

Maxmin Expected Utility on a Subjective State Space: Convex Preferences under Risk*

JOB MARKET PAPER

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Abstract

We study convex preferences over lotteries and over menus of lotteries. We consider a set of consequences C and we characterize complete, transitive, and convex binary relations over lotteries on the set C . We prove that convex preferences correspond to a decision criterion in which the Decision Maker reveals pessimism and a lack of confidence in the evaluation of consequences or his future tastes. We show in a context of choice over menus of lotteries how convex preferences translate into Maxmin Expected Utility on a Subjective State Space. Finally, we show how convex preferences can be interpreted as a cautious criterion of completion.

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1 Introduction

The object of our study are preferences over lotteries that exhibit a preference toward randomization or diversification. This is a classical topic in Decision Theory, partially overlooked by the literature of choice under risk. We show that such feature is characterized by cautiousness and pessimism of the Decision Maker (DM). Moreover, we argue and show that this pessimism can be interpreted as a lack of confidence about one or all of the following aspects: value of outcomes, future tastes, degree of risk aversion.

In order to fix ideas, consider two probability distributions p and q over a set of consequences C . For example, C could be monetary outcomes and p and q could be monetary lotteries. Assume further that the DM expresses the following two rankings: he deems p and q indifferent and he strictly

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prefers the mixture $\frac{1}{2}p + \frac{1}{2}q$ to any of the two lotteries. That is, this DM exhibits a preference toward randomization or diversification. Examples of this pattern of choice are not uncommon. For example, Prelec [31] reports overwhelming evidence for preference toward randomization. Some of the most prominent models in Decision Theory under risk cannot account for this behaviour (see, e.g., von Neumann and Morgenstern [36], Fishburn [16], Dekel [9], Gul [22]). Actually, a preference toward randomization constitutes a violation of these models. For example, the Expected Utility (EU) model would imply that the mixture $\frac{1}{2}p + \frac{1}{2}q$ is indifferent to any of the two lotteries p and q . Such violation of EU is classical since it shares a common feature with most of the well known violations of EU. Indeed, it is a violation of the Axiom of Independence on which the EU model rests. The Axiom of Independence consists in assuming that

$$p \succsim q \Rightarrow \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r \quad \forall \lambda \in [0, 1], \forall r.$$

It is not hard to check that Independence implies that randomization is neither detrimental nor beneficial, that is,

$$p \sim q \Rightarrow p \sim \lambda p + (1 - \lambda)q \sim p \quad \forall \lambda \in [0, 1]. \quad (1)$$

Assumption (1) is not violated by the Allais paradox and it is called *Betweenness*. For this reason, a preference toward randomization is a stronger violation of EU whence compared to the Allais paradox, since it constitutes a violation of *Betweenness*. The assumption of *Betweenness* was often considered by the literature of choice under risk in order to provide models consistent with the Allais paradox. Similarly, most of the recent literature of choice over *menus of lotteries* followed a similar path: by assuming that *randomization over lotteries* carries no or little value.¹

However, as we argue later, *Betweenness* (and a fortiori *Independence*) has been consistently rejected experimentally, see Camerer and Ho [5]. That is, DMs in a situation of choice under risk are not indifferent to randomization. Moreover, as we argue in Subsection 1.1, normatively a preference toward randomization, even for a context of choice under risk, might be desirable. For this reason, we study complete and transitive preferences that are suitably continuous and satisfy the following assumption of *Convexity* or *Mixing*:

$$p \sim q \Rightarrow \lambda p + (1 - \lambda)q \succsim p \quad \forall \lambda \in [0, 1].$$

This is very much in line with the path chosen by a consistent part of the literature of choice under ambiguity. In a setting à la Anscombe and Aumann [3], this literature weakened the assumption of *Independence* toward *Uncertainty Aversion* (see Gilboa and Schmeidler [21], Schmeidler [34]). Nevertheless, this path was consistently left unexplored for problems of choice under risk.

Convexity has been an important feature both in the theoretical and experimental literature on preferences over lotteries.

In the theoretical literature of choice under risk, *Convexity* was often the consequence of some stronger assumption, for example, *Weak Independence* (see Gul [22]), *Betweenness* (see, e.g., Dekel [9], Fishburn [16]), or it was paired with some extra assumptions (see Maccheroni [27]).

Similarly, in the experimental literature, *Convexity* has been mainly the object of indirect study in experiments that tested *Betweenness*. One of the most important contributions in this direction is the paper of Camerer and Ho [5]. Camerer and Ho review nine different studies that tested *Betweenness* and conduct their own experiments testing for this property. Almost all of the twenty experiments reviewed report that *Betweenness* is consistently violated and, for half of them, most of the violations observed are consistent with *Convexity*.²

¹See Gul and Pesendorfer [23], Dekel, Lipman Rustichini [10], Epstein, Marinacci, and Seo [14].

²For a short review of these facts, see Chapter 8 of Camerer in [24].

We study convex preferences under risk and propose a characterization for such preferences, with the minimum extra structure required. We do this for two reasons. First, Convexity is a central feature of the theory of choice under risk, but it has never been studied in its full generality. Second, Convexity is a less stringent requirement than Betweenness and it is more consistent with the experimental evidence. We study three main interpretations for Convexity as a hedging property for uncertainty about: the value of outcomes, future tastes, and the degree of risk aversion. Although, these are not mutually exclusive interpretations, each of them is better understood in a suitable and distinct environment. As we argue in the rest of the Introduction, the first and the second type of uncertainty arise naturally in problems of choice over, respectively, lotteries and menus of lotteries. The third interpretation arises in a setting where preferences are incomplete and Convexity cautiously completes them.

1.1 Convex Preferences over Lotteries

We assume that the DM's preferences are represented by a complete and transitive binary relation \succsim . We adopt a classical definition of Convexity. That is, we assume that if a lottery p is deemed equivalent to a lottery q then their mixture, $\lambda p + (1 - \lambda)q$, is at least as good as any of the two.^{3,4}

In the context of choice under ambiguity of Anscombe and Aumann [3], Convexity takes the usual interpretation of Uncertainty Aversion, or equivalently, of preference toward diversification. Here, p and q are simple random variables thus the mixture operation reduces the variability in the outcomes received.⁵ Therefore, Convexity imposes that a DM prefers prospects with less uncertainty.

On the other hand, in the context of choice under risk, a (strict) preference toward randomization seems to move toward the opposite direction. The DM appears to prefer more uncertain prospects. At first sight, this appears to be in contrast with much of the (applied) literature where it is typically assumed that the DM has von Neumann and Morgenstern EU preferences that further are risk averse. Intuitively, such a DM prefers less uncertain lotteries. Thus, Convexity might seem a counterintuitive assumption. In the next few lines, we argue this is not the case.

To fix ideas, consider a DM with preferences \succsim over the set of lotteries $\Delta(C)$ where C is a finite set of consequences. Moreover, assume that the DM has EU preferences, that is,

$$p \succsim q \Leftrightarrow \sum_{x \in C} v(x)p(x) \geq \sum_{x \in C} v(x)q(x)$$

where $v \in \mathbb{R}^C$. Here, v is uniquely determined (up to an affine transformation). Thus, it follows that the DM is sure about the relative value of consequences. Similarly, if $C \subseteq \mathbb{R}$, he is sure about his tastes, for example, his degree of risk aversion. The first aspect is captured by the fact that v represents the preferences over C , while the second is captured by the curvature of v .

Nevertheless, it is not hard to imagine situations where the consequences of different actions (deterministic lotteries) are of unsure value or situations where there is uncertainty about risk aversion. Here, we discuss the first case while the discussion of the second is postponed to Subsection 1.3.

³It is not hard to show that under a minimal assumption of continuity this property is equivalent to assuming that if p is weakly preferred to q then $\lambda p + (1 - \lambda)q$ is weakly preferred to q . This is a weaker version of the assumption of Convexity that the aforementioned experimental and theoretical works were referring to. There, weakly was replaced with strictly. For this reason, patterns of choice consistent with the stronger form of Convexity are consistent with our notion of Convexity.

⁴Concavity implies that the mixture, $\lambda p + (1 - \lambda)q$, is at most as good as any of the two. As for Convexity, the version of Concavity tested involved strict preferences. Betweenness is the assumption that implies that preferences satisfy both Convexity and Concavity.

⁵See Debreu [8, p. 101] and Schmeidler [34].

We start with a somewhat artificial but clear example. Consider a situation in which the DM can choose between two degenerate lotteries that pay, respectively, x and y . The DM is uncertain about his evaluation of consequences. Suppose that x is indifferent to y because, for instance, the evaluation of x is high when the one of y is low and vice versa. If the DM is cautious or pessimistic then he is more prone to consider the negative aspects of his choices. In pondering a deterministic choice between x and y , he might overvalue the possibility of making the wrong choice, that is, of selecting the element with lowest value. In other words, such a DM, being unsure of the value of outcomes, always gives more weight to the worst of all possible evaluations. For this reason, it is then sensible for him to opt for the lottery $\frac{1}{2}\delta_x + \frac{1}{2}\delta_y$. For sure, by flipping a coin, the DM exposes himself to the possibility of getting the ex post worst outcome. But, this occurrence was already something that subjectively he perceived was going to happen, if he opted for a deterministic choice. By randomizing instead, he leaves himself open to the possibility of getting the ex post best outcome.

The essence of this example is the uncertainty faced by the DM about his (future) evaluation of consequences and the preference toward randomization or diversification to cope with such uncertainty. This is a pervasive economic fact. For instance, choosing a particular environmental policy might lead to probabilistically very clear results but the economic value of such results can be unclear. Similarly, present choices over future consumption must deal with possible uncertainty that the DM faces today about his future preferences. Again, in both cases, the DM is unsure about his ranking over consequences as well as he is unsure about his future tastes.

More precisely, we provide a representation result for convex preferences \succsim on the set, $\Delta(C)$, of simple lotteries over a generic set C of consequences. Studies about preferences on simple lotteries are widely common in the literature (see, e.g., Dekel [9], Gilboa [19], Gul [22], and Maccheroni [27]). We adopt minimal assumptions over \succsim : completeness, transitivity, Convexity, and some form of continuity.⁶ These assumptions imply the existence of a utility function u and we obtain the following representation of \succsim : there exist a set of normalized Bernoulli utility functions \mathcal{V} and a function $U : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$ such that

$$u(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C). \quad (2)$$

Our DM values a lottery p as if he is unsure about the relative value of outcomes or as if he has multiple selves. The multiplicity of selves is captured by a normalized, closed, and convex family, \mathcal{V} , of different evaluation functions v of the outcomes.⁷ Given the family of evaluation functions, \mathcal{V} , the DM can compute for each v the expected utility of the lottery p . That is, the DM can compute the value $\mathbb{E}_v(p) = \sum_{x \in C} v(x)p(x)$ for all $v \in \mathcal{V}$. Since U is increasing in the first component, $\{U(\mathbb{E}_v(p), v)\}_{v \in \mathcal{V}}$ is a family of distorted EU evaluations. Of all possible distorted EU evaluations, the “worst” one is the final value that the DM attaches to p . This reflects a sort of pessimism in the DM behaviour, or, in the words of Maccheroni [27], it implies that “the most pessimist of [his] selves gets the upper hand over the others”.

Two objects are prominent in the decision criterion of (2): U and \mathcal{V} . For this reason, we study those two objects more in depth. First, U has the important feature of being essentially unique and it satisfies several properties of regularity, quasiconvexity, and continuity. Second, in the particular case of monetary lotteries, it can be shown that the function U is an index of risk aversion. Finally, in order to better interpret \mathcal{V} and U , we extend our analysis to preferences over menus of lotteries. In this context and in light of the previous literature we argue that \mathcal{V} can be interpreted as a Subjective

⁶The form of continuity assumed will depend on the properties of the set C .

⁷We fix an arbitrary consequence $x \in C$ and we define $\mathcal{V} = \mathcal{V}_1(x) = \{v \in \mathbb{R}^C : v(x) = 1\}$.

State Space. \mathcal{V} represents the DM's future tastes and $U(\mathbb{E}_v(p), v)$ can be seen as the state dependent utility of lottery p .

Our characterization generalizes the findings of Maccheroni [27] where (2) becomes

$$u(p) = \min_{v \in \mathcal{W}} \mathbb{E}_v(p) \quad \forall p \in \Delta(C),$$

\mathcal{W} is a closed and convex subset of \mathcal{V} . In terms of our representation, U can be chosen to be such that

$$U(t, v) = \begin{cases} t & v \in \mathcal{W} \\ \infty & v \notin \mathcal{W} \end{cases}.$$

Our main contributions (Theorems 1, 9, 13, and 20) show that under the assumption of Convexity *any model* of choice under risk, representing complete, transitive, and continuous preferences, can be viewed as adhering to the same kind of pessimistic decision rule found in [27].

1.2 Convex Preferences over Menus

One of our main motivations in introducing Convexity was the indecisiveness of the DM about his future tastes. The latter has been extensively studied by the literature of choice over menus, particularly, by the literature of choice over menus of lotteries, where a menu is defined to be a (closed) subset of lotteries. The seminal works of Dekel, Lipman, and Rustichini [10] and Gul and Pesendorfer [23] exactly analyzed preferences over menus of lotteries.⁸ In such works, following the standard interpretation of the literature, there are two periods: time 0 and time 1. The DM has ex ante preferences, \succsim , over menus. Then, he chooses a menu P at time 0, while consumption is chosen at time 1 from P .

To gain intuition, let us start by considering a DM who is sure about his preferences over lotteries at time 1. Therefore, he can be represented by an EU function. Then, he should rank menus according to the criterion

$$P \succsim Q \Leftrightarrow \max_{p \in P} \mathbb{E}_v(p) \geq \max_{p \in Q} \mathbb{E}_v(p). \quad (3)$$

Dekel, Lipman, and Rustichini [10] and Kreps [25], argue that the lack of confidence about tastes at time 1 is what renders the previous criterion unfeasible for the DM. They address this concern by exploring axiomatically decision criteria for a problem of choice over menus. In terms of assumptions, the criterion of choice represented in (3) imposes that preferences other than being complete, transitive, and suitably continuous, satisfy the assumptions of Independence, Flexibility, and (Strategic) Rationality. Independence for menus is an obvious generalization of the well known Independence for lotteries, that is,

$$P \succsim Q \Rightarrow \lambda P + (1 - \lambda) R \succsim \lambda Q + (1 - \lambda) R$$

where $\lambda \in [0, 1]$ and R is a third menu.⁹ Flexibility states that if menu P contains menu Q then $P \succsim Q$ while Rationality requires that if $P \succsim Q$ then $P \sim P \cup Q$. Moreover, those assumptions are known to be necessary and sufficient for the decision criterion represented in (3).¹⁰ In [10] and [25], the assumption that comes under attack is the one of Rationality. This is the assumption that forces the DM to rank menus in terms of the best element contained in each of them. Dekel, Lipman, and

⁸See also the seminal work in the literature of choice over generic menus: Kreps [25].

⁹The set $\lambda P + (1 - \lambda) R$ contains all the lotteries q such that $q = \lambda p + (1 - \lambda) r$ for some p in P and r in R . We refer the reader to [10] and [23] for an interpretation of the Axiom of Independence in this context.

¹⁰See, for example, [23, pag. 1408].

Rustichini, in their most central model, drop Rationality and show that the DM's preferences can be represented in the following way

$$P \succsim Q \Leftrightarrow \int_{\mathcal{W}} \max_{p \in P} \mathbb{E}_v(p) d\mu(v) \geq \int_{\mathcal{W}} \max_{p \in Q} \mathbb{E}_v(p) d\mu(v) \quad (4)$$

where \mathcal{W} is a set of Bernoulli utility functions and μ is a probability over \mathcal{W} . The lack of confidence about future tastes translates into a family of (rational) evaluations $\{\max_{p \in P} \mathbb{E}_v(p)\}_{v \in \mathcal{W}}$. Such evaluations are then aggregated through a subjective average.

Our work starts from the observation that a cautious DM might deem the average a too optimistic criterion to aggregate all these independent evaluations and he might want to declare

$$P \succsim Q \Leftrightarrow \inf_{v \in \mathcal{W}} \max_{p \in P} \mathbb{E}_v(p) \geq \inf_{v \in \mathcal{W}} \max_{p \in Q} \mathbb{E}_v(p). \quad (5)$$

Proposition 4, Proposition 5, and Corollary 41 show that, when we retain Rationality but we weaken Independence to Convexity of preferences, \succsim admits a representation in the spirit of (5).

More precisely, we consider a binary relation \succsim over the class of nonempty and closed subsets (menus) of lotteries over a finite set of consequences \mathcal{C} . We assume that \succsim represents the DM's preferences.¹¹ We provide a representation result for convex preferences \succsim that further satisfy standard assumptions of continuity as well as Flexibility and Rationality. Theorem 3 shows that a binary relation \succsim satisfies the previous assumptions if and only if there exists an essentially unique function $U : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$ such that

$$P \succsim Q \Leftrightarrow \max_{p \in P} \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \geq \max_{p \in Q} \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad (6)$$

where \mathcal{V} is a family of normalized Bernoulli utility functions. Following the current literature, the set of ex post preferences, \mathcal{V} , takes the interpretation of a Subjective State Space (see Subsection 3.2, for a more detailed discussion). Proposition 4 and Proposition 5 show that whenever we consider preferences \succsim restricted or defined only on convex menus we obtain that

$$P \succsim Q \Leftrightarrow \inf_{v \in \mathcal{V}} \max_{p \in P} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{V}} \max_{p \in Q} U(\mathbb{E}_v(p), v), \quad (7)$$

or equivalently,

$$P \succsim Q \Leftrightarrow \inf_{v \in \mathcal{V}} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right) \geq \inf_{v \in \mathcal{V}} U\left(\max_{p \in Q} \mathbb{E}_v(p), v\right). \quad (8)$$

Proposition 5 is both economically and mathematically nontrivial. Economically, by looking just at convex menus, we allow for the possibility that the DM can costlessly randomize without imposing, a priori, any further behavioural assumption on the value of randomization. Mathematically, finite menus play an essential role in the construction of the utility function for \succsim (see [23]). By restricting \succsim to convex menus, the only menus that are convex and finite are the ones with one element. This makes the construction of the utility function significantly harder.¹²

Our main contribution is to show that uncertainty about future tastes can be introduced by retaining Rationality but by weakening Independence. This is at odds with the reviewed literature where weakening Rationality is the main road chosen to introduce uncertainty about future tastes. On

¹¹This is basically the setting of Dekel, Lipman, Rustichini [10]. Indeed, they allow for non closed menus as well. Nevertheless, their hypothesis of continuity renders them virtually irrelevant for the derivation of their results.

