_\varepsilon, g_\varepsilon \in L$ such that $\|f - f_\varepsilon\| < \varepsilon$, $\|g - g_\varepsilon\| < \varepsilon$, and $f_\varepsilon + g_\varepsilon \in L$.*

In other words, a Stone lattice is comonotonic if it is suitably dense (in the sense of comonotonicity) in a Stone vector lattice. In particular, Stone vector lattices are automatically comonotonic. Moreover, if $L$ is a comonotonic Stone lattice then it is easy to check that

$$L \subseteq E \subseteq \bar{L} = \bar{E}, \tag{2}$$

where $\overline{L}$ is the supnorm closure of $L$ in the space of all bounded functions $f : S \to \mathbb{R}$.

For a given collection of functions $L$, consider the collections of subsets $\Sigma_L = \{(f \geq t) : f \in L \text{ and } t \in \mathbb{R}\}$ and $\Sigma'_L = \{(f > t) : f \in L \text{ and } t \in \mathbb{R}\}$.[3]

**Lemma 2** *If $L$ is a Stone lattice then both $\Sigma_L$ and $\Sigma'_L$ are lattices.*

**Proof.** We just prove the statement for $\Sigma_L$. A similar proof delivers the result for $\Sigma'_L$. Since $L$ is a Stone lattice take $t_1 = 2$, $t_2 = 0$, and $f \in L$ such that $f = 1$. It is immediate to see that $\varnothing = (f \geq t_1)$ and $S = (f \geq t_2)$, proving that $\varnothing, S \in \Sigma_L$. Consider $A, B \in \Sigma_L$. Then, there exist $f_1, f_2 \in L$ and $t_1, t_2 \in \mathbb{R}$ such that $A = (f_1 \geq t_1)$ and $B = (f_2 \geq t_2)$. Wlog, suppose that $t_1 \geq t_2$. Define $f_3 = f_2 + t_1 - t_2$. Since $L$ is a Stone lattice, it follows that $f_3$, $f_1 \vee f_3$, and $f_1 \wedge f_3$ belong to $L$. Hence, $(f_3 \geq t_1) = (f_2 \geq t_2) = B$, $A \cup B = (f_1 \vee f_3 \geq t_1) \in \Sigma_L$, and $A \cap B = (f_1 \wedge f_3 \geq t_1) \in \Sigma_L$. ■

[2] Observe that $f + \beta$ denotes $f + \beta 1_S$. With a small abuse of notation, we denote by the same symbol a real number and the constant function that takes that value. By setting $\alpha = 0$, it follows that a Stone lattice contains all constant functions.

[3] Given $f \in L$ and $t \in \mathbb{R}$, we denote by $(f \geq t)$ and $(f > t)$ the sets $\{s \in S : f(s) \geq t\}$ and $\{s \in S : f(s) > t\}$.

**Example 3** Let $\Sigma$ be an algebra. A function $f : S \to \mathbb{R}$ is $\Sigma$-measurable if $\Sigma_{\{f\}} \cup \Sigma'_{\{f\}} \subseteq \Sigma$. We denote by $B(\Sigma)$ the set of all bounded $\Sigma$-measurable $f : S \to \mathbb{R}$. The collection $B(\Sigma)$ is a Stone lattice, but it is not a vector space in general unless $\Sigma$ is a $\sigma$-algebra. Its supnorm closure $\bar{B}(\Sigma)$ is a Stone vector lattice with the property that, given any two comonotonic $f, g \in \bar{B}(\Sigma)$, there exist two sequences $\{f_n\}_n, \{g_n\}_n \subseteq B(\Sigma)$ that supnorm converge to $f$ and $g$, respectively, such that $f_n$ and $g_n$ are comonotonic and $f_n + g_n \in B(\Sigma)$ for all $n \in \mathbb{N}$. Thus, $B(\Sigma)$ is a comonotonic Stone lattice. Finally, $\Sigma_{B(\Sigma)} = \Sigma = \Sigma'_{B(\Sigma)}$. ▲

**Example 4** If we endow $S$ with a topology, the collection of all bounded continuous functions $C(S)$ is easily seen to be a Stone vector lattice. ▲

Thus, the notion of comonotonic Stone lattice allows to cover classic spaces that, like $C(S)$, are already Stone vector lattices, as well as classic spaces that, like $B(\Sigma)$, are not Stone vector lattices but nicely (in the sense of comonotonicity) embed into Stone vector lattices. Without this notion these different types of spaces would require a separate analysis.

Let $L$ be a Stone lattice, a functional $V : L \to \mathbb{R}$ is:

(i) *monotone* if $f \geq g$ implies $V(f) \geq V(g)$;

(ii) *positively homogeneous* if $V(\alpha f) = \alpha V(f)$ for all $f \in L$ and $\alpha \geq 0$;

(iii) *comonotonic additive* if $V(f+g) = V(f) + V(g)$ for any comonotonic pair $f, g \in L$ such that $f + g \in L$;

(iv) *translation invariant* if $V(f+\lambda) = V(f) + \lambda V(1)$ for all $f \in L$ and $\lambda \in \mathbb{R}$;

(v) *supermodular* if $V(f \vee g) + V(f \wedge g) \geq V(f) + V(g)$ for all $f, g \in L$;

(vi) *submodular* if $V(f \vee g) + V(f \wedge g) \leq V(f) + V(g)$ for all $f, g \in L$;

(vii) *outer* (resp., *inner*) *continuous* if $\lim_{n\to\infty} V(f_n) = V(f)$ whenever $\{f_n\}_n \subseteq L$ and $f \in L$ are such that $f_n \downarrow f$ (resp., $f_n \uparrow f$);[4]

(viii) *superadditive* if $V(f+g) \geq V(f) + V(g)$ for any pair $f, g \in L$ such that $f + g \in L$;

(ix) *Lipschitz continuous* if there is $k > 0$ such that $|V(f) - V(g)| \leq k \|f - g\|$ for all $f, g \in L$.

In the sequel we will also consider functionals $V : L_+ \to \mathbb{R}$, where $L_+ = \{f \in L : f \geq 0\}$. For them the previous properties apply, up to the obvious modifications.

Finally, let $L$ be a Stone lattice. Given a functional $V : L \to \mathbb{R}$ and two functions $f, g \in L$ such that $f \leq g$, set

$$T(f,g) = \sup \sum_{i=1}^{n} |V(f_i) - V(f_{i-1})| \in [0, \infty],$$

where the supremum is taken over all finite chains $f = f_0 \leq f_1 \leq \cdots \leq f_n = g$. We say that $V$ is of *bounded variation* if $T(0,f) < \infty$ for all $f \in L_+$ (see, e.g., [12] and [13]).

Given a functional $V : L \to \mathbb{R}$ defined on a Stone lattice, its dual functional $\overline{V} : L \to \mathbb{R}$ is given by $\overline{V}(f) = -V(-f)$. Next, we collect few basic relations between $V$ and its dual $\overline{V}$. Their simple proofs are omitted.

[4] Given a sequence $\{f_n\}_n \subseteq L$ and $f \in L$ we say that $f_n \downarrow f$ (resp., $f_n \uparrow f$) if for each $s \in S$ we have $\lim_n f_n(s) = f(s)$ and $f_n \geq f_{n+1}$ (resp., $f_n \leq f_{n+1}$) for all $n \in \mathbb{N}$.

**Lemma 5** *Let $V : L \to \mathbb{R}$ be a functional defined on a Stone lattice. Then,*

*(i)* $\overline{(\overline{V})} = V$*;*

*(ii) $V$ is comonotonic additive if and only if $\overline{V}$ is;*

*(iii) $V$ is monotone if and only if $\overline{V}$ is;*

*(iv) $V$ is outer (resp., inner) continuous if and only if $\overline{V}$ is inner (resp., outer) continuous;*

*(v) $V$ is supermodular (submodular) if and only if $\overline{V}$ is submodular (resp., supermodular);*

*(vi) $V$ is translation invariant if and only if $\overline{V}$ is;*

*(vii) $V$ is positively homogeneous if and only if $\overline{V}$ is;*

*(viii) if $V$ is comonotonic additive, then $V$ is of bounded variation if and only if $\overline{V}$ is.*

# 3 Decomposition

In this section we study inner and outer variations that we will use to decompose set functions of bounded variation as differences of capacities. In turn, these decompositions will play an important role in the integral representation results of next section. Below, given a real number $a$ we denote $a^+ = \max\{0, a\}$ and $a^- = \max\{0, -a\}$.

We consider a lattice of sets $\Sigma$. Given a set function $\nu : \Sigma \to \mathbb{R}$ and two nested sets $A \subseteq B$, set

$$\nu^+(A, B) = \sup \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^+ ,$$

$$\nu^-(A, B) = \sup \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^- ,$$

$$|\nu|(A, B) = \sup \sum_{i=1}^{n} |\nu(S_i) - \nu(S_{i-1})| ,$$

where suprema are taken over all finite chains $A = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = B$.[5]

Following Aumann and Shapley [5], define $\|\nu\|$ by $\|\nu\| = |\nu|(\varnothing, S)$. This is the *variation norm* of $\nu$, which reduces to the classic total variation norm when $\nu$ is a charge. Denote by $bv(\Sigma)$ the collection of all set functions $\nu$ such that $\|\nu\| < \infty$. Set functions in $bv(\Sigma)$ are necessarily bounded. Indeed, $|\nu(A)| = |\nu(A) - \nu(\varnothing)| \leq |\nu|(\varnothing, A) \leq |\nu|(\varnothing, S) = \|\nu\|$ for all $A \in \Sigma$.

**Lemma 6 ([5, p. 28])** *If $\Sigma$ is a lattice then $(bv(\Sigma), \|\cdot\|)$ is a Banach space.*[6]

Given a set function $\nu$ in $bv(\Sigma)$, its

(i) *inner upper variation* $\nu^+ : \Sigma \to [0, \infty)$ is given by $\nu^+(A) = \nu^+(\varnothing, A)$;

(ii) *inner lower variation* $\nu^- : \Sigma \to [0, \infty)$ is given by $\nu^-(A) = \nu^-(\varnothing, A)$;

(iii) *outer upper variation* $\nu_+ : \Sigma \to [0, \infty)$ is given by $\nu_+(A) = \nu^+(\varnothing, S) - \nu^+(A, S)$;

(iv) *outer lower variation* $\nu_- : \Sigma \to [0, \infty)$ is given by $\nu_-(A) = \nu^-(\varnothing, S) - \nu^-(A, S)$;

[5] Notice that if $C \subseteq A \subseteq B \subseteq D$, then $0 = v(A, A) \leq v(A, B) \leq v(C, D) \leq v(\varnothing, S)$ for $v = \nu^+, \nu^+, |\nu|$.

