

# Sources of Uncertainty and Subjective Prices\*

V. Cappelli<sup>•</sup>, S. Cerreia-Vioglio<sup>\*</sup>, F. Maccheroni<sup>\*</sup>, M. Marinacci<sup>\*</sup>, S. Minardi<sup>•</sup>

<sup>\*</sup>Università Bocconi and IGIER and <sup>•</sup>HEC Paris

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## Abstract

We develop a general framework to study source-dependent preferences in economic contexts. We behaviorally identify two key features. First, we drop the assumption of uniform uncertainty attitudes and allow for *source-dependent attitudes*. Second, we introduce *subjective prices* to compare outcomes across different sources. Our model evaluates profiles source-wise, by computing the source-dependent certainty equivalents; the latter are converted into the unit of account of a common source and then aggregated into a unique evaluation. By viewing time and location as instances of sources, we show that subjective discount factors and subjective exchange rates are emblematic examples of subjective prices. Finally, we use the model to explore the implications on optimal portfolio allocations and home bias.

*Key words:* source preference, source-dependent uncertainty attitudes, subjective prices, competence hypothesis, home bias

## 1 Introduction

### 1.1 Reductionisms

In applications the consequences of different courses of action are often summarized through basic quantitative indicators, like amounts of money or casualties. Though convenient, these succinct expressions are reduced forms of genuine, but often complicated and elusive to come by, consequences that record “anything that may happen to the person” as Savage (1954, p. 13) prescribes. In the classic Savagean paradigm, utility functions are defined over such all inclusive consequences, and expected utilities are computed relative to a subjective probability on states of nature that captures agents’ beliefs. Reduced-form

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consequences may give rise to problematic issues within this standard setting. As Smith (1969, p. 325) eloquently writes, “[...] if a man loses a dice game bet he and his associates might consider that he was merely the victim of bad luck [...]. But if he loses from incorrectly predicting a rise in the Dow-Jones, he may perceive that his colleagues feel that he should have known better. [...] the utility of money or other rewards is not independent of the circumstances under which it is obtained.” A reduced-form consequence is, by its very nature, a crude description of all the relevant outcomes of a course of action. Altogether different consequences may, for example, end up being translated into the same amount of money, so into the same reduced-form consequence.<sup>1</sup>

This “consequence reductionism” impacts on another key feature of Savage’s approach, that is, the reduction of uncertainty to risk: expected utilities are, effectively, computed with respect to the distributions on consequences (lotteries) induced by the subjective probability via the actions. Risk attitudes are determined with respect to these induced distributions, regardless of the underlying state spaces. As Smith’s quote indicates, this “risk reductionism” becomes questionable when combined with reduced-form consequences that might well receive multiple evaluations according to the kind of contingencies that deliver them.

**Source dependence** We address this issue by extending the expected utility framework to accommodate *source-dependent preferences*. In line with the literature, a source is a collection of contingencies that correspond to the same “mechanism” of uncertainty; different sources correspond to different domains of uncertainty.<sup>2</sup> A paradigmatic example is the classic two-urn Ellsberg experiment where the two urns identify distinct sources that differ in the stochastic nature of the uncertainty faced.

Our framework allows us to go beyond this classic example by encompassing an array of factors that may account for the presence of multiple sources in a decision problem. First of all, sources may vary in the personal implications of the material consequences involved, as suggested by the *competence hypothesis* (Heath and Tversky, 1991, and Fox and Tversky, 1995): an individual may prefer gaining a certain amount of money in the domain of judgment rather than in the domain of chance if, in the former case, the gain is accompanied by a feeling of credit.<sup>3</sup> Source-dependent preferences may also arise in a deterministic setting. For instance, the possibly distinct social implications of the same material consequence may be rationalized in terms of different (deterministic) sources. A

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<sup>1</sup>Throughout the paper, the terms “consequence” and “outcome” are used interchangeably.

<sup>2</sup>Originally introduced by Tversky and Fox (1995) and Tversky and Wakker (1995), the notion of source preference is the object of study of more recent works including Chew and Sagi (2008), Abdellaoui, Baillon, Placido, and Wakker (2011), and Gul and Pesendorfer (2015).

<sup>3</sup>“Psychic payoffs of satisfaction or embarrassment can result from self-evaluation or from an evaluation by others. [...] In the domain of chance, both success and failure are attributed primarily to luck. The situation is different when a person bets on his or her judgment.” (Heath and Tversky, 1991, pp. 7-8)

policy maker may evaluate differently the threats to public security of different categories, even if they involve the same number of casualties. Slovic (1999, p. 691) puts this in clear terms: “Are the deaths of 50 passengers in separate automobile accidents equivalent to the deaths of 50 passengers in one airplane crash?” While the relevance of this type of queries for policy making is obvious, it begs the question of how one compares prospects that depend on different sources. Moreover, any answer involves inevitably a subjective assessment: different individuals would most likely provide different reasons as to whether or not the above two source-dependent outcomes are equivalent.<sup>4</sup> Some insights may come from seemingly unrelated contexts, as anticipated next.

**Dates and locations** In standard economic settings, emblematic examples of sources that do not involve uncertainty are given by the different dates or locations at which a material consequence is delivered. The idea that time and location constitute formal attributes of a good dates back, at least, to Debreu (1959), who argues that “a commodity is a good or a service completely specified physically, temporally, and spatially.” These classic analyses arm us with familiar tools to evaluate the same outcome at different dates or locations—namely, the discount factor and the exchange rate in temporal and allocation contexts, respectively. Since they identify the subjective value of an outcome in relative terms, they can be referred to as the *subjective prices* of receiving a monetary outcome at a given date or location from the perspective of another date or location. We will see that subjective discount factors and subjective exchange rates are just two instances of a more encompassing notion of subjective price which lies at the core of our general theory of source dependence.

## 1.2 Source-dependent attitudes and subjective prices

Our goal is to develop a unifying framework for the study of source dependence. Our axiomatic foundation shows that the multifaceted nature of source dependence can be reduced to two behavioral features: *intra-source uncertainty attitudes* and *inter-source tastes*.

Source dependence may originate from a multidimensional perception of uncertainty, as shown by the Ellsberg paradox. More generally, various evidence indicates that individuals may exhibit heterogeneous attitudes toward uncertainty.<sup>5</sup> We thus dispense with the standard assumption that there exist universal risk attitudes, portable across sources. We recognize, instead, that such attitudes may depend on the underlying source of uncertainty.

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<sup>4</sup>According to Savage, these outcomes would correspond to distinct consequences arising from different full descriptions.

<sup>5</sup>Such awareness has reached out to policy makers, as suggested by the European Securities and Markets Authority (May 2014): “We propose to clarify, with respect to clients’ risk-bearing capacity, that any particular consumer has not only one and overall risk attitude but different risk attitudes towards different investment targets.”

Within the expected utility framework, this means that the von Neumann-Morgenstern utility functions are themselves allowed to be source dependent. We adopt the standard notion of certainty equivalent to identify the uncertainty attitude on each source. Consider a collection  $I = \{1, \dots, n\}$  of sources and let  $f_i$  stand for an uncertain prospect dependent on source  $i \in I$ . The certainty equivalent  $c_i(f_i)$  is the monetary consequence that the agent finds equivalent to the uncertain prospect  $f_i$  on source  $i$ .<sup>6</sup> It thus captures the *intra-source uncertainty attitude*. The traditional approach continues to hold in comparing prospects that depend on the same source, but no longer otherwise.

This leads us to the second distinguishing feature of source dependence. After having factored out uncertainty through certainty equivalents, we argue that the evaluation of sure outcomes may still be source dependent. To compare actions that depend on different sources, we need to convert units on one source into units on another source via a subjective conversion rate. The latter can be viewed as the cost of transporting units of consumption across sources: as such, it is a subjective price and it captures *inter-source tastes* by expressing the relative value of a sure consequence on one source in terms of its equivalent value on another source (for instance, reflecting agents' different competence on them).

Formally, given any prospects  $f_i$  and  $g_j$  on two sources  $i$  and  $j$ , our agent first computes the source-dependent certainty equivalents,  $c_i(f_i)$  and  $c_j(g_j)$ ; then, he applies a rate  $\delta_{ij}(c_j(g_j))$  that converts  $c_j(g_j)$  into the unit of account of source  $i$ . This rate is the subjective price of  $c_j(g_j)$  on source  $i$ . Our agent prefers  $f_i$  to  $g_j$  if and only if  $c_i(f_i) \geq \delta_{ij}(c_j(g_j))$ . Note that in the two-urn Ellsberg experiment, the two sources differ only in terms of their stochastic nature, while being perfectly comparable otherwise; in this case, the subjective price is given simply by the identity function. However, this does not hold in general, as suggested by Slovic (1999) in the context of casualties. Furthermore, our theory analyzes a richer domain of preferences which involves not only comparisons between source-dependent prospects, but also between profiles of source-dependent prospects of the form  $\mathbf{f} = (f_1, \dots, f_n)$ . A profile  $\mathbf{f}$  is evaluated by converting each source-dependent certainty equivalent  $c_i(f_i)$  into the unit of account of a common source  $o$ , and then by aggregating the  $o$ -normalized valuations  $\delta_{oi}(c_i(f_i))$  as  $W_o(\delta_{o1}(c_1(f_1)), \dots, \delta_{on}(c_n(f_n)))$ . We will focus on the prominent case of quasi-arithmetic aggregators and apply it to revisit both temporal and allocation settings.

**Temporal settings** We show that temporal decision problems can be viewed as instances of decision problems with source dependence. In this context,  $\mathbf{f} = (f_1, \dots, f_n)$  takes the form of a temporal stream of consumption where source  $i \in \{1, \dots, n\}$  is the date at which bundle  $f_i$  is consumed. In turn, the discount factor is a classic example of conversion rate as it is the subjective price of consumption in the next period in terms

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<sup>6</sup>Because of their economic relevance, throughout the paper we focus on monetary consequences, so on “monetary reductionism”.

of consumption in the current period. Moreover, the source-dependent certainty equivalents act as time-dependent utilities and give rise to time-dependent uncertainty attitudes. Thus, temporal sources suggest a behavioral basis to explain the empirical evidence that risk aversion may be domain-dependent and vary with the time horizon and with the age.<sup>7</sup> Recognizing such a dependence may help explain some puzzling findings in asset pricing.<sup>8</sup>

**Allocation settings** Evoking Debreu’s (1959) ideas, standard allocation problems in international economics can also be viewed as instances of decision problems with source dependence: different locations at which the same material consequence can be delivered correspond to different sources. We apply our model to a simple portfolio allocation problem and show that our conversion rate plays the role of a subjective exchange rate that captures the subjective value of consumption in one market in terms of consumption in another market. We view it as the agent’s relative degree of familiarity about the value of an outcome in different markets. Our result on optimal allocation uncovers a relationship between the subjective exchange rate and the (market) real exchange rate which is the spatial counterpart of the well-known relationship between the subjective discount factor and the real interest rate in temporal settings.

Source dependence may help explain the large empirical evidence on under-diversification and home bias in capital markets.<sup>9</sup> Our theoretical results point to a recent empirical literature suggesting that one driving factor for under-diversification might be a familiarity bias: individual investors prefer local securities because they feel more knowledgeable about nearby markets, even if this sense of competence cannot be attributed to superior information.<sup>10</sup> Note that this notion of familiarity bias is distinct from having effective knowledge about the underlying mechanism of uncertainty. The latter case corresponds to Heath and Tversky’s competence hypothesis and is controlled by the source-dependent uncertainty attitudes (captured by the certainty equivalents). Thus, our model accommodates both forces and allows us to analyze the impact of each of them on home bias.

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<sup>7</sup>Baucells and Heukamp (2010), Abdellaoui, Diecidue, and Onculer (2011), and Eisenbach and Schmalz (2016) report evidence on increasing risk aversion as the source of risk approaches in time. On age, see, e.g., Deakin, Aitken, Robbins, and Sahakian (2004), and Dohmen et al. (2011).

<sup>8</sup>For instance, contrary to the standard predictions, van Binsbergen, Brandt, and Koijen (2012), Andries, Eisenbach, Schmalz, and Wang (2015), Andries, Eisenbach, and Schmalz (2018), and Weber (2018) report evidence in favor of a downward-sloping term structure of equity returns.

<sup>9</sup>This phenomenon has been initially documented by French and Poterba (1991). Pointing to Heath and Tversky’s competence hypothesis, they recognize that (p. 225) “Investors may not evaluate the risk of different investments based solely on the historical standard deviation of returns. They may impute extra “risk” to foreign investments because they know less about foreign markets, institutions, and firms.” For a comprehensive survey, see, e.g., Campbell (2006).

<sup>10</sup>Among others, Grinblatt and Keloharju (2001), Huberman (2001), Zhu (2002), Goetzmann and Kumar (2008), Graham, Harvey, and Huang (2009), Seasholes and Zhu (2010), and Boyle, Garlappi, Uppal, and Wang (2012) find that familiarity is unlikely to be driven by access to superior information about local markets.

**Outline** The rest of the paper is organized as follows. Next we discuss the related literature. Section 2 presents the formal setup. Section 3 states the basic axioms and representation results of preferences over prospect-source pairs. Section 4 contains our axiomatic characterization of preferences over prospect profiles, the quasi-arithmetic specification, and the temporal setting. Section 5 provides a comparative analysis of the notion of source smoothing. Section 6 applies our model to a portfolio allocation problem. Section 7 concludes with some remarks on the possible extension to interdependence between sources and on the problem of elicitation of sources. All proofs are collected in appendix. An online appendix contains preliminary results on Chisini means.

### 1.3 Related literature

The economic interest on source-dependent preferences was originally sparked by the Ellsberg paradox; by now there is a sizable body of theoretical and experimental works. Klibanoff, Marinacci, and Mukerji (2005), Nau (2006), Ergin and Gul (2009), and Seo (2009) develop models characterized by two “hierarchical” sources.<sup>11</sup> The closest works to the present paper are Chew and Sagi (2008), Abdellaoui, Baillon, Placido, and Wakker (2011), and Gul and Pesendorfer (2015). These papers identify a source by a restricted collection of events that share the same mechanism of uncertainty. The key novelty is the recognition that preferences can be compatible with a single probability distribution as long as acts depend on the same source; thus, probabilistic sophistication holds within each source but not necessarily across sources. Chew and Sagi (2008) provide sufficient axiomatic conditions for probabilistic sophistication to hold locally, within each source referred to as a “small world.” Abdellaoui et al. (2011) test for the empirical relevance of source dependence. Their experimental method assumes probabilistic sophistication on each source, and it is based on the estimate of three parameters: a utility index, a subjective probability over states, and a source function which transforms probabilities into decision weights. As in Chew and Sagi, only the latter is source dependent. They find robust experimental evidence in support of source-dependent uncertainty attitudes in different decision problems. Gul and Pesendorfer (2015) characterize a special form of  $\alpha$ -maxmin preferences which allows for source dependence. A source is subjectively identified by a non-atomic probability measure over a restricted  $\sigma$ -algebra of events relative to which probabilistic sophistication holds. Similarly to Abdellaoui et al., the act-induced lotteries are evaluated according to rank-dependent expected utility with a source-dependent probability transformation function identifying the uncertainty attitude toward specific sources. Note that in the three aforementioned papers sources are derived endogenously, whereas in the present work sources are a primitive.

In addition to Abdellaoui et al., substantial empirical evidence points to domain-dependent uncertainty attitudes. There has been a growing interest in studying not only

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<sup>11</sup>See Marinacci (2015) for an analysis of two-stage decision models.

attitudes toward artificial sources, such as the two Ellsberg urns, but also toward natural sources (e.g., the temperature in different cities or different stock indices).<sup>12</sup>

The present paper contributes to the above literature in two main directions. First, we introduce a focal element given by the notion of subjective price. Acting as rates of conversion, subjective prices allow us to compare source-dependent prospects (beyond the uncertainty contexts studied in the literature) as well as to compare entire prospect profiles. Moreover, they offer a novel behavioral perspective to study some classic applied settings and related empirical findings. Second, our model accommodates not only source-dependent probability distortions (i.e., weighting of probabilities that depends on sources) but also source-dependent utilities (i.e., weighting of consequences that depends on sources). The latter approach is in line with some early discussions of the Ellsberg paradox (Roberts, 1963; Smith, 1969; Chew, Li, Chark, and Zhong, 2008), as well as the evidence on domain dependence of risk aversion. Our model thus incorporates alternative approaches by putting them into perspective.

## 2 Setup

The basic elements of a decision problem with source dependence are a set  $X$  of *outcomes* and a collection  $I$  of *sources of uncertainty* which affect the outcomes resulting from the agent’s choices.