¹²On the other hand, for example Dekel, Lipman, and Rustichini [10] or Epstein, Marinacci, and Seo [14], construct the utility function for \succsim over convex menus and then they extend it to generic menus. Their techniques are neither applicable nor natural to our setting. They are not applicable because our state space, \mathcal{V} , is not compact. They are not natural because they deliver a representation in terms of probabilities over the Subjective State Space.

the other hand, we conform to the traditional and dominating theme of Decision Theory. We weaken Independence (to Convexity) of preferences over lotteries. This leads to a foundation of the Maxmin and Minmax decision criteria over a Subjective State Space and it allows to introduce uncertainty about future tastes.¹³

1.3 Convex Preferences as a Criterion of Completion

The DM's lack of confidence about his future tastes, in particular, about risk aversion was an important motivation for adopting the assumption of Convexity. For example, suppose that a DM has preferences over monetary lotteries and that he evaluates such lotteries through their Certainty Equivalent. He is sure he likes more money to less and, particularly, he does not want to violate First Order Stochastic Dominance. However, he is unsure about his degree of risk aversion. We might therefore assume that he has a family \mathcal{W} of possible candidates for his Bernoulli utility function.¹⁴ Given such lack of confidence, the DM has a preliminary ranking on monetary lotteries. Indeed, he is certain that a lottery p is better than a lottery q whenever the certainty equivalent of p is bigger than the certainty equivalent of q for each element in \mathcal{W} . We call such preliminary ranking \succsim' and we have that

$$p \succsim' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \Leftrightarrow v^{-1}(\mathbb{E}_v(p)) \geq v^{-1}(\mathbb{E}_v(q)) \quad \forall v \in \mathcal{W}. \quad (9)$$

Even though, the DM is certain about the ranking expressed by the criterion represented in (9), \succsim' might not be useful to make a decision. Indeed, \succsim' is highly incomplete and not all the prospects can be ranked. If the DM is then pessimist or cautious, he might want to use a complete ranking \succsim where p is declared at least as good as q if and only if the certainty equivalent of p in the worst case scenario is bigger than the certainty equivalent of q in the worst case scenario. This is equivalent to saying that

$$p \succsim q \Leftrightarrow \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_v(p)) \geq \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_v(q)). \quad (10)$$

Notice that the ranking \succsim is complete and it preserves \succsim' , that is, if $p \succsim' q$ then $p \succsim q$. Hence, \succsim is a completion of \succsim' . Moreover, \succsim satisfies Convexity. In particular, we can represent \succsim in terms of our representation in (2). It can be shown that

$$p \succsim q \Leftrightarrow \min_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \geq \min_{v \in \mathcal{V}} U(\mathbb{E}_v(q), v) \quad (11)$$

where U is such that

$$U(t, v) = \begin{cases} v^{-1}(t) & v \in \mathcal{W} \\ \infty & \text{otherwise} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}.$$

This example clarifies the meaning of the distortion function U in our representation result and the role of Convexity as a criterion of completion. The function U has three roles. First, U transforms the family of expected utility evaluations, $\{\mathbb{E}_v(p)\}_{v \in \mathcal{W}}$, into the same units of account, for example, dollars. Second, it makes certain Bernoulli utility functions more plausible than others. For example, if $v \notin \mathcal{W}$ then the evaluation under v of p is $U(\mathbb{E}_v(p), v) = \infty$ and it will never be considered in the computation of the infimum. Finally, by taking these distorted evaluations through U and opting for

¹³See also Epstein, Marinacci, and Seo [14] for an axiomatization toward weakening Independence. In their first model, which is the only one sharing our setting, they weaken Independence just to *Convexity over menus* but they still dispense with the assumption of Rationality. On the other hand, they maintain other assumptions that do not make our and their model easily comparable.

¹⁴For technical reasons, we assume that \mathcal{W} is a compact and convex set such that each element in v is a strictly increasing function, $v(1) = 1$, and $v(\mathbb{R}) = \mathbb{R}$. See Example 17 for technical details.

a “worst” case scenario approach, the DM forms a complete and convex ranking that preserves the initial ranking \succsim' . Notice that for each p

$$\min_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) = \min_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) \quad (12)$$

that is, in completing cautiously his preferences the DM just considers the Bernoulli utility functions that he deemed plausible since the beginning: the ones in \mathcal{W} .

More precisely, given a binary relation \succsim' as in (9), Proposition 7 shows that all preferences, \succsim , represented as in (11) and (12) and arising as a completion of \succsim' , are convex and preserve \succsim' . Proposition 8 and Proposition 22 show the opposite implication. That is, they show that each convex binary relation \succsim can be seen as a completion of an incomplete maximal binary relation \succsim' represented as in (9).

Our main contribution is to show how the decision criterion in (2), therefore Convexity, provides a cautious way to complete preferences \succsim' represented as in (9). More importantly, we are able to show that all continuous and convex preferences over lotteries can be seen as arising through a process of cautious completion.

1.4 Final Remarks and Organization

As known, the model of Maccheroni [27] can be seen as the dual counterpart of the model for preferences under ambiguity of Gilboa and Schmeidler [21]. Similarly, our model can be seen as the dual counterpart of the model proposed by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [6]. In the same vein, convex preferences as a criterion of completion share the same perspective of Gilboa, Maccheroni, Marinacci, and Schmeidler [20]. Nevertheless, the economic setting and the mathematical structure of these other contributions is radically different.

From a mathematical point of view, the representation result in (2) is based on the dual representation theory for evenly quasiconcave functions on locally convex topological vector spaces. The essential uniqueness of the function U is very much in line with the findings of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [7] and it is derived by extensively using some of the techniques adopted in [7]. However, there are elements of novelty that make the results in this work nontrivial. The results in [7] cannot be directly applied to our setting. Indeed, [7] develops a complete duality theory for monotone and quasiconcave functions defined over an M -space. Here, we miss two key assumptions that deliver the complete dualities discussed in [7]: monotonicity and the M -space domain. For similar reasons, we cannot apply the (complete) duality results developed by Diewert [11] and Martinez-Legaz [28].

Section 2 introduces the notation and discusses the mathematical preliminaries. Section 3 states all the main results for the case in which the set of consequences C is finite. Subsection 3.2 and Subsubsection 3.2.1 report our results on convex preferences over menus of lotteries. Section 4 extends most of the results contained in Section 3 and discuss some examples. The analysis is carried over, first, for countable C and then for arbitrary C . Finally, we consider the case of C compact metric space and preferences over the set of *all Borel probability measures* $\Delta_{\mathcal{B}}(C)$. Subsubsection 4.1.1 analyzes the impact of First Order Stochastic Dominance on the main representation result when C is countable. Subsubsection 4.2.1 shows that U , under minimal assumptions, can be interpreted as an index of risk aversion.

The proofs are relegated to the Appendices. Appendix A provides the main duality results. Appendix B contains the proofs for Section 3 and Section 4. Appendix B.1 contains the proofs for Subsection 3.2. Appendix B.2 contains the proofs for Subsubsection 3.2.1.

2 Notation and Mathematical Preliminaries

The object of our study is a binary relation, \succsim , on the particular mixture space $\Delta(C)$.¹⁵ $\Delta(C)$ is the set of all simple lotteries over a *generic* set C . We assume that \succsim represents the DM's preferences. We call p, q, r, s, t elements of $\Delta(C)$. We call x, y, z elements of C . Given an element $x \in C$, we call δ_x the lottery that with probability 1 delivers x . Notice that whenever we consider a binary relation, \succsim , over $\Delta(C)$ we can restrict \succsim to C since this set is embedded into $\Delta(C)$ by the map such that $x \mapsto \delta_x$. In light of this observation, we can justify the following abuses of notation: we often identify δ_x with x and we write $x \succsim y$ instead of $\delta_x \succsim \delta_y$. A function $u : \Delta(C) \rightarrow \mathbb{R}$ is said to represent \succsim or to be a utility function for \succsim if and only if for each $p, q \in \Delta(C)$

$$p \succsim q \iff u(p) \geq u(q).$$

If C is equal to \mathbb{R} then we say that $u : \Delta(C) \rightarrow \mathbb{R}$ is a certainty equivalent utility function if and only if u is a utility function and $u(x) = x$ for all $x \in C$. A function $u : \Delta(C) \rightarrow \mathbb{R}$ is said to be mixture continuous if and only if for each $p, q \in \Delta(C)$ and for each $\alpha \in \mathbb{R}$ the sets $\{\lambda \in [0, 1] : u(\lambda p + (1 - \lambda)q) \geq \alpha\}$ and $\{\lambda \in [0, 1] : \alpha \geq u(\lambda p + (1 - \lambda)q)\}$ are closed in $[0, 1]$.

2.1 Duality Toolbox

For our representation result, we need to discuss different forms of continuity. This requires a topology on the set $\Delta(C)$. We consider the same setting of Maccheroni [27]. For this purpose, notice that each element p of $\Delta(C)$ is an element of \mathbb{R}_0^C . That is, p is a function from C to \mathbb{R} such that the set $\text{supp}\{p\} = \{x \in C : p(x) \neq 0\}$ is finite. We call elements of \mathbb{R}_0^C : p, q, r, s, t . A generic element of \mathbb{R}^C is called v .

Given such observation, we consider the duality $\langle \mathbb{R}_0^C, \mathbb{R}^C \rangle$ where the evaluation duality, $\langle \cdot, \cdot \rangle : \mathbb{R}_0^C \times \mathbb{R}^C \rightarrow \mathbb{R}$, is defined by

$$\langle p, v \rangle = \sum_{x \in C} v(x) p(x).$$

We endow \mathbb{R}_0^C with the weak topology induced by the evaluation duality and we endow \mathbb{R}^C with the weak* topology. A net $\{p_\alpha\}_{\alpha \in A} \subset \mathbb{R}_0^C$ is said to converge to p ($p_\alpha \rightarrow p$) if and only if $\langle p_\alpha, v \rangle \rightarrow \langle p, v \rangle$ for all $v \in \mathbb{R}^C$. Similarly, a net $\{v_\alpha\}_{\alpha \in A} \subset \mathbb{R}^C$ is said to converge to v ($v_\alpha \rightarrow v$) if and only if $\langle p, v_\alpha \rangle \rightarrow \langle p, v \rangle$ for all $p \in \mathbb{R}_0^C$. It is well known that both topologies are linear, Hausdorff, and locally convex (see [1, Chapter 5]). The second topology is the topology of pointwise convergence and the topological dual of \mathbb{R}_0^C is the algebraic dual. Both topologies coincide with the usual Euclidean topology as soon as C is finite.¹⁶ When we consider the image of a pair $(p, v) \in \Delta(C) \times \mathbb{R}^C$ under the evaluation duality $\langle \cdot, \cdot \rangle$ we write, equivalently, $\mathbb{E}_v(p)$ in place of $\langle p, v \rangle$. We consider $\Delta(C)$ endowed with the relative topology.

We assume the convention that the supremum of the empty set is equal to $-\infty$. We fix a *generic* element $x \in C$. We define $\mathcal{V} = \mathcal{V}_1(x) = \{v \in \mathbb{R}^C : v(x) = 1\}$. We say that a subset D of \mathbb{R}_0^C is *evenly*

¹⁵We denote \succ and \sim , respectively, the asymmetric and the symmetric parts of \succsim .

¹⁶Indeed, if we call $|C|$ the cardinality of C then both \mathbb{R}_0^C and \mathbb{R}^C can be identified with $\mathbb{R}^{|C|}$ where the latter is the set of vectors with $|C|$ components.

convex if and only if for each $\bar{p} \notin D$ there exists $v \in \mathbb{R}^C$ such that $\langle \bar{p}, v \rangle < \langle p, v \rangle$ for all $p \in D$.¹⁷ Set $\mathbb{R}^\diamond = \mathbb{R} \setminus \{0\}$. We say that a subset D of $\mathbb{R} \times \mathcal{V}$ is \diamond -evenly convex if and only if for each $(\bar{t}, \bar{v}) \notin D$ there exists $(s, p) \in \mathbb{R}^\diamond \times \mathbb{R}_0^C$ such that $\langle p, \bar{v} \rangle + \bar{t}s < \langle p, v \rangle + ts$ for all $(t, v) \in D$. In the sequel, we deal extensively with functions $U : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$. Given U , we define $u_U : \mathbb{R}_0^C \rightarrow [-\infty, \infty]$ by $p \mapsto \inf_{v \in \mathcal{V}} U(\langle p, v \rangle, v)$. We say that U is linearly continuous (resp., linearly mixture continuous) if and only if the function u_U is a real valued and continuous function on $\Delta(C)$ (resp., real valued and mixture continuous). We say that U is \diamond -evenly quasiconvex if and only if all its lower contour sets are \diamond -evenly convex.¹⁸ Finally, given U , we define U^+ by

$$U^+(t, v) = \inf \{U(t', v) : t' > t\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}.$$

A function from $\mathbb{R} \times \mathcal{V}$ to $[-\infty, \infty]$ might have some of the following properties:

P.1 For each $v \in \mathcal{V}$ the function $U(\cdot, v) : \mathbb{R} \rightarrow [-\infty, \infty]$ is increasing.

P.2 $\lim_{t \rightarrow \infty} U(t, v) = \lim_{t \rightarrow \infty} U(t, v')$ for all $v, v' \in \mathcal{V}$.

P.3 U is \diamond -evenly quasiconvex.

P.4 U is linearly continuous.

P.5 U is linearly mixture continuous.

P.6 U is such that $u_U = u_{U^+}$ on $\Delta(C)$.

P.7 If $C = \mathbb{R}$, $\inf_{v \in \mathcal{V}} U(v(y), v) = y$ for all $y \in \mathbb{R}$.

We define $\mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ to be the class of functions from $\mathbb{R} \times \mathcal{V}$ to $[-\infty, \infty]$ that satisfy P.1-P.4 and $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ to be the class of functions from $\mathbb{R} \times \mathcal{V}$ to $[-\infty, \infty]$ that satisfy P.1-P.3 and P.5-P.6. Similarly, if $C = \mathbb{R}$ we define $\mathcal{U}_n^c(\mathbb{R} \times \mathcal{V})$ (resp., $\mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V})$). U belongs to $\mathcal{U}_n^c(\mathbb{R} \times \mathcal{V})$ (resp., $\mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V})$) if and only if $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ and it satisfies P.7 (resp., $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ and it satisfies P.7). Finally, consider one of these four classes of functions and call it \mathcal{U} . Consider a function $u : \Delta(C) \rightarrow \mathbb{R}$ such that

$$u(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C) \tag{13}$$

for some $U \in \mathcal{U}$. U is said to be essentially unique (in \mathcal{U}) if and only if given another function U' in \mathcal{U}

$$u_U = u_{U'} \Rightarrow U = U'.$$

That is, whenever U and U' induce the same function on the entire set \mathbb{R}_0^C , not just on $\Delta(C)$, they happen to coincide.

¹⁷It can be shown that an evenly convex set D is convex. Indeed, a set D is evenly convex if and only if it is the intersection of half open spaces. By usual separation arguments, closed and open convex sets of \mathbb{R}_0^C are evenly convex.

¹⁸Sufficient conditions for \diamond -even quasiconvexity are provided in Lemma 38 in Appendix B.

3 Convex Preferences over Lotteries

In this section we discuss the main results and contributions under the assumption that the set of consequences C is finite.

3.1 The Representation Result

We consider a binary relation \succsim . We assume that \succsim represents the preferences of the DM over the set of simple lotteries over a *generic* finite set C of consequences. Notice that, since C is assumed to be finite, $\Delta(C)$ can be identified with the usual simplex in a finite dimensional vector space. We require \succsim to satisfy the following two assumptions:

Axiom A. 1 (Weak Order) *The binary relation \succsim is complete and transitive.*

Axiom A. 2 (Mixture Continuity) *If $p, q, r \in \Delta(C)$ then $\{\lambda \in [0, 1] : \lambda p + (1 - \lambda) q \succsim r\}$ and $\{\lambda \in [0, 1] : r \succsim \lambda p + (1 - \lambda) q\}$ are closed sets.*

Weak Order is a common assumption of rationality. Moreover, since we are after a (utility) representation result, Weak Order is a necessary assumption. Mixture Continuity is a technical assumption, needed to represent \succsim through a utility function u . Next, we discuss the main axiom for our result.

Axiom A. 3 (Mixing) *For each $p, q \in \Delta(C)$, $p \sim q$ implies that*

$$\lambda p + (1 - \lambda) q \succsim p \quad \forall \lambda \in [0, 1].$$

Mathematically, it is not hard to see that, given A.1 and A.2, \succsim satisfies A.3 if and only if \succsim is convex. That is, for each $q \in \Delta(C)$ the set $\{p \in \Delta(C) : p \succsim q\}$ is a convex set in $\Delta(C)$. Economically, the Axiom of Mixing is an axiom of smoothening or cautiousness as discussed in the Introduction.

We are ready to state our main representation result.

Theorem 1 *Let $\Delta(C)$ be the space of simple lotteries over a finite set C and \succsim a binary relation on $\Delta(C)$. The following are equivalent facts:*

(i) \succsim satisfies A.1, A.2, and A.3;

(ii) there exists an essentially unique $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ such that

$$p \succsim q \Leftrightarrow \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(q), v). \quad (14)$$

Moreover, if $u : \Delta(C) \rightarrow \mathbb{R}$ is a mixture continuous utility function for \succsim then the function $U^* : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ and represents \succsim as in (14).

Remark 2 Recall that we fixed from the beginning a generic consequence $x \in C$ and we defined $\mathcal{V} = \mathcal{V}_1(x) = \{v \in \mathbb{R}^C : v(x) = 1\}$. Then, the statement contained in (ii) could be restated to be “for each $x \in C$ there exists an essentially unique function $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x)) \dots$ ”. Hence, the role of the consequence x is uniquely of normalization confirming that x can be chosen arbitrarily. The same Remark applies to all the other results in the paper with the exception of Propositions 11, 15, and 19. In the first two cases, the choice of x could be arbitrary but we opted for specific and natural values. In the last case, the choice of x is not free but forced by the premises.

It is worth observing that Theorem 1 is derived in the same exact setting of von Neumann and Morgenstern result on Expected Utility (see, e.g., [26, Chapter 5] or [29, Chapter 6]). It is somewhat surprising that the weakening of Independence to Mixing\Convexity translates into a multiplicity of distorted expected utility evaluations where for each lottery p the “worst” one constitutes the final value attached to p . In particular, since $U(\cdot, v)$ is increasing for each $v \in \mathcal{V}$, we can see that the ranking induced by $p \mapsto U(\mathbb{E}_v(p), v)$ does not revert the expected utility ordering induced by $p \mapsto \mathbb{E}_v(p)$.