[6] More precisely, Aumann and Shapley [5] prove the previous lemma when $\Sigma$ is a $\sigma$-algebra. However, their techniques apply when $\Sigma$ is a lattice. A similar observation applies to Proposition 7, for the equivalence between points (i), (ii), and (iv).

(v) *total variation* $|\nu| : \Sigma \to [0, \infty)$ is given by $|\nu|(A) = |\nu|(\varnothing, A)$.

Outer variations are the natural counterparts of inner variations, which were first studied by [5]. Notice that $\nu^+(\varnothing) = \nu_+(\varnothing) = \nu^-(\varnothing) = \nu_-(\varnothing) = 0$ and that $\nu^+(S) = \nu_+(S)$ as well as $\nu^-(S) = \nu_-(S)$. Moreover, all variations (i)-(iv) are capacities, provided $\nu \in bv(\Sigma)$. The following result summarizes these facts and extends a basic decomposition result proved in [5].

**Proposition 7** *Let $\Sigma$ be a lattice and $\nu : \Sigma \to \mathbb{R}$ a set function. The following conditions are equivalent:*

*(i) $\nu \in bv(\Sigma)$;*

*(ii) $\nu^+$ and $\nu^-$ are two capacities;*

*(iii) $\nu_+$ and $\nu_-$ are two capacities;*

*(iv) there exist two capacities $\nu_1$ and $\nu_2$ on $\Sigma$ such that $\nu = \nu_1 - \nu_2$.*

*Moreover,*

$$\nu = \nu^+ - \nu^- = \nu_+ - \nu_- \quad \textit{and} \quad |\nu| = \nu^+ + \nu^- \tag{3}$$

*and*

$$\nu^+ \leq \nu_1 \quad \textit{and} \quad \nu^- \leq \nu_2, \tag{4}$$

*whenever $\nu = \nu_1 - \nu_2$ is any decomposition with capacities $\nu_1$ and $\nu_2$.*

**Proof.** The equivalence of (i), (ii), and (iv) is proved in [5], as well as the equalities $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$. In their analysis, $\Sigma$ is a $\sigma$-algebra but their arguments are easily adapted to lattices.
(iii) implies (i). Suppose that $\nu_+$ and $\nu_-$ are two capacities on $\Sigma$. This implies that $\nu^+(S)$ and $\nu^-(S)$ are finite and so is $\|\nu\| = |\nu|(S)$.
(i) implies (iii). It immediately follows from Footnote 5.

Next, let $A \subseteq B$, then

$$\begin{aligned}
\nu^+(A, B) &= \sup \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^+ \\
&= \sup \left( \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^+ - \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^- + \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^- \right) \\
&= \sup \left( \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})] + \sum_{i=1}^{n} [\nu(S_i) - \nu(S_{i-1})]^- \right) = \nu(B) - \nu(A) + \nu^-(A, B)
\end{aligned}$$

if, moreover $\nu \in bv(\Sigma)$, then $\nu^+(A, B), \nu^-(A, B) \leq |\nu|(A, B)$ are finite and

$$\nu(B) - \nu(A) = \nu^+(A, B) - \nu^-(A, B)$$

which (taking $B = S$) delivers $\nu = \nu_+ - \nu_-$.
Finally, (4) is proven in [13]. ■

When $\Sigma$ is an algebra, inner and outer variations can be connected through dual set functions.

**Lemma 8** *If $\Sigma$ is an algebra and $\nu : \Sigma \to \mathbb{R}$ is a set function of bounded variation then*

$$\nu_+ = \overline{(\overline{\nu})^+} \textit{ and } \nu_- = \overline{(\overline{\nu})^-}.$$

**Proof.** We only prove that $\nu_+ = \overline{(\bar{\nu})^+}$, as the other equality can be similarly proved. Pick $A \in \Sigma$. It follows that $\varnothing = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = A$ if and only if $A^c = S_n^c \subseteq S_{n-1}^c \subseteq \cdots \subseteq S_0^c = S$. Moreover,

$$\sum_{i=1}^{n} \left[\bar{\nu}(S_i) - \bar{\nu}(S_{i-1})\right]^+ = \sum_{i=1}^{n} \left[\nu\left(S_{i-1}^c\right) - \nu\left(S_i^c\right)\right]^+. \tag{5}$$

This implies that $\bar{\nu}^+(\varnothing, A) = \nu^+(A^c, S)$. If $A = S$ then we have that $\bar{\nu}^+(S) = \nu^+(S)$. On the other hand, we have that

$$\overline{(\bar{\nu})^+}(A) = \bar{\nu}^+(S) - \bar{\nu}^+(A^c) = \nu^+(S) - \nu^+(A, S) = \nu_+(A),$$

as desired. ■

**Remark 9** In light of previous lemma, we observe that the second equality of (3) can be derived in a simpler way when $\Sigma$ is an algebra. For, assume $\nu \in bv(\Sigma)$. This implies that $\bar{\nu} \in bv(\Sigma)$ and so $\bar{\nu} = \bar{\nu}^+ - \bar{\nu}^-$. By Lemma 8, $\nu = \overline{(\bar{\nu})} = \overline{(\bar{\nu}^+)} - \overline{(\bar{\nu}^-)} = \nu_+ - \nu_-$.

It is useful to introduce the following order in $bv(\Sigma)$. Given two elements $\mu, \nu \in bv(\Sigma)$, say that $\nu \succeq \mu$ if and only if $\nu - \mu$ is a capacity. For instance, some of the results of Proposition 7 can be formulated through the order $\succeq$ as follows: $|\nu| \succeq \nu^+ \succeq \nu \succeq -\nu^- \succeq -|\nu|$ for each $\nu \in bv(\Sigma)$. In addition, we have $\nu_+ \succeq \nu \succeq -\nu_-$.

Nevertheless, when $\Sigma$ is an algebra, the ordered vector space $(bv(\Sigma), \succeq)$ is not a vector lattice unless $\Sigma$ is trivial (see, e.g., [13, Proposition 3.4]).

The next "sandwich" result provides a simple way to check the continuity of a set function $\nu \in bv(\Sigma)$.

**Lemma 10** *Let $\Sigma$ be a lattice and $\nu : \Sigma \to \mathbb{R}$ a set function of bounded variation. A set function $\nu$ is outer (resp., inner) continuous provided $\nu_1 \succeq \nu \succeq \nu_2$, where $\nu_1$ and $\nu_2$ are both outer (resp., inner) continuous.*

**Proof.** If $A \subseteq B$ then it follows that $\nu_1(B) - \nu_1(A) \geq \nu(B) - \nu(A) \geq \nu_2(B) - \nu_2(A)$. If $\{A_n\}_n \subseteq \Sigma$, $A \in \Sigma$, and $A_n \downarrow A$ then $\nu_1(A_n) - \nu_1(A) \geq \nu(A_n) - \nu(A) \geq \nu_2(A_n) - \nu_2(A)$ for all $n \in \mathbb{N}$. This implies that $\lim_n \nu(A_n) = \nu(A)$. A similar argument applies for inner continuity. ■

**Proposition 11** *Let $\Sigma$ be a lattice and $\nu : \Sigma \to \mathbb{R}$ a set function of bounded variation. Then,*

*(i) $\nu$ is inner continuous if and only if both $\nu^+$ and $\nu^-$ are;*

*(ii) $\nu$ is outer continuous if and only if both $\nu_+$ and $\nu_-$ are;*

*(iii) $|\nu|$ is continuous if and only if both $\nu^+$ and $\nu^-$ are continuous, which in turn implies that $\nu$ is continuous.*

**Proof.** In light of Proposition 7 and (3), the sufficiency part of points (i), (ii), and (iii) is immediate. The necessity part of point (i) and (ii) follows from routine arguments.[7] As to (iii), by the relations $|\nu| \succeq \nu^+ \succeq 0$ and $|\nu| \succeq \nu^- \succeq 0$ and by Lemma 10, if $|\nu|$ is continuous then $\nu^+$ and $\nu^-$ are continuous. Finally, in this case and given (3), we can conclude that $\nu$ is continuous. ■

Proposition 11 characterizes the inner and outer continuity of set functions in $bv(\Sigma)$ in terms of the inner and outer continuity of their variations. A natural question is whether the continuity of a set function has a similar characterization, that is, whether a continuous $\nu$ can be decomposed in two continuous $\nu^+$ and

[7] Proofs are available upon request. Point (i) was proven first by [14, Proposition 2.1] when $\Sigma$ is a $\sigma$-algebra. Particularly, in the case $\Sigma$ is an algebra, the necessity part of (ii) is an easy consequence of point (i) and Lemma 8.

$\nu^-$. The next negative result shows that, in general, this is not the case. In other words, the implication contained in point (iii) of Proposition 11 does not admit a converse: there exist continuous set functions $\nu$ for which $|\nu|$ is not continuous. In this case, we can only assert that $|\nu|$ is inner continuous by point (i) of Proposition 11.

To see why this is the case, say that $\Sigma$ is a *nonatomic* $\sigma$-algebra if it admits a nonatomic probability measure. For example, Borel $\sigma$-algebras of uncountable Polish spaces have this property (see, e.g., [1, Theorem 12.22]).

**Proposition 12** *If $\Sigma$ is a nonatomic $\sigma$-algebra then there exists a continuous $\nu \in bv(\Sigma)$ such that its inner variations $\nu^+$ and $\nu^-$ are not outer continuous.*

This negative result is important for our analysis since it shows that we cannot provide a unified decomposition for the continuous case, but only separately for inner and outer continuity.