We assume that  $X$  is a non-singleton interval of the real line containing the null outcome 0. In particular, having  $X = \mathbb{R}$  is natural when outcomes are monetary payoffs. On the other hand, if they represent quantities of a homogeneous good (or casualties), it is natural to allow only for non-negative real numbers. Note that here outcomes are state-independent. We thus abstract from state-dependence, a classic issue in decision making under uncertainty (see Karni, 1985, and Dreze, 1987). However, our setup can be viewed as a “space-dependent” one – where the spaces are given by the  $S_i$ ’s or the  $\Sigma_i$ ’s described below.

For simplicity, let  $I = \{1, \dots, n\}$  be a finite set. A source of uncertainty  $i \in I$  is a collection of contingencies that correspond to a common mechanism of uncertainty. A source-dependent prospect describes how these contingencies affect the outcomes of a chosen alternative. Next, we present two ways of modeling sources (and, therefore, source-dependent prospects) according to how such contingencies are formalized.

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<sup>12</sup>See, e.g., Fox and Tversky (1998), Kilka and Weber (2001), Weber, Blais, and Betz (2002), Baillon and Bleichrodt (2015), Abdellaoui, Bleichrodt, Kemel, and l’Haridon (2017), Baillon, Huang, Selim, and Wakker (2018), and Li, Müller, Wakker, and Wang (2018).

## 2.1 Sources and source-dependent prospects

**Naive definitions** As a first way of describing sources of uncertainty, consider a collection  $\{S_i\}_{i \in I}$  of finite sets of contingencies. Each source  $i \in I$  identifies a set  $S_i$ , with generic element denoted by  $s_i$ , that can be thought of as the set of contingencies which features the common source of uncertainty  $i$ . A *prospect dependent on source  $i$*  is a function  $f_i : S_i \rightarrow X$  which yields outcome  $f_i(s_i) \in X$  if contingency  $s_i$  occurs. We denote by  $\mathcal{F}_i := X^{S_i}$  the set of all prospects that depend on source  $i$ . When no confusion arises, we denote by  $x$  both the outcome  $x$  in  $X$  (a scalar) and the constant source-dependent prospect  $x1_{S_i} \in \mathcal{F}_i$  (a function) yielding  $x$  in every contingency in  $S_i$ .

A *prospect profile* is a vector  $\mathbf{f} = (f_1, \dots, f_n)$ , or shortly  $\mathbf{f} = (f_i)_{i \in I}$ , that specifies for each source  $i$  the corresponding source-dependent prospect  $f_i$ . Let  $\mathcal{F} := \times_{i \in I} \mathcal{F}_i$  denote the set of all prospect profiles. With little abuse, we denote by  $\mathbf{x}$  both the outcome profile  $\mathbf{x} = (x_i)_{i \in I} \in X^I$  and the corresponding profile  $(x_i 1_{S_i})_{i \in I}$  of constant prospects in  $\mathcal{F}$ .

For example, in the Ellsberg two-urn context,  $I = \{1, 2\}$  is the set of urns,  $S_i = \{\text{Red}_i, \text{Black}_i\}$  is the set of contingencies for each  $i \in I$ , and a prospect profile is a “portfolio” of bets

$$\mathbf{f} = ((\text{Red}_1, x_1; \text{Black}_1, y_1), (\text{Red}_2, x_2; \text{Black}_2, y_2))$$

that describes the payoffs of an agent who (potentially) gambles on both urns.

If  $f_j = 0$  for all  $j \neq i$ , then the prospect profile  $\mathbf{f}$  depends only on source  $i$  and it is denoted by  $(f_i, i)$ . Let  $\mathcal{P}_i := \{(f_i, i) : f_i \in \mathcal{F}_i\}$  be the family of all these prospects. Thus,  $\mathcal{P} = \cup_{i \in I} \mathcal{P}_i$  is the family of all prospects that may yield non-zero outcomes on only one source at a time; we refer to these prospects as *prospect-source pairs*. In the Ellsberg two-urn context,  $(f_i, i)$  corresponds to a bet  $f_i$  on urn  $i$ , and the famous paradox involves comparisons of bets that depend only on one urn at a time.

Preferences are described by a binary relation  $\succsim$  on the set  $\mathcal{F}$  of all prospect profiles. We will often focus on the restriction of  $\succsim$  to the smaller set  $\mathcal{P}$  of all prospect-source pairs. The further restriction of  $\succsim$  to  $\mathcal{P}_i$  induces a preference  $\succsim_i$  on  $\mathcal{F}_i$  defined by

$$f_i \succsim_i g_i \iff (f_i, i) \succsim (g_i, i) \quad \text{for all } i \in I.$$

**Sophisticated definitions** As a second way of describing sources of uncertainty, consider a collection  $\{\Sigma_i\}_{i \in I}$  of finite algebras on a given state space  $S$ .<sup>13</sup> Each source  $i$  corresponds to an algebra  $\Sigma_i$  of subsets of  $S$ . In turn, the atoms of each algebra can be thought of as the set of contingencies featured by a given source. A prospect dependent on source  $i$  is now a  $\Sigma_i$ -measurable function  $f_i : S \rightarrow X$  that yields outcome  $f_i(s) \in X$  if state  $s$  occurs. We denote by  $\mathcal{F}_i$  the collection of all these prospects. The definitions of

<sup>13</sup>We consider finite algebras for convenience. Our results can be extended to infinite algebras.



$\mathcal{F}$ ,  $(f_i, i)$ ,  $\mathcal{P}_i$ , and  $\mathcal{P}$  are formally unchanged.<sup>14</sup>

The relation with the previous framework is straightforward: it is sufficient to set  $S = \times_{i \in I} S_i$ , consider the algebras  $\Sigma_i = \{A_i \times S_{-i} : A_i \subseteq S_i\}$  for all  $i \in I$ , and note that the value  $f_i(s)$  of an element of  $\mathcal{F}_i$  depends only on the  $i$ -th component  $s_i$  of  $s$ . Throughout we adopt this framework since it is more general than the naive one and preserves tractability. However, our axiomatic derivation and related results do not depend on the specific formulation adopted.

### 3 Preferences and basic representation results

We refer to  $\succsim_i$  on  $\mathcal{F}_i$  for  $i \in I$  as *intra-source preferences* and to  $\succsim$  on  $\mathcal{P}$  as *inter-source preferences*. The entire relation  $\succsim$  on  $\mathcal{F}$  describes *preferences over prospect profiles*.

#### 3.1 Intra-source preferences

For every source  $i$ , we impose the following basic assumption on  $\succsim_i$  over  $\mathcal{F}_i$ .

**Axiom A. 1** For every  $i \in I$  and  $f_i, g_i, h_i \in \mathcal{F}_i$ ,

(i) if  $f_i \geq g_i$ , then  $f_i \succsim_i g_i$ . Moreover, if both  $f_i$  and  $g_i$  are constant and  $f_i > g_i$ , then  $f_i \succ_i g_i$ ;

(ii) if  $f_i \succ_i g_i \succ_i h_i$ , then there exist  $\alpha, \beta \in (0, 1)$  such that

$$(1 - \alpha) f_i + \alpha h_i \succ_i g_i \succ_i (1 - \beta) f_i + \beta h_i.$$

These properties have an obvious interpretation since only prospects depending on a given source  $i$  are considered. Condition (i) is a standard monotonicity assumption stating that the agent should prefer a source-dependent prospect that dominates another in every state. The converse holds, as well, if the prospects at hand are constant. Condition (ii) is a standard Archimedean property that guarantees the continuity of each preference  $\succsim_i$ .

**Definition 1** For every  $i \in I$ , a monotone and continuous functional  $c_i : \mathcal{F}_i \rightarrow X$  is a *certainty equivalent* for  $\succsim_i$  if, for all  $x \in X$ ,  $x = c_i(x)$  and, for all  $f_i \in \mathcal{F}_i$ ,

$$f_i \sim_i c_i(f_i).$$

The following routine lemma shows that Axiom A.1 guarantees the existence of a unique certainty equivalent.

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<sup>14</sup>This approach is adopted by Chew and Sagi (2008), Abdellaoui et al. (2011), and Gul and Pesendorfer (2015).

**Lemma 2** For every  $i \in I$ , let  $\succsim_i$  be a binary relation on  $\mathcal{F}_i$ . The following conditions are equivalent:

- (i)  $\succsim_i$  is a weak order that satisfies Axiom A.1;
- (ii) there exists a unique certainty equivalent functional  $c_i : \mathcal{F}_i \rightarrow X$  for  $\succsim_i$ . In this case, for all  $f_i, g_i \in \mathcal{F}_i$ ,

$$f_i \succsim_i g_i \iff c_i(f_i) \geq c_i(g_i).$$

Certainty equivalents play a key role in our analysis. Not only do they provide a utility representation of intra-source preferences; more importantly, they identify the attitude toward uncertainty on each source. We can distinguish between source-dependent utilities (consistent with the evidence on the multidimensionality of risk) and source-dependent probabilities (consistent with the evidence on non-uniform ambiguity attitudes), as illustrated by the following classic examples.

**Example 3 (Source-dependent expected utility)** For every  $i \in I$ , there are a strictly increasing and continuous function  $u_i : X \rightarrow \mathbb{R}$ , and a probability measure  $p_i$  on  $(S, \Sigma_i)$  such that, for all  $f_i \in \mathcal{F}_i$ ,

$$c_i(f_i) = u_i^{-1}(\mathbb{E}_{p_i}[u_i(f_i)]). \quad (1)$$

In the above expression,  $c_i(f_i)$  is the  $u_i$ -mean of  $f_i$  with respect to  $p_i$ . It corresponds to a basic example of a Chisini mean (1929, p. 113). More precisely, the means of this kind are called *quasi-linear* Chisini mean<sup>15</sup> and their characterization dates back to de Finetti (1931). Axiom A.2 guarantees the existence of a certainty equivalent functional as in (1); it amounts to saying that the agent is an expected utility maximizer source-by-source. There exist different ways of expressing this standard assumption in terms of binary relations. As they are all well known, we omit their formal exposition and refer to the literature.<sup>16</sup>

**Axiom A. 2** For every  $i \in I$ , there exists a quasi-linear Chisini mean  $c_i : \mathcal{F}_i \rightarrow X$  that represents  $\succsim_i$ .

A popular generalization of (1) is the following rank-dependent formulation of source dependence in a spirit similar to Abdellaoui et al. (2011).<sup>17</sup>

<sup>15</sup>The online appendix studies the link between the notion of Chisini mean and the preference representations proposed in this paper. See, e.g., Bullen (2003) and Grabisch, Marichal, Mesiar, and Pap (2011) for a modern treatment of Chisini means.

<sup>16</sup>See, e.g., Savage (1954) and Wakker (1988; 2010, Section 4.6) for infinite and finite algebras  $\Sigma$ , respectively.

<sup>17</sup>Compared to Abdellaoui et al. (2011), Example 4 presents a slightly more general formulation where also the utility index is source-dependent.

**Example 4 (Source-dependent probability-weighting)** For every  $i \in I$ , there are a strictly increasing and continuous function  $u_i : X \rightarrow \mathbb{R}$ , a probability measure  $p_i$  on  $(S, \Sigma_i)$ , and an increasing and onto function  $w_i : [0, 1] \rightarrow [0, 1]$  such that, for all  $f_i \in \mathcal{F}_i$ ,

$$c_i(f_i) = u_i^{-1}(\mathbb{E}_{w_i(p_i)}[u_i(f_i)]). \quad (2)$$

On each source  $i$ , the source function  $w_i$  transforms subjective probabilities into decision weights, giving rise to a rank-dependent utility evaluation with probabilistic sophistication. With this, we can think of  $c_i(f_i)$  as the  $u_i$ -mean of  $f_i$  with respect to  $w_i(p_i)$ : we refer to this type of means as *rank-dependent Chisini means*. Axiom A.3 guarantees the existence of a certainty equivalent functional as in (2). For a formulation of this assumption in terms of preferences, we need to guarantee that the preference  $\succsim_i$  be probabilistically sophisticated on each source  $i$ : this amounts to saying that beliefs on each source are represented by a single probability, although preferences do not necessarily abide by expected utility.<sup>18</sup> In addition to basic axioms (such as monotonicity and continuity), we should also impose the standard rank-tradeoff consistency property on  $\succsim_i$  to derive the rank-dependent evaluation.<sup>19</sup>

**Axiom A. 3** For every  $i \in I$ , there exists a rank-dependent Chisini mean  $c_i : \mathcal{F}_i \rightarrow X$  that represents  $\succsim_i$ .

Naturally, many other examples are possible: all is needed, up to now, for our theory of source dependence is the existence of certainty equivalent functionals to represent intra-source preferences. Thus, any model that admits certainty equivalents can be adopted to specify the functional form of  $c_i$ . For instance, one could use models of ambiguity-sensitive behavior. Below we suggest a source-dependent version of rational preferences under ambiguity (Cerreia-Vioglio et al., 2011).

**Example 5 (Source-dependent preferences under ambiguity)** For every  $i \in I$ , there are a strictly increasing and continuous function  $u_i : X \rightarrow \mathbb{R}$ , and a monotone, normalized, and continuous function  $I_i : u_i(X) \rightarrow \mathbb{R}$  such that, for all  $f_i \in \mathcal{F}_i$ ,

$$c_i(f_i) = u_i^{-1}(I_i(u_i(f_i))). \quad (3)$$

This general structure encompasses various models of decision-making under ambiguity. For instance, a special case of (3) is given by the following source-dependent specification of the maxmin model (Gilboa and Schmeidler, 1989):

$$c_i(f_i) = \min_{p_i \in C_i} u_i^{-1}(\mathbb{E}_{p_i}[u_i(f_i)]),$$

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<sup>18</sup>Probabilistic sophistication was introduced by Machina and Schmeidler (1992) by extending Savage's axiomatization. Among others, Chew and Sagi (2006, 2008) provide an alternative characterization in terms of exchangeability of events.

<sup>19</sup>We refer to the literature for a formal exposition of this axiom, its interplay with probabilistic sophistication, and its counterpart in terms of revealed preferences. (See, e.g., Wakker, 2010, Sections 10.3.2 and 10.5.3.)

where  $C_i$  is a set of probabilities that reflects the perception of ambiguity on source  $i$ .<sup>20</sup>

We could also study source dependence in non-expected utility models that do not involve ambiguity. For instance, if we replace the utility  $u_i$  in Example 3 with a compact set of utilities  $U_i$ , we obtain the following source-dependent specification of the cautious expected utility model (Cerreia-Vioglio, Dillenberger, and Ortoleva, 2015), characterized by heterogeneity of the utility sets across sources:

$$c_i(f_i) = \min_{u_i \in U_i} u_i^{-1}(\mathbb{E}_{p_i}[u_i(f_i)]).$$

Note that an agent might be fully confident in a single utility function for a specific source, thereby behaving as a standard expected utility maximizer over that source.

### 3.2 Inter-source preferences

Before turning to the overall relation  $\succsim$  on  $\mathcal{F}$ , we focus on its restriction to the space  $\mathcal{P} := \{(f_i, i) : f_i \in \mathcal{F}_i\}$ . This subsection provides a characterization of preferences between prospect-source pairs in  $\mathcal{P}$ . The next axiom is key in relating monetary outcomes generated by distinct sources.

**Axiom B. 1** (*Inter-source tastes*) *For every  $i, j \in I$  and  $x \in X$ , there exists  $y \in X$  such that  $(x, i) \sim (y, j)$ .*

This is a comparability assumption across sources. It requires that, for any two sources  $i$  and  $j$ , and any sure outcome  $x$  generated by source  $i$ , the agent is always able to identify a sure outcome  $y$  generated by source  $j$  which is equivalent to  $x$ . For example, a policy maker may evaluate differently the same number of (sure) casualties due to a terror attack or to a seasonal flu: he may regard, say, 100 sure casualties from the first source as having the same impact as 1,000 sure casualties from the second (see, e.g., Slovic, 1999, for similar examples). In terms of the representation, these inter-source comparisons will be captured by functions dubbed *rates*, as defined next.

**Definition 6** *A self-map  $\delta_{ij} : X \rightarrow X$  is a **rate** if it is strictly increasing, with  $\delta_{ik} = \delta_{ij} \circ \delta_{jk}$  and  $\delta_{ii} = \text{id}_X$ , for all  $i, j, k \in I$ .*

For any  $x \in X$ , the rate  $\delta_{ij}(x) \in X$  identifies the outcome on source  $i$  that makes the agent indifferent to outcome  $x$  on source  $j$ . With this notion, we can state our behavioral characterization of preferences over prospect-source pairs.<sup>21</sup>

<sup>20</sup>By applying the comparative notion of Ghirardato and Marinacci (2002), we can say that an agent is more ambiguity averse on source  $i$  than on source  $j$  if  $C_i \supseteq C_j$ , provided that  $u_i$  and  $u_j$  are cardinally equivalent.