3.2 Maxmin Criterion over a Subjective State Space

In this subsection, we extend preferences, \succsim , to be over menus of lotteries. We consider two classes of menus. More precisely, given C , we define

$$\mathcal{M} = \{P \subseteq \Delta(C) : \emptyset \neq P \text{ is closed}\} \text{ and } \mathcal{C} = \{P \subseteq \Delta(C) : \emptyset \neq P \text{ is closed and convex}\}.$$

Clearly, we have $\mathcal{C} \subseteq \mathcal{M}$. We call P, Q, R elements of \mathcal{M} . We refer to them as menus and we refer to them as convex menus if they further belong to \mathcal{C} . Given a menu $P \in \mathcal{M}$, we denote by $co(P)$ the convex hull of P . Clearly, we have that $co(P) \in \mathcal{M}$. Since C is finite, we identify \mathbb{R}_0^C and \mathbb{R}^C with $\mathbb{R}^{|C|}$. The weak and weak* topology induced by the evaluation duality are the Euclidean topology. We endow \mathcal{M} and \mathcal{C} with the Hausdorff Metric and the Hausdorff Metric Topology. This makes \mathcal{M} and \mathcal{C} Polish spaces. Notice that we can identify an element p of $\Delta(C)$ with the element $\{p\}$ of \mathcal{M} or \mathcal{C} . With a small abuse of notation, we use $\{p\}$ and p indifferently and we treat $\Delta(C)$ as a closed subclass of elements of \mathcal{M} or \mathcal{C} . In this subsection, object of our study is a binary relation \succsim on \mathcal{M} while in the next subsection it is a binary relation \succsim over the smaller class \mathcal{C} . Call \mathcal{P} one of these classes of sets, we assume in this section and the next one that \succsim satisfies the following axioms:

Axiom B. 1 (Weak Order) *The binary relation \succsim is complete and transitive.*

Axiom B. 2 (Upper Semicontinuity) *For each $Q \in \mathcal{P}$ the set $\{P \in \mathcal{P} : P \succsim Q\}$ is a closed set.*

Axiom B. 3 (Lower Singleton Semicontinuity) *For each $q \in \Delta(C)$ the set $\{p \in \Delta(C) : q \succsim p\}$ is a closed set.*

Axiom B. 4 (Flexibility) *If $P \supseteq Q$ then $P \succsim Q$.*

Axiom B. 5 (Rationality) *If $P \succsim Q$ and $P \cup Q \in \mathcal{P}$ then $P \sim P \cup Q$.¹⁹*

¹⁹Notice that if $\mathcal{P} = \mathcal{M}$ then the requirement $P \cup Q \in \mathcal{M}$ in B.5 is redundant and it can be withdrawn.

Before discussing the main axioms, we need to discuss the setting in which \succsim is considered. We adopt the standard interpretation in the literature of choice over menus of lotteries (see, e.g., [10], [14], and [23]). \succsim is interpreted as ex ante preferences over menus at time 0. Whenever a menu is selected, an object from it will be chosen for consumption at time 1. In this setting, it is then possible to discuss the possibility of uncertainty about ex post preferences, about future tastes. It is feasible to introduce a Subjective State Space that sums up the possible different states of the world through ex post preferences. Indeed, many events and states of the world might realize between time 0, the time when a menu P is chosen, and time 1, the time when consumption is chosen from P . Many of these events might not even be conceivable by the DM at time 0. In the most extreme case, he might just know that something will happen between time 0 and time 1. Those events are not irrelevant since they might have an effect on his choice at time 1. Nevertheless, it will not be important what these events are but what ex post preferences they induce, in other words, how these events will make feel the DM at time 1 and therefore how they will influence his choice at time 1. For this reason, the set of ex post preferences constitutes a natural summary of all the possible states of the world that might realize and therefore a parsimonious Subjective State Space.²⁰

We next discuss the axioms for the setting $\mathcal{P} = \mathcal{M}$ since they can be better related to the literature. Their interpretation stays unchanged for the case $\mathcal{P} = \mathcal{C}$. That said, since we are after a (utility) representation result, Weak Order is a necessary assumption. As usual, we have some technical assumptions of continuity: Upper Semicontinuity and Lower Singleton Semicontinuity. These two assumptions are continuity assumptions that are standard in the theory of choice over menus (see, e.g., [10, pag. 904] or [23, pag. 1412]). Flexibility is an hypothesis that exactly captures the fact that the DM might be unsure about his ex post preferences. It imposes that the DM prefers bigger menus to smaller ones, since in this way the possibilities of consumption at time 1 are wider and so they can accommodate more easily different future preferences. Finally, (Strategic) Rationality is an axiom that was first weakened by Kreps [25].²¹ This assumption apparently seems to be in contrast with the possibility that the DM considers the fact that he is unsure about his future tastes. Indeed, given the other axioms, it is not hard to show that the extra assumption of Rationality implies that $P \succsim Q$ if and only if P and Q are evaluated according to their best element, with respect to the restriction of \succsim to $\Delta(C)$. A tension between Rationality and the possibility of different ex post preferences over lotteries seems to arise by the previous equivalence, since it appears that there is no room for uncertainty about the ranking of lotteries. However, this tension is only apparent. Indeed, it might be that \succsim restricted to lotteries already incorporates the uncertainty on future tastes. For example, when \succsim restricted to $\Delta(C)$ is represented by a criterion like the one in (14).

Before discussing the main representation result, we introduce the two main axioms of this subsection:

Axiom B. 6 (Mixing) *For each $p, q \in \Delta(C)$, $p \sim q$ implies that $\lambda p + (1 - \lambda) q \succsim p$ for all $\lambda \in [0, 1]$.*

Axiom B. 7 (Menu Mixing) *For each $P, Q \in \mathcal{P}$, $P \sim Q$ implies that $\lambda P + (1 - \lambda) Q \succsim P$ for all $\lambda \in [0, 1]$.*²²

²⁰For a similar discussion see also [10] and [14].

²¹See [26, pag. 184] for a textbook exposure.

²²Given $P, Q \in \mathcal{P}$ and $\lambda \in (0, 1)$,

$$\lambda P + (1 - \lambda) Q = \{r \in \Delta(C) : \exists p \in P, \exists q \in Q \text{ such that } r = \lambda p + (1 - \lambda) q\}.$$

If $\lambda = 0$ then $\lambda P + (1 - \lambda) Q = Q$ and if $\lambda = 1$ then $\lambda P + (1 - \lambda) Q = P$. It is immediate to see that, given our setting, $\lambda P + (1 - \lambda) Q \in \mathcal{P}$.

Obviously, Menu Mixing implies Mixing. We will consider these two assumptions separately. The appeal of B.6 is the same as the one of previous subsection and potentially, in this setting, might be reinforced since there could be explicit uncertainty about future tastes or ex post preferences. The interpretation of B.7 instead is of a preference toward diversification of menus. Epstein, Marinacci, and Seo, [14] justify B.7 exactly along these lines and we refer the interest reader to their paper. Mathematically, B.7 is a weakening of the usual Independence assumption that can be found in the literature of preferences over menus of lotteries (see, e.g., [10] and [23]).

Theorem 3 *Let C be a finite set and let \succsim be a binary relation on \mathcal{M} . The following are equivalent facts:*

- (i) \succsim satisfies B.1-B.5 and B.7;
- (ii) \succsim satisfies B.1-B.6;
- (iii) *there exists an essentially unique and upper semicontinuous $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ such that the function $V : \mathcal{M} \rightarrow \mathbb{R}$, defined by*

$$V(P) = \max_{p \in P} \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), p) \quad \forall P \in \mathcal{M}, \quad (15)$$

represents \succsim .

Moreover, if we define $u : \Delta(C) \rightarrow \mathbb{R}$ such that $u(p) = V(p)$ for all $p \in \Delta(C)$ then $U^ : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by*

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

is upper semicontinuous, belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V})$, and represents \succsim as in (15).

Proof Sketch. (i) implies (ii) is immediate as well as (iii) implies (i) is routine. If we assume B.1-B.6 and we consider \succsim restricted to menus of one lottery, \succsim is a complete, transitive, and continuous binary relation on $\Delta(C)$. Moreover, it satisfies A.3. Therefore, it can be represented as in Theorem 1. If we define $u : \Delta(C) \rightarrow \mathbb{R}$ to be such that

$$u(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C) \quad (16)$$

then B.1-B.5 allow us to extend the utility function u from the set of single lotteries to \mathcal{M} , by imposing, $V(P) = \max_{p \in P} u(p)$ for all $P \in \mathcal{M}$. By (16), the statement follows. \blacksquare

Next proposition suggests that \mathcal{V} is a Subjective State Space. It is arguable that if the DM can freely and costlessly randomize then the only menus of interests for his ranking are the convex ones. Moreover, there are two extra forces that justify the restriction to convex menus. Indeed, assume the DM is presented with a menu P and he can freely randomize among the elements of P . Since he satisfies B.4 and B.6, he will prefer $co(P)$ to P and act as if he is evaluating $co(P)$ instead of P . This is *not* to impose that $co(P)$ is indifferent to P . Indeed, if the DM is forbidden to randomize then when he is facing P he will evaluate it just for the elements contained in P . Furthermore, if he has a strict preference for randomization, he will say that $co(P)$ is strictly preferred to P .

Proposition 4 *Let C be a finite set and let \succsim be a binary relation on \mathcal{M} that satisfies B.1-B.6. If U and V are as in Theorem 3 then it follows that*

$$V(P) = \inf_{v \in \mathcal{V}} \max_{p \in P} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right)$$

for all convex $P \in \mathcal{M}$.

In other words, we can interpret each element v in \mathcal{V} as a future possible subjective state of the world for the DM, that is, as a future possible ex post ranking over lotteries in $\Delta(C)$. Indeed, in evaluating a menu P , depending on his future tastes v in \mathcal{V} , the DM evaluates it by looking at the best lottery in P with respect to expected utility. In fact, since U is increasing in the first component, we have that $\max_{p \in P} U(\mathbb{E}_v(p), v) = U(\max_{p \in P}(\mathbb{E}_v(p), v))$. However, the novelty is exactly how all the evaluations $\{\max_{p \in P} U(\mathbb{E}_v(p), v)\}_{v \in \mathcal{V}}$ are condensed to provide the final evaluation $V(P)$. Instead of being averaged as in [10], the “worst” of such distorted evaluations delivers $V(P)$. Therefore, Theorem 3 is characterizing a DM that in evaluating a menu P values it according to its best ex ante element p but under the assumption that the ex post value of p will be the “worst” possible. That is, the DM maximizes over his Subjective State Space \mathcal{V} . Moreover, Proposition 4 proves the equivalence of these two approaches when the menus considered are convex. Notice that this result is obtained by retaining B.5 but by weakening Independence.

Proposition 4 is partially unsatisfactory. Indeed, its interpretation is based on the claim that since the DM can costlessly randomize then the only menus of interests are the convex ones but it requires the DM to have preferences over all \mathcal{M} . A standard assumption in the literature (see, e.g., [10] and [14]) that guarantees that the DM only cares about convex menus is the following one:

Axiom B. 8 (Indifference to Randomization) *For each $P \in \mathcal{M}$ we have that $P \sim co(P)$.*

Corollary 41 in the Appendix extends the result of Proposition 4 to all menus P in \mathcal{M} . That is, it shows that the Maxmin decision criterion and the Minmax decision criterion coincide for each menu P if and only if \succsim further satisfies B.8. Nevertheless, even in this case, we might have reached a result that is partially unsatisfactory. Axiom B.8 reflects more than asking that the DM can randomize costlessly. In the words of Dekel, Lipman, and Rustichini [10], it implies that randomization “has no value or cost to” the DM. But it is exactly the motivation that randomization can be of strict value to the DM that gives normative and descriptive interest to the assumption of Mixing. For this reason, in the next subsection we consider preferences that are just defined over convex menus.

3.2.1 Convex Menus

In the previous subsection we argued that if the DM can costlessly randomize then the only menus of interest are the convex ones and we derived the result (Proposition 4) that suggested that a DM with convex preferences over menus could be seen as a DM that minimizes over his Subjective State Space. Corollary 41 confirms that this decision criterion can apply to all menus, provided that \succsim further satisfies Axiom B.8. The shortcoming of Proposition 4 is that while it only applies to convex menus it requires the DM to have preferences even over nonconvex menus. On the other hand, Corollary 41 can account for all menus but by imposing that randomization is of no cost and no value either. For this reason in this subsection, we restrict preferences \succsim to be over nonempty, closed, and convex subsets of lotteries. In this way, we naturally make convex menus the unique object of choice by not imposing, a priori, any assumption on the value of randomization.

Proposition 5 *Let C be a finite set and let \succsim be a binary relation on \mathcal{C} . The following are equivalent facts:*

- (i) \succsim satisfies B.1-B.6;

(ii) there exists an essentially unique and upper semicontinuous $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ such that the function $V : \mathcal{C} \rightarrow \mathbb{R}$, defined by

$$V(P) = \inf_{v \in \mathcal{V}} \max_{p \in P} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right) \quad \forall P \in \mathcal{C}, \quad (17)$$

represents \succsim .

Moreover, if we define $u : \Delta(C) \rightarrow \mathbb{R}$ such that $u(p) = V(p)$ for all $p \in \Delta(C)$ then $U^* : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

is upper semicontinuous, belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V})$, and represents \succsim as in (17).

We already discussed the economic relevance of restricting preferences to convex menus. Mathematically, it is important to notice that this result is not a corollary of Proposition 4. Indeed, in Theorem 3 and Proposition 4, the construction of the utility function V is first done over single menus, then, by induction and B.5, it is extended to finite menus, and finally, by continuity, it is extended to all menus. By considering preferences over \mathcal{C} , we rule out the possibility of using finite menus. The techniques used by Dekel, Lipman, and Rustichini [10] and Epstein, Marinacci, and Seo [14] seem to be not applicable. [10] and [14] construct the utility function V just over convex menus and then extend it to other nonconvex menus. Nevertheless, their techniques seem to rely heavily on the compactness of the Subjective State Space and they obtain representations in terms of subjective probabilities over the Subjective State Space, which is not our case or goal.

3.3 Convex Preferences Under Risk as a Criterion of Completion

One way in which the DM might show multiple selves or a lack of confidence in the evaluation of consequences or in his future tastes might arise in the incompleteness of his preferences (see Dubra, Maccheroni, and Ok [12]). That is, the DM might be sure that his future tastes or the relative ranking of consequences lie in a closed and convex set \mathcal{W} of \mathcal{V} . But he might not be sure which exact element is. Therefore, his original preferences can be represented by an a priori incomplete binary relation, \succsim' , such that

$$p \succsim' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \mathcal{W}. \quad (18)$$

The fact that \succsim' is typically incomplete could not help him in making a choice. For this reason, the DM might need to complete his preferences. That is why he needs to form a complete ranking \succsim over elements of $\Delta(C)$. Reasonably, the DM will choose a complete ranking \succsim such that

$$p \succsim' q \Rightarrow p \succsim q. \quad (19)$$

Indeed, if $p \succsim' q$ the DM knows that no matter what his future tastes will be he will prefer p to q and for this reason he will feel compelled to declare $p \succsim q$.

Definition 6 Let \succsim and \succsim' be two binary relations on $\Delta(C)$. \succsim is a completion of \succsim' if and only if \succsim is complete and \succsim and \succsim' satisfy (19). Equivalently, we say that \succsim preserves \succsim' .

If the DM is cautious then one way in which he can choose \succsim is to take a function $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ and define his new complete preferences to be such that

$$p \succsim q \Leftrightarrow \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(q), v). \quad (20)$$

Notice that just the elements in \mathcal{W} are considered in computing the infimum in (19) and not the entire set \mathcal{V} . The example in the Introduction, formally discussed in Example 17, shows a very important case of this instance when $C = \mathbb{R}$. Next proposition shows that preferences, \succsim , arising from (18) and (20) are convex preferences over lotteries where \succsim is actually a completion of \succsim' . Before discussing the results of this subsection, we need to strengthen our continuity assumption into the following one.

Axiom A. 4 (Continuity) *For each $q \in \Delta(C)$ the sets $\{p \in \Delta(C) : p \succsim q\}$ and $\{p \in \Delta(C) : q \succsim p\}$ are closed sets in $\Delta(C)$.*

Notice that A.4 is a standard assumption of continuity (see Debreu [8, pag. 56]). We then have:

Proposition 7 *Let $\Delta(C)$ be the space of simple lotteries over a finite set C and \succsim and \succsim' two binary relations on $\Delta(C)$. If \succsim' is represented as in (18), \succsim is represented as in (20) and*

$$u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C) \quad (21)$$

then \succsim satisfies A.1, A.3, and A.4 and \succsim is a completion of \succsim' .

The rest of the subsection is devoted to prove the opposite implication. In other words, we prove that if C is finite then each convex binary relation can be interpreted as a completion, of the kind represented in (20), of a binary relation \succsim' represented as in (18). In order to do that we need to introduce few notions. If \succsim'' is a binary relation on $\Delta(C)$, we say that \succsim'' is a stochastic order if and only if \succsim'' satisfy (18) for a closed and convex subset $\mathcal{W}'' \subseteq \mathcal{V}$ and \mathcal{W}'' is maximal in representing \succsim'' . That is, if there exists $\bar{\mathcal{W}} \supseteq \mathcal{W}''$ such that $\mathcal{V} \supseteq \bar{\mathcal{W}}$ and

$$p \succsim'' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \bar{\mathcal{W}}$$

then $\bar{\mathcal{W}} = \mathcal{W}''$.

Given a binary relation \succsim , we can construct an auxiliary binary relation \succsim' defined by

$$p \succsim' q \Leftrightarrow \lambda p + (1 - \lambda) r \succsim \lambda q + (1 - \lambda) r \quad \forall \lambda \in (0, 1], \forall r \in \Delta(C). \quad (22)$$

Intuitively, \succsim' captures the (largest) part of the ranking \succsim for which randomizing does not carry any benefit.²³

Proposition 8 *Let $\Delta(C)$ be the space of simple lotteries over a finite set C and \succsim a binary relation on $\Delta(C)$. The following facts are equivalent:*

- (i) \succsim satisfies A.1, A.3, and A.4;
- (ii) there exist a closed and convex set $\mathcal{W} \subseteq \mathcal{V}$ and an essentially unique $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ such that

$$p \succsim' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \mathcal{W}, \quad (23)$$

\succsim is a completion of \succsim' , and $u : \Delta(C) \rightarrow \mathbb{R}$ such that

$$u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C) \quad (24)$$

represents \succsim .

²³Notice that \succsim' , defined as in (22), can be seen as the dual counterpart of the revealed unambiguous preference of Ghirardato, Maccheroni, and Marinacci [18].