**Proof.** Let $\mu$ be the nonatomic probability measure on $\Sigma$. Let $A$ be such that $\mu(A) = \mu(A^c) = 1/2$, and define $\mu_1, \mu_2 : \Sigma \to [0,1]$ by $\mu_1(B) = \mu(A^c \cap B)/\mu(A^c)$ and $\mu_2(B) = \mu(A \cap B)/\mu(A)$. Clearly, $\mu_1$ and $\mu_2$ are mutually singular nonatomic probability measures. By the Lyapunov Theorem,

$$\{(\mu_1(B), \mu_2(B)) : B \in \Sigma\} = [0,1]^2 . \tag{6}$$

Consider the function $f : [0,1]^2 \to \mathbb{R}$ defined in [5, p. 55–56] and define $\nu : \Sigma \to \mathbb{R}$ by $\nu(B) = f(\mu_1(B), \mu_2(B))$. By the properties of this function proved by [5], $\nu$ belongs to $bv(\Sigma)$, $\nu$ is continuous, and $\nu^+(B) = f^+(\mu_1(B), \mu_2(B))$.[8] However, $\lim_m f^+(2^{-m}, 1) \neq f^+(0,1)$.

By (6) and since $\mu_1$ and $\mu_2$ are mutually singular, there exists a sequence $\{A_m\}_m \supseteq A$ such that $\mu_1(A_m) = 2^{-m}$ and $\mu_2(A_m) = 1$ for all $m \in \mathbb{N}$. Set $A' = \bigcap_{m \in \mathbb{N}} A_m$. We have $\mu_1(A') = 0$ and $\mu_2(A') = 1$. Hence,

$$\nu^+(A') = f^+(0,1) \neq \lim_m f^+\left(\frac{1}{2^m}, 1\right) = \lim_m f^+(\mu_1(A_m), \mu_2(A_m)) = \lim_m \nu^+(A_m),$$

which shows that $\nu^+$ is not outer continuous. A similar argument shows that also $\nu^-$ is not outer continuous. ■

# 4 Integral Representation of Comonotonic Additive Functionals

Let $L$ be a Stone lattice. Given an element $\nu \in bv(\Sigma_L)$ and an element $\nu' \in bv(\Sigma'_L)$, we define $V_c : L \to \mathbb{R}$ and $V_{sc} : L \to \mathbb{R}$ the Choquet functionals given by

$$V_c(f) = \int_0^\infty \nu(f \geq t)\, dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(S)]\, dt \qquad \forall f \in L, \tag{7}$$

and

$$V_{sc}(f) = \int_0^\infty \nu'(f > t)\, dt + \int_{-\infty}^0 [\nu'(f > t) - \nu'(S)]\, dt \qquad \forall f \in L. \tag{8}$$

The Riemann integrals on the right hand sides are well defined. Indeed, the scalar functions $\varphi(t) = \nu(f \geq t)$ and $\varphi'(t) = \nu'(f > t)$ are of bounded variation over $[-\|f\| - 1, \|f\| + 1]$. For, if $t_0 \leq t_1 \leq \cdots \leq t_n$, $t_0 = -\|f\| - 1$, and $t_n = \|f\| + 1$ then

$$\sum_{i=1}^n |\varphi(t_i) - \varphi(t_{i-1})| = \sum_{i=1}^n |\nu(f \geq t_i) - \nu(f \geq t_{i-1})| \leq \|\nu\| . \tag{9}$$

[8] Here $f^+$ is defined as in [5, p. 50].

A similar argument holds for $\varphi'$. Hence, the two integrands of (7) are of bounded variations on the interval $[-\|f\|-1, \|f\|+1]$ and zero on the rest of their respective domains of integration. When $\nu$ is defined over the entire $\sigma$-algebra generated by $\Sigma_L$ we write alternatively $V_c(f) = \int f d\nu$ for all $f \in L$.

We can now state and prove our first main result.

**Theorem 13** *Let $V : L \to \mathbb{R}$ be a functional defined on a comonotonic Stone lattice. The following conditions are equivalent:*

*(i) $V$ is comonotonic additive, of bounded variation, and outer continuous;*

*(ii) there exists an outer continuous set function $\nu \in bv(\Sigma_L)$ such that*

$$V(f) = \int_0^\infty \nu(f \geq t)\, dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(S)]\, dt \qquad \forall f \in L; \tag{10}$$

*(iii) there exist two outer continuous capacities $\nu^1$ and $\nu^2$ over $\Sigma_L$ such that*

$$V(f) = V_c^1(f) - V_c^2(f) \qquad \forall f \in L \tag{11}$$

*where $V_c^i(f) = \int_0^\infty \nu^i(f \geq t)\, dt + \int_{-\infty}^0 \left[\nu^i(f \geq t) - \nu^i(S)\right] dt$ for all $f \in L$ and $i \in \{1, 2\}$.*

*Moreover,*

*(a) the outer continuous set function $\nu : \Sigma_L \to \mathbb{R}$ for which (10) holds is unique;*

*(b) $\nu$ is a capacity if and only if $V$ is monotone;*

*(c) $\nu$ is supermodular if and only if $V$ is supermodular.*

The proof of this theorem relies on few lemmas.

**Lemma 14** *Let $V : L \to \mathbb{R}$ be a comonotonic additive functional of bounded variation defined on a Stone lattice $L$. Then,*

*(i) there exist two functionals $V_1, V_2 : L \to \mathbb{R}$ that are monotone, translation invariant, positively homogeneous, and such that $V = V_1 - V_2$ on $L$;*

*(ii) $V$ is inner (resp., outer) continuous if and only if both $V_1$ and $V_2$ can be chosen to be inner (resp., outer) continuous;*

*(iii) $V$ is Lipschitz continuous, translation invariant, and positively homogeneous on $L$.*

**Proof.** (i) By proceeding as in [12, p. 69], it is easy to see that comonotonic additivity implies that

$$V(\alpha f + \beta) = \alpha V(f) + V(\beta) \tag{12}$$

for all $\alpha \in \mathbb{Q}_{++}$ and all $\beta \in \mathbb{R}$.[9] Being $V$ of bounded variation, we have that $T(0, f) < \infty$ for all $f \in L_+$. Define $V_1 : L_+ \to \mathbb{R}$ by $V_1(f) = T(0, f)$. Since $V$ is of bounded variation, $V_1$ is well defined. Clearly, it is monotone.

**Claim 1** $T(0, f + \lambda) = T(-\lambda, f)$ for all $f \in L_+$ and for all $\lambda \in \mathbb{R}_+$.

[9] We denote by $\mathbb{Q}_+$ (resp., $\mathbb{Q}_{++}$) the positive (resp., strictly positive) rational numbers. $\mathbb{R}_+$ and $\mathbb{R}_{++}$ are defined analogously.

**Proof of the Claim.** Fix $f \in L_+$ and $\lambda \in \mathbb{R}_+$. Notice that $\{g_i\}_{i=0}^n$ is a chain in $L$ such that $-\lambda = g_0 \leq \cdots \leq g_n = f$ if and only if there exists a chain $\{f_i\}_{i=0}^n$ in $L$ such that $0 = f_0 \leq \cdots \leq f_n = f + \lambda$ and $f_i = g_i + \lambda$ for all $i \in \{0, ..., n\}$. In view of (12), it follows that

$$\begin{aligned} T(-\lambda, f) &= \sup \sum_{i=1}^n |V(g_i) - V(g_{i-1})| = \sup \sum_{i=1}^n |V(f_i - \lambda) - V(f_{i-1} - \lambda)| \\ &= \sup \sum_{i=1}^n |V(f_i) - V(f_{i-1})| = T(0, f + \lambda). \end{aligned}$$

□

**Claim 2** $T(-\lambda, f) = T(-\lambda, 0) + T(0, f)$ for all $f \in L_+$ and $\lambda \in \mathbb{Q}_{++}$.

**Proof of the Claim.** Fix $f \in L_+$ and $\lambda \in \mathbb{Q}_{++}$. Since $V$ is comonotonic and of bounded variation and by definition and Claim 1, $\infty > T(-\lambda, f) \geq T(-\lambda, 0) + T(0, f)$. We are now left to prove the opposite inequality. Fix $\varepsilon > 0$. Then, there exists a finite chain $\{f_i\}_{i=0}^n \subseteq L$, with $-\lambda = f_0$ and $f_n = f$, such that

$$\sum_{i=1}^n |V(f_i) - V(f_{i-1})| \geq T(-\lambda, f) - \varepsilon.$$

Since $L$ is a Stone lattice, $\{f_i^+\}_{i=0}^n$ and $\{-f_i^-\}_{i=0}^n$ are chains in $L$. Moreover, observe that $f_i^+$ and $-f_i^-$ are comonotonic for all $i \in \{0, ..., n\}$, since $\left[f_i^+(s_1) - f_i^+(s_2)\right]\left[f_i^-(s_2) - f_i^-(s_1)\right] = f_i^+(s_1) f_i^-(s_2) + f_i^+(s_2) f_i^-(s_1) \geq 0$.[10] So that $V(f_i) = V\left(f_i^+\right) + V\left(-f_i^-\right)$ for all $i \in \{0, ..., n\}$.

Finally, from $-\lambda = -f_0^- \leq \cdots \leq -f_n^- = 0$ and $0 = f_0^+ \leq \cdots \leq f_n^+ = f$, it follows that

$$\begin{aligned} T(-\lambda, 0) + T(0, f) &\geq \sum_{i=1}^n \left|V\left(-f_i^-\right) - V\left(-f_{i-1}^-\right)\right| + \sum_{i=1}^n \left|V\left(f_i^+\right) - V\left(f_{i-1}^+\right)\right| \\ &\geq \sum_{i=1}^n \left|V\left(-f_i^-\right) - V\left(-f_{i-1}^-\right) + V\left(f_i^+\right) - V\left(f_{i-1}^+\right)\right| \\ &= \sum_{i=1}^n |V(f_i) - V(f_{i-1})| \geq T(-\lambda, f) - \varepsilon. \end{aligned}$$

Since $\varepsilon$ was arbitrarily chosen, the statement follows. □

**Claim 3** $T(0, \lambda f) = \lambda T(0, f)$ for all $f \in L_+$ and $\lambda \in \mathbb{Q}_{++}$.

**Proof of the Claim.** Fix $f \in L_+$ and $\lambda \in \mathbb{Q}_{++}$. Given $\varepsilon > 0$, there is a chain in $L$ such that $0 = f_0 \leq \cdots \leq f_n = f$ and for which $\sum_{i=1}^n |V(f_i) - V(f_{i-1})| \geq T(0, f) - \varepsilon$. Consider the chain $0 = \lambda f_0 \leq \cdots \leq \lambda f_n = \lambda f$. In view of (12), we have that

$$T(0, \lambda f) \geq \sum_{i=1}^n |V(\lambda f_i) - V(\lambda f_{i-1})| \geq \lambda T(0, f) - \lambda\varepsilon.$$

It follows that $T(0, \lambda f) \geq \lambda T(0, f)$. Since $\lambda$ was generic, particularly, we have that $T\left(0, \lambda^{-1} f\right) \geq \lambda^{-1} T(0, f)$ for all $\lambda \in \mathbb{Q}_{++}$. By replacing $f$ with $\lambda f$, we obtain that $T(0, f) \geq \lambda^{-1} T(0, \lambda f)$. Consequently, $T(0, \lambda f) = \lambda T(0, f)$. □

By construction, $V_1$ is monotone. Given Claims 1-3, if $\lambda \in \mathbb{Q}_{++}$ and $f \in L_+$ then we can conclude that

$$\begin{aligned} V_1(f + \lambda) &= T(0, f + \lambda) = T(-\lambda, f) = T(-\lambda, 0) + T(0, f) = T(0, \lambda) + T(0, f) \\ &= \lambda T(0, 1) + T(0, f) = V_1(f) + \lambda V_1(1). \end{aligned} \tag{13}$$

[10] More generally, $f \wedge a$ and $f \vee a$ are comonotonic for all $a \in \mathbb{R}$ (see, e.g., [13]).