<sup>21</sup>The domain of prospect-source pairs is reminiscent of the domain of act-probability set pairs in Gadjos, Hayashi, Tallon, and Vergnaud (2008): the set of probabilities represents the objective information given to

**Proposition 7** Let  $\succsim$  be a binary relation on  $\mathcal{P}$ . The following conditions are equivalent:

(i)  $\succsim$  is a weak order that satisfies Axioms A.1 and B.1;

(ii) there exist

(a) a family  $(c_i)_{i \in I}$  of **certainty equivalents** for  $(\succsim_i)_{i \in I}$ ,

(b) a family  $(\delta_{ij})_{i,j \in I}$  of **rates**,

such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$  and  $g_j \in \mathcal{F}_j$ ,

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)). \quad (4)$$

The elements of  $(c_i)_{i \in I}$  and  $(\delta_{ij})_{i,j \in I}$  are unique.

Representation (4) suggests a two-step procedure to compare prospect-source pairs. First, the pairs  $(f_i, i)$  and  $(g_j, j)$  are independently evaluated by computing their certainty equivalents,  $c_i(f_i)$  and  $c_j(g_j)$ , respectively. For every source  $i$ , the function  $c_i$  can be viewed as a unit of account for that source. Second, the certainty-equivalent evaluations are converted into the unit of account of one source. Such conversion is carried out by the function  $\delta_{ij}$  which reflects the rate of substitution of outcomes between sources  $i$  and  $j$ . In particular, representation (4) guarantees that, for any  $x, y \in X$  and  $i, j \in I$ ,

$$(x, i) \sim (y, j) \iff x = \delta_{ij}(y) \iff y = \delta_{ji}(x).$$

Thus,  $\delta_{ij}$  measures the value of a sure outcome on source  $j$  in terms of an equivalent outcome on source  $i$ . The content of Proposition 7 lies in recognizing that source preferences build on a distinction between *intra-source uncertainty attitudes* and *inter-source tastes*. Diagrammatically:

$$\text{Source Preferences}_i = \underbrace{\hspace{10em}}_{\delta_{ij}} \circ \underbrace{\hspace{10em}}_{c_j}$$

The notion of inter-source tastes—the first component in the diagram—is the core novelty. Different sources affect the desirability of outcomes, so that an agent may prefer receiving (the sure) outcome  $x$  on source  $i$  rather than the same outcome on another source  $j$ . Representation (4) states that inter-source tastes are identified by  $(\delta_{ij})_{i,j}$ . The rate  $\delta_{ij}$  can be thought of as the cost of transporting one unit of consumption from source  $j$  to source  $i$ , thereby capturing the (relative) subjective price of source  $j$ . Intra-source uncertainty

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an agent in evaluating an act and can be viewed as a specific instance of source. In their setup, the ranking of act-probability set pairs is always direct, i.e., the rate of substitution between sources is, implicitly, 1. Hence, there is no need to introduce certainty equivalents and to adopt our two-step procedure. Their approach captures the Ellsberg paradox in a way consistent with our related discussion.

attitudes—the second component in the diagram—correspond to a localized notion of preferences under uncertainty. They are characterized by standard axioms imposed on each source and are identified by the certainty equivalents.

Source preferences are determined by the composition of both features; yet, there are prominent cases where they are driven by only one of them. To illustrate, consider the two-urn Ellsberg paradox: as will be shown next, source preferences are determined solely by intra-source uncertainty attitudes. On the other hand, in certain deterministic contexts source preferences are generated solely by inter-source tastes: for instance, the comparison of human casualties generated by different hazards (e.g., earthquakes, epidemics, terror attacks, nuclear accidents) may be evaluated in different ways by policy makers taking prevention measures. In this case, rates can be interpreted as the relative prices of hazards. In general, both forces are at work. We bring to the fore two classic examples. First, agents discount consumption at future dates in different ways (inter-source tastes), but they might be more or less affected by outcomes' variability at different dates (intra-source attitudes). Second, agents attach different values to the same monetary payoff in different markets (inter-source tastes) and, at the same time, they exhibit different (intra-source) uncertainty attitudes on each market, as discussed in Section 6.

Two notable special cases easily follow from representation (4). We start by characterizing the case of source-dependent expected utility, as seen in Example 3.

**Corollary 8** *Let  $\succsim$  be a binary relation on  $\mathcal{P}$ . The following conditions are equivalent:*

(i)  $\succsim$  is a weak order that satisfies Axioms A.2 and B.1;

(ii) there exist

(a) a family  $(u_i)_{i \in I}$  of strictly increasing and continuous functions  $u_i : X \rightarrow \mathbb{R}$  and a family  $(p_i)_{i \in I}$  of probability measures  $p_i$  on  $\Sigma_i$ ,

(b) a family  $(\delta_{ij})_{i,j \in I}$  of rates,

such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$  and  $g_j \in \mathcal{F}_j$ ,

$$(f_i, i) \succsim (g_j, j) \iff u_i^{-1}(\mathbb{E}_{p_i}[u_i(f_i)]) \geq \delta_{ij} \left( u_j^{-1}(\mathbb{E}_{p_j}[u_j(g_j)]) \right) \quad (5)$$

The elements of  $(u_i)_{i \in I}$  are cardinally unique and those of  $(p_i)_{i \in I}$  are unique, provided that, for each  $i \in I$ , there are at least two non-null states in  $S_i$ .<sup>22</sup> The elements of  $(\delta_{ij})_{i,j \in I}$  are unique.

<sup>22</sup>As usual, a state  $s \in S_i$  is null if indifference holds between any two profiles that differ only in state  $s$ , that is, if  $x\{s\}f_i \sim_i y\{s\}f_i$  for all  $f_i \in \mathcal{F}_i$  and  $x, y \in X$ .

Next corollary extends the popular criterion of probabilistically sophisticated rank-dependent utility to allow for source dependence, as described in Example 4.<sup>23</sup>

**Corollary 9** *Let  $\succsim$  be a binary relation on  $\mathcal{P}$ . The following conditions are equivalent:*

- (i)  $\succsim$  is a weak order that satisfies Axioms A.3 and B.1;
- (ii) there exist

- (a) a family  $(u_i)_{i \in I}$  of strictly increasing and continuous functions  $u_i : X \rightarrow \mathbb{R}$ , a family  $(p_i)_{i \in I}$  of probability measures  $p_i$  on  $\Sigma_i$ , and a family  $(w_i)_{i \in I}$  of increasing and onto functions  $w_i : [0, 1] \rightarrow [0, 1]$ ,
- (d) a family  $(\delta_{ij})_{i, j \in I}$  of rates,

such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$  and  $g_j \in \mathcal{F}_j$ ,

$$(f_i, i) \succsim (g_j, j) \iff u_i^{-1}(\mathbb{E}_{w_i(p_i)}[u_i(f_i)]) \geq \delta_{ij} \left( u_j^{-1}(\mathbb{E}_{w_j(p_j)}[u_j(g_j)]) \right). \quad (6)$$

The elements of  $(u_i)_{i \in I}$  are cardinally unique and those of  $(p_i)_{i \in I}$  are unique, provided that, for each  $i \in I$ , there are at least two non-null states in  $S_i$ . The elements of  $(w_i)_{i \in I}$  and  $(\delta_{ij})_{i, j \in I}$  are unique.

Corollaries 8 and 9 generalize the two main strands of the literature (respectively, the utility approach and the probability-weighting approach) to situations where the comparison between source-dependent prospects is not direct and it requires the application of a conversion rate.

### 3.2.1 The expected utility case

An interesting special case occurs when the sure outcome  $x$  is on source  $j$  equivalent to itself on source  $i$ .

**Axiom B. 2** (*Pecunia non olet*) *For every  $i, j \in I$  and  $x \in X$ ,  $(x, i) \sim (x, j)$ .*

This axiom requires that, when deterministic outcomes are concerned, the agent does not discriminate between the sources generating them. Any difference in the evaluations of uncertain prospects must be due solely to the agent holding different uncertainty attitudes toward different sources. That is, the only perceived difference between sources is the quality of the information about the stochastic nature of the states of the world.

When monetary payoffs are involved, the fact that deterministic outcomes are unaffected by the stochastic nature of the underlying source rules out any consideration about

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<sup>23</sup>Note that one can derive other probabilistically sophisticated non-expected utility specifications by imposing suitable axioms on each intra-source preference  $\succsim_i$ .

the agent’s competence, thus making this assumption particularly transparent. Its behavioral content can be summarized by the time-honored claim of Vespasian that *pecunia non olet* — money has no smell. For instance, the value of a monetary outcome is the same on both Ellsberg urns. That said, one can think of situations in which this assumption would be too restrictive: it is likely that a person would not be indifferent between receiving the same monetary profit from legal versus illegal entrepreneurial activities.<sup>24</sup>

Replacing Axiom B.1 with Axiom B.2 in Proposition 7 is immediately seen to correspond to  $\delta_{ij} = \text{id}_X$  for all  $i, j \in I$ . Then, representation (4) reduces to

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq c_j(g_j). \quad (7)$$

In view of the importance of expected utility, we present a special case of Proposition 7 where certainty equivalents can be directly compared—without the mediation of the rates  $\delta_{ij}$  as in Corollary 8—and, at the same time, they conform to subjective expected utility.

**Proposition 10** *Let  $\succsim$  be a binary relation on  $\mathcal{P}$ . The following conditions are equivalent:*

- (i)  $\succsim$  is a weak order that satisfies Axioms A.2 and B.2;
- (ii) there exist a family  $(u_i)_{i \in I}$  of strictly increasing and continuous functions  $u_i : X \rightarrow \mathbb{R}$ , and a family  $(p_i)_{i \in I}$  of probability measures  $p_i$  on  $\Sigma_i$ , such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$  and  $g_j \in \mathcal{F}_j$ ,

$$(f_i, i) \succsim (g_j, j) \iff u_i^{-1}(\mathbb{E}_{p_i}[u_i(f_i)]) \geq u_j^{-1}(\mathbb{E}_{p_j}[u_j(g_j)]). \quad (8)$$

The elements of  $(u_i)_{i \in I}$  are cardinally unique and those of  $(p_i)_{i \in I}$  are unique, provided that, for each  $i \in I$ , there are at least two non-null states in  $S_i$ .

In representation (8), source dependence manifests itself only in the potentially different uncertainty attitudes, whereas the value of a deterministic outcome is source independent. This is an important special case of our model as it offers a formal treatment of the utility approach in alternative to the more traditional probability-weighting approach. We illustrate the differences between these two approaches by applying them to resolve the classic two-urn Ellsberg paradox.

**Two-urn Ellsberg experiment** Denote by  $I = \{1, 2\}$  the set of urns and by  $S_i = \{\text{Red}_i, \text{Black}_i\}$  the set of states for each  $i \in I$ . Urn 1 contains 50 red and 50 black balls, while Urn 2 has unknown composition. Let  $f_i$  (resp.,  $g_i$ ) be the bet that pays \$100 if the ball drawn from urn  $i \in I$  is red (resp., black) and \$0 otherwise. The typical pattern is  $f_1 \sim g_1 \succ g_2 \sim f_2$ . Most subjects are indifferent between betting on red or on black from either urn; yet, they strictly prefer to bet on Urn 1 rather than on Urn 2.

<sup>24</sup>As mentioned earlier, Slovic (1999) contains various examples, mostly in the context of casualties.



Source preferences generate a ‘paradox’ because indifference between betting on red or on black from either urn suggests that the two states are perceived as equally likely; but, if subjects assign also the same utilities to the same consequences generated by different sources, then the certainty equivalents of all the four bets must be the same. This is clearly inconsistent with a preference for betting on the known urn.

The source-dependent utility criterion developed in Proposition 10 can rationalize the observed pattern by maintaining the same equal probability of the states, while allowing for utilities to depend on sources as follows:

$$c_1(f_1) = u_1^{-1} \left[ \frac{1}{2} u_1(100) + \frac{1}{2} u_1(0) \right] \neq u_2^{-1} \left[ \frac{1}{2} u_2(100) + \frac{1}{2} u_2(0) \right] = c_2(f_2),$$

explaining the pattern if  $u_1$  is less concave than  $u_2$ . Early discussions of this approach appear in Fellner (1961), Roberts (1963), Smith (1969), and, more recently, in Chew et al. (2008). On the other hand, the dominant approach in the literature follows the seminal papers of Kahneman and Tversky (1979) and Schmeidler (1989) by focussing on source-dependent *probability distortions* as follows:

$$\begin{aligned} c_1(f_1) &= u^{-1} \left[ w_1 \left( \frac{1}{2} \right) u(100) + w_1 \left( \frac{1}{2} \right) u(0) \right] \\ &\neq u^{-1} \left[ w_2 \left( \frac{1}{2} \right) u(100) + w_2 \left( \frac{1}{2} \right) u(0) \right] = c_2(f_2), \end{aligned}$$

explaining the pattern if  $w_1 > w_2$ .

### 3.3 Preferences over prospect profiles

It remains to study the preferences  $\succsim$  over the entire domain  $\mathcal{F}$  of prospect profiles. We posit the following basic axiom.

**Axiom C. 1** *The binary relation  $\succsim$  on  $\mathcal{F}$  is a weak order such that:*

- (i) *(Monotonicity) if  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$  and  $f_i \succsim_i g_i$  for all  $i \in I$ , then  $\mathbf{f} \succsim \mathbf{g}$ ;*
- (ii) *(Strict Monotonicity) if both  $\mathbf{x}$  and  $\mathbf{y}$  in  $X^I$  are constant and  $\mathbf{x} > \mathbf{y}$ , then  $\mathbf{x} \succ \mathbf{y}$ ;*
- (iii) *(Archimedean Continuity) if  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^I$  and  $\mathbf{x} \succ \mathbf{y} \succ \mathbf{z}$ , then there exist  $\alpha, \beta \in (0, 1)$  such that*

$$(1 - \alpha) \mathbf{x} + \alpha \mathbf{z} \succ \mathbf{y} \succ (1 - \beta) \mathbf{x} + \beta \mathbf{z}.$$

Axiom C.1 can be viewed as the analogue of Axiom A.1 over the richer domain of prospect profiles. Property (i) maintains that the agent should prefer a profile  $\mathbf{f}$  to another profile  $\mathbf{g}$  whenever  $\mathbf{f}$  consists of better source-dependent prospects than  $\mathbf{g}$ , source-by-source. Property (ii) is a form of strict monotonicity imposed on fully constant prospect

profiles. Property (iii) is a standard continuity property which, together with the other assumptions in Axiom C.1, guarantees that, for any  $\mathbf{f} \in \mathcal{F}$ , the agent can find a fully constant profile  $\mathbf{x}$  which is indifferent to  $\mathbf{f}$ .

## 4 Representation results

This section provides behavioral characterizations of source-dependent preferences over prospect profiles—the domain of central interest of our theory. We start with a general utility structure that captures the essential features of source-dependent preferences.

### 4.1 General representation result

The following terminology will be useful to state our results.

**Definition 11** *A monotone and continuous functional  $D : X^I \rightarrow X$  is*

- (i) *an **aggregator** if it is normalized, that is,  $D(x1_I) = x$  for all  $x \in X$ ;*
- (ii) *an **aggregator at the numeraire**  $\mathbf{i} \in I$  if it is  $i$ -normalized, that is,  $D(x1_i) = x$  for all  $x \in X$ .*

We are now ready to state our general representation result. It disentangles the two behavioral features that may give rise to source dependence, namely intra-source attitudes and inter-source tastes, within the rich domain of prospect profiles.

**Theorem 12** *Let  $\succsim$  be a binary relation on  $\mathcal{F}$ . The following conditions are equivalent:*

- (i)  *$\succsim$  satisfies Axioms A.1, B.1, and C.1;*
- (ii) *there exist*
  - (a) *a family  $(c_i)_{i \in I}$  of **certainty equivalents** for  $(\succsim_i)_{i \in I}$ ,*
  - (b) *a family  $(\delta_{ij})_{i,j \in I}$  of **rates**,*  
*such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$ , and  $g_j \in \mathcal{F}_j$ ,*

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)),$$

- (c) *an **aggregator at the numeraire**  $\mathbf{o} \in I$ ,  $W_o : X^I \rightarrow X$ , such that, for every  $\mathbf{f} = (f_i)_{i \in I}$ ,*

$$V_o(\mathbf{f}) = W_o(\delta_{o1}(c_1(f_1)), \dots, \delta_{on}(c_n(f_n))) \quad (9)$$

*represents  $\succsim$  on  $\mathcal{F}$ .*

According to representation (9), a prospect profile is first evaluated source-wise, with each source  $i$ 's own unit of account given by  $c_i$ . Each of these evaluations is then converted into the unit of account of a posited numeraire source  $o$  by applying the conversion rates  $\delta_{oi}$ . Finally, this profile of evaluations is aggregated in a unique evaluation given by  $W_o$  which preserves the normalization to source  $o$ . Thus,  $W_o$  is an aggregator of source-dependent deterministic payoffs evaluated through the lens of the numeraire  $o$ .<sup>25</sup>

By Theorem 12, we can represent a binary relation  $\succsim$  on  $\mathcal{F}$  that satisfies Axioms A.1, B.1, and C.1 by a triple  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_o)$ . We refer to such a triple as a *source-dependent representation* of  $\succsim$ . Our utility representation has standard uniqueness properties, as shown next.