Moreover, if \succsim preserves a stochastic order \succsim'' then $\mathcal{W}'' \supseteq \mathcal{W}$ and

$$u(p) = \inf_{v \in \mathcal{W}''} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C).$$

Given the previous discussion, we can interpret the DM as endowed with two binary relations, \succsim' and \succsim . \succsim' represents the part of the ranking that to the DM seems uncontroversial while \succsim represents the completion of \succsim' , that is, it represents the preferences of the DM if he is forced to choose. The last part of Proposition 8 confirms that \succsim' captures the largest part of the ranking \succsim that appears to him indisputable. In other words, Proposition 8 shows that each DM with continuous and convex preferences, \succsim , acts as if he has an original ranking \succsim' that can be represented through a multi-expected utility criterion. \succsim' captures through \mathcal{W} the Bernoulli utility functions, the tastes, of the DM that are deemed plausible. Finally, he completes his preferences with a cautious decision criterion and just by considering his tastes in \mathcal{W} .

4 Extensions and Special Cases

In this section we extend most of the results previously presented. There are two possible ways of proceeding: by generalizing C in terms of cardinality or by generalizing it in terms of topological structure. We explore both paths. Notice that the assumptions of Weak Order, Mixture Continuity, Continuity, and Mixing do not rely on any of the properties of C therefore, they do not need to be restated here. Similar discussion applies for the definition of \succsim' in (10), the notion of completion in Definition 6, and the notion of stochastic order.

4.1 The Case of Countable C

We return to the study of convex preferences, \succsim , over the set of simple lotteries $\Delta(C)$ but now we allow the set C to be infinite, particularly, we allow it to be at most countable. The characterization result is identical to the one reported for the finite case, although the proof is significantly more difficult.

Theorem 9 *Let $\Delta(C)$ be the space of simple lotteries over an at most countable set C and \succsim a binary relation on $\Delta(C)$. The following are equivalent facts:*

- (i) \succsim satisfies A.1, A.2, and A.3;
- (ii) there exists an essentially unique $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ such that

$$p \succsim q \Leftrightarrow \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(q), v). \quad (25)$$

Moreover, if $u : \Delta(C) \rightarrow \mathbb{R}$ is a mixture continuous utility function for \succsim then the function $U^* : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ and represents \succsim as in (25).

Similarly, we can partially extend the results of Convexity as a criterion of completion. In the Appendix, we prove Proposition 7 for the case C at most countable while next proposition constitutes a generalization of Proposition 8.

Proposition 10 *Let $\Delta(C)$ be the space of simple lotteries over an at most countable set C , \succsim a binary relation on $\Delta(C)$ that satisfies A.1, A.3, and A.4, and \succsim' a binary relation defined as in (22). The following facts are true:*

- (a) *There exists a closed and convex set $\mathcal{W} \subseteq \mathcal{V}$ such that $p \succsim' q$ if and only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$.*
- (b) *For each $p, q \in \Delta(C)$ if $p \succsim' q$ then $p \succsim q$.*
- (c) *If \succsim'' is another binary relation that satisfies (a) and (b) then $p \succsim'' q$ implies $p \succsim' q$.²⁴*
- (d) *If \succsim'' is a stochastic order and \succsim preserves \succsim'' then $\mathcal{W}'' \supseteq \mathcal{W}$.*

Again, we can interpret the DM as endowed with two binary relations, \succsim' and \succsim . \succsim' represents the part of the ranking that to the DM seems uncontroversial while \succsim represents the completion of \succsim' , that is, it represents the preferences of the DM if he is forced to choose. Point (c) (and (d)) of Proposition 10 confirms that \succsim' captures the largest part of the ranking \succsim that appears to him indisputable.

4.1.1 First Order Stochastic Dominance

In this subsection, we study First Order Stochastic Dominance and preferences \succsim that preserve it.

Axiom A. 5 (First Order Stochastic Dominance) *If $p, q \in \Delta(C)$ are such that*

$$\sum_{y \succsim x} p(y) \geq \sum_{y \succsim x} q(y) \quad \forall x \in C \quad (26)$$

then $p \succsim q$.

Observe that (26) is the obvious translation in a somewhat more generic setting of the usual assumption of First Order Stochastic Dominance. For the purpose of this subsection, we specialize the set of consequences C . We require that C is ordered with a maximum element. That is, we require that \succsim is an order with maximum element and that C is listed accordingly to \succsim . Notice that we do not just require that C is countable, that is $C = \{x_n\}_{n \in \mathbb{N}}$, but we also require that $x_n \succ x_{n+1}$ for all $n \in \mathbb{N}$. This requirement is pretty mild. Indeed, if we assume that a DM prefers strictly more money to less money and we assume that C are discrete monetary outcomes in dollars bounded from above, then C is ordered with a maximum element. Given an ordered set C with maximum element and an element $v \in \mathbb{R}^C$, we say that v is increasing if and only if $v(x_n) \geq v(x_{n+1})$ for all $n \in \mathbb{N}$. We define $\mathcal{V}_{inc} = \{v \in \mathbb{R}^C : v(x_1) = 1 \text{ and } v \text{ is increasing}\}$. We fix x to be equal to x_1 , that is, $\mathcal{V} = \mathcal{V}_1(x_1)$.

Proposition 11 *Let $\Delta(C)$ be the space of simple lotteries over C , \succsim a binary relation on $\Delta(C)$, and C an ordered set with maximum element. The following are equivalent facts:*

- (i) *\succsim satisfies A.1, A.2, A.3, and A.5;*
- (ii) *there exists an essentially unique $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ such that the function $u : \Delta(C) \rightarrow \mathbb{R}$*

$$u(p) = \inf_{v \in \mathcal{V}_{inc}} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C) \quad (27)$$

represents \succsim .

²⁴More generally, in Appendix B we show that \succsim' is the maximal binary relation that satisfies Independence and for which \succsim is a completion.

Remark 12 *It is not hard to check that p and q satisfy (26) if and only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{V}_{inc}$. Therefore, intuitively, Proposition 11 can be interpreted as a particular extension to the countable case of the result provided in Proposition 8.*

4.2 The Case of Generic C

We next consider the case of a generic set C of consequences. As before, we assume that \succsim satisfies A.1, A.2, and A.3. Moreover, in order to have an existence result for an upper semicontinuous utility function $u : \Delta(C) \rightarrow \mathbb{R}$ we need to introduce two extra assumptions:

Axiom A. 6 (Countable Boundedness) *There exists $\{p_k\}_{k \in \mathbb{Z}} \subset \Delta(C)$ such that for each $p \in \Delta(C)$ there exist $k, k' \in \mathbb{Z}$ such that $p_k \succsim p \succsim p_{k'}$.*

Axiom A. 7 (Upper Semicontinuity) *For each $q \in \Delta(C)$ the set $\{p \in \Delta(C) : p \succsim q\}$ is a closed set in $\Delta(C)$.*

Although A.6 might appear an unusual assumption, it is easily verified that any binary relation, \succsim , that satisfies Weak Order, Mixture Continuity, and is represented by a mixture continuous utility function must satisfy this axiom. Moreover, if the set of consequences are monetary outcomes, it is immediate to see that \succsim satisfies Countable Boundedness as soon as \succsim satisfies the usual First Order Stochastic Dominance.²⁵ On the other hand, A.7 is a standard technical assumption of continuity that allows us to derive the existence of an upper semicontinuous utility function for \succsim . We are ready to state our most general representation result.

Theorem 13 *Let $\Delta(C)$ be the space of simple lotteries over a set C and \succsim a binary relation on $\Delta(C)$. The following are equivalent facts:*

- (i) \succsim satisfies A.1, A.2, A.3, A.6, and A.7;
- (ii) there exists an essentially unique $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ such that

$$p \succsim q \Leftrightarrow \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(q), v). \quad (28)$$

Moreover, if $u : \Delta(C) \rightarrow \mathbb{R}$ is a mixture continuous utility function for \succsim then the function $U^* : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$ and represents \succsim as in (28).

4.2.1 Risk Aversion

In this subsection, we specialize the previous setting and show that U^* can be interpreted as an index of risk aversion and it can be derived just using certainty equivalents. We assume that $C = \mathbb{R}$ and we fix x to be equal to 1, that is, $\mathcal{V} = \mathcal{V}_1(1)$. We consider a binary relation \succsim on $\Delta(C)$ that satisfies A.1, A.2, A.3, and A.7. We replace A.6 with the following two axioms:

²⁵Furthermore, it should be noticed that Fishburn [17] (see also Monteiro [30, pag. 151]) provides an example of an uncountable set C and a binary relation \succsim on $\Delta(C)$ that cannot be represented by any utility function u . Surprisingly, \succsim satisfies A.1, A.2, and A.3 but it violates A.6.

Axiom A. 8 (Monotonicity) Given $x, y \in \mathbb{R}$, $x \succsim y$ if and only if $x \geq y$.

Axiom A. 9 (Certainty Equivalent) For each $p \in \Delta(\mathbb{R})$ there exists a unique $x_p \in \mathbb{R}$ such that $x_p \sim p$.

The notion of comparative risk aversion that we use is standard in the theory of choice under risk (see [29, Chapter 6]). We declare DM 1 more risk averse than DM 2 if and only if the certainty equivalent of 1 is smaller or equal than the one of 2 for each simple lottery.

Definition 14 Let \succsim_1 and \succsim_2 be two binary relations on $\Delta(\mathbb{R})$ that satisfy A.9. \succsim_1 is more risk averse than \succsim_2 if and only if $x_p^2 \geq x_p^1$ for all $p \in \Delta(\mathbb{R})$.

We can then specialize our existence result and provide a characterization of risk aversion in terms of U^* .

Proposition 15 Let $\Delta(\mathbb{R})$ be the set of simple lotteries over \mathbb{R} and \succsim a binary relation on $\Delta(\mathbb{R})$. The following are equivalent facts:

(i) \succsim satisfies A.1, A.2, A.3, A.7, A.8, and A.9;

(ii) there exists an essentially unique $U \in \mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V})$ such that $u : \Delta(C) \rightarrow \mathbb{R}$, defined by

$$u(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(\mathbb{R}), \quad (29)$$

is a (certainty equivalent) utility function for \succsim .

Moreover, $U^* : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{x_p : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V})$ and represents \succsim as in (29).

As a consequence, we can characterize attitudes toward risk aversion in terms of U^* .

Proposition 16 Let $\Delta(\mathbb{R})$ be the set of simple lotteries over \mathbb{R} and \succsim_1 and \succsim_2 two binary relations on $\Delta(\mathbb{R})$ that satisfy (i) of Proposition 15. The following are equivalent facts:

(i) \succsim_1 is more risk averse than \succsim_2 ;

(ii) $U_1^* \leq U_2^*$.

In the context of monetary simple lotteries, the next example shows an instance of preferences satisfying A.1, A.2, A.3, A.7, A.8, and A.9 and arising as a criterion of completion.

Example 17 Consider $\Delta(\mathbb{R})$. Assume that the DM is unsure about his (future) attitudes on risk aversion therefore his original preferences, \succsim' , are represented by the decision criterion

$$p \succsim' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \mathcal{W} \quad (30)$$

where, $\mathcal{W} \subseteq \mathcal{V}$ is a nonempty, compact, and convex set,²⁶ and each element $v \in \mathcal{W}$ is a strictly increasing function over the real line such that $v(\mathbb{R}) = \mathbb{R}$. This last fact translates into saying that

²⁶That is, for each $x \in \mathbb{R}$ there exist $a_x, b_x \in \mathbb{R}$ such that $v(x) \in [a_x, b_x]$ for all $v \in \mathcal{W}$.

the DM always prefers more money to less money and he does not violate First Order Stochastic Dominance. Given the assumptions on \mathcal{W} , it is immediate to see that

$$p \succsim' q \Leftrightarrow v^{-1}(\mathbb{E}_v(p)) \geq v^{-1}(\mathbb{E}_v(q)) \quad \forall v \in \mathcal{W}. \quad (31)$$

This translates into saying that the DM prefers surely lottery p to lottery q if and only if the certainty equivalent of lottery p is higher than the one for lottery q for all his possible attitudes. Although mathematically equivalent, economically, the criterion in (31) looks more reasonable than the one in (30). Indeed, the representation in (31) allows to compare the different evaluations in monetary terms, that is, in a cardinal way. If the DM has to make a final choice whenever he faces two lotteries then it is sensible to think that he has to complete his preferences with a complete binary relation \succsim . If he is prudent he might want to complete them in a cautious way, that is, in evaluating a lottery p he might want to trust the worst of his selves, the worst of his evaluations. This implies that

$$p \succsim q \Leftrightarrow \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_v(p)) \geq \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_v(q)).$$

It is not hard to prove, as we show in Appendix B, that \succsim satisfies A.1, A.2, A.3, A.7, A.8, and A.9. Moreover, the function U , defined by

$$U(t, v) = \begin{cases} v^{-1}(t) & v \in \mathcal{W} \\ \infty & \text{otherwise} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V})$ and represents \succsim as in (29) of Proposition 15.

4.2.2 Some Example

In this subsection, we study how our representation result specialize for two models in the literature. The first model we study is the well known Expected Utility model of von Neumann and Morgenstern. The key assumption behind such model is Independence:

Axiom A. 10 (Independence) For each $p, q, r \in \Delta(C)$ we have that

$$p \succsim q \Rightarrow \frac{1}{2}p + \frac{1}{2}r \succsim \frac{1}{2}q + \frac{1}{2}r.$$

Proposition 18 Let $\Delta(C)$ be the set of simple lotteries over a set C and \succsim a binary relation on $\Delta(C)$. The following are equivalent facts:

(i) \succsim satisfies A.1, A.2, and A.10;

(ii) there exists $\bar{v} \in \mathcal{V}$ such that $U : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U(t, v) = \begin{cases} t & v = \bar{v} \\ \infty & v \neq \bar{v} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ (and $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$) and represents \succsim as in (28) of Theorem 13.

The second model we characterize in terms of our representation is the one proposed by Maccheroni [27, Theorem 1]. His assumptions are richer than ours. Indeed, other than having A.1, A.3, and A.4, he considers the following two assumptions:

Axiom A. 11 (Best Outcome) *There exists $\bar{x} \in C$ such that $\delta_{\bar{x}} \succ p$ for all $p \in \Delta(C)$.*

Axiom A. 12 (Best Outcome Independence) *For each $p, q \in \Delta(C)$ and $\lambda \in (0, 1)$ we have that*

$$p \succ q \Leftrightarrow \lambda p + (1 - \lambda) \delta_{\bar{x}} \succ \lambda q + (1 - \lambda) \delta_{\bar{x}}.$$

We fix x to be equal to \bar{x} , that is, $\mathcal{V} = \mathcal{V}_1(\bar{x})$.

Proposition 19 *Let $\Delta(C)$ be the set of simple lotteries over a set C and \succsim a binary relation on $\Delta(C)$ that satisfies A.11. The following are equivalent facts:*

(i) \succsim satisfies A.1, A.3, A.4, and A.12;

(ii) there exists a closed and convex set $\mathcal{W} \subseteq \mathcal{V}$ such that $U : \mathbb{R} \times \mathcal{V} \rightarrow [-\infty, \infty]$, defined by

$$U(t, v) = \begin{cases} t & v \in \mathcal{W} \\ \infty & v \notin \mathcal{W} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V},$$

belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V})$ (and $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V})$) and represents \succsim as in (28) of Theorem 13.

4.3 The Case of Compact C

In this subsection, we consider another familiar set of lotteries (see, e.g., Dekel [9], Dubra, Maccheroni, and Ok [12], and Gul and Pesendorfer [23]). We assume that the set of consequences is a compact metric space and we consider a binary relation \succsim over $\Delta_{\mathcal{B}}(C)$ where $\Delta_{\mathcal{B}}(C)$ is the set of *all Borel probability measures* (lotteries) over C . The mathematical setting and terminology is partially different from the one discussed in Section 2. In the next short subsection, we just discuss the major departures. The results reported here rely on a different dual pair but they can easily be proven by adjusting the arguments reported in the Appendices. For this reason, we simply state them.²⁷

4.3.1 Notation and Mathematical Preliminaries: the Compact Case

The object of our study is a binary relation, \succsim , on the mixture space $\Delta_{\mathcal{B}}(C)$. We call elements of $\Delta_{\mathcal{B}}(C)$: p, q, r . We call x an element of C . We define \mathbb{R}_{cont}^C to be the set of all continuous functions from C to \mathbb{R} endowed with the supnorm. It follows that the space of all signed and finite Borel measures, endowed with the total variation norm, $ca(C)$, is isometrically isomorphic to the norm dual of \mathbb{R}_{cont}^C . We call p, q, r the elements of $ca(C)$. We call v a generic element of \mathbb{R}_{cont}^C . The evaluation duality, $\langle \cdot, \cdot \rangle : ca(C) \times \mathbb{R}_{cont}^C \rightarrow \mathbb{R}$, is defined to be such that

$$\langle p, v \rangle = \int v dp$$

where the latter is a standard Lebesgue integral. A net $\{p_\alpha\}_{\alpha \in A} \subseteq ca(C)$ is said to converge to p ($p_\alpha \rightarrow p$) if and only if $\int v dp_\alpha \rightarrow \int v dp$ for all $v \in \mathbb{R}_{cont}^C$. We consider $\Delta_{\mathcal{B}}(C)$ endowed with the relative topology. Notice that the relative topology is metrizable and it coincides with the topology of weak convergence. When we consider a pair $(p, v) \in \Delta_{\mathcal{B}}(C) \times \mathbb{R}_{cont}^C$ we write $\mathbb{E}_v(p)$ instead of $\int v dp$ or $\langle p, v \rangle$. Fix a generic element $x \in C$. Then, we define $\mathcal{V}_c = \mathcal{V}_1(x) = \{v \in \mathbb{R}_{cont}^C : v(x) = 1\}$. Set $\mathbb{R}^\diamond = \mathbb{R} \setminus \{0\}$. We say that a subset D of $\mathbb{R} \times \mathcal{V}_c$ is \diamond -evenly convex if for each $(\bar{t}, \bar{v}) \notin D$ there exists $(s, p) \in \mathbb{R}^\diamond \times ca(C)$, such that $\langle p, \bar{v} \rangle + \bar{t}s < \langle p, v \rangle + ts$ for all $(t, v) \in D$. In the sequel, we deal

²⁷Proofs are available upon request.

with functions $U : \mathbb{R} \times \mathcal{V}_c \rightarrow [-\infty, \infty]$. Given U , we define $u_U : ca(C) \rightarrow [-\infty, \infty]$ to be such that $p \mapsto \inf_{v \in \mathcal{V}} U(\langle p, v \rangle, v)$. We say that U is linearly continuous if and only if the function u_U is a real valued and continuous function on $\Delta_{\mathcal{B}}(C)$. We say that U is \diamond -evenly quasiconvex if and only if all its lower contour sets are \diamond -evenly convex. We define $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_c)$ to be the class of functions from $\mathbb{R} \times \mathcal{V}$ to $[-\infty, \infty]$ that satisfy P.1-P.4 adjusted to the new setting. We define essential uniqueness in the same way we defined it in Section 2.