Let $f, g \in L_+$. Since $L$ is a Stone lattice, $g + \|f-g\| \in L_+$. Let $\{r_n\}_n \subseteq \mathbb{Q}_{++}$ be such that $r_n \downarrow \|f-g\|$. By (13) and since $f \leq g + \|f-g\|$, we have that

$$V_1(f) \leq V_1(g + \|f-g\|) \leq V_1(g + r_n) = V_1(g) + r_n V_1(1) \qquad \forall n \in \mathbb{N}. \tag{14}$$

By a symmetric argument, we can interchange the roles of $f$ and $g$ in (14). Passing to the limit, we get $|V_1(f) - V_1(g)| \leq V_1(1)\|f-g\|$, which shows that $V_1$ is Lipschitz continuous. Given Claims 1-3, we have that for each $f \in L_+$ and for each $\alpha, \beta \in \mathbb{Q}_{++}$

$$V_1(\alpha f + \beta) = \alpha V_1(f) + \beta V_1(1). \tag{15}$$

Since $V_1$ is Lipschitz continuous, it follows that (15) holds for all $f \in L_+$ and $\alpha, \beta \geq 0$.

Define now $V_2 = V_1 - V$ on $L_+$. Consider $f, g \in L_+$ such that $f \geq g$. Since $V$ is of bounded variation, we have that

$$V(f) - V(g) \leq |V(f) - V(g)| \leq T(g,f) \leq T(0,f) - T(0,g) = V_1(f) - V_1(g).$$

In turn, this implies that $V_2$ is monotone. By (12) and (15), we have that for each $f \in L_+$ and for each $\alpha, \beta \in \mathbb{Q}_{++}$

$$V_2(\alpha f + \beta) = \alpha V_2(f) + \beta V_2(1). \tag{16}$$

Since $V_2$ is monotone and by the same argument used for $V_1$, it follows that $V_2$ is Lipschitz continuous. Finally, by Lipschitz continuity, we can conclude that (16) holds for all $\alpha, \beta \geq 0$.

In sum, we have proved that there exist two monotone functionals, $V_1$ and $V_2$, from $L_+$ to $\mathbb{R}$ such that $V = V_1 - V_2$ on $L_+$ and such that for each $i \in \{1,2\}$

$$V_i(\alpha f + \beta) = \alpha V_i(f) + \beta V_i(1) \qquad \forall f \in L_+, \forall \alpha, \beta \geq 0.$$

We complete the proof by extending $V_1$ and $V_2$ to $L$. To this end, observe that $L = \{f + k : f \in L_+ \text{ and } k \in \mathbb{R}\}$. For $i = 1, 2$, define $\widehat{V}_i : L \to \mathbb{R}$ by $\widehat{V}_i(f) = V_i(f + \lambda) - \lambda V_i(1)$ where $\lambda$ is any nonnegative scalar such that $f + \lambda \in L_+$. The functionals $\widehat{V}_i$ are easily seen to be well defined with $\widehat{V}_i(f) = V_i(f)$ for all $f \in L_+$. It is also easy to check that they are monotone, translation invariant, and positively homogeneous. Thus, it remains to prove that $V = \widehat{V}_1 - \widehat{V}_2$. Let $f \in L$ and define $k = \lfloor \|f\| \rfloor + 1$. Notice that $f + k \in L_+$. By (12) and since $V = V_1 - V_2$ on $L_+$, it follows that

$$\begin{aligned}
V(f) + kV(1) = V(f+k) = V_1(f+k) - V_2(f+k) &= \widehat{V}_1(f+k) - \widehat{V}_2(f+k) \\
&= \widehat{V}_1(f) - \widehat{V}_2(f) + k\left(\widehat{V}_1(1) - \widehat{V}_2(1)\right) = \widehat{V}_1(f) - \widehat{V}_2(f) + k(V_1(1) - V_2(1)) \\
&= \widehat{V}_1(f) - \widehat{V}_2(f) + kV(1).
\end{aligned}$$

This completes the proof of (i).

(ii) The sufficiency part of the statement is obvious. We next prove the necessity part. We first show the inner continuity of $V_1 : L_+ \to \mathbb{R}$. Let $\{f_m\}_m \subseteq L_+$ be such that $f_m \uparrow f$. Since $V_1$ is monotone, $\lim_m V_1(f_m)$ is well defined, with $\lim_m V_1(f_m) \leq V_1(f)$. As to the converse inequality, pick $\varepsilon > 0$ and consider a chain $0 = g_0 \leq g_1 \leq \cdots \leq g_n = f$ such that

$$V_1(f) - \varepsilon = T(0,f) - \varepsilon \leq \sum_{i=1}^{n} |V(g_i) - V(g_{i-1})|.$$

Define $f_i^m = g_i \wedge f_m$ for all $m \in \mathbb{N}$ and for all $i \in \{0, ..., n\}$. Since $L$ is a Stone lattice, we have that $f_i^m \in L$ for all $m \in \mathbb{N}$ and for all $i \in \{0, ..., n\}$, moreover, $f_i^m \uparrow g_i$ for all $i \in \{0, ..., n\}$ and $0 = f_0^m \leq$

$f_{i-1}^m \leq f_i^m \leq f_n^m = f_m$ for all $i \in \{1, ..., n\}$ and for all $m \in \mathbb{N}$. Since $V$ is inner continuous, it follows that $\lim_m \left|V(f_i^m) - V(f_{i-1}^m)\right| = |V(g_i) - V(g_{i-1})|$ for each $i \in \{1, ..., n\}$. Therefore,

$$\lim_m \sum_{i=1}^{n} \left|V(f_i^m) - V(f_{i-1}^m)\right| = \sum_{i=1}^{n} |V(g_i) - V(g_{i-1})|.$$

By definition, for each $m \in \mathbb{N}$ we have that $V_1(f_m) = T(0, f_m) \geq \sum_{i=1}^{n} \left|V(f_i^m) - V(f_{i-1}^m)\right|$. This implies that

$$\lim_m V_1(f_m) \geq \lim_m \sum_{i=1}^{n} \left|V(f_i^m) - V(f_{i-1}^m)\right| \geq V_1(f) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen, this proves the statement.

It remains to show that the extension $\widehat{V}_1 : L \to \mathbb{R}$ is also inner continuous. Consider $\{f_m\}_m \subseteq L$ and $f \in L$ such that $f_m \uparrow f$. Define $k = \|f_1\|$. Then, $\{f_m + k\}_m \subseteq L_+$, $f + k \in L_+$, and $f_m + k \uparrow f + k$. Since $V_1$ is inner continuous, this implies that

$$\widehat{V}_1(f + k) = V_1(f + k) = \lim_m V_1(f_m + k) = \lim_m \widehat{V}_1(f_m + k).$$

Hence, $\widehat{V}_1(f) = \widehat{V}_1(f + k) - k\widehat{V}_1(1) = \lim_m \left(\widehat{V}_1(f_m + k) - k\widehat{V}_1(1)\right) = \lim_m \widehat{V}_1(f_m)$. Clearly, since $\widehat{V}_2 = \widehat{V}_1 - V$ and $V$ and $\widehat{V}_1$ are inner continuous, it follows that $\widehat{V}_2$ is inner continuous as well.

Finally, we prove the outer continuous case. If $V$ is outer continuous, by Lemma 5 it follows that $\bar{V}$ is an inner continuous and comonotonic additive functional of bounded variation. Therefore, by the previous part of the proof $\bar{V} = V_1 - V_2$, where $V_1, V_2 : L \to \mathbb{R}$ are monotone, translation invariant, positively homogeneous, and inner continuous functionals. By Lemma 5, $V = \overline{(\bar{V})} = \bar{V}_1 - \bar{V}_2$, where $\bar{V}_1, \bar{V}_2 : L \to \mathbb{R}$ are monotone, translation invariant, positively homogeneous, and outer continuous functionals.

(iii) It follows from (i) since $V$ is the difference of two functionals that share these properties. ■

In the sequel, we still consider a comonotonic functional of bounded variation, $V$, defined on a Stone lattice $L$. We will need to extend $V$ to the supnorm closure $\bar{L}$ of $L$. The next result tells us that we can extend $V$ to $\bar{L}$ maintaining some of its properties, particularly and surprisingly, the property of outer continuity. Given the mappings $V, V_1, V_2 : L \to \mathbb{R}$ as in Lemma 14, we denote by $W, W_1, W_2 : \bar{L} \to \mathbb{R}$ their unique continuous extensions to $\bar{L}$.[11] By definition, $W(f) = \lim_m V(f_m)$ for all $f \in \bar{L}$ where $\{f_m\}_m \subseteq L$ and $\|f_m - f\| \to 0$. Clearly, this implies that since the functional is Lipschitz continuous so is its extension and if the functional is monotone so is its extension.