**Proposition 13** *Let  $\succsim$  be a binary relation on  $\mathcal{F}$  that satisfies the assumptions of Theorem 12. Then, the following statements hold:*

- (i) *for every  $o \in I$ , the source-dependent representation  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_o)$  of  $\succsim$  is unique;*
- (ii) *if  $o \neq \bar{o}$  and  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_o)$  and  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_{\bar{o}})$  are two source-dependent representations of  $\succsim$ , then  $W_o = \delta_{o\bar{o}}(W_{\bar{o}})$ .*

Theorem 12 serves the purpose of providing a general framework and an axiomatic foundation that captures the core features of source-dependent preferences. It encompasses numerous special cases which vary in the functional specifications of the aggregator  $W_o$  and the rates  $\delta_{ij}$  (as well as the forms of the certainty equivalents, as illustrated in Examples 3-5). For instance, one could think of non-additive aggregators (in the spirit of Choquet expected utility (Schmeidler, 1989) or maxmin preferences (Gilboa and Schmeidler, 1989) to introduce considerations of ambiguity in the process of evaluating sources. On the other hand, Theorem 12 does not suggest an operational criterion to adopt in applied problems. We next address this point by introducing important special cases.

## 4.2 Quasi-arithmetic aggregators

This section considers a structured specification of the aggregator  $W_o$  and focuses on the well-known special case of quasi-arithmetic means. We will see that the class of quasi-arithmetic aggregators is analytically tractable and accommodates classic utility specifications used in applied economic settings.

For the rest of this section, we assume that  $X = \mathbb{R}$  and impose few additional properties on  $\succsim$  restricted to the domain of constant prospect profiles. We begin with a standard continuity assumption.

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<sup>25</sup>In Appendix A.2, Theorem 20 provides an alternative utility representation according to which the agent evaluates prospect profiles without committing to a specific numeraire.

**Axiom D. 1** (*Continuity*) For each  $\mathbf{x} \in X^I$ , the sets  $\{\mathbf{y} \in X^I : \mathbf{y} \succsim \mathbf{x}\}$  and  $\{\mathbf{y} \in X^I : \mathbf{x} \succsim \mathbf{y}\}$  are closed.

Next axiom is a classic consistency property. It is a form of separability (in particular, a strengthening of the standard Coordinate Independence property) that is needed to obtain the desired quasi-arithmetic aggregator.<sup>26</sup>

**Axiom D. 2** (*Tradeoff consistency*) For each  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in X^I$ , each  $x, y, z, w \in X$ , and each  $i, j \in I$ , the condition

$$\mathbf{b}_{-i}y \succsim \mathbf{a}_{-i}x \quad \mathbf{a}_{-i}z \succsim \mathbf{b}_{-i}w \quad \mathbf{c}_{-j}x \succsim \mathbf{d}_{-j}y$$

implies  $\mathbf{c}_{-j}z \succsim \mathbf{d}_{-j}w$ .

Next proposition shows that Axioms D.1 and D.2, together with the basic properties of the general theorem, deliver the classic quasi-arithmetic representation.

**Proposition 14** Let  $\succsim$  be a binary relation on  $\mathcal{F}$  that satisfies the axioms of Theorem 12. The following conditions are equivalent:

(i)  $\succsim$  satisfies Axioms D.1 and D.2;

(ii) there exist a collection of strictly positive weights  $\{\alpha_i\}_{i \in I} \subseteq \Delta(I)$  and a strictly increasing, continuous function  $v : X \rightarrow \mathbb{R}$  with  $v(0) = 0$  and  $v(1) = 1$ , such that, for all  $\mathbf{f} \in \mathcal{F}$ ,

$$V_o(\mathbf{f}) = v^{-1} \left( \frac{1}{\alpha_o} \sum_{i \in I} \alpha_i v(c_i(f_i)) \right). \quad (10)$$

The collection  $\{\alpha_i\}_{i \in I}$  is unique and the function  $v$  is unique.

By this proposition, *quasi-arithmetic source-dependent preferences* are uniquely represented by triples  $((c_i)_{i \in I}, \{\alpha_i\}_{i \in I}, v)$ . Certainty equivalents are aggregated by means of a weighted sum of utilities. Each weight  $\alpha_i$  can be viewed as a measure of the relevance of source  $i$  and, since  $\alpha_i > 0$  for all  $i$ , all sources affect the evaluation albeit with a different degree. Furthermore, all weights  $\alpha_i$  are normalized with respect to some source  $o$ , so the overall evaluation of any prospect profile is computed using the unit of account of that source. Note that (10) is a special case of (9) where the rates and the aggregator have the following form:

$$\delta_{ij}(x) = v^{-1} \left( \frac{\alpha_j}{\alpha_i} v(x) \right) \quad \text{and} \quad W_o((x_i)_{i \in I}) = v^{-1} \left( \sum_{i \in I} v(x_i) \right),$$

---

<sup>26</sup>Continuous subjective expected utility and quasi-arithmetic mean models are characterized by Wakker (1988, Theorem 6.2).

for all  $x \in X$  and all  $(x_i)_{i \in I} \in X^I$ . Finally, the form of the utility  $v$  captures a preference for smoothing across sources, as shown by our comparative statics analysis later on.

Before moving on, let us recall that the class of quasi-arithmetic means is closely related to classic utility models in economics. Indeed, representation (10) conforms essentially to subjective expected utility if the certainty equivalent functionals are quasi-linear as in Example 3, and the utilities  $u_i$ 's in that example are such that  $u_i = v$  for all  $i \in I$ .

Standard temporal and allocation settings can be cast as special cases of the quasi-arithmetic source-dependent model. Importantly, we will next see that, in these applied settings, the numeraire earns precise economic content.

### 4.3 Temporal sources

By viewing consumption dates as temporal sources, this section studies intertemporal preferences within our framework of source dependence. A key advantage of taking a source perspective is to allow for time-dependent uncertainty attitudes. The same source of risk (e.g., financial risk) evaluated at different ages or time horizons may affect agents in different ways; this may naturally generate different risk attitudes in a way consistent with the empirical evidence, as argued in the Introduction.

Let  $I = \{0, 1, \dots, T\}$  be interpreted as a set of dates. A prospect profile  $\mathbf{f} = (f_0, \dots, f_T)$  is a temporal stream of consumption where  $f_t$  denotes the bundle consumed at time  $t \in I$ . We impose the following stationarity assumption.

**Axiom D. 3** (*Stationarity*) *If  $s, t \in I \setminus \{T\}$  and  $x, y \in X$ , then*

$$(x, s) \sim (y, t) \implies (x, s + 1) \sim (y, t + 1).$$

The above axiom preserves the well-known behavioral content of stationarity: the indifference between two time-dependent consequences hinges on the dates  $s$  and  $t$ , whereas it is not affected by postponing consumption of both consequences by the same amount of time. We next show that familiar exponential discounting utility models can be recovered as special cases of our quasi-arithmetic source-dependent representation.

**Proposition 15** *Let  $\succsim$  be a binary relation on  $\mathcal{F}$  that satisfies the axioms of Theorem 12. The following conditions are equivalent:*

- (i)  $\succsim$  satisfies Axioms D.1, D.2, and D.3;
- (ii) there exist a scalar  $\beta > 0$  and a strictly increasing, continuous function  $v : X \rightarrow \mathbb{R}$  with  $v(0) = 0$  and  $v(1) = 1$ , such that, for all  $\mathbf{f} \in \mathcal{F}$ ,

$$V_0(\mathbf{f}) = v^{-1} \left( \sum_{t=0}^T \beta^t v(c_t(f_t)) \right). \tag{11}$$

The scalar  $\beta$  and the function  $v$  are unique.

According to (11), the agent evaluates  $\mathbf{f}$  by first computing the time- $t$  certainty equivalent of each component  $f_t$ . Time- $t$  certainty equivalent captures the risk attitude exhibited over time- $t$  horizon. The overall evaluation is given by the discounted sum of utilities of (time-dependent) certainty equivalents. From our source perspective, the discount factor is the price of time: in particular, it can be viewed as the cost of transporting one unit of consumption between different dates. Moreover, the choice of the numeraire  $o$  is simple since, as usual in applied temporal contexts, the value of a stream is discounted at time 0. Indeed, (11) is a special case of (9) where

$$\delta_{ij}(x) = v^{-1}(\beta^{j-i}v(x)) \quad \text{and} \quad W_0((x_i)_{i \in I}) = v^{-1}\left(\sum_{i \in I} v(x_i)\right),$$

for all  $x \in X$  and all  $(x_i)_{i \in I} \in X^I$ . The aggregator  $W_0(\mathbf{x})$  can be thought of as the value of a perpetuity that is exactly as good as  $\mathbf{x}$  itself.

Note that representation (11) reduces to standard discounted utility over constant prospect profiles. Furthermore, one can specify the certainty equivalents as in Example 3 and set  $c_t(f_t) = u_t^{-1}(\int_S u_t(f_t) dp_t)$ , where  $u_t$  is a utility function on  $X$  and  $p_t$  is a probability measure on  $S$  for all  $t = 0, \dots, T$ . In the extreme case in which  $u_t = v$  for all  $t = 0, \dots, T$ , we recover the standard discounted expected utility criterion with no source effects and constant risk aversion:

$$V_0(\mathbf{f}) = v^{-1}\left(\sum_{t=0}^T \beta^t \int_S v(f_t) dp_t\right).$$

We conclude this section by briefly relating our model to some existing works on horizon-dependent risk attitudes. If we consider a set  $I = \{0, 1\}$  of only two dates, representation (11) can be rewritten as

$$v^{-1}((1 - \beta)v(c_0(f_0)) + \beta v(c_1(f_1))),$$

which corresponds to the case studied in Andries et al. (2018) who distinguish between two levels of risk aversion toward immediate and delayed uncertainty, respectively. By allowing for levels of risk aversion that are decreasing with the time horizon, they generate predictions consistent with the empirical evidence on the term structure of equity returns.

## 5 Comparative attitudes

The additional structure of the quasi-arithmetic model allows us to develop a comparative statics analysis of the effects of inter-source smoothing on choice behavior.

Consider an agent with quasi-arithmetic source-dependent preferences represented by  $((c_i)_{i \in I}, \{\alpha_i\}_{i \in I}, v)$ . We say that the agent has a *preference for source smoothing* if, for

any  $\mathbf{x} = (x_i)_{i \in I} \in X^I$ , he prefers the profile that yields the quasi-arithmetic  $v$ -mean of  $\mathbf{x}$  on all sources to  $\mathbf{x}$  itself. That is, if we denote by  $z = v^{-1} \left( \frac{1}{\alpha_o} \sum_{i \in I} \alpha_i v(x_i) \right) \in X$  such a mean, we must have  $(z, z, \dots, z) \succsim \mathbf{x}$ .<sup>27</sup> This definition is reminiscent of the classic notion of risk aversion with an important caveat: our notion of source smoothing applies to prospect profiles that do not involve any risk as they are constant within each source. Risk aversion, instead, pertains to intra-source attitudes and is captured by the source-dependent certainty equivalents. The latter do not play any role here.

Consider two agents with preferences given by  $\succsim_1$  and  $\succsim_2$ . We posit that  $\succsim_1$  has a *higher preference for source smoothing* than  $\succsim_2$  if and only if, for all  $\mathbf{x} \in X^I$  and  $z \in X$ ,

$$\mathbf{x} \succsim_1 (z, z, \dots, z) \implies \mathbf{x} \succsim_2 (z, z, \dots, z).$$

In words,  $\succsim_1$  has a higher preference for source smoothing than  $\succsim_2$  if, whenever  $\succsim_1$  prefers a non-constant outcome profile across sources to a constant one, then the same is true for  $\succsim_2$ . Next result provides a behavioral characterization and shows that the comparative attitudes toward source smoothing are controlled by the function  $v$ .

**Proposition 16** *Let  $\succsim_1$  and  $\succsim_2$  be two quasi-arithmetic source-dependent preferences represented by  $(\{\alpha_i^1\}_{i \in I}, v_1)$  and  $(\{\alpha_i^2\}_{i \in I}, v_2)$ , respectively. The following conditions are equivalent:*

- (i)  $\succsim_1$  has a higher preference for source smoothing than  $\succsim_2$ ;
- (ii)  $\{\alpha_i^1\}_{i \in I} = \{\alpha_i^2\}_{i \in I}$  and  $v_1 = \psi \circ v_2$  for some concave, strictly increasing, and continuous transformation  $\psi : v_2(X) \rightarrow v_1(X)$ .

This proposition states that higher preferences for source smoothing are characterized by more concave utility functions  $v$ . As anticipated earlier, this result resembles the standard comparative characterization of risk aversion.<sup>28</sup> In turn, our result captures a form of aversion to the variance of outcomes across sources.

When  $I = \{0, 1, \dots, T\}$  is interpreted as a set of dates, the notion of higher preference for source smoothing corresponds to the familiar notion of *higher preference for intertemporal smoothing*. As an immediate corollary of the above proposition, we conclude that in the intertemporal setting of Proposition 15,  $\succsim_1$  has a higher preference for intertemporal smoothing than  $\succsim_2$  if and only if  $\beta_1 = \beta_2$  and  $v_1$  is more concave than  $v_2$ .

## 6 A portfolio allocation problem

This section adopts our quasi-arithmetic model to introduce source dependence into a simple financial allocation problem. We examine the implications on the optimal portfolio allocation and the relationship with some well-known empirical puzzles.

<sup>27</sup>Here  $(z, z, \dots, z) \in X^I$  is the constant profile that yields outcome  $z$  on all sources.

<sup>28</sup>See, e.g., Wakker (1989, Definition VII.6.4).

## 6.1 Market and subjective prices

Consider an agent who faces the following investment problem: there are two assets  $(f_1, f_2)$  in two different markets, say Eurozone and US, with prices  $(p_1, p_2) \in \mathbb{R}_{++}^2$  and he has to decide how to allocate his wealth  $w = (w_1, w_2) \in \mathbb{R}_{++}^2$  across these two assets (prices and wealth are denominated in the local currencies). Each asset is modeled as a random variable  $f_i : S_i \rightarrow [0, \infty)$  that pays in the local currency and depends on a local source of uncertainty. In view of our theory, we can set  $I = \{1, 2\}$ , where source 1 (resp., source 2) identifies the Eurozone market (resp., the US market).

The agent can buy  $\xi_i \geq 0$  units of asset  $i$  (short-sales are not possible), forming a portfolio  $\xi = (\xi_1, \xi_2)$ . He can move money from one market to the other, once converted in the same currency through a market exchange rate. We denote by  $\rho_{ij}$  the market (*nominal*) *exchange rate* from currency  $j$  to currency  $i$ ; so,  $\rho_{12}$  says how many euros the market trades for 1 dollar. The budget constraint is then:

$$\xi_1, \xi_2 \geq 0, \xi_1 p_1 + \xi_2 \rho_{12} p_2 \leq w_1 + \rho_{12} w_2. \quad (12)$$

Here, the *real exchange rate* is  $e_{12} = \rho_{12} p_2 / p_1$ , which states how many units of the Eurozone asset the market trades for 1 unit of the US asset.