4.3.2 The Results

We consider a binary relation \succsim . We assume that \succsim represents the preferences of the DM over the set of lotteries over a generic *compact metric space* C of consequences. We require \succsim to satisfy the assumptions A.1, A.3, and A.4. The translation of these axioms to this setting is straightforward as well as their interpretation stays unchanged.

Theorem 20 *Let $\Delta_{\mathcal{B}}(C)$ be the set of lotteries over a compact metric space C and \succsim a binary relation on $\Delta_{\mathcal{B}}(C)$. The following are equivalent facts:*

- (i) \succsim satisfies A.1, A.3, and A.4;
- (ii) there exists an essentially unique $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_c)$ such that

$$p \succsim q \Leftrightarrow \inf_{v \in \mathcal{V}_c} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{V}_c} U(\mathbb{E}_v(q), v). \quad (32)$$

Moreover, if $u : \Delta_{\mathcal{B}}(C) \rightarrow \mathbb{R}$ is a continuous utility function for \succsim then the function $U^* : \mathbb{R} \times \mathcal{V}_c \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_c,$$

belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_c)$ and represents \succsim as in (32).

Remark 21 *Notice that one major difference with the result contained in Theorem 13 is that the Bernoulli utility functions considered in the representation are continuous functions on C .*

Even, for the case of C compact metric space, we can interpret each convex binary relation as a completion of a maximal incomplete preorder that admits a multi-expected utility representation. Recall that for this purpose, given a binary relation \succsim , we can construct an auxiliary binary relation \succsim' defined by

$$p \succsim' q \Leftrightarrow \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r \quad \forall \lambda \in (0, 1], \forall r \in \Delta_{\mathcal{B}}(C). \quad (33)$$

Again, we obtain that if \succsim satisfies A.1, A.3, and A.4 then \succsim' is the maximal binary relation that admits a multi-expected utility representation and for which \succsim is a completion. More importantly, we obtain that:

Proposition 22 *Let $\Delta_{\mathcal{B}}(C)$ be the space of lotteries over a compact metric space C and \succsim a binary relation on $\Delta_{\mathcal{B}}(C)$. The following facts are equivalent:*

- (i) \succsim satisfies A.1, A.3, and A.4;

(ii) there exist a closed and convex set $\mathcal{W} \subseteq \mathcal{V}_c$ and an essentially unique $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_c)$ such that

$$p \succsim' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \mathcal{W}, \quad (34)$$

\succsim is a completion of \succsim' , and $u : \Delta_{\mathcal{B}}(C) \rightarrow \mathbb{R}$ such that

$$u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_c} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta_{\mathcal{B}}(C) \quad (35)$$

represents \succsim .

Moreover, if \succsim preserves a stochastic order \succsim'' then $\mathcal{W}'' \supseteq \mathcal{W}$ and

$$u(p) = \inf_{v \in \mathcal{W}''} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C).$$

5 Conclusions

We have characterized preferences over lotteries and over menus of lotteries that exhibit a propensity for randomization. The Axiom of Mixing, or equivalently Convexity of preferences, formally captures an inclination toward randomization. We argued that this feature is able to accommodate for more patterns of choice than most of the existing models can. Moreover, this property of preferences is appealing as soon as the DM is unsure about: value of outcomes, his future tastes, or the degree of risk aversion. We showed that Convexity is equivalent to cautiousness about any of these three kinds of uncertainty. More precisely, we proved that a DM has convex preferences if and only if his preferences are represented by a utility function u , defined by

$$u(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C). \quad (36)$$

This is equivalent to say that a DM with convex preferences in evaluating a lottery p takes a “minimum” of quasi-EU evaluations. That is, he takes a minimum of monotonic transformations of EU evaluations. Moreover, the transformation function U is essentially unique. In a context of monetary lotteries, the function U can be interpreted as an index of risk aversion. We argued and showed that the set \mathcal{V} has three different interpretations: set of outcomes evaluations, set of future tastes of the DM, Subjective State Space. In a context of choice over menus of lotteries, we provided a foundation of the Maxmin EU and Minmax EU criteria over a Subjective State Space. Finally, we proved Convexity being a cautious criterion of completion thereby showing that our representation is well behaved with respect to stochastic orders. Indeed, a DM that has convex preferences and preserves first or second order stochastic dominance, in computing the infimum, will consider just elements in \mathcal{V} that are increasing or increasing and concave.

A Quasiconcave Duality

In this appendix we provide the basic mathematical results behind our representation. Most of the facts found here heavily rely on the techniques developed in [7]. The essential uniqueness of the function U is very much in line with the findings of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [7] and it is derived by extensively using some of the techniques adopted in [7]. However, there are elements of novelty that make the results in this work nontrivial. Two assumptions are crucial for the arguments of [7]. First, the quasiconcave functions studied are monotone. Second, the underlying domain is an M -space with unit. The existence of a unit element in the domain plays a key role in deriving results of complete duality. On the contrary, in our case the domain of our

utility function u is $\Delta(C)$ as subset of \mathbb{R}_0^C . We consider the latter set endowed with the topology induced by the algebraic dual. Therefore, not only our domain is not a vector space but, as soon as C is not finite, \mathbb{R}_0^C is not even a normed vector space. Moreover, neither \mathbb{R}_0^C has a unit element with respect to the usual pointwise order nor u is monotone with respect to such order. For similar reasons, we cannot apply the usual duality results stemming from the research on direct and indirect utility functions (see, e.g., Diewert [11] and Martinez-Legaz [28]), since again we miss at least a key assumption: monotonicity.

Before discussing the main results of this section we introduce some notation, given a function $\hat{u} : \mathbb{R}_0^C \rightarrow [-\infty, \infty]$ (resp., $\hat{u} : \Delta(C) \rightarrow \mathbb{R}$), we say that

1. \hat{u} is upper semicontinuous if and only if $\{\hat{u} \geq \alpha\} = \{p \in \mathbb{R}_0^C : \hat{u}(p) \geq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$ (resp., $\{\hat{u} \geq \alpha\} = \{p \in \Delta(C) : \hat{u}(p) \geq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$)
2. \hat{u} is (evenly) quasiconcave if and only if $\{\hat{u} \geq \alpha\}$ is (evenly) convex for all $\alpha \in \mathbb{R}$.

It is important to notice that an evenly quasiconcave function is quasiconcave. Conversely, we have that an upper semicontinuous quasiconcave function is evenly quasiconcave. Given an element $v \in \mathbb{R}^C$ and $t \in \mathbb{R}$, we denote $\{v \leq t\} = \{p \in \mathbb{R}_0^C : \langle p, v \rangle \leq t\}$ and $\{v = t\} = \{p \in \mathbb{R}_0^C : \langle p, v \rangle = t\}$. Given $\alpha \in \mathbb{R}$, we denote with α both the real number α as well as the element of \mathbb{R}^C that is equal to α for each $x \in C$: no confusion should arise.

Object of our study in this appendix is a function $u : \Delta(C) \rightarrow \mathbb{R}$. We define $\bar{u} : \mathbb{R}_0^C \rightarrow [-\infty, \infty)$ to be such that

$$p \longmapsto \begin{cases} u(p) & p \in \Delta(C) \\ -\infty & p \notin \Delta(C) \end{cases} . \quad (37)$$

We fix an arbitrary $x \in C$. Recall that we defined $\mathcal{V} = \mathcal{V}_1(x) = \{v \in \mathbb{R}^C : v(x) = 1\}$. Given $\hat{u} : \mathbb{R}_0^C \rightarrow [-\infty, \infty)$, we define an auxiliary map from $\mathbb{R} \times \mathcal{V}_1(x)$ to $[-\infty, \infty]$ to be such that

$$(t, v) \longmapsto U_v(t) = \sup \{\hat{u}(p) : \langle p, v \rangle \leq t\} . \quad (38)$$

Lemma 23 *Let u be a function from $\Delta(C)$ to \mathbb{R} . The following facts are true:*

- (a) u is quasiconcave (resp., evenly quasiconcave) if and only if \bar{u} is quasiconcave (resp., evenly quasiconcave);
- (b) u is upper semicontinuous if and only if \bar{u} is upper semicontinuous.

Proof.

Before proving (a) and (b), notice that $\Delta(C)$ is a closed and convex set.²⁸ It follows that $\Delta(C)$ is evenly convex. Pick a generic $\alpha \in \mathbb{R}$ and notice that

$$\{p \in \mathbb{R}_0^C : \bar{u}(p) \geq \alpha\} = \{p \in \mathbb{R}_0^C : \bar{u}(p) \geq \alpha\} \cap \Delta(C) = \{p \in \Delta(C) : u(p) \geq \alpha\} . \quad (39)$$

²⁸Convexity is obvious. Consider a net $\{p_\alpha\}_{\alpha \in A} \subseteq \Delta(C)$ and suppose that $p_\alpha \rightarrow p \in \mathbb{R}_0^C$. This is equivalent to say that $\langle p_\alpha, v \rangle \rightarrow \langle p, v \rangle$ for all $v \in \mathbb{R}^C$. Pick a generic $x \in C$ and consider $\delta_x \in \mathbb{R}^C$. Then,

$$0 \leq p_\alpha(x) = \langle p_\alpha, \delta_x \rangle \rightarrow \langle p, \delta_x \rangle = p(x) .$$

Finally, if we define $v \in \mathbb{R}^C$ to be such that $v(x) = 1$ for all $x \in C$ then it follows that

$$1 = \langle p_\alpha, v \rangle \rightarrow \langle p, v \rangle .$$

This proves that $\Delta(C)$ is closed.

From (39), (a) and (b) follow. ■

We next show that the map $(t, v) \mapsto U_v(t)$ satisfies P.1-P.3.

Lemma 24 *Let \hat{u} be a function from \mathbb{R}_0^C to $[-\infty, \infty)$. The map $(t, v) \mapsto U_v(t)$ satisfies P.1-P.3.*

Proof.

P.1 follows by definition. P.2 follows from the fact that for each $v \in \mathcal{V}_1(x)$ we have that $\lim_{t \rightarrow \infty} U_v(t) = \sup_{p \in \mathbb{R}_0^C} \hat{u}(p)$. Indeed, by definition, it follows that for each $(t, v) \in \mathbb{R} \times \mathcal{V}_1(x)$ we have that $U_v(t) \leq \sup_{p \in \mathbb{R}_0^C} \hat{u}(p)$. If we fix v then we have that $\lim_{t \rightarrow \infty} U_v(t) \leq \sup_{p \in \mathbb{R}_0^C} \hat{u}(p)$. Conversely, consider $\{p_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_0^C$ such that $\hat{u}(p_n) \uparrow \sup_{p \in \mathbb{R}_0^C} \hat{u}(p)$. Since $\hat{u}(p_n) \leq U_v(\langle p_n, v \rangle) \leq \lim_{t \rightarrow \infty} U_v(t)$ for all $n \in \mathbb{N}$ and for all $v \in \mathcal{V}_1(x)$, if we fix v then we have that $\lim_{t \rightarrow \infty} U_v(t) \geq \sup_{p \in \mathbb{R}_0^C} \hat{u}(p)$. Finally, we show that P.3 is satisfied. Fix $\alpha \in \mathbb{R}$ and define $L_\alpha = \{(t, v) \in \mathbb{R} \times \mathcal{V}_1(x) : U_v(t) \leq \alpha\}$. Suppose that L_α is neither empty nor $\mathbb{R} \times \mathcal{V}_1(x)$. Pick $(\bar{t}, \bar{v}) \in \mathbb{R} \times \mathcal{V}_1(x)$ such that $(\bar{t}, \bar{v}) \notin L_\alpha$. It follows that $U_{\bar{v}}(\bar{t}) > \alpha$. This implies that there exists a point \bar{p} such that $\langle \bar{p}, \bar{v} \rangle \leq \bar{t}$ and $\hat{u}(\bar{p}) > \alpha$. But $U_v(t) \leq \alpha$ for all $(t, v) \in L_\alpha$, which implies that $\langle \bar{p}, v \rangle > t$ for all $(t, v) \in L_\alpha$. This, in turn, implies that

$$\langle \bar{p}, v \rangle - t > 0 \geq \langle \bar{p}, \bar{v} \rangle - \bar{t} \quad \forall (t, v) \in L_\alpha. \quad \blacksquare$$

The next result tell us that given an evenly quasiconcave function, $u : \Delta(C) \rightarrow \mathbb{R}$, we can use the mapping $(t, v) \mapsto U_v(t)$ to reconstruct u .

Theorem 25 *A function $u : \Delta(C) \rightarrow \mathbb{R}$ is evenly quasiconcave if and only if there exists $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ that satisfies P.1 and P.2 such that*

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v) \in \mathbb{R} \quad \forall p \in \Delta(C). \quad (40)$$

Moreover:

- (i) U can be chosen to be such that $U(t, v) = U_v(t)$ for all $(t, v) \in \mathbb{R} \times \mathcal{V}_1(x)$, where $\hat{u} = \bar{u}$.
- (ii) u is upper semicontinuous if (resp., only if) in (40) we can replace U with U^+ (resp., U_v with U_v^+).

Proof.

“Only if.” Recall $x \in C$ is fixed. Suppose u is evenly quasiconcave. Define \bar{u} as in (37), and choose U to be defined as in (38) with $\hat{u} = \bar{u}$. Then, by Lemma 23, \bar{u} is evenly quasiconcave. By construction, $\bar{u} \neq -\infty$, $\bar{u}(p) = u(p) \in \mathbb{R}$ for all $p \in \Delta(C)$, and $\bar{u}(p) = -\infty$ otherwise. By definition of $U_v(t)$, it follows that

$$\bar{u}(p) \leq U_v(\langle p, v \rangle) \quad \forall p \in \Delta(C), \forall v \in \mathcal{V}_1(x) \quad (41)$$

and so

$$\bar{u}(p) \leq \inf_{v \in \mathcal{V}_1(x)} U_v(\langle p, v \rangle) \quad \forall p \in \Delta(C). \quad (42)$$

Pick $\bar{p} \in \Delta(C)$. If \bar{p} is a global maximum for \bar{u} on $\Delta(C)$ (hence for u) then equality holds in (41) and equality holds in (42). Next, assume that $\bar{p} \in \Delta(C)$ and it is not a global maximum. Since \bar{p} belongs to $\Delta(C)$, then $\bar{u}(\bar{p}) \in \mathbb{R}$. Since \bar{p} is not a global maximum, there is $\varepsilon > 0$ such that $\{\bar{u} \geq \bar{u}(\bar{p}) + \varepsilon\} \neq \emptyset$ for all $\varepsilon \in (0, \bar{\varepsilon}]$. Moreover, for each $\varepsilon \in (0, \bar{\varepsilon}]$, $\bar{p} \notin \{\bar{u} \geq \bar{u}(\bar{p}) + \varepsilon\} \subseteq \Delta(C)$. Since this upper contour set is evenly convex, there is $\bar{v} \in \mathbb{R}^C \setminus \{0\}$ such that $\langle \bar{p}, \bar{v} \rangle < \langle p, \bar{v} \rangle$ for all

$p \in \{\bar{u} \geq \bar{u}(\bar{p}) + \varepsilon\}$. We can consider $\bar{v} \in \mathcal{V}_1(x)$.²⁹ That is, $\{\bar{u} \geq \bar{u}(\bar{p}) + \varepsilon\} \subseteq \{\bar{v} > \langle \bar{p}, \bar{v} \rangle\}$. Namely, $\{\bar{v} \leq \langle \bar{p}, \bar{v} \rangle\} \subseteq \{\bar{u} < \bar{u}(\bar{p}) + \varepsilon\}$. Thus, $U_{\bar{v}}(\langle \bar{p}, \bar{v} \rangle) \leq \bar{u}(\bar{p}) + \varepsilon$ and

$$\bar{u}(\bar{p}) \leq \inf_{v \in \mathcal{V}_1(x)} U_v(\langle \bar{p}, v \rangle) \leq U_{\bar{v}}(\langle \bar{p}, \bar{v} \rangle) \leq \bar{u}(\bar{p}) + \varepsilon \quad (43)$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$. This implies equality in (42). By construction and Lemma 24, it follows that U satisfies P.1 and P.2.

“If.” Suppose (40) holds. Define \bar{u} as in (37). Then, $\bar{u}(p) = u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v) \in \mathbb{R}$ for all $p \in \Delta(C)$ and $\bar{u}(p) = -\infty$ otherwise. Pick $\alpha \in \mathbb{R}$. We prove that $\{\bar{u} \geq \alpha\}$ is evenly convex. By construction, $\Delta(C) \supseteq \{\bar{u} \geq \alpha\} \neq \mathbb{R}_0^C$. If $\{\bar{u} \geq \alpha\} = \emptyset$ then there is nothing to prove. Otherwise, let $\bar{p} \notin \{\bar{u} \geq \alpha\}$. We have two cases $\bar{p} \in \Delta(C)$ or $\bar{p} \notin \Delta(C)$. In the first case, we have that $\alpha > u(\bar{p}) \in \mathbb{R}$. Then, there exists $\bar{v} \in \mathcal{V}_1(x)$ for which $U(\langle \bar{p}, \bar{v} \rangle, \bar{v}) < \alpha$. Let $p \in \{\bar{u} \geq \alpha\}$. By contradiction, suppose that $\langle p, \bar{v} \rangle \leq \langle \bar{p}, \bar{v} \rangle$. Then, since U is increasing in the first argument, $\bar{u}(p) = u(p) \leq U(\langle p, \bar{v} \rangle, \bar{v}) \leq U(\langle \bar{p}, \bar{v} \rangle, \bar{v}) < \alpha$, a contradiction. In the second case, $\bar{p} \notin \Delta(C)$ and the latter set is closed and convex. By construction, it follows that $\{\bar{u} \geq \alpha\} \subseteq \Delta(C)$. By [33, Theorem 3.4], we have that there exists $\bar{v} \in \mathbb{R}^C \setminus \{0\}$ for which $\langle p, \bar{v} \rangle > \langle \bar{p}, \bar{v} \rangle$ for all $p \in \Delta(C)$. Therefore, we can conclude that if $\bar{p} \notin \{\bar{u} \geq \alpha\}$ then there exists $\bar{v} \in \mathbb{R}^C \setminus \{0\}$ such that $\langle \bar{p}, \bar{v} \rangle < \langle p, \bar{v} \rangle$ for all $p \in \{\bar{u} \geq \alpha\}$ and $\{\bar{u} \geq \alpha\}$ is evenly convex. By Lemma 23, it follows that u is evenly quasiconcave.

(i) The statement follows by the necessity part of previous proof.