**Lemma 15** *Let $L$ be a Stone lattice. If $V : L \to \mathbb{R}$ is a comonotonic additive and outer continuous functional of bounded variation then $W$ is an outer continuous functional of bounded variation.*

**Proof.** By Lemma 14, we have that $V = V_1 - V_2$ where $V_1$ and $V_2$ are monotone, translation invariant, positively homogeneous, and outer continuous functionals. It follows that $W = W_1 - W_2$ where $W_1$ and $W_2$ are monotone. Indeed, consider a generic $f \in \bar{L}$ and $\{f_m\}_m \subseteq L$ such that $\|f_m - f\| \to 0$. Then, we have that

$$W(f) = \lim_m V(f_m) = \lim_m \{V_1(f_m) - V_2(f_m)\} = \lim_m V_1(f_m) - \lim_m V_2(f_m) = W_1(f) - W_2(f). \tag{17}$$

Monotonicity of $W_1$ and $W_2$ follows similarly. Since $W$ is a difference of two monotone functionals, it follows that $W$ is of bounded variation (on $\bar{L}$). We are left to prove that $W$ is outer continuous. We proceed by proving few facts. Fix $i \in \{1, 2\}$.

[11] Since $V$, $V_1$, and $V_2$ are Lipschitz continuous, these extensions exist and are unique.

**Claim 1** For each $f \in \bar{L}$ there exists $\{f_m\}_m \subseteq L$ (resp., $\{f'_m\}_m \subseteq L$) such that $\|f_m - f\| \to 0$ and $f_m \geq f$ for all $m \in \mathbb{N}$ (resp., $\|f'_m - f\| \to 0$ and $f'_m \leq f$ for all $m \in \mathbb{N}$).

**Proof of the Claim.** The proof follows from standard arguments. □

**Claim 2** For each $f \in \bar{L}$ and for each $\{g_m\}_m \subseteq \bar{L}$ such that $g_m \downarrow f$ there exists $\{f_m\}_m \subseteq L$ such that $f_m \downarrow f$ and $\|f_m - g_m\| \leq \frac{1}{m}$.

**Proof of the Claim.** By Claim 1, for each $m \in \mathbb{N}$ there exists $h_m \in L$ such that $h_m \geq g_m$ and $\|h_m - g_m\| \leq \frac{1}{m}$. Define $f_m = \wedge_{k=1}^m h_k$ for all $m \in \mathbb{N}$. It is immediate to see that $\{f_m\}_m$ is a nonincreasing sequence of functions. Moreover, since $L$ is a Stone lattice, we have that $\{f_m\}_m \subseteq L$. Furthermore, we have that $f_m \geq g_m \geq f$ for all $m \in \mathbb{N}$. Indeed, we have that $h_k \geq g_k \geq g_m$ for all $m \in \mathbb{N}$ and for all $k \leq m$. It follows that

$$|f_m(s) - g_m(s)| = f_m(s) - g_m(s) \leq h_m(s) - g_m(s) = |h_m(s) - g_m(s)| \leq \frac{1}{m} \qquad \forall m \in \mathbb{N}, \forall s \in S.$$

This implies that $\|f_m - g_m\| \leq \frac{1}{m}$ for all $m \in \mathbb{N}$. Finally, we have that

$$g_m(s) + \frac{1}{m} \geq f_m(s) \geq f(s) \qquad \forall m \in \mathbb{N}, \forall s \in S.$$

This implies that $f_m \downarrow f$. □

**Claim 3** If $f \in \bar{L}$ and $\{f_m\}_m \subseteq L$ is such that $f_m \downarrow f$ then $\lim_m V_i(f_m) = W_i(f)$.

**Proof of the Claim.** By monotonicity of $V_i$ and since $L$ is a Stone lattice, it follows that $\{V_i(f_m)\}_m$ is a nonincreasing sequence which is bounded from below by $V_i(-\|f\|) \in \mathbb{R}$. Hence, $\lim_m V_i(f_m)$ is well defined. Since $W_i$ is monotone as well, it follows that $\lim_m V_i(f_m) = \lim_m W_i(f_m) \geq W_i(f)$.

Viceversa, by Claim 1, there exists a sequence $\{g_k\}_k \subseteq L$ such that $g_k \geq f$ and $\|g_k - f\| \to 0$. Notice that, by definition of $W_i$, we have that $\lim_k V_i(g_k) = W_i(f)$.

Define for each $k \in \mathbb{N}$ the sequence $\left\{f_m^k\right\}_m$ such that $f_m^k = f_m \vee g_k$ for all $m \in \mathbb{N}$. Since $L$ is a Stone lattice, $\left\{f_m^k\right\}_m \subseteq L$. By construction, $f_m^k \downarrow g_k \in L$ for all $k \in \mathbb{N}$. By monotonicity and outer continuity of $V_i$, this implies that $\lim_m V_i(f_m) \leq \lim_m V_i\left(f_m^k\right) = V_i(g_k)$ for all $k \in \mathbb{N}$. This implies that $\lim_m V_i(f_m) \leq \lim_k V_i(g_k) = W_i(f)$, proving the statement. □

**Claim 4** $W_i$ is outer continuous.

**Proof of the Claim.** Consider $f \in \bar{L}$ and $\{g_m\}_m \subseteq \bar{L}$ such that $g_m \downarrow f$. We want to show that $\lim_m W_i(g_m) = W_i(f)$. By Claim 2, there exists $\{f_m\}_m \subseteq L$ such that $f_m \downarrow f$ and $\|f_m - g_m\| \leq \frac{1}{m}$. It follows that

$$\begin{aligned}|W_i(g_m) - W_i(f)| &\leq |W_i(f_m) - W_i(f)| + |W_i(g_m) - W_i(f_m)| \\ &\leq |V_i(f_m) - W_i(f)| + \frac{1}{m} W_i(1) \qquad \forall m \in \mathbb{N}.\end{aligned}$$

The second inequality follows since $W_i$ is the unique continuous extension of $V_i$ to $\bar{L}$ and $W_i$ is Lipschitz of order $V_i(1) = W_i(1)$ given that $V_i$ is. By Claim 3, it follows that $|V_i(f_m) - W_i(f)| \to 0$, proving the statement. □

By Claim 4 and (17), it follows that $W_1$ and $W_2$ are outer continuous, and so is $W$. This completes the proof of the lemma. ■

The next Lemma can be proved by using the same techniques of [18, Lemma 1 and Theorem 1].

**Lemma 16** *Let $V : L \to \mathbb{R}$ be a monotone, translation invariant, positively homogeneous, and outer continuous functional defined on a Stone vector lattice $L$. For any $A \in \Sigma_L$ there exists a sequence $\{f_m\}_m$ in $L_+$ such that $f_m \downarrow 1_A$. Moreover, the set function $\nu : \Sigma_L \to \mathbb{R}$ given by $\nu(A) = \lim_m V(f_m)$, where $\{f_m\}_m$ is a generic sequence in $L_+$ such that $f_m \downarrow 1_A$, is a well defined outer continuous capacity.*

We can now prove Theorem 13.

**Proof of Theorem 13.** (i) implies (ii). Suppose first that $L$ is a Stone vector lattice. By Lemma 14 and since $V$ is comonotonic additive, of bounded variation, and outer continuous, there exist two functionals $V_1, V_2 : L \to \mathbb{R}$ that are monotone, translation invariant, positively homogeneous, outer continuous, and such that $V = V_1 - V_2$. Define $\nu : \Sigma_L \to \mathbb{R}$ by $\nu(A) = \nu_1(A) - \nu_2(A)$ for all $A \in \Sigma_L$ where $\nu_1$ and $\nu_2$ are defined as in Lemma 16 via the functionals $V_1$ and $V_2$. By Theorem 7, Lemma 14 point (ii), and Lemma 16, $\nu$ is an outer continuous set function of bounded variation.

We now prove that (10) holds. Suppose that $f \in L_+$ and define $k = \|f\| + 1$. By (9), $\int_0^\infty \nu(f \geq t)\,dt$ is well defined. Let $\varepsilon > 0$. There exists a partition $\{t_i\}_{i=0}^n$ such that $0 = t_0 < \cdots < t_n = k$, $k/n < \varepsilon$, and

$$\left|\int_0^\infty \nu(f \geq t)\,dt - \sum_{i=1}^n \nu(f \geq t_{i-1})(t_i - t_{i-1})\right| = \left|\int_0^k \nu(f \geq t)\,dt - \sum_{i=1}^n \nu(f \geq t_{i-1})(t_i - t_{i-1})\right| < \varepsilon. \quad (18)$$

By [18, pag. 1815], for each $i \in \{1, ..., n\}$ there exists $f_{i-1} \in L_+$ such that

(a) $|\nu(f \geq t_{i-1}) - V(f_{i-1})| < \varepsilon/k$;

(b) $f_{i-1}(t_i - t_{i-1})$ and $\Sigma_{j=i+1}^n f_{j-1}(t_j - t_{j-1})$ are comonotonic for each $i \in \{1, ..., n-1\}$;

(c) $f \leq \Sigma_{i=1}^n f_{i-1}(t_i - t_{i-1}) \leq f + 2\varepsilon$.

By (c) and Lemma 14, it follows that

$$V_j(f) \leq V_j\left(\sum_{i=1}^n f_{i-1}(t_i - t_{i-1})\right) \leq V_j(f) + 2\varepsilon V_j(1) \qquad \text{for } j \in \{1, 2\}.$$

This implies that

$$\left|V_j\left(\sum_{i=1}^n f_{i-1}(t_i - t_{i-1})\right) - V_j(f)\right| \leq 2\varepsilon V_j(1) \qquad \text{for } j \in \{1, 2\}.$$

By (18), (a), and (b), it follows that

$$\begin{aligned}
\left|\int_0^\infty \nu(f \geq t)\,dt - V(f)\right| &\leq \left|\int_0^\infty \nu(f \geq t)\,dt - V\left(\sum_{i=1}^n f_{i-1}(t_i - t_{i-1})\right)\right| + \left|V\left(\sum_{i=1}^n f_{i-1}(t_i - t_{i-1})\right) - V(f)\right| \\
&\leq \left|\int_0^\infty \nu(f \geq t)\,dt - \sum_{i=1}^n V(f_{i-1})(t_i - t_{i-1})\right| + 2\varepsilon V_1(1) + 2\varepsilon V_2(1) \\
&\leq \left|\int_0^\infty \nu(f \geq t)\,dt - \sum_{i=1}^n \nu(f \geq t_{i-1})(t_i - t_{i-1})\right| \\
&\quad + \left|\sum_{i=1}^n \nu(f \geq t_{i-1})(t_i - t_{i-1}) - \sum_{i=1}^n V(f_{i-1})(t_i - t_{i-1})\right| + 2\varepsilon(V_1(1) + V_2(1)) \\
&\leq 2\varepsilon(1 + V_1(1) + V_2(1)).
\end{aligned}$$

Since $\varepsilon$ was arbitrarily chosen, this proves the statement. If $f \notin L_+$ then $f + \|f\| \in L_+$. It follows that

$$\begin{aligned}
V(f) + \|f\| V(1) = V(f + \|f\|) &= \int_0^\infty \nu(f + \|f\| \geq t)\,dt \\
&= \int_0^\infty \nu(f \geq t)\,dt + \int_{-\|f\|}^0 \nu(f \geq t)\,dt \\
&= \int_0^\infty \nu(f \geq t)\,dt + \int_{-\|f\|}^0 [\nu(f \geq t) - \nu(S)]\,dt + \|f\|\,\nu(S) \\
&= \int_0^\infty \nu(f \geq t)\,dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(S)]\,dt + \|f\| V(1),
\end{aligned}$$

proving the statement when $L$ is a Stone vector lattice.