Each portfolio  $\xi$  determines the random profile  $(\xi_1 f_1, \xi_2 f_2)$ . The agent evaluates each payoff  $\xi_i f_i$  separately, according to the source-dependent criterion

$$c_i(\xi_i f_i) = u_i^{-1} \left( \int_{S_i} u_i(\xi_i f_i) dq_i \right), \quad (13)$$

where  $u_i$  is a strictly increasing, strictly concave, and continuous function, and  $q_i$  is a probability measure on  $S_i$ . As seen next, we posit that each  $u_i$  has a CRRA form so that  $c_i(\xi_i f_i) = \xi_i c_i(f_i)$ .<sup>29</sup> Using the quasi-arithmetic specification of Proposition 14, the agent aggregates the evaluations in (13) according to the function  $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  given by

$$W(\xi_1, \xi_2) = v^{-1}(\alpha_1 v(\xi_1 c_1(f_1)) + \alpha_2 v(\xi_2 c_2(f_2))),$$

where  $\alpha_1, \alpha_2 \in (0, 1)$  and  $\alpha_1 + \alpha_2 = 1$ . Assume that, for each  $i$ , there exists  $\gamma_i \in (0, 1)$  such that

$$u_i(x) = \frac{x^{1-\gamma_i}}{1-\gamma_i} \quad \text{for all } x \geq 0,$$

and there exists  $k \in (0, 1)$  such that

$$v(x) = \frac{x^{1-k}}{1-k} \quad \text{for all } x \geq 0.$$

Let  $\delta_{21} = \delta_{21}(1)$  stand for how many dollars are subjectively equivalent to a payoff of 1 euro. Thus,  $\delta_{ij}$  is a subjective price and we refer to it as the (nominal) *subjective exchange*

<sup>29</sup>To make the problem nontrivial, we also assume that  $c_1(f_1), c_2(f_2) > 0$ .



rate. By quantifying the subjective value of one currency in terms of the other,  $\delta_{ij}$  is interpreted as a measure of familiarity toward one market relative to another (which in turn determines how outcomes are evaluated in the two markets). We next show that subjective exchange rates play a key role in determining the optimal portfolio allocation.

**Proposition 17** *The unique optimal portfolio  $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$ , provided it is an interior solution, is such that*

$$\frac{p_1 \hat{\xi}_1}{\rho_{12} p_2 \hat{\xi}_2} = \left( e_{12} \frac{\delta_{21} c_1(f_1)}{c_2(f_2)} \right)^{\frac{1-k}{k}}, \quad (14)$$

where  $\delta_{21} = \left( \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1-k}}$ . Moreover,  $\frac{\partial W}{\partial \xi_2}(\hat{\xi}) / \frac{\partial W}{\partial \xi_1}(\hat{\xi}) = \rho_{12} p_2 / p_1 = e_{12}$ .

The left-hand side of (14) is the ratio of the optimal investments in each market, expressed in euros via the nominal exchange rate. The right-hand side is a function of the product, also expressed in euros, of the objective  $e_{12}$  and subjective  $\delta_{21} c_1(f_1) / c_2(f_2)$  terms of trade between the two assets. In particular,  $\delta_{21} c_1(f_1) / c_2(f_2)$  is the ratio of the subjective values of each euro invested in each market.

The relation between subjective and market exchange rates in (14) reminds the classic relation between subjective discount rates and market interest rates that characterizes Euler equations in intertemporal consumption problems. Indeed, (14) is a spatial counterpart of the classic temporal relation. Across time or space, optimality imposes a tradeoff between market prices and subjective traits.

## 6.2 Comparative analysis and home bias

The last proposition derives the optimal portfolio allocation of an agent with quasi-arithmetic source-dependent preferences and formalizes the way subjective exchange rates affect financial decisions. In the special case  $\delta_{21} = 1$ , the agent perceives a payoff of 1 euro as subjectively equivalent to a payoff of 1 dollar and, therefore, he is equally familiar about the value of euros versus dollars. This means that he may still exhibit source-dependent uncertainty attitudes, reflected in the certainty equivalents  $c_1$  and  $c_2$ ; however, once uncertainty is factored out, he finds monetary outcomes generated by either source as perfectly comparable, analogously to what we have seen in the two-urn Ellsberg experiment.<sup>30</sup> Condition (14) then reduces to

$$\frac{p_1 \hat{\xi}_1}{\rho_{12} p_2 \hat{\xi}_2} = \left( e_{12} \frac{c_1(f_1)}{c_2(f_2)} \right)^{\frac{1-k}{k}}, \quad (15)$$

and the ratio of the optimal investments depends only on the market exchange rate  $e_{12}$ . On the other side, if  $\delta_{21} > 1$ , then a payoff of 1 euro is worth more for the agent than a payoff of 1 dollar. By comparing (14) to (15), we conclude that, *ceteris paribus*, the

<sup>30</sup>Note that  $\delta_{21} = 1$  does not necessarily imply that the market (nominal) exchange rate  $\rho_{21}$  is 1, as well.

more  $\delta_{21}$  exceeds 1, the more the agent will invest in asset 1. This observation suggests an explanation of the evidence on home bias and under-diversification in terms of familiarity. Taking the perspective of a Eurozone investor, the higher is the degree of familiarity for the Eurozone in comparison to the US zone (i.e., the higher  $\delta_{21}$  is), the higher will be the inclination to invest in the Eurozone asset.

The role of  $\delta_{21}$  is analogous to the one played by a subjective discount factor  $\beta$  in temporal settings. The lower the subjective discount factor is, the higher the (market) real interest rate must be to compensate for the investor's impatience. Here, the higher the subjective exchange rate  $\delta_{21}$  is, the lower the (real) market exchange rate  $e_{12}$  needs to be to compensate for the familiarity bias toward the Eurozone. In a vein akin to the classic temporal Euler equation, these opposing forces can be seen in action in condition (14).

We conclude with a comparative analysis of how each parameter affects the optimal portfolio allocation. First, note that the higher the coefficient  $k$  is, the more concave the function  $v$  is. By Proposition 16, it follows that  $k$  is an index of the agent's preference for source smoothing (i.e., preference for constant profiles across sources). Now, denote by  $\tilde{\xi}_i = \frac{\hat{\xi}_i}{w_1 + \rho_{12} w_2}$  the optimal share of asset  $i$  which measures the fraction of wealth invested in asset  $i$ .

**Proposition 18** *Let  $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$  be the unique optimal portfolio satisfying condition (14) of Proposition 17 and  $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$  be the resulting vector of asset shares. Then,*

$$\frac{\partial \tilde{\xi}_i}{\partial \alpha_i} > 0, \quad \frac{\partial \tilde{\xi}_i}{\partial \gamma_i} < 0, \quad \text{and} \quad \frac{\partial \tilde{\xi}_i}{\partial p_i} < 0 \quad \text{for} \quad i = 1, 2.$$

A higher weight  $\alpha_i$  reflects a stronger familiarity bias for source  $i$  and induces a higher share of asset  $i$ . Moreover, a higher coefficient of uncertainty aversion  $\gamma_i$  within source  $i$  yields a lower share of asset  $i$ . Finally, a higher price  $p_i$  yields a lower share of asset  $i$ .

## 7 Concluding remarks

This paper develops a unifying framework that incorporates existing approaches to study choice problems in the presence of multiple sources. We show that source preference originates from two behavioral features represented by source-dependent certainty equivalents and subjective prices. The notion of subjective price is the key novelty. It has a precise economic content in terms of subjective rates of substitution between sources. Their role is to enable comparisons of prospects across different sources. We illustrate the distinction between these two features in the settings of intertemporal and allocation choices, as well as comparisons of hazards. We close with two remarks on the limitation of our framework and indicate possible directions for future research.

**Source interdependence** Our setup does not explicitly account for source-dependent prospects that depend simultaneously on multiple sources. This is compatible with many contexts (notably with intertemporal settings), but it can be restrictive on occasion. While a systematic analysis of this issue is beyond the scope of this paper, our setup can easily be extended to accommodate some forms of multiple dependence. To illustrate, consider the allocation problem of Section 6 where each asset is a source-dependent prospect that depends only on one source, the Eurozone market (source 1) or the US market (source 2). Suppose that an additional source, say the Asian market (source 3), influences the performance of both assets. Indeed, a financial crisis in Asia is likely to affect both the Eurozone and the US. This issue can be addressed within our setup by viewing it as a form of interdependence between sources. Let  $\hat{S}_i = S_i \times S_3$  stand for the set of effective contingencies for market  $i = 1, 2$ . That is,  $\hat{S}_i$  describes the relevant uncertainty on market  $i$  by taking into account all possible combinations of contingencies featured by the local source  $i$  and the Asian source. Asset  $i$  is then defined as  $\hat{f}_i : \hat{S}_i \rightarrow [0, \infty)$  for  $i = 1, 2$ . As before,  $\hat{f}_i$  pays in the local currency, but now it depends on the interactions between the local source (Eurozone or US) and the external source (Asia) that, together, determine the payoff structure of the asset. By affecting the underlying mechanism of uncertainty, this interdependence may alter the source-dependent uncertainty attitudes: the certainty equivalent in (13) is now computed with respect to a probability measure defined on the extended set of contingencies  $\hat{S}_i$ . The definition of the remaining parameters is formally unchanged. In particular, all exchange rates are still defined between euros and dollars. Any influence of the Asian source on the Eurozone and the US is captured by the respective certainty equivalents and assessed using one of the two currencies. Hence, the analysis of the last section remains valid within this extended setup. A natural direction for future research is the study of more complicated forms of interdependence and their potential impact on the familiarity perception of markets as captured by subjective prices.

**Source elicitation** Our work does not address the question of how sources could be elicited from choices—they are exogenously given. This is certainly a limitation of our approach. In more specific settings, other works in the literature, notably Chew and Sagi (2008), Abdellaoui, Baillon, Placido, and Wakker (2011), and Gul and Pesendorfer (2015) identify sources subjectively, through collections of events that satisfy certain properties.

Nevertheless, we should remark that, in many decision problems, sources are naturally defined by the framework itself. Moreover, agents can often choose which source to act upon in a way consistent with our representation of preferences over prospect-source pairs. We study several contexts of this sort: the two urns in the Ellsberg paradox, dates in temporal settings, and locations in portfolio allocation settings. We leave the more decision-theoretic question of elicitation as a direction for future research.

## A Proofs and related analysis

### A.1 Proofs of the results of Section 3

**Proof of Lemma 2.** It follows directly from Lemma 24 (in Online Appendix) by letting  $\Omega = S$  and  $F = \mathcal{F}_i$  in that lemma.  $\blacksquare$

**Proof of Proposition 7.** (i) implies (ii). Since  $\succsim$  satisfies Axiom A.1, it follows from Lemma 2 that there exists a family  $(c_i)_{i \in I}$  of certainty equivalents for  $(\succsim_i)_{i \in I}$ .

Fix  $i, j \in I$ . By Axiom B.1, for every  $x \in X$ , there exists  $y \in X$  such that  $(x, i) \sim (y, j)$ . Note that the number  $y$  is unique. In fact, by transitivity of  $\succsim$  on  $\mathcal{P}$ , if  $\hat{y} \in X$  were such that  $(x, i) \sim (\hat{y}, j)$ , then  $(\hat{y}, j) \sim (y, j)$ . That is,  $(\hat{y}, j) \sim_j (y, j)$ , yielding that  $y = c_j(y) = c_j(\hat{y}) = \hat{y}$ . By the same reasoning, note that if  $i = j$ , then  $y = x$ . Thus, we can define a function  $\delta_{ji} : X \rightarrow X$  as

$$\delta_{ji}(x) = y \text{ where } y \text{ is such that } (x, i) \sim (y, j). \quad (16)$$

Note that if  $i = j$ , then  $\delta_{ii} = \text{id}_X$ . Since  $i, j \in I$  were arbitrarily chosen, we have that  $\delta_{ji} : X \rightarrow X$ , defined as in (16), is a well-defined function. Next, we show that, for all  $i, j \in I$ ,  $\delta_{ji}$  is a rate and, hence, it satisfies the properties described in Definition 6.

- If  $x > z$ , then  $(x, i) \succ (z, i)$  by Axiom A.1(i) and the definition of  $\succsim_i$ . By contradiction, assume that  $\delta_{ji}(x) \leq \delta_{ji}(z)$ . Then, by Axiom A.1(i), it follows that  $(\delta_{ji}(z), j) \succ (\delta_{ji}(x), j)$ . By construction, we also have that  $(z, i) \sim (\delta_{ji}(z), j) \succ (\delta_{ji}(x), j) \sim (x, i)$ , yielding, by transitivity, that  $(z, i) \succ (x, i)$ , a contradiction. This proves that  $\delta_{ji}$  is strictly increasing.
- By construction, for every  $y \in X$ ,  $(y, j) \sim (\delta_{ij}(y), i)$ , that is,  $y = \delta_{ji}(\delta_{ij}(y))$ . Thus,  $\delta_{ji}$  is onto (and  $\delta_{ij} = \delta_{ji}^{-1}$ ).
- By construction, for every  $x \in X$  and  $k \in I$ ,  $(x, i) \sim (\delta_{ji}(x), j)$  and  $(\delta_{ji}(x), j) \sim (\delta_{kj}(\delta_{ji}(x)), k)$ . By transitivity, this implies that  $(x, i) \sim (\delta_{kj}(\delta_{ji}(x)), k)$ , that is,  $\delta_{kj}(\delta_{ji}(x)) = \delta_{ki}(x)$ . Thus,  $\delta_{ki} = \delta_{kj} \circ \delta_{ji}$ .

Now, let  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$ , and  $g_j \in \mathcal{F}_j$ . By the previous part of the proof, we have

$$(f_i, i) \sim (c_i(f_i), i), (g_j, j) \sim (c_j(g_j), j), \text{ and } (c_j(g_j), j) \sim (\delta_{ij}(c_j(g_j)), i).$$

By transitivity of  $\succsim$  on  $\mathcal{P}$  and Lemma 2, we obtain that

$$\begin{aligned} (f_i, i) \succsim (g_j, j) &\iff (c_i(f_i), i) \succsim (c_j(g_j), j) \\ &\iff (c_i(f_i), i) \succsim (\delta_{ij}(c_j(g_j)), i) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)), \end{aligned}$$

(ii) implies (i). Let  $i \in I$  and  $f_i, g_i \in \mathcal{F}_i$ . Then,

$$(f_i, i) \succsim (g_i, i) \iff c_i(f_i) \geq \delta_{ii}(c_i(g_i)) \iff c_i(f_i) \geq c_i(g_i).$$

Since  $i$  was arbitrarily chosen, it follows that  $\succsim$  on  $\mathcal{P}$  is a complete binary relation. Next, if  $(f_i, i) \succsim (g_j, j)$  and  $(g_j, j) \succsim (h_k, k)$ , then representation (4) implies that

$$c_i(f_i) \geq \delta_{ij}(c_j(g_j)) \text{ and } c_j(g_j) \geq \delta_{jk}(c_k(h_k)).$$

Since  $\delta_{ij}$  is increasing and onto and  $\delta_{ki} = \delta_{kj} \circ \delta_{ji}$ , we have

$$c_i(f_i) \geq \delta_{ij}(c_j(g_j)) \geq \delta_{ij}(\delta_{jk}(c_k(h_k))) = \delta_{ik}(c_k(h_k)),$$

that is,  $(f_i, i) \succsim (h_k, k)$ , proving that  $\succsim$  on  $\mathcal{P}$  is transitive, too.

Since  $c_i$  is a certainty equivalent functional, we can apply Lemma 2 and directly conclude that  $\succsim_i$  on  $\mathcal{F}_i$  satisfies Axiom A.1.

Finally, for every  $i, j \in I$  and  $x \in X$ , let  $y = \delta_{ji}(x)$ . Then,

$$c_j(y) = y = \delta_{ji}(x) = \delta_{ji}(c_i(x)),$$

proving that  $(y, j) \sim (x, i)$ . Thus, Axiom B.1 is satisfied.

The uniqueness of  $(c_i)_{i \in I}$  follows from Lemma 2. Next, assume that  $(\hat{\delta}_{ji})_{j, i \in I}$  is another family of functions that, together with  $(c_i)_{i \in I}$ , represents  $\succsim$  on  $\mathcal{P}$  as in (4). Fix arbitrary  $i, j \in I$ . It follows that, for all  $x \in X$ ,

$$(x, i) \sim (\delta_{ji}(x), j) \text{ and } (x, i) \sim (\hat{\delta}_{ji}(x), j).$$

By transitivity of  $\succsim$ , we have  $(\delta_{ji}(x), j) \sim (\hat{\delta}_{ji}(x), j)$  which, by using the representation, implies  $\delta_{ji}(x) = \hat{\delta}_{ji}(x)$ . We conclude that  $\delta_{ij} = \hat{\delta}_{ij}$  for all  $i, j \in I$ .  $\blacksquare$

**Proofs of Corollaries 8 and 9.** (i) implies (ii). By Proposition 7, there exist a family  $(c_i)_{i \in I}$  of certainty equivalents for  $(\succsim_i)_{i \in I}$ , and a family  $(\delta_{ij})_{i, j \in I}$  of rates, such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$  and  $g_j \in \mathcal{F}_j$ ,  $(f_i, i) \succsim (g_j, j)$  iff  $c_i(f_i) \geq \delta_{ij}(c_j(g_j))$ . Moreover, for each  $i \in I$ ,  $c_i : \mathcal{F}_i \rightarrow X$  is unique. Thus, we can apply Axioms A.2 and A.3 which directly yield representations (5) and (6), respectively.