(ii) Suppose that further $u : \Delta(C) \rightarrow \mathbb{R}$ is upper semicontinuous. By Lemma 23, \bar{u} is upper semicontinuous and \bar{u} is evenly quasiconcave. By the previous part of the proof, (40) holds for the map $(t, v) \mapsto U_v(t)$. Let $\bar{p} \in \Delta(C)$. If \bar{p} is a global maximum for \bar{u} on $\Delta(C)$, then, by (41) and the definition of upper semicontinuous envelope,

$$\bar{u}(\bar{p}) \leq U_v(\langle \bar{p}, v \rangle) \leq U_v^+(\langle \bar{p}, v \rangle) \leq U_v(\langle \bar{p}, v \rangle + 1) \leq \bar{u}(\bar{p}) \quad \forall v \in \mathcal{V}_1(x),$$

and $u(p) = \bar{u}(\bar{p}) = \inf_{v \in \mathcal{V}_1(x)} U_v^+(\langle \bar{p}, v \rangle)$.

If \bar{p} is not a global maximum for \bar{u} on $\Delta(C)$, then, $\bar{u}(\bar{p}) \in \mathbb{R}$ and there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $\lambda_n \downarrow u(\bar{p})$ and $\bar{p} \notin \{\bar{u} \geq \lambda_n\} \neq \emptyset$ for all $n \in \mathbb{N}$. Since each set $\{\bar{u} \geq \lambda_n\}$ is nonempty, closed, and convex, by [33, Theorem 3.4], there is a sequence $\{v_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^C \setminus \{0\}$ such that $\langle \bar{p}, v_n \rangle + \varepsilon_n < \langle p, v_n \rangle$ for all $p \in \{\bar{u} \geq \lambda_n\}$, where $\varepsilon_n > 0$. As argued in the previous part of the proof, we can consider $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{V}_1(x)$. Hence, $\{\bar{u} \geq \lambda_n\} \subset \{v_n > \langle \bar{p}, v_n \rangle + \varepsilon_n\}$ for all $n \in \mathbb{N}$. That is, $\{v_n \leq \langle \bar{p}, v_n \rangle + \varepsilon_n\} \subset \{\bar{u} < \lambda_n\}$ for all $n \in \mathbb{N}$. This implies that $U_{v_n}(\langle \bar{p}, v_n \rangle + \varepsilon_n) \leq \lambda_n$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$

$$\begin{aligned} u(\bar{p}) &\leq \inf_{v \in \mathcal{V}_1(x)} U_v^+(\langle \bar{p}, v \rangle) \leq U_{v_n}^+(\langle \bar{p}, v_n \rangle) \\ &\leq U_{v_n}(\langle \bar{p}, v_n \rangle + \varepsilon_n) \leq \lambda_n \end{aligned}$$

²⁹Indeed, if \bar{v} was not, define $\tilde{v} = \bar{v} + (1 - \bar{v}(x))$. Then, $\tilde{v} \in \mathcal{V}_1(x)$. Since $\{\bar{u} \geq \bar{u}(\bar{p}) + \varepsilon\} \subseteq \Delta(C)$ and $\bar{p} \in \Delta(C)$, it follows that for each $p \in \{\bar{u} \geq \bar{u}(\bar{p}) + \varepsilon\}$

$$\begin{aligned} \langle \bar{p}, \bar{v} \rangle < \langle p, \bar{v} \rangle &\iff \langle \bar{p}, \bar{v} \rangle + (1 - \bar{v}(x)) < \langle p, \bar{v} \rangle + (1 - \bar{v}(x)) \\ &\iff \langle \bar{p}, \bar{v} + (1 - \bar{v}(x)) \rangle < \langle p, \bar{v} + (1 - \bar{v}(x)) \rangle \\ &\iff \langle \bar{p}, \tilde{v} \rangle < \langle p, \tilde{v} \rangle. \end{aligned}$$

which yields the result. Conversely, if in (40) we can replace U with U^+ , we have that u is the lower envelope of a family of upper semicontinuous and quasiconcave functions on $\Delta(C)$, that is, $p \mapsto U^+(\langle p, v \rangle, v)$ for all $v \in \mathcal{V}_1(x)$. Therefore, u is upper semicontinuous and quasiconcave on $\Delta(C)$. \blacksquare

We next proceed by proving two ancillary lemmas that will deliver the uniqueness part in our main representation results.

Lemma 26 *If $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ satisfies P.1 and P.2 and $(\bar{t}, \bar{v}) \in \mathbb{R} \times \mathcal{V}_1(x)$ then*

$$U(\bar{t}, \bar{v}) = \min_{v \in \mathcal{V}_1(x)} \left(\sup_{p \in \{\bar{v} \leq \bar{t}\}} U(\langle p, v \rangle, v) \right).$$

Proof.

Consider the program

$$\rho(v, \bar{v}, \bar{t}) = \sup_{p \in \{\bar{v} \leq \bar{t}\}} U(\langle p, v \rangle, v).$$

It is sufficient to show that $\rho(v, \bar{v}, \bar{t}) \geq \rho(\bar{v}, \bar{v}, \bar{t}) = U(\bar{t}, \bar{v})$ for all $v \in \mathcal{V}_1(x)$. For the second equality, just notice that, by P.1, $\rho(\bar{v}, \bar{v}, \bar{t}) = \sup_{p \in \{\bar{v} \leq \bar{t}\}} U(\langle p, \bar{v} \rangle, \bar{v}) \leq U(\bar{t}, \bar{v})$. Furthermore, since $\bar{v} \neq 0$, there exists $\bar{p} \in \mathbb{R}_0^C$ such that $\langle \bar{p}, \bar{v} \rangle = \bar{t}$. This implies the inverse inequality. To prove the first inequality, fix $v \in \mathcal{V}_1(x)$. We have two cases:

(i) Suppose $v \in \text{span}(\bar{v})$. Then, $v = \beta \bar{v}$ for some $\beta \in \mathbb{R}$. Since $v, \bar{v} \in \mathcal{V}_1(x)$, we have that $1 = v(x) = \beta \bar{v}(x) = \beta$. Then, it follows that $v = \bar{v}$ and $\rho(v, \bar{v}, \bar{t}) = \rho(\bar{v}, \bar{v}, \bar{t}) = U(\bar{t}, \bar{v})$.

(ii) Suppose $v \notin \text{span}(\bar{v})$. By the Fundamental Theorem of Duality (see, e.g., [1, Theorem 5.91]), $\ker(\bar{v}) \not\subseteq \ker(v)$. That is, there exists $p \in \mathbb{R}_0^C$ such that $\langle p, \bar{v} \rangle = 0$ and $\langle p, v \rangle \neq 0$. Choose $\bar{p} \in \mathbb{R}_0^C$ such that $\langle \bar{p}, \bar{v} \rangle = \bar{t}$, then the straight line $\bar{p} + \alpha p$ is thus included into the hyperplane $\{\bar{v} = \bar{t}\}$. By P.1 and P.2, it follows that

$$\rho(v, \bar{v}, \bar{t}) \geq \lim_{t \rightarrow \infty} U(t, v) = \lim_{t \rightarrow \infty} U(t, \bar{v}) \geq U(\bar{t}, \bar{v}).$$

In sum, $\rho(v, \bar{v}, \bar{t}) \geq U(\bar{t}, \bar{v})$ for all $v \in \mathcal{V}_1(x)$ and $\rho(\bar{v}, \bar{v}, \bar{t}) = U(\bar{t}, \bar{v})$. \blacksquare

Lemma 27 *If $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ satisfies P.1-P.3 then*

$$\sup \{\hat{u}(p) : \langle p, v \rangle \leq t\} = U(t, v) \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x) \quad (44)$$

where $\hat{u}(p) = \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v)$ for all $p \in \mathbb{R}_0^C$.

Proof.

Fix $(\bar{t}, \bar{v}) \in \mathbb{R} \times \mathcal{V}_1(x)$. Observe that

$$\sup \{\hat{u}(p) : \langle p, \bar{v} \rangle \leq \bar{t}\} = \sup_{p \in \{\bar{v} \leq \bar{t}\}} \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v).$$

By Lemma 26, we have that

$$\inf_{v \in \mathcal{V}_1(x)} \sup_{p \in \{\bar{v} \leq \bar{t}\}} U(\langle p, v \rangle, v) = U(\bar{t}, \bar{v}).$$

Since it is well known that

$$\sup_{p \in \{\bar{v} \leq \bar{t}\}} \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v) \leq \inf_{v \in \mathcal{V}_1(x)} \sup_{p \in \{\bar{v} \leq \bar{t}\}} U(\langle p, v \rangle, v) = U(\bar{t}, \bar{v}), \quad (45)$$

it remains to prove the converse inequality. If $U(\bar{t}, \bar{v}) = \inf_{(t,v) \in \mathbb{R} \times \mathcal{V}_1(x)} U(t, v)$ then equality in (45) is easily checked. Otherwise, set $\alpha = U(\bar{t}, \bar{v})$. We have that $\alpha > -\infty$. Moreover, for each scalar $\beta < \alpha$, $L_\beta = \{(t, v) \in \mathbb{R} \times \mathcal{V}_1(x) : U(t, v) \leq \beta\}$ is \diamond -evenly convex and $(\bar{t}, \bar{v}) \notin L_\beta$. If β is large enough, L_β is neither empty nor $\mathbb{R} \times \mathcal{V}_1(x)$. Therefore, there is $\bar{p} \in \mathbb{R}_0^C$ and $\bar{s} \neq 0$ such that,

$$\langle \bar{p}, v \rangle + \bar{s}t > \langle \bar{p}, \bar{v} \rangle + \bar{s}\bar{t} \quad \forall (t, v) \in L_\beta. \quad (46)$$

Since U is increasing in the first component, it is easy to see that $\bar{s} < 0$.³⁰ Define $\lambda = \bar{t} - \left\langle \frac{\bar{p}}{|\bar{s}|}, \bar{v} \right\rangle$ and $\hat{p} = \frac{\bar{p}}{|\bar{s}|} + \lambda \delta_x$. It follows that $\langle \hat{p}, \bar{v} \rangle = \left\langle \frac{\bar{p}}{|\bar{s}|} + \lambda \delta_x, \bar{v} \right\rangle = \left\langle \frac{\bar{p}}{|\bar{s}|}, \bar{v} \right\rangle + \lambda \langle \delta_x, \bar{v} \rangle = \left\langle \frac{\bar{p}}{|\bar{s}|}, \bar{v} \right\rangle + \lambda = \bar{t}$ and for all $(t, v) \in L_\beta$

$$\begin{aligned} \langle \bar{p}, v \rangle + \bar{s}t > \langle \bar{p}, \bar{v} \rangle + \bar{s}\bar{t} &\implies \left\langle \frac{\bar{p}}{|\bar{s}|} + \lambda \delta_x, v \right\rangle - t > \left\langle \frac{\bar{p}}{|\bar{s}|} + \lambda \delta_x, \bar{v} \right\rangle - \bar{t} \\ &\implies \langle \hat{p}, v \rangle - t > \langle \hat{p}, \bar{v} \rangle - \bar{t} \\ &\implies \langle \hat{p}, v \rangle - t > 0. \end{aligned}$$

This implies that if $\langle \hat{p}, v \rangle - t \leq 0$ then $(t, v) \notin L_\beta$. If for each $v \in \mathcal{V}_1(x)$ we pick $t_v = \langle \hat{p}, v \rangle$ then $\langle \hat{p}, v \rangle - t_v = 0$. Therefore, $(t_v, v) = (\langle \hat{p}, v \rangle, v) \notin L_\beta$ for all $v \in \mathcal{V}_1(x)$. It follows that $U(\langle \hat{p}, v \rangle, v) > \beta$ for all $v \in \mathcal{V}_1(x)$. Since $\hat{p} \in \{\bar{v} \leq \bar{t}\}$, we have that

$$\alpha \geq \sup_{p \in \{\bar{v} \leq \bar{t}\}} \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v) \geq \inf_{v \in \mathcal{V}_1(x)} U(\langle \hat{p}, v \rangle, v) \geq \beta.$$

This is true for each β in a left neighborhood of α and close enough to α , thus

$$\sup_{p \in \{\bar{v} \leq \bar{t}\}} \inf_{v \in \mathcal{V}_1(x)} U(\langle p, v \rangle, v) = \alpha,$$

as desired. ■

B Proofs

In this Appendix, we prove all the results of the paper. The results contained in Subsection 3.2 are proven in Subsection B.1 and the result contained in Subsubsection 3.2.1 is proven in Subsection B.2. We start by providing the main existence result for a utility function on which our representation result rests.

Theorem 28 (Monteiro, 1987, Theorem 3) *Let \succsim be a binary relation on $\Delta(C)$. The following are equivalent facts:*

- (i) \succsim satisfies A.1, A.2, and A.6 (resp., A.1, A.4, and A.6);
- (ii) there exists a mixture continuous (resp., continuous) function $u : \Delta(C) \rightarrow \mathbb{R}$ such that for all $p, q \in \Delta(C)$

$$p \succsim q \iff u(p) \geq u(q).$$

³⁰By contradiction, assume that $\bar{s} > 0$. Take $(t', v') \in L_\beta$, then, by monotonicity of U , $(t' - n, v') \in L_\beta$ for all $n \in \mathbb{N}$. Therefore, it would follow that $\langle \bar{p}, v' \rangle + \bar{s}t' - \bar{s}n > \langle \bar{p}, \bar{v} \rangle + \bar{s}\bar{t}$ for all $n \in \mathbb{N}$, which is a contradiction.

Notice that this result differs from Eilenberg [13] (see also Debreu [8, (1) pag. 56]) for the reason that no assumption is made on the topology of $\Delta(C)$ in terms of separability of the set over which the binary relation is taken. Particularly, if C is uncountable then such usual existence result cannot be applied. Corollary 29 shows that u can be assumed to be upper semicontinuous and quasiconcave too as soon as the binary relation is assumed to satisfy A.3 and A.7. The natural existence result to look at, in presence of upper semicontinuity, is the one proposed by Rader [32] and correctly proven by Bosi and Mehta [4, Corollary 5]. But, again in the case C is uncountable, $\Delta(C)$ is not separable and the result of [32] cannot be applied. Our concern regarding properties of upper semicontinuity is deeply connected to the dual theory we use to represent the utility function. Furthermore, we would like to retain the assumption of mixture continuity, instead of A.4, as much as we could, since it is pretty weak and more in the spirit of the literature of choice under risk.

Corollary 29 *Let \succsim be a binary relation on $\Delta(C)$. The following are equivalent facts:*

- (i) \succsim satisfies A.1, A.2, A.3, A.6, and A.7;
- (ii) there exists a mixture continuous, upper semicontinuous, and quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ such that for all $p, q \in \Delta(C)$

$$p \succsim q \iff u(p) \geq u(q).$$

Remark 30 *It is immediate to see that if we replace in (i) the Axioms A.2 and A.7 with Axiom A.4 then we can replace in (ii) mixture continuous and upper semicontinuous with continuous.*

In order to prove Corollary 29, we need an ancillary fact.

Lemma 31 *Let \succsim be a binary relation on $\Delta(C)$. If \succsim satisfies A.1, A.2, and A.3 then for each $q \in \Delta(C)$ we have that*

$$\{p \in \Delta(C) : p \succsim q\}$$

is a convex set.

Proof.

The proof follows by using the same techniques of [6, Lemma 66]. ■

Proof of Corollary 29.

(ii) implies (i). The proof is standard and therefore left to the reader.

(i) implies (ii). Assume that A.1, A.2, A.3, A.6, and A.7 are satisfied by \succsim . By Theorem 28, there exists a mixture continuous function u that represents \succsim . Since u is mixture continuous and $\Delta(C)$ is convex, the set $u(\Delta(C))$ is connected, that is, $u(\Delta(C))$ is an interval. This implies that for a generic $\alpha \in \mathbb{R}$ the set $\{p \in \Delta(C) : u(p) \geq \alpha\} \in \{\emptyset, \{p \in \Delta(C) : u(p) \geq u(q)\}, \Delta(C)\}$ where $q \in \Delta(C)$ and $u(q) = \alpha$. In all three cases, by Lemma 31 and since u represents \succsim and \succsim satisfies A.7, it follows that the upper contour set of u is closed and convex, proving that u is upper semicontinuous and quasiconcave. ■

Proof of Theorem 13.

(ii) implies (i). Since $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$ then U is linearly mixture continuous and, by assumption, $u = u_U$ represents \succsim on $\Delta(C)$. It follows that u on $\Delta(C)$ is real valued, mixture continuous,

and it represents \succsim . Since $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$, it follows that $u = u_{U^+} = u_U$ on $\Delta(C)$. Therefore, by Theorem 25, u is upper semicontinuous and quasiconcave on $\Delta(C)$ as well. By Corollary 29, it follows that \succsim satisfies A.1, A.2, A.3, A.6, and A.7.

(i) implies (ii). Assume that \succsim satisfies A.1, A.2, A.3, A.6, and A.7. By Corollary 29, there exists a mixture continuous, upper semicontinuous, and (evenly) quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ that represents \succsim . Define \bar{u} as in (37). By Theorem 25 part (i), it follows that $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$U(t, v) = \sup \{ \bar{u}(p) : \langle p, v \rangle \leq t \} = U_v(t) \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x), \quad (47)$$

satisfies (28) and it belongs to $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$. Indeed, by Theorem 25, U is such that

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C).$$

Hence, (28) is satisfied. By (47) and Lemma 24, it follows that U satisfies P.1-P.3. By (47), Theorem 25, and since u is real valued and mixture continuous, it follows that U satisfies P.5. By (47) and Theorem 25 point (ii), it follows that U satisfies P.6.

Next, assume that two generic functions $U, U' \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$ are such that U, U' represent \succsim as in (28). Moreover, assume that $u_U = \hat{u} = u_{U'}$. Then, by Lemma 27, it follows that $U(t, v) = \sup \{ \hat{u}(p) : \langle p, v \rangle \leq t \} = U'(t, v)$ for all $(t, v) \in \mathbb{R} \times \mathcal{V}_1(x)$, proving that U is essentially unique.

Finally, assume that (i) or equivalently (ii) is satisfied and that $u : \Delta(C) \rightarrow \mathbb{R}$ is mixture continuous and it represents \succsim . By the first assumption, it follows that \succsim satisfies A.3 and A.7. By A.7 and Lemma 31 and since u is mixture continuous, it follows that u is upper semicontinuous and quasiconcave. Hence, u is mixture continuous, upper semicontinuous, and quasiconcave function that represents \succsim . By the proof of (i) implies (ii) and (47), we have that $u = u_U$ on $\Delta(C)$ where $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$. Moreover, by (47), it follows that $U^* = U$. Hence, U^* belongs to $\mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$ and it represents \succsim as in (28). \blacksquare

We next prove Theorem 1 and Theorem 9. It is enough to notice that the only axioms missing with respect to Theorem 13 are A.6 and A.7. However, the assumption that C is at most countable is sufficient to obtain from the other axioms the upper semicontinuity of \succsim and its countable boundedness.