Suppose that $L$ is not a vector space. By Lemma 14, $V : L \to \mathbb{R}$ is Lipschitz continuous. By Lemma 15, this implies the existence and uniqueness of its extension to $\bar{L}$. We still denote by $V$ this extension. Moreover, this extension is of bounded variation and outer continuous. In view of (2), the restriction of $V$ on $E$ is also comonotonic additive. For, given any comonotonic pair $f, g \in E$, there exist two sequences $\{f_n\}_n, \{g_n\}_n \subseteq L$ that supnorm converge to $f$ and $g$, respectively, and such that $f_n$ and $g_n$ are comonotonic and $f_n + g_n \in L$ for each $n \in \mathbb{N}$. Then, by the Lipschitz continuity of $V$, we have that $V(f+g) = \lim_n V(f_n + g_n) = \lim_n (V(f_n) + V(g_n)) = V(f) + V(g)$. Since $E$ is a Stone vector lattice, by the first part of the proof there exists an outer continuous $\nu \in bv(\Sigma_E)$ such that (10) holds on $E$ for the extension of $V$. In turn, this implies the existence of an outer continuous $\nu \in bv(\Sigma_L)$ such that (10) holds on $L$.

(ii) implies (iii). From $\nu = \nu_+ - \nu_-$ and Proposition 11, we get (11).

(iii) implies (i). It is easy to check that the functional $V : L \to \mathbb{R}$ defined by (10) is comonotonic additive, of bounded variation, and outer continuous since it is difference of functionals sharing these properties.

(a). Assume that $L$ is a Stone vector lattice and let $\nu$ be defined as in the previous part of the proof. Consider an outer continuous set function $\nu'$ in $bv(\Sigma_L)$ that satisfies (10). Given any $A = (f \geq t) \in \Sigma_L$, following [18, p. 1814] set

$$f_n = 1 - \left[1 \wedge n(t-f)^+\right].$$

We have $f_n(s) \in [0,1]$ for all $s \in S$ and the nonincreasing sequence $\{f_n\}_n$ is such that $f_n \downarrow 1_A$. In particular, $A = (f_n \geq 1)$ for all $n \in \mathbb{N}$ and

$$(f_n \geq t) \downarrow A \qquad \forall t \in (0,1]. \tag{19}$$

Define $g_n : [0,1] \to \mathbb{R}$ by $g_n(t) = \nu'(f_n \geq t)$ for all $n \in \mathbb{N}$. We have that $\{g_n\}_n$ is a sequence of functions of bounded variation, uniformly bounded by $\|\nu'\|$. By (19) and since $\nu'$ is outer continuous, $\lim_n g_n(t) = \nu'(A)$ for all $t \in (0,1]$. By the Arzelà Dominated Convergence Theorem (see, e.g., [11]), $\lim_n \int_0^1 g_n(t)\,dt = \nu'(A)$. By (10) and by definition of $\nu$, we have $\nu(A) = \lim_n V(f_n) = \lim_n \int_0^1 g_n(t)\,dt = \nu'(A)$, thus proving the uniqueness of $\nu$. If $L$ is a comonotonic Stone lattice, then $\nu$ is constructed on $\Sigma_E \supseteq \Sigma_L$.[12] By following the same technique, it follows that any outer continuous $\nu' \in bv(\Sigma_L)$ must coincide with $\nu_{|\Sigma_L}$.

(b). Necessity follows from a routine argument. On the other hand, sufficiency follows by noticing that $V = V_1$ and $V_2 = 0$. By Lemma 16, this implies that $\nu = \nu_1$ is an outer continuous capacity on $\Sigma_L$.

(c). If $\nu$ is supermodular then we have that

$$\begin{aligned}\nu((f \wedge g) \geq t) + \nu((f \vee g) \geq t) &= \nu((f \geq t) \cap (g \geq t)) + \nu((f \geq t) \cup (g \geq t)) \\ &\geq \nu(f \geq t) + \nu(g \geq t) \qquad \forall f, g \in L, \forall t \in \mathbb{R}\end{aligned}$$

By (10), we have that

$$V(f \wedge g) + V(f \vee g) \geq V(f) + V(g) \qquad \forall f, g \in L.$$

Viceversa, assume that $V$ is further supermodular. Pick $A, B \in \Sigma_L$. Define $\nu$ as in the initial part of the proof. Consider $\{f_n\}_n, \{g_n\}_n \subseteq L$ such that $f_n \downarrow 1_A$ and $g_n \downarrow 1_B$. We have that $f_n \vee g_n \downarrow 1_{A \cup B}$ and $f_n \wedge g_n \downarrow 1_{A \cap B}$. By Lemma 16 and since $V$ is supermodular, this implies that

$$\nu(A \cup B) + \nu(A \cap B) = \lim_n V(f_n \vee g_n) + \lim_n V(f_n \wedge g_n) \geq \lim_n V(f_n) + \lim_n V(g_n) = \nu(A) + \nu(B),$$

proving the statement. ■

[12] This observation is useful in the next two points as well.

## 4.1 Inner Continuous Representation

We now use the previous results to provide a characterization in terms of Choquet integral of *inner continuous* and comonotonic additive functionals of bounded variation from $L$ to $\mathbb{R}$.

**Proposition 17** *Let $V : L \to \mathbb{R}$ be a functional defined on a comonotonic Stone lattice. The following conditions are equivalent:*

*(i) $V$ is comonotonic additive, of bounded variation, and inner continuous;*

*(ii) there exists an inner continuous set function $\nu \in bv(\Sigma'_L)$ such that*

$$V(f) = \int_0^\infty \nu(f > t)\,dt + \int_{-\infty}^0 [\nu(f > t) - \nu(S)]\,dt \qquad \forall f \in L; \tag{20}$$

*(iii) there exist two inner continuous capacities $\nu^1$ and $\nu^2$ over $\Sigma'_L$ such that*

$$V(f) = V^1_{sc}(f) - V^2_{sc}(f) \qquad \forall f \in L \tag{21}$$

*where $V^i_{sc}(f) = \int_0^\infty \nu^i(f > t)\,dt + \int_{-\infty}^0 \left[\nu^i(f > t) - \nu^i(S)\right] dt$ for all $f \in L$ and $i \in \{1, 2\}$.*

*In particular, the inner continuous set function $\nu$ for which (20) holds is unique.*

**Proof.** (i) implies (ii) Given a set function $\nu : \Sigma_L \to \mathbb{R}$, define $\overline{\nu} : \Sigma'_L \to \mathbb{R}$ by $\overline{\nu}(A) = \nu(S) - \nu(A^c)$. The set function $\overline{\nu}$ is well defined since, being $L$ a Stone lattice, it is easy to check that $A \in \Sigma_L$ if and only if $A^c \in \Sigma'_L$. Moreover, $\nu$ is outer continuous and of bounded variation if and only if $\overline{\nu}$ is inner continuous and of bounded variation.

Since $V$ is comonotonic additive, of bounded variation, and inner continuous, by Lemma 5 the functional $\overline{V}$ is comonotonic additive, of bounded variation, and outer continuous. By Theorem 13, there exists a unique outer continuous set function $\nu \in bv(\Sigma_L)$ such that

$$\overline{V}(f) = \int_0^\infty \nu(f \geq t)\,dt + \int_{-\infty}^0 [\nu(f \geq t) - \nu(S)]\,dt, \qquad \forall f \in L.$$

Then,

$$\begin{aligned}
V(f) = -\overline{V}(-f) &= -\left(\int_0^\infty \nu(-f \geq t)\,dt + \int_{-\infty}^0 [\nu(-f \geq t) - \nu(S)]\,dt\right) \\
&= \int_{-\infty}^0 [\nu(S) - \nu(-f \geq t)]\,dt - \int_0^\infty \nu(-f \geq t)\,dt \\
&= \int_0^\infty [\nu(S) - \nu(f \leq t)]\,dt - \int_{-\infty}^0 \nu(f \leq t)\,dt \\
&= \int_0^\infty [\nu(S) - \nu(f \leq t)]\,dt + \int_{-\infty}^0 [\nu(S) - \nu(f \leq t) - \nu(S)]\,dt \\
&= \int_0^\infty \overline{\nu}(f > t)\,dt + \int_{-\infty}^0 [\overline{\nu}(f > t) - \overline{\nu}(S)]\,dt
\end{aligned}$$

where $\bar{\nu}$ is inner continuous and of bounded variation. This proves (20).

(ii) implies (iii). From $\nu = \nu^+ - \nu^-$ and Proposition 11, we get (21).

(iii) implies (i). It is easy to check that the Choquet functional $V : L \to \mathbb{R}$ defined by (21) is comonotonic additive, of bounded variation, and inner continuous since it is difference of functionals sharing these properties.

A suitable modification of the arguments used to prove uniqueness in Theorem 13 shows that $\nu$ is unique even in this case. ■

## 5 Two Special Cases

In this section we show what form Theorem 13 takes in the two classic comonotonic Stone lattices of Examples 3 and 4, that is, $B(\Sigma)$ and $C(S)$. In so doing, we both illustrate the unifying power of Theorem 13 and generalize two classic integral representation results. Throughout this section, the notation adopted in Theorem 13 that relates $V_c^i$ with $\nu^i$ for $i \in \{1,2\}$ is maintained. Similarly, the relation between $V_{sc}^i$ and $\nu^i$ for $i \in \{1,2\}$ is the one introduced in Proposition 17.

We begin with the collection $B(\Sigma)$ of measurable functions. An early version of this result was stated in [13] without any continuity assumption on $V$.