It is immediate to verify that (ii) implies (i) for both corollaries. The uniqueness parts follow from Proposition 7 and standard arguments (see, e.g., Wakker, 1989, Observation IV.2.7, and Wakker, 2010, Observation 10.5.6, respectively).  $\blacksquare$

**Proof of Proposition 10.** (i) implies (ii). By Axiom A.2, for every  $i \in I$ , there are a strictly increasing and continuous function  $u_i : X \rightarrow \mathbb{R}$  and a probability measure  $p_i$  on  $(S, \Sigma_i)$  such that  $c_i : \mathcal{F}_i \rightarrow X$  represents  $\succsim_i$  where  $c_i(f_i) = u_i^{-1}(\int_S u_i(f_i) dp_i)$  for all  $f_i \in \mathcal{F}_i$ . Fix arbitrary  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$ , and  $g_j \in \mathcal{F}_j$  and suppose that  $(f_i, i) \succsim (g_j, j)$ . By Axiom A.2, we have  $(f_i, i) \sim (c_i(f_i), i)$  and  $(g_j, j) \sim (c_j(g_j), j)$ . By Axiom B.2,  $(c_i(f_i), i) \sim (c_i(f_i), j)$ . Then, transitivity implies  $(c_i(f_i), j) \succsim (c_j(g_j), j)$ , that is,  $c_i(f_i)1_{S_j} \succsim_j c_j(g_j)1_{S_j}$ . Since  $c_j$  represents  $\succsim_j$ , we conclude that  $c_i(f_i) \geq c_j(g_j)$ .

It is immediate to verify that (ii) implies (i). The uniqueness part follows from standard arguments (see, e.g., Wakker, 1989, Observation IV.2.7).  $\blacksquare$

## A.2 Proofs of the results of Section 4

In the rest of the appendix, we will refer to

$$\mathcal{F}_c = \{\mathbf{f} \in \mathcal{F} : \mathbf{f} = (x_i 1_{S_i})_{i \in I} \text{ for } x_i \in X\}$$

as the set of constant prospect profiles. We endow  $\mathcal{F}_c$  with the product topology. Define the map  $P : \mathcal{F}_c \rightarrow X^I$  by

$$P(\mathbf{f}) = \varphi \quad \text{where} \quad f_i = x_i 1_{S_i} \quad \text{and} \quad \varphi_i = x_i \quad \forall i \in I,$$

and note that  $P$  is a well-defined affine homeomorphism. Define a binary relation  $\succsim$  on  $X^I$  as

$$\varphi \succsim \psi \iff \mathbf{f} \succsim \mathbf{g}, \tag{17}$$

where  $\mathbf{f}, \mathbf{g} \in \mathcal{F}_c$ ,  $\varphi = P(\mathbf{f})$ , and  $\psi = P(\mathbf{g})$ .

We will also make use of the following notion for an aggregator.

**Definition 19** *Let  $D : X^I \rightarrow X$  be an **aggregator**.*

- (i) *For every  $i \in I$ , define  $D_i : X \rightarrow X$  as  $D_i(x) = D(x 1_i)$  for all  $x \in X$ . We say that  $D_i$  is a **component** of  $D$ .*
- (ii) *We say that  $D : X^I \rightarrow X$  is a **commensurable aggregator** if  $D_i$  is strictly increasing and  $D_i(X) = D_j(X)$  for all  $i, j \in I$ .*

Next, we present an alternative formulation of Theorem 12 where the aggregator is not normalized with respect to a specific source. We then show that Theorem 12 follows from Theorem 20.

**Theorem 20** *Let  $\succsim$  be a binary relation on  $\mathcal{F}$ . The following conditions are equivalent:*

- (i)  $\succsim$  *satisfies Axioms A.1, B.1, and C.1;*
- (ii) *there exist*

- (a) *a family  $(c_i)_{i \in I}$  of **certainty equivalents** for  $(\succsim_i)_{i \in I}$ ,*
- (b) *a family  $(\delta_{ij})_{i, j \in I}$  of **rates**,*

*such that, for all  $i, j \in I$ ,  $f_i \in \mathcal{F}_i$ , and  $g_j \in \mathcal{F}_j$ ,*

$$(f_i, i) \succsim (g_j, j) \iff c_i(f_i) \geq \delta_{ij}(c_j(g_j)),$$

- (c) *a **commensurable aggregator**  $D : X^I \rightarrow X$ , such that, for all  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ ,*

$$\mathbf{f} \succsim \mathbf{g} \iff D(c_1(f_1), \dots, c_n(f_n)) \geq D(c_1(g_1), \dots, c_n(g_n)).$$

In this case,  $\delta_{ij} = D_i^{-1} \circ D_j$  for all  $i, j \in I$ . Moreover, the elements of  $(c_i)_{i \in I}$  and  $(\delta_{ij})_{i, j \in I}$  are unique, and so is  $D$ .

**Proof of Theorem 20.** (i) implies (ii). Parts (a) and (b) follow from Proposition 7. Define the map  $\tilde{T} : \mathcal{F} \rightarrow \mathcal{F}_c$  such that  $\mathbf{f} \mapsto \tilde{\mathbf{f}}$  where  $\tilde{f}_i = c_i(f_i) 1_{S_i}$  for all  $i \in I$ . Clearly,  $\tilde{T}$  is onto. Since  $\succsim$  on  $\mathcal{F}$  satisfies Axiom C.1(i), we have that, for all  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ ,

$$\mathbf{f} \succsim \mathbf{g} \iff \tilde{\mathbf{f}} \succsim \tilde{\mathbf{g}}. \quad (18)$$

Define the map  $T = P \circ \tilde{T} : \mathcal{F} \rightarrow X^I$  and note that it is also onto. Moreover,

$$T(\mathbf{f}) = T(\mathbf{g}) \iff c_i(f_i) = c_i(g_i) \quad \forall i \in I.$$

Since  $\succsim$  satisfies Axioms A.1 and C.1, it follows that  $T(\mathbf{f}) = T(\mathbf{g})$  implies  $\mathbf{f} \sim \mathbf{g}$ . Hence,  $T$  is a well-defined function. Using conditions (17) and (18), we obtain that, for all  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ ,

$$\mathbf{f} \succsim \mathbf{g} \iff \tilde{T}(\mathbf{f}) \succsim \tilde{T}(\mathbf{g}) \iff T(\mathbf{f}) \succcurlyeq T(\mathbf{g}). \quad (19)$$

Next, we show that  $\succcurlyeq$  on  $X^I$  satisfies properties  $(\pi.1)$ - $(\pi.3)$  of Lemma 24 (of Online Appendix). Set  $F = X^I$  in that lemma. Clearly,  $\succcurlyeq$  satisfies property  $(\pi.1)$  as  $\succsim$  on  $\mathcal{F}$  is a weak order. Let  $\mathbf{x}, \mathbf{y} \in X^I$  such that  $x_i \geq y_i$  for all  $i \in I$ . Axiom A.1(i) yields  $x_i 1_{S_i} \succsim_i y_i 1_{S_i}$  for all  $i \in I$ , and Axiom C.1(i) together with (17) imply  $\mathbf{x} \succcurlyeq \mathbf{y}$ . Thus, the first part of property  $(\pi.2)$  is satisfied. If  $\mathbf{x}, \mathbf{y} \in X^I$  are constant and  $\mathbf{x} \succcurlyeq \mathbf{y}$ , then Axiom C.1(ii) gives  $\mathbf{x} \geq \mathbf{y}$ . Hence, property  $(\pi.2)$  holds, too. Property  $(\pi.3)$  follows directly from Axiom C.1(iii).

By Lemma 24 (Online Appendix), there exists a Chisini mean  $D : X^I \rightarrow X$  that represents  $\succcurlyeq$  on  $X^I$  and such that  $\mathbf{x} \sim D(\mathbf{x}) 1_I$  for all  $\mathbf{x} \in X^I$ . Together with condition (19), it implies that

$$\mathbf{f} \succsim \mathbf{g} \iff D(T(\mathbf{f})) \geq D(T(\mathbf{g})) \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{F}.$$

By Proposition 23 (Online Appendix),  $D$  is monotone, normalized, and continuous. Thus,  $D$  is an aggregator.

It remains to show that  $D$  is commensurable and that  $\delta_{ij} = D_i^{-1} \circ D_j$  for all  $i, j \in I$ . Let  $x, y \in X$  such that  $x > y$  and fix an arbitrary  $i \in I$ . By Axiom A.1(i), we have  $x 1_{S_i} \succ_i y 1_{S_i}$ . Then,  $D(0, \dots, x, \dots, 0) > D(0, \dots, y, \dots, 0)$ , that is,  $D_i(x) > D_i(y)$ , which proves that  $D_i$  is strictly increasing for all  $i \in I$ .

Let  $x \in X$  and  $i, j \in I$ . Parts (a) and (b) give  $(x, j) \sim (\delta_{ij}(x), i)$ . Then,

$$D_j(x) = D(0, \dots, x, \dots, 0) = D(0, \dots, \delta_{ij}(x), \dots, 0) = D_i(\delta_{ij}(x)).$$

We conclude that  $\delta_{ij} = D_i^{-1} \circ D_j$ . Finally, since  $\delta_{ij}$  is onto, it directly follows that  $D_i(X) = D_j(X) = X$  for all  $i, j \in I$ . Part (c) is now proven.

(ii) implies (i). By Proposition 7, parts (a) and (b) imply that  $\succsim$  satisfies Axioms A.1 and B.1. Since for every  $f \in \mathcal{F}$ , we can compute  $D((c_i(f_i))_{i \in I})$ , it follows from part (c) that  $\succsim$  on  $\mathcal{F}$  is a weak order. Let  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$  be such that  $f_i \succsim_i g_i$  for all  $i \in I$ . Then, parts (a) and (b) imply that  $c_i(f_i) \geq c_i(g_i)$  for all  $i \in I$ . Since  $D$  is monotone, we can conclude that  $D((c_i(f_i))_{i \in I}) \geq D((c_i(g_i))_{i \in I})$ , that is,  $\mathbf{f} \succsim \mathbf{g}$ . Thus,  $\succsim$  satisfies Axiom C.1(i). Let  $\mathbf{x}$  and  $\mathbf{y}$  in  $X^I$  be constant and such that  $\mathbf{x} > \mathbf{y}$ . Then,  $x = D(\mathbf{x}) > D(\mathbf{y}) = y$ , implying that  $\mathbf{x} \succ \mathbf{y}$ , and Axiom C.1(ii) holds, too. Since  $D$  is a Chisini mean, we can apply Lemma 24 (Online Appendix) with  $F = X^I$  and conclude that  $\succsim$  satisfies Axiom C.1(iii).

Finally, the uniqueness of the elements of  $(c_i)_{i \in I}$  and  $(\delta_{ij})_{i,j \in I}$  follows from Proposition 7. The functional  $D$  is unique by Lemma 24 (Online Appendix).  $\blacksquare$

We now prove Theorem 12 as a corollary of Theorem 20.

**Proof of Theorem 12.** (i) implies (ii). Parts (a) and (b) follow from Theorem 20. Fix  $o \in I$  and define  $\tilde{T}_o : \mathcal{F} \rightarrow \mathcal{F}_c$  such that  $\mathbf{f} \mapsto \tilde{\mathbf{f}}$  where  $\tilde{f}_i = \delta_{oi}(c_i(f_i))1_{S_i}$  for all  $i \in I$ . Since each  $\delta_{oi}$  is onto, it follows that  $\tilde{T}_o$  is also onto. Define the map  $T_o = P \circ \tilde{T}_o : \mathcal{F} \rightarrow X^I$  and note that  $T_o$  is onto. Since each  $\delta_{oi}$  is strictly increasing, we have

$$T_o(\mathbf{f}) = T_o(\mathbf{g}) \iff c_i(f_i) = c_i(g_i) \quad \forall i \in I.$$

Since  $\succsim$  satisfies Axioms A.1 and C.1, we have that  $T_o(\mathbf{f}) = T_o(\mathbf{g})$  yields  $\mathbf{f} \sim \mathbf{g}$ . Thus, the map  $T_o$  is well defined. By Axiom C.1(i) and condition (17), it follows that  $\mathbf{f} \succsim \mathbf{g} \iff T_o(\mathbf{f}) \succcurlyeq T_o(\mathbf{g})$ , for all  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ .

By Theorem 20, there exists a commensurable aggregator  $D : X^I \rightarrow X$  such that

$$\mathbf{f} \succsim \mathbf{g} \iff D(T_o(\mathbf{f})) \geq D(T_o(\mathbf{g})) \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{F}.$$

Since  $D$  is commensurable, we have that  $D_o^{-1}$  is strictly increasing. Define  $W_o : X^I \rightarrow X$  as  $W_o(\mathbf{x}) = (D_o^{-1}(D(\mathbf{x})))$ . Then,

$$\mathbf{f} \succsim \mathbf{g} \iff W_o(T_o(\mathbf{f})) \geq W_o(T_o(\mathbf{g})) \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{F}.$$

Note that  $W_o$  is monotone because  $D$  is monotone. Moreover, for any  $\mathbf{x} \in X^I$  such that  $x_j = 0$  for all  $j \neq o$ , we have that  $W_o(\mathbf{x}) = D_o^{-1}(D(\mathbf{x})) = x_o$ . Thus,  $W_o$  is  $o$ -normalized.

(ii) implies (i). Omitted as it is analogous to the proof of the sufficiency part of Theorem 20.  $\blacksquare$

**Proof of Proposition 13.** (i) Let  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_o)$  and  $((c'_i)_{i \in I}, (\delta'_{ij})_{i,j \in I}, W'_o)$  be two source-dependent representations of  $\succsim$ . By Proposition 7, it follows that  $c'_i = c_i$  and  $\delta_{ij} = \delta'_{ij}$  for all  $i, j \in I$ . Moreover, uniqueness of  $W_o$  follows directly by construction. Indeed, recall that  $W_o = D_o^{-1} \circ D$  and the aggregator  $D$  is unique by Theorem 20.

(ii) Let  $o \neq \bar{o}$  and consider  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_o)$  and  $((c_i)_{i \in I}, (\delta_{ij})_{i,j \in I}, W_{\bar{o}})$ . By construction, we have that  $W_o = D_o^{-1} \circ D$  and  $W_{\bar{o}} = D_{\bar{o}}^{-1} \circ D$ . Since  $\delta_{o\bar{o}} = D_o^{-1} \circ D_{\bar{o}}$  by Theorem 20, we conclude that  $W_o = \delta_{o\bar{o}}(W_{\bar{o}})$ .  $\blacksquare$



**Proof of Proposition 14.** (i) implies (ii). Note that  $X^I$  is a convex set. Moreover, since for each  $i \in I$ ,  $x \geq y$  if and only if  $x1_{S_i} \succsim_i y1_{S_i}$ , it follows that each component is essential (non-null). By Wakker (1988), there exist a collection  $\{\alpha_i\}_{i \in I} \subseteq \Delta(I)$  and a continuous function  $v : X \rightarrow \mathbb{R}$  such that, for all  $\mathbf{f}, \mathbf{g} \in \mathcal{F}_c$ ,

$$\mathbf{f} \succsim \mathbf{g} \iff \boldsymbol{\varphi} \succ \boldsymbol{\psi} \iff \sum_{i=1}^I \alpha_i v(x_i) \geq \sum_{i=1}^I \alpha_i v(y_i),$$

where  $f_i = x_i 1_{S_i}$ ,  $g_i = y_i 1_{S_i}$ ,  $\varphi_i = x_i$ , and  $\psi_i = y_i$  for all  $i \in I$ . After a positive and affine transformation, we can normalize  $v$  by setting  $v(0) = 0$  and  $v(1) = 1$ . Using condition (19), we can conclude that, for arbitrary  $\mathbf{f}, \mathbf{g} \in \mathcal{F}$ ,

$$\mathbf{f} \succsim \mathbf{g} \iff \tilde{T}(\mathbf{f}) \succsim \tilde{T}(\mathbf{g}) \iff T(\mathbf{f}) \succ T(\mathbf{g}) \iff \sum_{i=1}^I \alpha_i v(c_i(f_i)) \geq \sum_{i=1}^I \alpha_i v(c_i(g_i)),$$

where  $T$  and  $\tilde{T}$  are the maps defined in the Proof of Theorem 20. Let  $i \in I$ . Observe that

$$x \geq y \iff x1_{S_i} \succsim_i y1_{S_i} \iff x1_{S_i} \succ y1_{S_i} \iff \alpha_i v(x) \geq \alpha_i v(y).$$

Since  $\alpha_i \in [0, 1]$ , it follows necessarily that  $\alpha_i > 0$ . This implies that

$$x \geq y \iff \alpha_i v(x) \geq \alpha_i v(y) \iff v(x) \geq v(y).$$

Since  $i$  was arbitrarily chosen, we have that  $v$  is strictly increasing and  $\{\alpha_i\}_{i \in I} \subseteq \Delta(I)$  is a collection of strictly positive weights. Define  $\gamma_j = \frac{1}{\alpha_j} \in (0, \infty)$  for some  $j \in I$ . Then,

$$\mathbf{f} \succsim \mathbf{g} \iff \gamma_j \sum_{i=1}^I \alpha_i v(c_i(f_i)) \geq \gamma_j \sum_{i=1}^I \alpha_i v(c_i(g_i)). \quad (20)$$

Finally, define  $\hat{W}_j : X^I \rightarrow X$  as

$$\hat{W}_j(\boldsymbol{\varphi}) = v^{-1} \left( \gamma_j \sum_{i=1}^I \alpha_i v(\varphi_i) \right) \quad \forall \boldsymbol{\varphi} \in X^I.$$

Given (20), it is immediate to verify that  $\hat{W}_j$  is  $j$ -normalized, monotone, and  $V_j = \hat{W}_j \circ T$  is a utility function for  $\succsim$ .