Lemma 32 *Let $\Delta(C)$ be the space of simple lotteries over a set C and \succsim a binary relation on $\Delta(C)$ that satisfies A.1, A.2, and A.3. The following statements are true:*

- (i) *If C is finite then \succsim satisfies A.7.*
- (ii) *If C is countable then \succsim satisfies A.7.*

Proof.

If C is finite recall that we can identify \mathbb{R}_0^C and \mathbb{R}^C with $\mathbb{R}^{|C|}$ and the topology we consider on \mathbb{R}_0^C coincides with the usual Euclidean topology. Conversely, if C is countable then $C = \{x_k\}_{k \in \mathbb{N}}$ and, for the purpose of this proof, we consider a different topology on \mathbb{R}_0^C : the inductive limit topology \mathcal{E} generated by the family of all finite dimensional vector subspaces of \mathbb{R}_0^C . This topology is the finest locally convex topology on \mathbb{R}_0^C . It turns out that \mathbb{R}^C is the topological dual of $(\mathbb{R}_0^C, \mathcal{E})$ (for further details on \mathcal{E} see [2]). The proof of part (i) follows from an adaptation of the arguments contained in Dubra, Maccheroni, and Ok [12, Proof of Proposition 1]. We report just the parts of the proof where our arguments are different from the ones of the aforementioned authors.

(i). Fix $q \in \Delta(C)$. By Lemma 31 and since \succsim satisfies A.1, A.2, and A.3, it follows that $\{p \in \Delta(C) : p \succsim q\}$ is a convex set. By using the same techniques contained in [12, Proof of Proposition 1], it follows that

$$\{p \in \Delta(C) : p \succsim q\}$$

is closed in $\Delta(C)$.

(ii). Fix an upper contour set, $\{p \in \Delta(C) : p \succsim q\}$, and consider $C = \{x_k\}_{k \in \mathbb{N}}$. Since C is countable, it is immediate to see that $\{\delta_{x_k}\}_{k \in \mathbb{N}}$ is a countable Hamel basis for \mathbb{R}_0^C . Define $F_n = \text{span}\{\delta_{x_k}\}_{k=1}^n$ for all $n \in \mathbb{N}$. Notice that the family $\{F_n\}_{n \in \mathbb{N}}$ is increasing, with respect to set inclusion, and given any finite dimensional vector space F in \mathbb{R}_0^C there exists an n big enough such that $F_n \supseteq F$.

Next, fix $n \in \mathbb{N}$. F_n is a finite dimensional vector space endowed with the relative topology induced by \mathcal{E} . It follows that F_n is isomorphic, algebraically and topologically, to \mathbb{R}^n . Let us call $i_n : F_n \rightarrow \mathbb{R}^n$ such isomorphism. It is immediate to see that we can choose it to be such that $i_n^{-1}(\Delta(\{x_k\}_{k=1}^n)) \subseteq \Delta(C)$. In light of such fact, we can consider the binary relation \succsim^n on the set $\Delta(\{x_k\}_{k=1}^n)$ defined by

$$p' \succsim^n q' \iff i_n^{-1}(p') \succ i_n^{-1}(q') \quad \forall p', q' \in \Delta(\{x_k\}_{k=1}^n).$$

Since i_n^{-1} is a bijection, \succsim^n is a well defined, complete, and transitive binary relation. Similarly, since i_n^{-1} is an isomorphism of topological vector spaces, \succsim^n turns out to satisfy A.2 and A.3. Therefore, by (i), it follows that $\{p' \in \Delta(\{x_k\}_{k=1}^n) : p' \succsim^n q'\}$ is closed. If we define $q' = i_n(q)$ for each $q \in \Delta(C) \cap F_n$ then

$$\{p \in \Delta(C) : p \succsim q\} \cap F_n = i_n^{-1}(\{p' \in \Delta(\{x_k\}_{k=1}^n) : p' \succsim^n q'\})$$

is closed. Finally, since n was picked to be generic, we have that $\{p \in \Delta(C) : p \succsim q\} \cap F_n$ is a closed subset of F_n for all $n \in \mathbb{N}$. By [2, Corollary 2.1], we can conclude that $\{p \in \Delta(C) : p \succsim q\}$ is closed in \mathbb{R}_0^C with respect to \mathcal{E} . By Lemma 31, $\{p \in \Delta(C) : p \succsim q\}$ is convex. By [33, Theorem 3.12], it follows that $\{p \in \Delta(C) : p \succsim q\}$ is closed even with respect to the weak topology induced by \mathbb{R}^C and the statement follows. \blacksquare

Lemma 33 *Let $\Delta(C)$ be the space of simple lotteries over a set C and \succsim a binary relation on $\Delta(C)$ that satisfies A.1, A.2, and A.3. The following statements are true:*

- (i) *If C is finite then \succsim satisfies A.6.*
- (ii) *If C is countable then \succsim satisfies A.6.*

Proof.

We use the notation of Lemma 32.

(i). By assumption, \succsim satisfies A.1, A.2, and A.3 and C is finite. By Lemma 32, it follows that \succsim satisfies A.7. Since C is finite, it follows that $\Delta(C)$ is compact. Therefore, since \succsim satisfies A.1 and A.7, there exists \bar{q} such that $\bar{q} \succsim p$ for all $p \in \Delta(C)$. Since C is finite and \succsim satisfies A.1, it follows that there exists \bar{x} such that $y \succsim \bar{x}$ for all $y \in C$. By Lemma 31, the set $\left\{p \in \Delta(C) : p \succsim \bar{\delta}_x\right\}$ is convex. Since the latter set contains \bar{C} , it follows that $p \succsim \bar{\delta}_x$ for all $p \in \Delta(C)$. If we define $\{p_k\}_{k \in \mathbb{Z}}$ such that $p_k = \bar{\delta}_x$ for all $k \leq 0$ and $p_k = \bar{q}$ for all $k > 0$ then it is immediate to see that \succsim satisfies A.6.

(ii). By assumption, \succsim satisfies A.1, A.2, and A.3 and C is countable. Fix $n \in \mathbb{N}$ and consider \succsim restricted to $\Delta(C) \cap F_n$. By the same arguments of point (i), it follows that there exist \bar{q}_n and δ_{z_n} such that $\bar{q}_n \succsim p$ and $p \succsim \delta_{z_n}$ for all $p \in \Delta(C) \cap F_n$. Define $\{p_k\}_{k \in \mathbb{Z}}$ such that

$$p_k = \begin{cases} \delta_{z_{-k}} & k < 0 \\ \delta_{x_1} & k = 0 \\ \bar{q}_k & k > 0 \end{cases} .$$

Pick $p \in \Delta(C)$ and define $\bar{n} = \max\{n \in \mathbb{N} : x_n \in \text{supp}\{p\}\}$. By construction, it follows that $p_{\bar{n}} \succsim p \succsim p_{-\bar{n}}$. That is, \succsim satisfies A.6. \blacksquare

Proof of Theorem 1.

(i) implies (ii). By assumption, \succsim satisfies A.1, A.2, A.3. By Lemma 33 part (i), it follows that \succsim satisfies A.6. By Lemma 32 part (i), it follows that \succsim satisfies A.7. Hence, the statement follows by Theorem 13.

(ii) implies (i). The statement follows by Theorem 13.

Same argument of Theorem 13 applies for U^* . \blacksquare

Proof of Theorem 9.

We need just to prove the equivalence of (i) and (ii) under the case C is countable, since if C is finite then it descends from Theorem 1.

(i) implies (ii). By assumption, \succsim satisfies A.1, A.2, A.3. By Lemma 33 part (ii), it follows that \succsim satisfies A.6. By Lemma 32 part (ii), it follows that \succsim satisfies A.7. Hence, the statement follows by Theorem 13.

(ii) implies (i). The statement follows by Theorem 13.

Same argument of Theorem 13 applies for U^* . \blacksquare

Remark 34 *It is immediate to see that if in (i) of Theorem 1, Theorem 9, and Theorem 13 we replace Axiom A.2 (and A.7) with Axiom A.4 then we can require in (ii) that $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ (resp., u continuous and $U^* \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$).*

Proof of Proposition 7.

Since \succsim is represented by the utility function defined in (21) and $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$, by Theorem 9 and Remark 34, \succsim satisfies A.1, A.3, and A.4. Finally, suppose that $p \succsim' q$. By assumption and (18), this is equivalent to say that $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$. Since $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$, this implies that $U(\mathbb{E}_v(p), v) \geq U(\mathbb{E}_v(q), v)$ for all $v \in \mathcal{W}$. By (21), this implies that $u(p) \geq u(q)$, hence that $p \succsim q$.

Finally, notice that the arguments used above do not rely on C being finite. Therefore, the statement of Proposition 7 holds true even for C countable, as claimed in the main text. \blacksquare

In order to prove Proposition 8 and Proposition 10, we prove a pair of ancillary lemmas.

Lemma 35 *Let $\Delta(C)$ be the space of simple lotteries over an at most countable set C and \succsim a binary relation on $\Delta(C)$. If \succsim satisfies A.1, A.3, and A.4 then \succsim' , defined as in (22), is a preorder such that*

(a) *for each $p, q, r, s \in \Delta(C)$ the set $\{\lambda \in [0, 1] : \lambda p + (1 - \lambda)r \succsim' \lambda q + (1 - \lambda)s\}$ is closed;*

(b) *for each $p, q, r \in \Delta(C)$ and for each $\lambda \in [0, 1]$ if $p \succsim' q$ then $\lambda p + (1 - \lambda)r \succsim' \lambda q + (1 - \lambda)r$;*

(c) for each $p, q \in \Delta(C)$ if $p \succsim' q$ then $p \succsim q$;

(d) if \succsim'' is a binary relation that satisfies (b) and (c) then $p \succsim'' q$ implies $p \succsim' q$.

Proof.

It is immediate to verify that \succsim' is a preorder, that is, \succsim' is a reflexive and transitive binary relation.

(a). We consider four generic elements $p, q, r, s \in \Delta(C)$. Notice that the set

$$\{\lambda \in [0, 1] : \lambda p + (1 - \lambda)r \succsim' \lambda q + (1 - \lambda)s\}$$

is equal to the set

$$\bigcap_{\substack{t \in \Delta(C) \\ \mu \in (0, 1]}} \{\lambda \in [0, 1] : \lambda(\mu p + (1 - \mu)t) + (1 - \lambda)(\mu r + (1 - \mu)t) \succsim \lambda(\mu q + (1 - \mu)t) + (1 - \lambda)(\mu s + (1 - \mu)t)\}.$$

Since \succsim satisfies A.1 and A.4, this is an intersection over closed sets proving that

$$\{\lambda \in [0, 1] : \lambda p + (1 - \lambda)r \succsim' \lambda q + (1 - \lambda)s\}$$

is closed.³¹

(b). Consider $p, q, r \in \Delta(C)$ and $\lambda \in [0, 1]$ such that $p \succsim' q$. If $\lambda = 0, 1$ then it follows trivially that $\lambda p + (1 - \lambda)r \succsim' \lambda q + (1 - \lambda)r$. Otherwise, since $p \succsim' q$, observe that for each $s \in \Delta(C)$ and $\mu \in (0, 1]$ we have that

$$\begin{aligned} \mu(\lambda p + (1 - \lambda)r) + (1 - \mu)s &= \\ &= (\mu\lambda)p + (1 - \mu\lambda)\left(\frac{\mu(1 - \lambda)}{1 - \mu\lambda}r + \frac{1 - \mu}{1 - \mu\lambda}s\right) \\ &\succsim (\mu\lambda)q + (1 - \mu\lambda)\left(\frac{\mu(1 - \lambda)}{1 - \mu\lambda}r + \frac{1 - \mu}{1 - \mu\lambda}s\right) \\ &= \mu(\lambda q + (1 - \lambda)r) + (1 - \mu)s. \end{aligned}$$

The statement then follows by definition of \succsim' .

(c). The statement follows trivially by definition of \succsim' and by choosing in such definition $\lambda = 1$.

(d). Suppose that $p \succsim'' q$. Since \succsim'' satisfies (b), we have that

$$\lambda p + (1 - \lambda)r \succsim'' \lambda q + (1 - \lambda)r \quad \forall \lambda \in (0, 1], \forall r \in \Delta(C).$$

Since \succsim'' satisfies (c), it follows that $\lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r$ for all $\lambda \in (0, 1]$ and for all $r \in \Delta(C)$, implying the statement. ■

We next show that a *generic* preorder \succsim' on $\Delta(C)$ that satisfies the properties (a) and (b) of Lemma 35 can be represented by a multi-utility representation as in (18). Such result is very much related and close to [12]. When C is finite, it is exactly [12, Proposition 1]. On the other hand, when C is countable, the result is novel and it provides an alternative answer to an open question contained in [12, Remark 1] (see also [15]).³²

³¹Recall that if \succsim satisfies A.1 and A.4, it follows that if $\{p_\alpha\}_{\alpha \in A}$ and $\{q_\alpha\}_{\alpha \in A}$ are such that $p_\alpha \rightarrow p$, $q_\alpha \rightarrow q$, and $p_\alpha \succsim q_\alpha$ for all $\alpha \in A$ then $p \succsim q$.

³²In the case C is countable the result of Evren [15, Theorem 2] cannot be invoked. Indeed, although we can make C a σ -compact metric space, by endowing it with the discrete metric, our requirement of continuity is significantly weaker.

Lemma 36 Let $\Delta(C)$ be the space of simple lotteries over an at most countable set C and \succsim' a preorder on $\Delta(C)$ that satisfies (a) and (b) of Lemma 35. If we define

$$C(\succsim') = \{\lambda(p - q) : \lambda > 0 \text{ and } p \succsim' q\}$$

then $C(\succsim')$ is a closed and convex cone.

Proof.

We use the notation of Lemma 32.

It is a routine argument to verify that the arguments contained in [12, Lemma 1 and Lemma 2] work in our setting as well. Therefore, it follows that $C(\succsim')$ is a convex cone, $p - q \in C(\succsim')$ if and only if $p \succsim' q$, and if $\lambda p + (1 - \lambda)r \succsim' \lambda q + (1 - \lambda)r$ where $\lambda \in (0, 1]$ and $r \in \Delta(C)$ then $p \succsim' q$.

Next, we show that $C(\succsim')$ is closed. If C is finite then the statement follows from the proof of [12, Proposition 1]. If C is countable then $C = \{x_k\}_{k \in \mathbb{N}}$. Since C is countable, it is immediate to see that $\{\delta_{x_k}\}_{k \in \mathbb{N}}$ is a countable Hamel basis for \mathbb{R}_0^C . Define $F_n = \text{span}\{\delta_{x_k}\}_{k=1}^n$ for all $n \in \mathbb{N}$. Notice that the family $\{F_n\}_{n \in \mathbb{N}}$ is increasing, with respect to set inclusion, and given any finite dimensional vector space F in \mathbb{R}_0^C there exists an n big enough such that $F_n \supseteq F$. Next, fix $n \in \mathbb{N}$. F_n is a finite dimensional vector space and it is isomorphic, algebraically and topologically, to \mathbb{R}^n . Let us call $i_n : F_n \rightarrow \mathbb{R}^n$ such isomorphism. It is immediate to see that we can choose it to be such that $i_n^{-1}(\Delta(\{x_k\}_{k=1}^n)) \subseteq \Delta(C)$. In light of such fact, we can consider the binary relation \succsim^n on the set $\Delta(\{x_k\}_{k=1}^n)$ defined by

$$p' \succsim^n q' \iff i_n^{-1}(p') \succsim' i_n^{-1}(q') \quad \forall p', q' \in \Delta(\{x_k\}_{k=1}^n).$$

Since i_n^{-1} is a bijection and \succsim' is a preorder, \succsim^n is a well defined preorder. Similarly, since i_n^{-1} is an isomorphism of topological vector spaces, \succsim^n turns out to satisfy (a) and (b) of Lemma 35. It follows that

$$\begin{aligned} C(\succsim') \cap F_n &= \{\lambda(p - q) : \lambda > 0, p \succsim' q, \text{ and } p - q \in F_n\} \\ &= \{\lambda(p - q) : \lambda > 0, p \succsim' q, \text{ and } p, q \in F_n\} \\ &= i_n^{-1}(C(\succsim^n)). \end{aligned} \tag{48}$$

The first and third equality follow by definition. For the second one, it is immediate to see that $\{\lambda(p - q) : \lambda > 0, p \succsim' q, \text{ and } p, q \in F_n\} \subseteq \{\lambda(p - q) : \lambda > 0, p \succsim' q, \text{ and } p - q \in F_n\}$. For the opposite inclusion, suppose that $\lambda > 0, p \succsim' q$, and $p - q \in F_n$. If $p, q \in F_n$ then there is nothing to prove. If p or q do not belong to F_n then we have two cases: $\lambda(p - q) = 0$ or $\lambda(p - q) \neq 0$. In the first case, it is obvious that $\lambda(p - q)$ belongs to $\{\lambda(p - q) : \lambda > 0, p \succsim' q, \text{ and } p, q \in F_n\}$. In the second case, we have that there exists $\bar{n} \in \mathbb{N}$ such that $\text{supp}\{p\}, \text{supp}\{q\} \subseteq \{x_k\}_{k=1}^{\bar{n}}$. Since $\text{supp}\{p\}, \text{supp}\{q\} \subseteq \{x_k\}_{k=1}^{\bar{n}}$ and $p - q \in F_n$, it follows that $p(x_k) = q(x_k) = 0$ for all $k > \bar{n}$, $p(x_k) = q(x_k)$ for all $n < k \leq \bar{n}$, and $p(x_{k'}) = q(x_{k'}) > 0$ for some $n < k' \leq \bar{n}$. Define $c = \sum_{n < k \leq \bar{n}} p(x_k) = \sum_{n < k \leq \bar{n}} q(x_k)$. Since $0 \neq p - q \in F_n$ and $p, q \notin F_n$, it is immediate to see that $0 < c < 1$ and $0 < 1 - c < 1$. Given c , define \hat{p} and \hat{q} to be such that

$$\hat{p}(x_k) = \begin{cases} \frac{p(x_k)}{1-c} & 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{q}(x_k) = \begin{cases} \frac{q(x_k)}{1-c} & 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, define \hat{r} to be such that

$$\hat{r}(x_k) = \begin{cases} \frac{p(x_k)}{c} = \frac{q(x_k)}{c} & n < k \leq \bar{n} \\ 0 & \text{otherwise} \end{cases}.$$

It is immediate to see that $\hat{p}, \hat{q} \in F_n \cap \Delta(C)$ and similarly that $\hat{r} \in \Delta(C)$. By construction, we have that $\hat{p}, \hat{q} \in F_n$, $\hat{p} - \hat{q} = \frac{1}{1-c}(p - q)$, and

$$(1-c)\hat{p} + c\hat{r} = p \succsim' q = (1-c)\hat{q} + c\hat{r}.$$

By the initial part of the proof, this implies that $\hat{p} \succsim' \hat{q}$. It follows that

$$\lambda(p - q) = [\lambda(1-c)](\hat{p} - \hat{q}) \in \{\lambda(p - q) : \lambda > 0, p \succsim' q, \text{ and } p, q \in F_n\},$$

proving the second inequality.