**Corollary 18** *Let $V : B(\Sigma) \to \mathbb{R}$ be a functional. The following conditions are equivalent:*

*(i) $V$ is comonotonic additive, of bounded variation, and outer (resp., inner) continuous;*

*(ii) there exists an outer (resp., inner) continuous set function $\nu \in bv(\Sigma)$ such that*

$$V(f) = V_c(f) \ \ (resp., \ = V_{sc}(f)) \qquad \forall f \in B(\Sigma); \tag{22}$$

*(iii) there exist two outer (resp., inner) continuous capacities $\nu^1$ and $\nu^2$ over $\Sigma$ such that*

$$V(f) = V_c^1(f) - V_c^2(f) \ \ (resp., \ = V_{sc}^1(f) - V_{sc}^2(f)) \qquad \forall f \in B(\Sigma).$$

*The unique $\nu$ that satisfies (22) is given by $\nu(A) = V(1_A)$.*

**Proof.** The result follows from Theorem 13 (resp., Proposition 17) since $\Sigma = \Sigma_{B(\Sigma)}$ (resp., $= \Sigma'_{B(\Sigma)}$), as observed in Example 3. ■

**Remark.** Rébillé [14] proves a version of Corollary 18 where he does not assume bounded variation and, as a result, the right hand side of (22) is a Lebesgue integral (without bounded variation the function $\varphi(t) = \nu(f \geq t)$ may not be Riemann integrable).

Endow now $S$ with a topology and consider the classic Stone vector lattice $C(S)$ of bounded continuous functions. When $S$ is compact, Theorem 13 takes the following stark form, where thank to Dini's Theorem we no longer need to require the outer continuity of $V$.

**Corollary 19** *Let $V : C(S) \to \mathbb{R}$ be a functional where $S$ is a compact topological space. The following conditions are equivalent:*

*(i) $V$ is comonotonic additive and of bounded variation;*

*(ii) there exists a unique outer continuous set function $\nu \in bv\left(\Sigma_{C(S)}\right)$ such that*

$$V(f) = \int_0^\infty \nu(f \geq t)\, dt + \int_{-\infty}^0 \left[\nu(f \geq t) - \nu(S)\right] dt \qquad \forall f \in C(S); \tag{23}$$

*(iii) there exist two outer continuous capacities $\nu^1$ and $\nu^2$ over $\Sigma_{C(S)}$ such that*

$$V(f) = V_c^1(f) - V_c^2(f) \qquad \forall f \in C(S).$$

**Proof.** Let $V$ be comonotonic additive and of bounded variation. By Lemma 14, it is Lipschitz continuous. Suppose $\{f_n\}_n \subseteq C(S)$ is such that $f_n \downarrow f \in C(S)$. By Dini's Theorem (see, e.g., [1, p. 54]), $\|f_n - f\| \to 0$, so that $\lim_n V(f_n) = V(f)$. This shows that $V$ is outer continuous. In view of this observation, the result now follows from Theorem 13. ■

## 6 A Daniell-Stone Theorem for Comonotonic Additive Functionals

In this section we assume that $L$ is a Stone *vector* lattice. Since $L$ is endowed with the supnorm, $L$ is a normed vector space and we denote by $L^*$ the norm dual of $L$. It follows that $L^*$ endowed with the dual norm $\|\cdot\|_*$ is an $AL$-space (see, e.g., [3, Theorem 4.1] and [2, Theorem 3.38]).[13] We denote by $\mathcal{A}$ the smallest $\sigma$-algebra such that each function in $L$ is measurable. It is immediate to see that $\mathcal{A} = \sigma(\Sigma'_L) = \sigma(\Sigma_L)$. We denote by $ca(\mathcal{A})$ the class of set functions on $\mathcal{A}$ that are countably additive. We endow $ca(\mathcal{A})$ with the total variation norm, $\|\cdot\|_{var}$. Notice that $(ca(\mathcal{A}), \|\cdot\|_{var})$ is a normed Riesz space, particularly, it is an $AL$-space (see, e.g., [1, Theorem 10.56]). Finally, we define $L' \subseteq L^*$ to be such that

$$L' = \left\{ I \in L^* : \lim_n I(f_n) = 0 \text{ if } f_n \downarrow 0 \right\}.$$

**Proposition 20** *$L'$ is a Riesz subspace of $L^*$.*

**Proof.** By definition of $L'$, it is immediate to see that $L'$ is a vector subspace of $L^*$. We are just left to show that $L'$ is a lattice as well. Notice that for each $I \in L'$ and for each $f \in L_+$ we have that

$$I^+(f) = \sup\{I(g) : 0 \leq g \leq f\} \geq 0. \tag{24}$$

Consider $\{f_n\}_{n\in\mathbb{N}} \subseteq L_+$ such that $f_n \downarrow 0$. For each $n \in \mathbb{N}$ define $g_n \in L$ to be such that $I(g_n) + \frac{1}{n} \geq I^+(f_n)$ and $0 \leq g_n \leq f_n$. By (24) and since $I \in L'$ and $g_n \downarrow 0$, we have that

$$0 \leq \liminf_n I^+(f_n) \leq \limsup_n I^+(f_n) \leq \lim_n \left\{ I(g_n) + \frac{1}{n} \right\} = 0.$$

It follows that $I^+$ belongs to $L'$, provided $I \in L'$. Given the equality $I = I^+ - I^-$ and since $L'$ is a vector space, we have that $I^-$ belongs to $L'$ as well. Hence, we can conclude that $|I| \in L'$ and that $L'$ is a Riesz subspace of $L^*$. ■

Given a functional $V : L \to \mathbb{R}$, we say that $V$ is (bounded) pointwise continuous at $f \in L$ if and only if $V(f_n) \to V(f)$ whenever $f_n(s) \to f(s)$ for all $s \in S$ and $\{f_n\}_n$ is uniformly bounded. We say that $V$ is pointwise continuous if and only if $V$ is pointwise continuous at each $f \in L$. Notice that if $V$ is pointwise continuous then it is inner and outer continuous. Moreover, $V$ is pointwise continuous at 0 if and only if $\bar{V}$ is.

In the theory of integration, elements in $L'_+$ are usually called Daniell integrals (see, e.g., [15, Chapter 16]). By the celebrated Daniell-Stone Theorem, they turn out to be pointwise continuous.

**Theorem 21 (Daniell-Stone)** *Let $V : L \to \mathbb{R}$ be a functional defined on a Stone vector lattice. The following conditions are equivalent:*

*(i) $V$ is monotone, linear, and pointwise continuous;*

*(ii) $V$ is monotone, linear, and pointwise continuous at 0;*

*(iii) $V$ is monotone, linear, and outer continuous at 0;*

*(iv) there exists a unique $\mu \in ca_+(\mathcal{A})$ such that*

$$V(f) = \int f d\mu \qquad \forall f \in L.$$

[13] Recall that for each $I \in L^*$ we have that $\|I\|_* = \sup\{|I(f)| : \|f\| \leq 1\} = \sup\{|I(f)| : -1 \leq f \leq 1\}$. Moreover, if $I \geq 0$ then $\|I\|_* = I(1)$.

In this section, we propose a generalization of the Daniell-Stone Theorem in which linearity is replaced by comonotonic additivity and superadditivity, while monotonicity is replaced by bounded variation. This is the second main result of the paper.

**Theorem 22** *Let $V : L \to \mathbb{R}$ be a functional defined on a Stone vector lattice. The following conditions are equivalent:*

*(i) $V$ is comonotonic additive, superadditive, pointwise continuous, and of bounded variation;*

*(ii) $V$ is comonotonic additive, superadditive, pointwise continuous at $0$, and of bounded variation;*

*(iii) there exists a unique continuous and supermodular $\nu \in bv(\mathcal{A})$ such that*

$$V(f) = \int f d\nu \qquad \forall f \in L.$$

*Moreover, $V$ is monotone if and only if $\nu$ is a capacity.*

As before, we prove few ancillary lemmas before proving the main theorem. First observe that, by the Daniell-Stone Theorem, for each $I \in L'_+$ there exists a unique element $\mu_I \in ca_+(\mathcal{A})$ such that

$$I(f) = \int f d\mu_I \qquad \forall f \in L. \tag{25}$$

Define the map $S : L'_+ \to ca_+(\mathcal{A})$ to be such that $I \mapsto \mu_I$. Moreover, without loss of generality, define by $\bar{S}$ the map from $L'$ to $ca(\mathcal{A})$ such that

$$\bar{S}(I) = S(I^+) - S(I^-) \qquad \forall I \in L'.$$

**Lemma 23** *Let $S$ and $\bar{S}$ be defined as above. The following statements are true:*

1. *$S$ is well defined, additive, and bijective;*
2. *$\bar{S}$ is a lattice isomorphism;*
3. *$\bar{S}$ is an isometry;*
4. *$\bar{S}$ is continuous when $L'$ and $ca(\mathcal{A})$ are endowed with their respective weak topologies.*

**Proof.** 1. By the Daniell-Stone Theorem, it follows that $S$ is well defined. Consider $I_1, I_2 \in L'_+$. With the previous notation, it follows that for each $f \in L$:

$$(I_1 + I_2)(f) = \int f d\mu_{I_1+I_2} \text{ and } (I_1 + I_2)(f) = I_1(f) + I_2(f) = \int f d\mu_{I_1} + \int f d\mu_{I_2} = \int f d(\mu_{I_1} + \mu_{I_2}).$$

By the uniqueness part of the Daniell-Stone Theorem, $S(I_1 + I_2) = \mu_{I_1+I_2} = \mu_{I_1} + \mu_{I_2} = S(I_1) + S(I_2)$. The fact that $S$ is injective follows easily from (25). Finally, since each $\mu \in ca_+(\mathcal{A})$ induces a linear, monotone, and outer continuous functional on $L$, it follows that $S$ is surjective.