(ii) implies (i). We omit it as it follows from routine arguments.

By applying standard results (see Wakker, 1988, 1989), the collection  $\{\alpha_i\}_{i \in I}$  is unique and the function  $v$  is unique up to positive affine transformations. Moreover, the normalization  $v(0) = 0$  and  $v(1) = 1$  implies that  $v$  must be unique.  $\blacksquare$

**Proof of Proposition 15.** (i) implies (ii). By Proposition 7 and since  $\succsim$  satisfies Axiom D.3, we have that, for every  $i, j \in I \setminus \{T\}$  and every  $x, y \in X$ ,

$$\delta_{ji}(x) = y \iff (x, i) \sim (y, j) \implies (x, i+1) \sim (y, j+1) \implies \delta_{j+1, i+1}(x) = y = \delta_{ji}(x),$$

that is,  $\delta_{j+1} \ i_{+1} = \delta_{ji}$ . If we set  $j = t$  and  $i = t + 1$ , we have that, for each  $t \in I \setminus \{T\}$ ,

$$\delta_{01}(x) = \delta_{12}(x) = \dots = \delta_{t \ t+1}(x) \quad \forall x \in X.$$

In particular, set  $\hat{\delta} = \delta_{01}(1) = \delta_{12}(1) = \dots = \delta_{t \ t+1}(1)$ . Recall also that for every  $i, j \in I$  and for every  $x \in X$  the number  $\delta_{ji}(x)$  is the unique element in  $X$  such that

$$(x, i) \sim (\delta_{ji}(x), j). \quad (21)$$

By Proposition 14, there exist a collection of strictly positive weights  $\{\alpha_i\}_{i \in I} \subseteq \Delta(I)$  and a continuous and strictly increasing function  $v : X \rightarrow \mathbb{R}$  with  $v(0) = 0$  and  $v(1) = 1$ , and  $\gamma_0 > 0$  such that

$$V_0(\mathbf{f}) = v^{-1} \left( \gamma_0 \sum_{t=0}^T \alpha_t v(c_t(f_t)) \right) \quad \forall \mathbf{f} \in \mathcal{F} \quad (22)$$

is a utility function for  $\succsim$ . Let  $x = 1$ . For every  $i, j \in I$ , we have

$$(\delta_{ji}(1), j) \sim (1, i) \iff \alpha_j v(\delta_{ji}(1)) = \alpha_i v(1) \iff \alpha_j v(\delta_{ji}(1)) = \alpha_i.$$

By setting  $i = t + 1$  and  $j = t$ , we have

$$\alpha_{t+1} = \alpha_t v(\delta_{t \ t+1}(1)) \quad \forall t \in I \setminus \{T\}.$$

Since  $\delta_{t \ t+1}(1) = \delta_{01}(1) = \hat{\delta} > 0$  for all  $t \in I \setminus \{T\}$ , we can define  $\beta = v(\hat{\delta}) > 0$  and obtain  $\alpha_{t+1} = \alpha_t \beta$  for all  $t \in I \setminus \{T\}$ . It follows that  $\alpha_{t+1} = \alpha_0 \beta^{t+1}$  for all  $t \in I \setminus \{T\}$ .<sup>31</sup> By the proof of Proposition 14, we have that  $\gamma_0 = \frac{1}{\alpha_0}$ . Thus, we can conclude that

$$\begin{aligned} V_0(\mathbf{f}) &= v^{-1} \left( \gamma_0 \sum_{t=0}^T \alpha_t v(c_t(f_t)) \right) = v^{-1} \left( \sum_{t=0}^T \gamma_0 \alpha_t v(c_t(f_t)) \right) \\ &= v^{-1} \left( \sum_{t=0}^T \gamma_0 \alpha_0 \beta^t v(c_t(f_t)) \right) \\ &= v^{-1} \left( \sum_{t=0}^T \beta^t v(c_t(f_t)) \right), \end{aligned}$$

proving the statement.

(ii) implies (i). We omit it as it is routine.

Finally,  $\beta$  and  $v$  are unique by the uniqueness properties of Proposition 14. ■

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<sup>31</sup>Note that  $\alpha_{t+1} = \alpha_t v(\delta_{t \ t+1}(1)) = \alpha_t v(\hat{\delta})$  implies that  $v(\hat{\delta}) = \frac{\alpha_{t+1}}{\alpha_t} > 0$ . Since  $v(0) = 0$  and  $v$  is strictly increasing, we have that  $\hat{\delta} > 0$ .

### A.3 Proofs of the results of Sections 5 and 6

**Proof of Proposition 16.** (i) implies (ii). Using the implication with the same name in Wakker (1989, Theorem VII.6.5), observe that the fact that  $\{\alpha_i^j\}_{i \in I} \subseteq \Delta(I)$  for  $j = 1, 2$  is a collection of strictly positive weights prevents case (i.b) in that theorem; moreover, the fact that  $v_2$  is strictly increasing prevents case (i.c) there. We conclude that  $\{\alpha_i^1\}_{i \in I} = \{\alpha_i^2\}_{i \in I}$  and that there exists a convex, nondecreasing, and continuous  $\varphi : v_1(X) \rightarrow \mathbb{R}$  such that  $v_2 = \varphi \circ v_1$ . Note that, for each  $x \in X$ ,  $v_2(x) = v_2(v_1^{-1}(v_1(x)))$ . Then, for each  $t = v_1(x_t) \in v_1(X)$ ,

$$\varphi(t) = \varphi(v_1(x_t)) = v_2(x_t) = v_2(v_1^{-1}(v_1(x_t))) = v_2(v_1^{-1}(t))$$

and  $\varphi = v_2 \circ v_1^{-1}$ . As a consequence,  $\varphi : v_1(X) \rightarrow v_2(X)$  is bijective, strictly increasing, and continuous, and so is its inverse  $\psi = v_1 \circ v_2^{-1} : v_2(X) \rightarrow v_1(X)$ , which satisfies  $v_1 = \psi \circ v_2$ . Convexity of  $\varphi$  implies

$$\varphi(\alpha t + (1 - \alpha)z) \leq \alpha \varphi(t) + (1 - \alpha)\varphi(z) \quad \forall t, z \in u_1(X), \alpha \in [0, 1]$$

$$\iff$$

$$\alpha t + (1 - \alpha)z \leq \varphi^{-1}(\alpha \varphi(t) + (1 - \alpha)\varphi(z)) \quad \forall t, z \in u_1(X), \alpha \in [0, 1].$$

Now, taking any  $x, y \in v_2(X)$ , set  $t = \psi(x), z = \psi(y) \in v_1(X)$  so that, for each  $\alpha \in [0, 1]$ ,

$$\alpha \psi(x) + (1 - \alpha)\psi(y) \leq \psi(\alpha \varphi(\psi(x)) + (1 - \alpha)\varphi(\psi(y))) = \psi(\alpha x + (1 - \alpha)y).$$

Thus,  $\psi = v_1 \circ v_2^{-1} : v_2(X) \rightarrow \mathbb{R}$  is concave, strictly increasing, continuous, and such that  $v_1 = \psi \circ v_2$ .

(ii) implies (i). Assume that  $\{\alpha_i^1\}_{i \in I} = \{\alpha_i^2\}_{i \in I}$  and  $v_1 = \psi \circ v_2$  for some concave transformation  $\psi$ . For each  $i \in I$ , let  $\alpha_i := \alpha_i^1 = \alpha_i^2$ . Consider  $\mathbf{x} \in X^I$  and  $z \in X$  such that  $\mathbf{x} \succsim_1(z, z, \dots, z)$ . By using the representation in Proposition 14,  $\sum_{i \in I} \alpha_i v_1(x_i) \geq v_1(z)$ . Since  $v_1 = \psi \circ v_2$ , we have  $\sum_{i \in I} \alpha_i \psi(v_2(x_i)) \geq \psi(v_2(z))$ . By concavity of  $\psi$ , it follows that  $\psi(\sum_{i \in I} \alpha_i v_2(x_i)) \geq \psi(v_2(z))$ . Hence, we conclude that  $\mathbf{x} \succsim_2(z, z, \dots, z)$ . ■

**Proof of Proposition 17.** Maximizing  $W$ , given the constraint, is equivalent to maximizing  $\tilde{W} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  defined by  $\tilde{W}(\xi_1, \xi_2) = \alpha_1 v(\xi_1 c_1(f_1)) + \alpha_2 v(\xi_2 c_2(f_2))$  subject to (12). Since  $\tilde{W}$  is continuous, strictly monotone, and strictly concave,<sup>32</sup>  $\tilde{W}$  has a unique maximum  $\hat{\xi}$  which satisfies (12) with equality. Let  $\rho = \rho_{12}$ . Set up the Lagrangian:

$$\mathcal{L}(\xi_1, \xi_2, \lambda_1, \lambda_2, \mu) = \tilde{W}(\xi_1, \xi_2) + \lambda_1 \xi_1 + \lambda_2 \xi_2 + \mu(w_1 + \rho w_2 - \xi_1 p_1 - \xi_2 \rho p_2),$$

and find  $\hat{\xi}, \hat{\lambda}_1, \hat{\lambda}_2$ , and  $\hat{\mu}$  that satisfy the following conditions:

$$(i) \quad \alpha_1 v'(\hat{\xi}_1 c_1(f_1)) c_1(f_1) + \hat{\lambda}_1 - \hat{\mu} p_1 = 0;$$

<sup>32</sup>To avoid technicalities, we also assume that  $c_i(f_i) > 0$ .

$$(ii) \quad \alpha_2 v' \left( \hat{\xi}_2 c_2(f_2) \right) c_2(f_2) + \hat{\lambda}_2 - \hat{\mu} \rho p_2 = 0;$$

$$(iii) \quad \hat{\lambda}_i \geq 0, \hat{\mu} \geq 0 \text{ and } \hat{\lambda}_i \hat{\xi}_i = 0 \text{ for } i = 1, 2 \text{ as well as } \hat{\xi}_1 p_1 + \hat{\xi}_2 \rho p_2 = w_1 + \rho w_2.$$

Then,  $\hat{\xi}$  is the maximizer we are looking for. We start our search by assuming  $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$  and  $\hat{\mu} > 0$ . Since  $v(x) = \frac{x^{1-k}}{1-k}$ , we have that

$$\alpha_1 \left( \hat{\xi}_1 c_1(f_1) \right)^{-k} c_1(f_1) + \hat{\lambda}_1 = \hat{\mu} p_1 \iff \hat{\xi}_1^k = \frac{\alpha_1 c_1(f_1)^{1-k}}{\hat{\mu} p_1} \iff \hat{\xi}_1 = \left( \frac{\alpha_1 c_1(f_1)^{1-k}}{\hat{\mu} p_1} \right)^{\frac{1}{k}},$$

and

$$\alpha_2 \left( \hat{\xi}_2 c_2(f_2) \right)^{-k} c_2(f_2) + \hat{\lambda}_2 = \hat{\mu} \rho p_2 \iff \hat{\xi}_2^k = \frac{\alpha_2 c_2(f_2)^{1-k}}{\hat{\mu} \rho p_2} \iff \hat{\xi}_2 = \left( \frac{\alpha_2 c_2(f_2)^{1-k}}{\hat{\mu} \rho p_2} \right)^{\frac{1}{k}}.$$

Call  $c_1(f_1) = c_1$  and  $c_2(f_2) = c_2$ . We have that

$$\left( \frac{\alpha_1 c_1^{1-k}}{\hat{\mu} p_1} \right)^{\frac{1}{k}} p_1 + \left( \frac{\alpha_2 c_2^{1-k}}{\hat{\mu} \rho p_2} \right)^{\frac{1}{k}} \rho p_2 = w_1 + \rho w_2.$$

It follows that

$$\left( \frac{1}{\hat{\mu}} \right)^{\frac{1}{k}} = \frac{w_1 + \rho w_2}{\left( \frac{\alpha_1 c_1^{1-k}}{p_1} \right)^{\frac{1}{k}} p_1 + \rho \left( \frac{\alpha_2 c_2^{1-k}}{\rho p_2} \right)^{\frac{1}{k}} p_2}$$

We can conclude that

$$\hat{\xi}_1 = \left( \frac{\alpha_1 c_1^{1-k}}{p_1} \right)^{\frac{1}{k}} \frac{w_1 + \rho w_2}{\left( \frac{\alpha_1 c_1^{1-k}}{p_1} \right)^{\frac{1}{k}} p_1 + \rho \left( \frac{\alpha_2 c_2^{1-k}}{\rho p_2} \right)^{\frac{1}{k}} p_2} > 0 \quad (23)$$

and

$$\hat{\xi}_2 = \left( \frac{\alpha_2 c_2^{1-k}}{\rho p_2} \right)^{\frac{1}{k}} \frac{w_1 + \rho w_2}{\left( \frac{\alpha_1 c_1^{1-k}}{p_1} \right)^{\frac{1}{k}} p_1 + \rho \left( \frac{\alpha_2 c_2^{1-k}}{\rho p_2} \right)^{\frac{1}{k}} p_2} > 0. \quad (24)$$

This implies that

$$\frac{\hat{\xi}_1}{\hat{\xi}_2} = \frac{\left( \frac{\alpha_1 c_1^{1-k}}{p_1} \right)^{\frac{1}{k}}}{\left( \frac{\alpha_2 c_2^{1-k}}{\rho p_2} \right)^{\frac{1}{k}}} = \left( \frac{\alpha_1 \rho p_2 c_1^{1-k}}{\alpha_2 p_1 c_2^{1-k}} \right)^{\frac{1}{k}} = \left( \frac{\alpha_1 \rho p_2}{\alpha_2 p_1} \right)^{\frac{1}{k}} \left( \frac{c_1}{c_2} \right)^{\frac{1-k}{k}}. \quad (25)$$

Finally, recall that by Proposition 14,  $\delta_{21}(x) = v^{-1} \left( \frac{\alpha_1}{\alpha_2} v(x) \right)$  and  $v^{-1}(x) = ((1-k)x)^{\frac{1}{1-k}}$ .

Hence,  $\delta_{21} := \delta_{21}(1) = \left( \frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1-k}}$ . It follows that

$$\frac{\hat{\xi}_1}{\hat{\xi}_2} = \left( \delta_{21} \frac{c_1}{c_2} \right)^{\frac{1-k}{k}} \left( \frac{\rho p_2}{p_1} \right)^{\frac{1}{k}}, \quad (26)$$

proving condition (14).

Now, recall that for  $i \in \{1, 2\}$ ,

$$\frac{\partial \tilde{W}}{\partial \xi_i}(\hat{\xi}) = \alpha_i c_i (\hat{\xi}_i c_i)^{-k}.$$

Hence,

$$\frac{\frac{\partial \tilde{W}}{\partial \xi_2}(\hat{\xi})}{\frac{\partial \tilde{W}}{\partial \xi_1}(\hat{\xi})} = \frac{\alpha_2 c_2 (\hat{\xi}_2 c_2)^{-k}}{\alpha_1 c_1 (\hat{\xi}_1 c_1)^{-k}} = \frac{\alpha_2}{\alpha_1} \left(\frac{c_2}{c_1}\right)^{1-k} \left(\frac{\hat{\xi}_2}{\hat{\xi}_1}\right)^{-k}.$$

Using (26) and rearranging, we obtain that

$$\frac{\frac{\partial \tilde{W}}{\partial \xi_2}(\hat{\xi})}{\frac{\partial \tilde{W}}{\partial \xi_1}(\hat{\xi})} = \frac{\rho p_2}{p_1}.$$

Since  $W = g \circ \tilde{W}$  where  $g$  is such that  $g(x) = [(1-k)x]^{\frac{1}{1-k}}$  for all  $x \in \mathbb{R}$ , it follows that

$$\frac{\partial W}{\partial \xi_i}(\hat{\xi}) = g'(\tilde{W}(\hat{\xi})) \frac{\partial \tilde{W}}{\partial \xi_i}(\hat{\xi}) \quad \forall i \in \{1, 2\}.$$

Note that  $\tilde{W}(\hat{\xi}) > 0$ , thus  $g'(\tilde{W}(\hat{\xi})) > 0$ . We can conclude that

$$\frac{\frac{\partial W}{\partial \xi_2}(\hat{\xi})}{\frac{\partial W}{\partial \xi_1}(\hat{\xi})} = \frac{g'(\tilde{W}(\hat{\xi})) \frac{\partial \tilde{W}}{\partial \xi_2}(\hat{\xi})}{g'(\tilde{W}(\hat{\xi})) \frac{\partial \tilde{W}}{\partial \xi_1}(\hat{\xi})} = \frac{\rho p_2}{p_1},$$

proving the second part of the proposition. ■

**Proof of Proposition 18.** The optimal portfolio allocation  $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2)$  is given by expressions (23) and (24) in the proof of Proposition 17. Then, the optimal portfolio share  $\tilde{\xi}_i = \frac{\hat{\xi}_i}{w_1 + \rho_1 w_2}$  for each asset  $i$  is given by

$$\tilde{\xi}_1 = \frac{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}}}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2} \quad \text{and} \quad \tilde{\xi}_2 = \frac{\left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}}}{\left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2}$$

To ease notation, set  $d = \left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 + \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2$  and note that  $d > 0$ . Then

$$\begin{aligned} \frac{\partial \tilde{\xi}_1}{\partial \alpha_1} &= \frac{1}{d^2} \left[ \frac{1}{k} \alpha_1^{\frac{1-k}{k}} \left(\frac{c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} d - \frac{1}{k} \alpha_1^{\frac{1-k}{k}} \left(\frac{c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} p_1 \left(\frac{\alpha_1 c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} \right] \\ &= \frac{1}{d^2} \left[ \frac{1}{k} \left(\frac{\alpha_1^{1-k} c_1^{1-k}}{p_1}\right)^{\frac{1}{k}} \left(\frac{\alpha_2 c_2^{1-k}}{\rho p_2}\right)^{\frac{1}{k}} \rho p_2 \right] > 0. \end{aligned}$$

Since  $\frac{\partial c_1}{\partial \gamma_1} < 0$ , we have that

$$\frac{\partial \tilde{\xi}_1}{\partial \gamma_1} = \frac{1}{d^2} \left[ \frac{k}{1-k} c_1^{\frac{2k-1}{1-k}} \left( \frac{\alpha_1 k}{p_1} \right)^{\frac{1}{1-k}} \left( \frac{k \alpha_2 c_2^k}{\rho p_2} \right)^{\frac{1}{1-k}} \rho p_2 \right] \frac{\partial c_1}{\partial \gamma_1} < 0.$$

Analogously, it can be shown that  $\frac{\partial \tilde{\xi}_2}{\partial \alpha_2} > 0$  and  $\frac{\partial \tilde{\xi}_2}{\partial \gamma_2} < 0$ .

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## Online Appendix: Preliminaries on Chisini means

This online appendix introduces the notion of Chisini mean and provides preliminary results relating this notion to our preference representations. These results are useful to prove the representation theorems in Appendix A.

We start with some terminology. Let  $\Omega$  be a nonempty set (in this appendix,  $\Omega$  can be either  $S$  or  $I$ ),  $\Sigma$  an algebra of subsets of  $\Omega$  (in this appendix,  $\Sigma$  can be either one of the  $\Sigma_i$ 's or the power set  $\wp(I)$ ),  $F = B_0(\Omega, \Sigma, X)$  the space of all simple  $\Sigma$ -measurable functions  $f : \Omega \rightarrow X$ , (endowed with the supnorm topology  $\|\cdot\|$ ), and  $F_c = \{x1_\Omega : x \in X\}$  the set of constant elements of  $F$ .

A functional  $V : F \rightarrow \mathbb{R}$  is said to:

- be *monotone* if  $f \geq g$  implies  $V(f) \geq V(g)$ ;
- be *normalized* if  $V(x1_\Omega) = x$  for all  $x \in X$ ;
- *represent* a binary relation  $\succsim$  on  $F$  if  $V(f) \geq V(g) \iff f \succsim g$ .

Denote by  $\mathcal{V}$  the class of all functionals  $V : F \rightarrow \mathbb{R}$  with the following two properties:

( $\chi$ .1) the restriction of  $V$  to  $F_c$  is one-to-one;

( $\chi$ .2)  $V$  is monotone and continuous.

**Remark 21** *First note that*

$$x_n 1_\Omega \rightarrow x 1_\Omega \text{ in } B_0(\Omega, \Sigma, X) \iff \|x_n 1_\Omega - x 1_\Omega\| \rightarrow 0 \iff x_n \rightarrow x \text{ in } \mathbb{R}.$$

*Therefore property ( $\chi$ .1) is equivalent to require the function*

$$\begin{aligned} v : X &\rightarrow \mathbb{R} \\ x &\mapsto V(x1_\Omega) \end{aligned}$$

*to be one-to-one on  $X$ . Together with property ( $\chi$ .2), this implies that  $v$  is strictly increasing and continuous. Moreover, denoting by  $\bar{f}$  and  $\underline{f}$  the pointwise maximum and minimum of any function  $f \in F$  on  $\Omega$ , monotonicity of  $V$  guarantees that  $v(\underline{f}) \leq V(f) \leq v(\bar{f})$ . Hence,  $V(F) = v(X)$ .*

**Definition 22** *Given  $V \in \mathcal{V}$ , a **Chisini mean** of an element  $f \in F$  with respect to  $V$  is the (unique) number  $M = M_V(f)$  such that*

$$V(M1_\Omega) = V(f).$$

*That is,  $M \in X$  is the unique solution*

$$M_V(f) = v^{-1}(V(f))$$

*of the equation  $v(M) = V(f)$ .*

The next simple proposition is a characterization of *Chisini means*:

**Proposition 23** *The following conditions are equivalent for a functional  $C : F \rightarrow X$ :*

- *there exists  $V \in \mathcal{V}$  such that*

$$C(f) = M_V(f) \quad \forall f \in F; \quad (27)$$

- *the functional  $C$  is monotone, normalized, and continuous.*

Moreover,  $V$  is unique up to strictly increasing and continuous transformations.<sup>33</sup>

**Proof of Proposition 23.** If there exists  $V \in \mathcal{V}$  such that  $C = M_V$ , then

$$f \geq g \implies V(f) \geq V(g) \implies M_V(f) = v^{-1}(V(f)) \geq v^{-1}(V(g)) = M_V(g)$$

and, so,  $C$  is monotone. As for normalization, note that, for all  $x \in X$ ,

$$C(x1_\Omega) = v^{-1}(V(x1_\Omega)) = v^{-1}(v(x)) = x.$$

Moreover, continuity of  $C$  follows directly from continuity of  $V$ .

Conversely, if  $C$  is monotone, normalized, and continuous, then  $C \in \mathcal{V}$ , that is, it satisfies properties  $(\chi.1)$  and  $(\chi.2)$ . Define  $c : X \rightarrow X$  as  $c(x) = C(x1_\Omega)$ . Then,  $c(x) = x$  for all  $x \in X$ , implying that

$$M_C(f) = c^{-1}(C(f)) = C(f) \quad \forall f \in F.$$

Hence,  $C$  is a Chisini mean with respect to itself.

Finally, assume  $W$  is another element of  $\mathcal{V}$  such that  $C = M_W$ . Then, for all  $f \in F$ ,

$$v^{-1}(V(f)) = M_V(f) = M_W(f) = w^{-1}(W(f)) \implies W(f) = (w \circ v^{-1})(V(f)).$$

By Remark 21, both  $v^{-1} : v(X) = V(F) \rightarrow X$  and  $w : X \rightarrow \mathbb{R}$  are strictly increasing and continuous, so is  $\tau = w \circ v^{-1}$ . Conversely, let  $\tau : V(F) \rightarrow \mathbb{R}$  be strictly increasing and continuous, and  $W = \tau \circ V$ . Then, for all  $x \in X$ ,

$$w(x) = W(x1_\Omega) = \tau(V(x1_\Omega)) = \tau(v(x)) \quad (28)$$

is strictly increasing and continuous because  $v$  and  $\tau$  are. Thus,  $W$  satisfies property  $(\chi.1)$ . Monotonicity of  $W$  follows immediately from that of  $V$  through  $\tau$ , so that  $W$  satisfies property  $(\chi.2)$ . Finally, (28) implies that  $\tau = w \circ v^{-1}$ . Hence, for all  $f \in F$ ,

$$M_W(f) = w^{-1}(W(f)) = w^{-1}(\tau(V(f))) = w^{-1}((w \circ v^{-1})(V(f))) = v^{-1}(V(f)) = M_V(f)$$

as desired. ■

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<sup>33</sup>Another element  $W \in \mathcal{V}$  is such that  $C = M_W$  if and only if there exists a strictly increasing and continuous transformation  $\tau : V(F) \rightarrow \mathbb{R}$  such that  $W = \tau \circ V$ . In particular,  $C$  is the only normalized element of  $W \in \mathcal{V}$  such that  $C(f) = M_W(f)$  for all  $f \in F$ .

**Lemma 24** *Let  $\succsim$  be a binary relation on  $F$ . The following conditions are equivalent:*

(i) *there exists a Chisini mean  $C : F \rightarrow X$  that represents  $\succsim$ ;*

(ii) *the binary relation  $\succsim$  has the following properties:*

( $\pi.1$ )  *$\succsim$  is complete and transitive;*

( $\pi.2$ ) *if  $f, g \in F$  and  $f \geq g$  then  $f \succsim g$ . The converse is true if both  $f$  and  $g$  are constant;*

( $\pi.3$ ) *if  $f, g, h \in F$  and  $f \succ g \succ h$ , then there exist  $\alpha, \beta \in (0, 1)$  such that  $(1 - \alpha)h + \alpha g \succ f \succ (1 - \beta)h + \beta g$ .*

*Moreover,  $C$  is the unique Chisini mean representing  $\succsim$  on  $F$  and it is such that*

$$f \sim C(f)1_\Omega \quad \forall f \in F. \quad (29)$$

When the elements of  $F$  are interpreted as single uncertain payoffs (in this paper, when  $F = \mathcal{F}_i$ ), equation (29) says that the Chisini mean  $C(f)$  of  $f$  is the single sure payoff which is equivalent to  $f$ , that is  $C$  is a *certainty equivalent*.

When the elements of  $F$  are interpreted as streams of sure payoffs (in this paper, when  $F = X^I$ ), equation (29) says that the Chisini mean  $C(f)$  of  $f$  is the constant payoff a stream of which is equivalent to  $f$ , that is  $C$  is an *aggregator*.<sup>34</sup>

**Proof of Lemma 24.** (i) implies (ii). Let  $C$  be a Chisini mean that represents  $\succsim$  on  $F$ . By Proposition 23, the functional  $C$  is monotone, normalized, and continuous. Since  $X$  is a convex subset of  $\mathbb{R}$ , we can apply Proposition 1 of Cerreia-Vioglio et al. (2011), and immediately conclude that  $\succsim$  satisfies properties ( $\pi.1$ ) and ( $\pi.3$ ). Moreover, monotonicity of  $C$  implies that

$$h \geq g \implies C(h) \geq C(g) \implies h \succsim g.$$

If  $h = x1_\Omega$  and  $g = y1_\Omega$  are constant, we have, by normalization of  $C$ , that

$$h \succsim g \implies C(x1_\Omega) \geq C(y1_\Omega) \implies x \geq y \implies h \geq g.$$

Hence, property ( $\pi.2$ ) holds, too.

(ii) implies (i). We claim that, for all  $f \in F$ , there exists a unique  $x_f \in X$  such that  $f \sim x_f1_\Omega$ . Assume  $f \sim x1_\Omega$  and  $f \sim y1_\Omega$ . Then, by transitivity,  $x1_\Omega \sim y1_\Omega$ , and, by property ( $\pi.2$ ), it must be that  $x1_\Omega = y1_\Omega$ , i.e.,  $x = y$ . This proves uniqueness.

We next prove existence. For all  $f \in F$ , let  $\bar{f}, \underline{f} \in X$  be such that  $\bar{f}1_\Omega \geq f \geq \underline{f}1_\Omega$ . Then, property ( $\pi.2$ ), again, yields  $\bar{f}1_\Omega \succsim f \succsim \underline{f}1_\Omega$ . If either  $f \sim \bar{f}1_\Omega$  or  $f \sim \underline{f}1_\Omega$

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<sup>34</sup>Note however that being a Chisini mean is a property of the functional unrelated to whether the functional represents a preference. Instead being a certainty equivalent is indissolubly tied to the existence of a corresponding preference relation.

existence is guaranteed; otherwise,  $\bar{f}1_\Omega \succ f \succ \underline{f}1_\Omega$ . The set  $X_f = \{x \in X : x1_\Omega \succsim f\}$  contains  $\bar{f}$  but not  $\underline{f}$  and, if  $y \in X_f$  all elements  $x \geq y$  also belong to  $X_f$  (by property  $(\pi.2)$  and transitivity). Let  $x_f = \inf X_f$ . If  $x_f1_\Omega \succ f$ , then  $x_f1_\Omega \succ f \succ \underline{f}1_\Omega$  and, by property  $(\pi.3)$ , there exists  $\alpha \in (0, 1)$  such that  $(1 - \alpha)x_f1_\Omega + \alpha\underline{f}1_\Omega \succ f \succ \underline{f}1_\Omega$ . Hence,  $(1 - \alpha)x_f + \alpha\underline{f} \in X_f$ . But, by property  $(\pi.2)$ , we have  $x_f > \underline{f}$ , which implies  $(1 - \alpha)x_f + \alpha\underline{f} < x_f$ , contradicting the fact that  $x_f$  is a lower bound of  $X_f$ . If  $f \succ x_f1_\Omega$ , then  $\bar{f}1_\Omega \succ f \succ x_f1_\Omega$  and, by property  $(\pi.3)$ , there exists  $\beta \in (0, 1)$  such that  $\bar{f}1_\Omega \succ f \succ (1 - \beta)\bar{f}1_\Omega + \beta x_f1_\Omega$ ; but, by property  $(\pi.2)$ ,  $\bar{f} > x_f$ , so  $(1 - \beta)\bar{f} + \beta x_f > x_f$ . On the other hand, for every  $x \in X_f$ ,  $x1_\Omega \succsim f \succ (1 - \beta)\bar{f}1_\Omega + \beta x_f1_\Omega$ . Thus, transitivity and property  $(\pi.2)$  imply  $x > (1 - \beta)\bar{f}1_\Omega + \beta x_f$ , contradicting the fact that  $x_f$  is the greatest lower bound of  $X_f$ . In conclusion, it must be the case that  $f \sim x_f1_\Omega$ .

Next, we show that there exists a Chisini mean that represents  $\succsim$ . Let  $x, y \in X$  be such that  $x1_\Omega \succ y1_\Omega$ . Then, property  $(\pi.2)$  implies that  $x > y$  and, by convexity of  $X$ , we have that  $(\alpha x + (1 - \alpha)z)1_\Omega > (\alpha y + (1 - \alpha)z)1_\Omega$  for all  $z \in X$  and  $\alpha \in (0, 1)$ . Again by property  $(\pi.2)$ , we obtain  $(\alpha x + (1 - \alpha)z)1_\Omega \succ (\alpha y + (1 - \alpha)z)1_\Omega$ , and so,  $\succsim$  restricted to  $F_c$  satisfies Independence. Thus, the axioms of Proposition 1 in Cerreia-Vioglio et al. (2011) are satisfied, implying that there exists a monotone, normalized and continuous functional  $C : F \rightarrow \mathbb{R}$  that represents  $\succsim$  on  $F$ . Since  $C$  is normalized and monotone,  $C(F) = X$ . We conclude, by Proposition 23, that the functional  $C : F \rightarrow X$  is a Chisini mean.

It remains to prove the uniqueness of  $C$ . For all  $f \in F$  and  $x \in X$ , normalization of  $C$  implies that  $f \sim x1_\Omega$  if and only if  $C(f) = C(x1_\Omega) = x$ . By the above argument, for all  $f \in F$ , there exists a unique  $x_f \in X$  such that  $f \sim x_f1_\Omega$ . Hence,  $f \sim C(f)1_\Omega$  for all  $f \in F$ . ■