2. Since $(ca(\mathcal{A}), \|\cdot\|_{var})$ is a Banach lattice and by [1, Theorem 8.43], we have that $(ca(\mathcal{A}), \|\cdot\|_{var})$ is an Archimedean Riesz space. Since $ca(\mathcal{A})$ is an Archimedean Riesz space, $L'$ is a Riesz space, and by point 1. and the Kantorovich theorem (see, e.g., [3, Theorem 1.10]), it follows that $S$ admits a unique extension to a positive operator from $L'$ to $ca(\mathcal{A})$. Moreover, this extension is $\bar{S}$. For each $I \in L'$ define $\mu_I = \bar{S}(I) \in ca(\mathcal{A})$. From the previous part of the proof and the definition of $S$, it follows that for each $I \in L'$

$$I(f) = \int f d\mu_I \qquad \forall f \in L.$$

This implies that $\bar{S}(I) = 0$ only if $I = 0$. It follows that $\bar{S}$ is injective. On the other hand, take $\mu \in ca(\mathcal{A})$. Define $I_1 = S^{-1}(\mu^+)$ and $I_2 = S^{-1}(\mu^-)$. Notice that $I = I_1 - I_2 \in L'$. Since $\bar{S}$ is linear and $S$ is bijective, it follows that

$$\bar{S}(I) = \bar{S}(I_1 - I_2) = \bar{S}(I_1) - \bar{S}(I_2) = S(I_1) - S(I_2) = \mu^+ - \mu^- = \mu,$$

proving that $\bar{S}$ is surjective. Finally, observe that if $\mu \in ca_+(\mathcal{A})$ then $\left(\bar{S}\right)^{-1}(\mu) = (S)^{-1}(\mu) \in L'_+$. It follows that $\bar{S}$ and its inverse are positive operators. By [3, Theorem 2.15], it follows that $\bar{S}$ is a lattice isomorphism.

3. First, notice that if $\mu \in ca_+(\mathcal{A})$ then we have that $\|\mu\|_{var} = \mu(S)$. It follows that

$$\left\|\bar{S}(I)\right\|_{var} = \|\mu_I\|_{var} = \mu_I(S) = I(1) = \|I\|_* \qquad \forall I \in L'_+. \tag{26}$$

Finally, since $(L', \|\cdot\|_*)$ is a normed Riesz space, we have that $I = I^+ - I^-$, $|I| = I^+ + I^-$, and $\|I\|_* = \||I|\|_*$ for all $I \in L'$. Since $\bar{S}$ is a lattice isomorphism and by (26), we have that

$$\left\|\bar{S}(I)\right\|_{var} = \left\|\left|\bar{S}(I)\right|\right\|_{var} = \left\|\bar{S}(|I|)\right\|_{var} = \||I|\|_* = \|I\|_* \qquad \forall I \in L',$$

proving the statement.

4. Since $\bar{S}$ is a linear isometry, $\bar{S}$ is norm continuous. By [1, Theorem 6.17], it follows that $\bar{S}$ is weakly continuous. ■

**Lemma 24** *Let $V : L \to \mathbb{R}$ be a comonotonic additive and superadditive functional of bounded variation defined on a Stone vector lattice. The following conditions are equivalent:*

*(i) $V$ is pointwise continuous;*

*(ii) $V$ is pointwise continuous at $0$;*

*(iii) there exists a unique convex and weak compact set $C \subseteq L'$ such that $I(1) = V(1)$ for all $I \in C$ and*

$$V(f) = \min_{I \in C} I(f) \qquad \forall f \in L;$$

*(iv) there exists a unique convex and weak compact set $D \subseteq ca(\mathcal{A})$ such that $\mu(S) = V(1)$ for all $\mu \in D$ and*

$$V(f) = \min_{\mu \in D} \int f d\mu \qquad \forall f \in L.$$

**Proof.** Since $V$ is a comonotonic additive and superadditive functional of bounded variation, we have that $V$ is translation invariant, superlinear, and Lipschitz continuous.

(i) implies (ii). It is obvious.

(ii) implies (iii). By [10] and since $V$ is translation invariant and superlinear, it follows that there exists a unique convex and weak* compact set $C \subseteq L^*$ such that $I(1) = V(1)$ for all $I \in C$ and

$$V(f) = \min_{I \in C} I(f) \qquad \forall f \in L.$$

Notice that $\bar{V}(f) = \max_{I \in C} I(f)$ for all $f \in L$. Next, we show that $C \subseteq L'$. Consider a bounded sequence $\{f_n\}_{n \in \mathbb{N}}$ in $L$ such that $f_n \to 0$. Notice that this is the case if either $f_n \downarrow 0$ or $\{f_n\}_{n \in \mathbb{N}}$ is bounded and order disjoint. It follows that

$$V(f_n) \le I(f_n) \le \bar{V}(f_n) \qquad \forall n \in \mathbb{N}, \forall I \in C. \tag{27}$$

Hence

$$\sup_{I \in C} |I(f_n)| \le \max\left\{\bar{V}(f_n), -V(f_n)\right\} \qquad \forall n \in \mathbb{N}.$$

Since $V$ is pointwise continuous at 0, it follows that $\lim_n \left(\sup_{I \in C} |I(f_n)|\right) = 0$. This implies that $I \in L'$ for all $I \in C$ and, by using the same arguments contained in the proof of [3, Theorem 4.41], $C$ is weak compact.

(iii) implies (iv). Given the set $C$, it is enough to define $D = \bar{S}(C)$. By Lemma 23, it follows that $D \subseteq ca(\mathcal{A})$ is a convex and weak compact set such that $\mu(S) = V(1)$ for all $\mu \in D$. Uniqueness follows from a standard separation argument.

(iv) implies (i). Define $\hat{V} : B(\mathcal{A}) \to \mathbb{R}$ by

$$\hat{V}(f) = \min_{\mu \in D} \int f d\mu \qquad \forall f \in B(\mathcal{A}).$$

It is immediate to see that $\hat{V}_{|L} = V$. Since $D$ is a weak compact subset of $ca(\mathcal{A})$, it follows that $\hat{V}$ is pointwise continuous. Hence, $V$ is pointwise continuous. ■

**Proof of Theorem 22.** (i) implies (ii). It is trivial.

(ii) implies (iii). By Lemma 24, we have that $V$ is even inner and outer continuous. Moreover, there exists a weak compact set $D \subseteq ca(\mathcal{A})$ such that

$$V(f) = \min_{\mu \in D} \int f d\mu \qquad \forall f \in L. \tag{28}$$

Define $\nu : \mathcal{A} \to \mathbb{R}$ by $\nu(A) = \min_{\mu \in D} \mu(A)$ for all $A \in \mathcal{A}$. It is not hard to show that $\nu$ is an inner and outer continuous bounded set function. On the other hand, by Theorem 13 and its proof, we have that there exists a unique outer continuous set function $\eta \in bv(\Sigma_L)$ such that

$$V(f) = \int_0^\infty \eta(f \geq t)\, dt + \int_{-\infty}^0 [\eta(f \geq t) - \eta(S)]\, dt \qquad \forall f \in L.$$

Moreover, for each $E \in \Sigma_L$ there exists $\{f_n\}_n \subseteq L$ such that $f_n \downarrow 1_E$ and $\lim_n V(f_n) = \eta(E)$. It follows that

$$\eta(E) = \lim_n V(f_n) = \lim_n \left( \min_{\mu \in D} \int f_n d\mu \right) = \min_{\mu \in D} \mu(E) = \nu(E) \qquad \forall E \in \Sigma_L. \tag{29}$$

Next, define $\hat{V} : B(\mathcal{A}) \to \mathbb{R}$ by

$$\hat{V}(f) = \int f d\nu \qquad \forall f \in B(\mathcal{A}).$$

Since $\nu$ is a continuous and bounded set function, it is easy to show that $\hat{V}$ is a well defined functional (see, e.g., [14, Corollary 2.2]). Define $\hat{L} = \left\{ f \in B(\mathcal{A}) : \hat{V}(f) = \min_{\mu \in D} \int f d\mu \right\}$. By (28) and (29), we have that

$$\hat{V}(f) = V(f) = \min_{\mu \in D} \int f d\mu \qquad \forall f \in L.$$

It follows that $L \subseteq \hat{L}$. Next, consider $\{f_n\}_n \subseteq \hat{L}$ such that $\{f_n\}_n$ is bounded and $f_n \downarrow f$ (resp., $f_n \uparrow f$). Since $\nu$ is outer (resp., inner) continuous and $D$ is convex and weak compact, it is immediate to see that

$$\hat{V}(f) = \lim_n \hat{V}(f_n) = \lim_n \left( \min_{\mu \in D} \int f_n d\mu \right) = \min_{\mu \in D} \int f d\mu.$$

By [6, Theorem 22.3], it follows that $\hat{L} = B(\mathcal{A})$. This implies that $\hat{V}$ is superadditive, hence $\nu$ is supermodular. By [12, Theorem 4.7] and since $\nu$ is bounded, $\nu$ belongs to $bv(\mathcal{A})$, proving the statement.

We are left to prove uniqueness. Consider two continuous supermodular set functions $\nu_1, \nu_2 \in bv(\mathcal{A})$ such that $V(f) = \int f d\nu_i$ for all $f \in L$ and for $i \in \{1, 2\}$. For each $i \in \{1, 2\}$ define $\hat{V}_i$ to be the the functional from $B(\mathcal{A})$ to $\mathbb{R}$ such that $\hat{V}_i(f) = \int f d\nu_i$ for all $f \in B(\mathcal{A})$. By [12, Theorem 4.7], for each $i \in \{1,2\}$ there exists a convex and weak compact set $D_i \subseteq ca(\mathcal{A})$ such that

$$\hat{V}_i(f) = \min_{\mu \in D_i} \int f d\mu \qquad \forall f \in B(\mathcal{A}).$$

In particular, notice that $\nu_i(A) = \min_{\mu \in D_i} \mu(A)$ for all $A \in \mathcal{A}$ and for all $i \in \{1,2\}$.

Define $C_i = \bar{S}^{-1}(D_i)$ for $i \in \{1,2\}$. By Lemma 23, we have that $C_i$ is a weak compact and convex subset of $L'$. Since $\hat{V}_i(f) = V(f)$ for all $f \in L$ and for $i \in \{1,2\}$, it follows that for each $i \in \{1,2\}$

$$V(f) = \min_{I \in C_i} I(f) \qquad \forall f \in L.$$

By Lemma 24, it follows that $C_1 = C_2$. By Lemma 23, we have that $D_1 = \bar{S}(C_1) = \bar{S}(C_2) = D_2$, proving that $\nu_1 = \nu_2$.

(iii) implies (i). It follows from routine arguments.

Finally, if $\nu$ is a capacity trivially $V$ is monotone. Viceversa, since $\bar{S}$ is a positive operator, if $V$ is monotone then $D$ is a subset of $ca_+(\mathcal{A})$ since $C$ in Lemma 24 can be chosen to be a subset of $L'_+$. This implies that $\nu$ is a capacity. ■

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