By the proof of [12, Proposition 1] and since \succsim^n is a preorder that satisfies (a) and (b) of Lemma 35, we have that $C(\succsim^n)$ is a closed set for all $n \in \mathbb{N}$. By (48), this implies that for each $n \in \mathbb{N}$ the set $C(\succsim^n) \cap F_n$ is closed. By [2, Corollary 2.1], we have that $C(\succsim')$ is closed in the inductive limit topology \mathcal{E} . By [33, Theorem 3.12] and since $C(\succsim')$ is convex, it follows that it is closed in the topology induced by the topological dual, \mathbb{R}^C , proving the statement. \blacksquare

The proof of next Proposition follows from the same argument used in [12] and [15, Lemma 2]. For completeness and since we recur to a different normalization used later, we report it here. Recall that the key step and contribution in our result consists in the fact that, by Lemma 36, even if the set of consequences is infinite but countable, $C(\succsim')$ is closed.

Proposition 37 *Let $\Delta(C)$ be the space of simple lotteries over an at most countable set C and \succsim' a preorder on $\Delta(C)$. The following facts are equivalent:*

- (i) \succsim' satisfies (a) and (b) of Lemma 35;
- (ii) there exists a closed and convex set $\mathcal{W} \subseteq \mathcal{V}_1(x)$ such that $p \succsim' q$ if and only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$.

Proof.

(ii) implies (i). It follows from a routine argument.

(i) implies (ii). Consider the set

$$\mathcal{Z} = \{v \in \mathbb{R}^C : \langle r, v \rangle \geq 0 \text{ for all } r \in C(\succsim')\}.$$

We next show that $p \succsim' q$ if and only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{Z}$. By definition, if $p \succsim' q$ then $p - q \in C(\succsim')$ and so $\langle p - q, v \rangle \geq 0$ for all $v \in \mathcal{Z}$. That is, $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{Z}$. Conversely, by contradiction, assume that there exist $\hat{p}, \hat{q} \in \Delta(C)$ such that $\hat{p} \not\succeq' \hat{q}$ and that $\mathbb{E}_v(\hat{p}) \geq \mathbb{E}_v(\hat{q})$ for all $v \in \mathcal{Z}$. By the first part of the proof of Lemma 36, it follows that $\hat{p} - \hat{q} \notin C(\succsim')$. By Lemma 36, $C(\succsim')$ is a closed and convex cone in a locally convex topological vector space. By [33, Theorem 3.4], it follows that there exists $\bar{v} \in \mathbb{R}^C \setminus \{0\}$ such that $\langle \hat{p} - \hat{q}, \bar{v} \rangle < \gamma \leq \langle r, \bar{v} \rangle$ for all $r \in C(\succsim')$. Since $C(\succsim')$ is a cone and $0 \in C(\succsim')$, it follows that we can choose $\gamma = 0$. This implies that $\bar{v} \in \mathcal{Z}$ and $\mathbb{E}_{\bar{v}}(\hat{p}) < \mathbb{E}_{\bar{v}}(\hat{q})$, a contradiction. Finally, we are left to prove that we can replace \mathcal{Z} with a closed and convex set $\mathcal{W} \subseteq \mathcal{V}_1(x)$. Recall that x is fixed and arbitrary. Define the set $\mathcal{W} = \{v \in \mathbb{R}^C : v \in \mathcal{Z} \text{ and } v(x) = 1\}$. It is immediate to see that, by construction, $\mathcal{W} \subseteq \mathcal{V}_1(x)$, $\mathcal{W} \subseteq \mathcal{Z}$, and \mathcal{W} is closed and convex. In turn, this implies that $p \succsim' q$ only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$. Conversely, suppose that $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$ and consider a generic $v' \in \mathcal{Z}$. It follows that $v' + (1 - v'(x)) \in \mathcal{W}$. By assumption, we have that $\mathbb{E}_{v' + (1 - v'(x))}(p) \geq \mathbb{E}_{v' + (1 - v'(x))}(q)$. This implies that $\mathbb{E}_{v'}(p) + 1 - v'(x) \geq \mathbb{E}_{v'}(q) + 1 - v'(x)$, which in turn implies that $\mathbb{E}_{v'}(p) \geq \mathbb{E}_{v'}(q)$. Since v' was chosen to be generic, we have that $p \succsim' q$ if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$, proving the statement. \blacksquare

Proof of Proposition 10.

We use the notation of Proposition 37.

- (a). By Lemma 35 and Proposition 37, the statement follows.
- (b). It follows from Lemma 35 point (c).
- (c). It follows from Proposition 37 and Lemma 35 point (d).
- (d). Consider the stochastic order \succsim'' such that

$$p \succsim'' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \mathcal{W}'', \quad (49)$$

$\mathcal{W}'' \subseteq \mathcal{V}_1(x)$ is maximal, and \succsim is a completion of \succsim'' . By Proposition 37 and its proof, we have that the set

$$\mathcal{Z}'' = \{v \in \mathbb{R}^C : \langle r, v \rangle \geq 0 \text{ for all } r \in C(\succsim'')\}$$

represents \succsim'' as in (49). By Proposition 37 and its proof, we have that the set

$$\mathcal{V}'' = \{v \in \mathbb{R}^C : v \in \mathcal{Z}'' \text{ and } v(x) = 1\}$$

represents \succsim'' as in (49). Notice that, by construction, $\mathcal{W}'' \subseteq \mathcal{Z}''$. By assumption, we have that $v(x) = 1$ for all $v \in \mathcal{W}''$. Therefore, we have that $\mathcal{W}'' \subseteq \mathcal{V}''$. Since, by assumption, \mathcal{W}'' is maximal then it follows that $\mathcal{W}'' = \mathcal{V}''$. Finally, notice that by point (c) of this proposition, we have that $C(\succsim'') \subseteq C(\succsim')$. This implies that $\mathcal{Z} \subseteq \mathcal{Z}''$. That is, this implies that $\mathcal{W} \subseteq \mathcal{W}''$. \blacksquare

Proof of Proposition 8.

- (ii) implies (i). It follows from Theorem 1 and Remark 34.
- (i) implies (ii). We proceed by Steps.

Step 1: There exists a closed and convex set $\mathcal{W} \subseteq \mathcal{V}_1(x)$ such that \succsim' is represented as in (23) and \succsim is a completion of \succsim' .

Proof of the Step.

By Proposition 10 point (a) and (b), it follows that there exists a closed and convex set $\mathcal{W} \subseteq \mathcal{V}_1(x)$ such that \succsim' is represented as in (23) and \succsim is a completion of \succsim' . Choose \mathcal{W} as in the proof of Proposition 37 \square

Define $O = \{r \in \mathbb{R}_0^C : \langle r, v \rangle \geq 0 \text{ for all } v \in \mathcal{Z}\}$ where \mathcal{Z} is chosen as in the proof of Proposition 37. Recall that $\mathcal{W} = \{v \in \mathbb{R}^C : v \in \mathcal{Z} \text{ and } v(x) = 1\}$ and that $p \succsim' q$ if and only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{Z}$ if and only if $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{W}$. It is easy to see that $O = C(\succsim')$. We next show an ancillary fact.

Step 2: Given $q \in \Delta(C)$, the set $\{p \in \Delta(C) : p \succsim q\} + O$ is closed and convex.

Proof of the Step.

Since \succsim satisfies A.4 and C is finite, it follows that the set $\{p \in \Delta(C) : p \succsim q\}$ is compact. By Lemma 31, the latter set is convex. On the other hand, it is immediate to check that O is a convex and closed cone. Therefore, the algebraic sum of these two sets is closed and convex. \square

Step 3: Let q be an element of $\Delta(C)$. If $\hat{p} \notin \{p \in \Delta(C) : p \succsim q\}$ then there exists $v \in \mathcal{W}$ such that $\langle \hat{p}, v \rangle < \langle p, v \rangle$ for all $p \succsim q$.

Proof of the Step.

First observe that $\hat{p} \notin \{p \in \Delta(C) : p \succsim q\} + O$. Otherwise, by contradiction, there would exist $p \succsim q$ and $r \in O$ such that $\hat{p} = p + r$. This would imply that $\langle \hat{p} - p, v \rangle = \langle r, v \rangle \geq 0$ for all $v \in \mathcal{Z}$. By

Step 1 and since \succsim satisfies A.1, this would imply that $\hat{p} \succsim' p$, hence that $\hat{p} \succsim q$, a contradiction. By Step 2 and [33, Theorem 3.4], it follows that there exists $v \in \mathbb{R}^C \setminus \{0\}$ such that

$$\langle \hat{p}, v \rangle < \langle r', v \rangle \quad \forall r' \in \{p \in \Delta(C) : p \succsim q\} + O. \quad (50)$$

We next show that v can be chosen to belong to \mathcal{Z} and then \mathcal{W} . Fix $p' \in \{p \in \Delta(C) : p \succsim q\}$. Consider a generic $r \in O$. By (50), for each $n \in \mathbb{N}$ it follows that

$$\langle \hat{p}, v \rangle < \langle p' + nr, v \rangle \Rightarrow \langle r, v \rangle > \frac{\langle \hat{p}, v \rangle - \langle p', v \rangle}{n} \rightarrow 0.$$

Therefore, for each $r \in O = C(\succsim')$ we have that $\langle r, v \rangle \geq 0$. This implies that $v \in \mathcal{Z}$. By (50) and since $0 \in O$, if we define $\bar{v} = v + (1 - v(x))$ then it follows that $\bar{v} \in \mathcal{W}$ and $\langle \hat{p}, \bar{v} \rangle < \langle p, \bar{v} \rangle$ for all $p \succsim q$ \square

Step 4: There exists $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that the function $u : \Delta(C) \rightarrow \mathbb{R}$

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C)$$

represents \succsim .

Proof of the Step.

By Lemma 33 and since \succsim satisfies A.1, A.3, and A.4, it follows that \succsim satisfies A.6. By Corollary 29 and Remark 30, it follows that there exists a continuous and quasiconcave utility function $u : \Delta(C) \rightarrow \mathbb{R}$ such that $u(p) \geq u(q)$ if and only if $p \succsim q$. This implies that u is evenly quasiconcave. Choose then $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ such that

$$U(t, v) = \sup \{ \bar{u}(p) : \langle p, v \rangle \leq t \} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x)$$

where \bar{u} is defined as in (37). Since u is continuous and quasiconcave, by Lemma 24 and Theorem 25, we have that $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ and

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C).$$

Fix $p \in \Delta(C)$. By a careful inspection of the proof of Theorem 25 (“Only if” part), it follows that if p is a global maximum then $u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v)$. Otherwise, by Step 3, we can choose the separating functional \bar{v} in (43) to belong to \mathcal{W} . This proves that

$$u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C)$$

and it concludes the proof, since essential uniqueness follows from the same argument provided in the proof of Theorem 13.

Finally, assume that (i) or equivalently (ii) is satisfied and \succsim preserves a stochastic order \succsim'' . By the proof of (i) implies (ii), it follows that there exist $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ and a closed and convex subset, \mathcal{W} , of $\mathcal{V}_1(x)$ such that the function $u : \Delta(C) \rightarrow \mathbb{R}$, defined by

$$u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C),$$

represents \succsim and \mathcal{W} represents \succsim' as in (23). By Proposition 10 point (d), it follows that $\mathcal{W} \subseteq \mathcal{W}'' \subseteq \mathcal{V}_1(x)$. Therefore, we have that for each $p \in \Delta(C)$

$$u(p) = \inf_{v \in \mathcal{W}} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{W}''} U(\mathbb{E}_v(p), v) \geq \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) = u(p),$$

proving the statement. ■

Proof of Proposition 11.

We use the notation of Lemma 32. Fix $x = x_1$.

(i) implies (ii). We proceed by Steps. By assumption, $C = \{x_n\}_{n \in \mathbb{N}}$ where $x_n \succ x_{n+1}$ for all $n \in \mathbb{N}$. Consider $F : \mathbb{R}_0^C \rightarrow \mathbb{R}_0^C$ defined by

$$F(p)(x) = \sum_{x \succ y} p(y) \quad \forall x \in C.$$

Step 1: F is well defined.

Proof of the Step.

Define $\bar{n} = \max \{n \in \mathbb{N} : x_n \in \text{supp}\{p\}\}$ if $\text{supp}\{p\} \neq \emptyset$ and $\bar{n} = 1$, otherwise. Since $p \in \mathbb{R}_0^C$, it turns out that \bar{n} is well defined. Indeed, pick $n > \bar{n}$. Then, $F(p)(x_n) = \sum_{x_n \succ y} p(y) = 0$. □

Step 2: F is linear.

Proof of the Step.

Pick $p, q \in \mathbb{R}_0^C$ and $\lambda, \mu \in \mathbb{R}$. Then,

$$\begin{aligned} F(\lambda p + \mu q)(x) &= \sum_{x \succ y} (\lambda p + \mu q)(y) = \sum_{x \succ y} (\lambda p(y) + \mu q(y)) \\ &= \lambda \sum_{x \succ y} p(y) + \mu \sum_{x \succ y} q(y) = \lambda F(p)(x) + \mu F(q)(x). \end{aligned}$$

□

For each $v \in \mathbb{R}^C$ define $v' \in \mathbb{R}^C$ such that $v'(x_n) = \sum_{k=1}^n v(x_k)$ for all $n \in \mathbb{N}$. Notice that

$$v(x_n) = \begin{cases} v'(x_n) - v'(x_{n-1}) & n \geq 2 \\ v'(x_n) & n = 1 \end{cases}.$$

Step 3: $\langle F(p), v \rangle = \langle p, v' \rangle$ for all $v \in \mathbb{R}^C$ and for all $p \in \mathbb{R}_0^C$.

Proof of the Step.

First, notice that

$$\begin{aligned} F(p)(x_n) - F(p)(x_{n+1}) &= \sum_{x_n \succ y} p(y) - \sum_{x_{n+1} \succ y} p(y) \\ &= \sum_{x_n \succ y \succ x_{n+1}} p(y) + \sum_{x_{n+1} \succ y} p(y) - \sum_{x_{n+1} \succ y} p(y) = p(x_n) \quad \forall n \in \mathbb{N}. \end{aligned}$$

Then, we have that

$$\begin{aligned}
\langle F(p), v \rangle &= \sum_{n=1}^{\infty} F(p)(x_n) v(x_n) = \sum_{n=2}^{\infty} F(p)(x_n) v(x_n) + F(p)(x_1) v(x_1) \\
&= \sum_{n=2}^{\infty} F(p)(x_n) [v'(x_n) - v'(x_{n-1})] + F(p)(x_1) v'(x_1) \\
&= \sum_{n=2}^{\infty} F(p)(x_n) v'(x_n) - \sum_{n=2}^{\infty} F(p)(x_n) v'(x_{n-1}) + F(p)(x_1) v'(x_1) \\
&= \sum_{n=1}^{\infty} F(p)(x_n) v'(x_n) - \sum_{n=1}^{\infty} F(p)(x_{n+1}) v'(x_n) \\
&= \sum_{n=1}^{\infty} [F(p)(x_n) - F(p)(x_{n+1})] v'(x_n) = \sum_{n=1}^{\infty} p(x_n) v'(x_n) = \langle p, v' \rangle.
\end{aligned}$$

□

Step 4: F is continuous.

Proof of the Step.

Pick $\{p_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}_0^C$ and $p \in \mathbb{R}_0^C$ such that $p_\alpha \rightarrow p$. By Step 3, for each $v \in \mathbb{R}^C$ we have that

$$\langle F(p_\alpha), v \rangle = \langle p_\alpha, v' \rangle \rightarrow \langle p, v' \rangle = \langle F(p), v \rangle,$$

proving the statement. □

Step 5: F is a bijection.

Proof of the Step.

We first show that F is injective. Pick $p_1, p_2 \in \mathbb{R}_0^C$ and assume that $F(p_1) = F(p_2)$. By the proof of Step 3, it follows that

$$p_1(x_n) = F(p_1)(x_n) - F(p_1)(x_{n+1}) = F(p_2)(x_n) - F(p_2)(x_{n+1}) = p_2(x_n) \quad \forall n \in \mathbb{N}.$$

By routine arguments, we can show that F is surjective. □

Step 6: F^{-1} is a well defined, linear, and continuous function.

Proof of the Step.

By Step 5 and Step 2, it follows that F^{-1} is well defined and linear. Next consider, $\{r_\alpha\}_{\alpha \in A} \subseteq \mathbb{R}_0^C$ and $r \in \mathbb{R}_0^C$ such that $r_\alpha \rightarrow r$. It follows that for each $\alpha \in A$ there exists a unique $p_\alpha \in \mathbb{R}_0^C$ such that $F(p_\alpha) = r_\alpha$ and there exists a unique $p \in \mathbb{R}_0^C$ such that $F(p) = r$. Fix $v \in \mathbb{R}^C$ and define $v'' \in \mathbb{R}^C$ such that

$$v''(x_n) = \begin{cases} v(x_n) - v(x_{n-1}) & n \geq 2 \\ v(x_n) & n = 1 \end{cases}.$$

Then, it follows that $v(x_n) = \sum_{k=1}^n v''(x_k)$ for all $n \in \mathbb{N}$. By Step 3, it follows that

$$\langle F^{-1}(r_\alpha), v \rangle = \langle p_\alpha, v \rangle = \langle r_\alpha, v'' \rangle \rightarrow \langle r, v'' \rangle = \langle p, v \rangle = \langle F^{-1}(r), v \rangle.$$

□

Define $\mathcal{F} = F(\Delta(C))$. By Step 6, observe that \mathcal{F} is closed and convex. Fix $s \in \mathcal{F}$. Define $U = \{r \in \mathcal{F} : F^{-1}(r) \succsim F^{-1}(s)\}$. By Lemma 31, Lemma 32, Step 5, Step 6, and since \succsim satisfies A.1, A.2, and A.3, it follows that U is convex and closed.

Step 7: Given $m \in \mathbb{N}$, $F(p) \in F_m$ if and only if $p \in F_m$.

Proof of the Step.

First, we prove sufficiency. $F(p)(x_n) = \sum_{x_n \lesssim y} p(y) = 0$ if $n > m$, since $p(y) = 0$ if $x_n \lesssim y$ and $n > m$. We prove necessity. By assumption, $F(p)(x_n) = F(p)(x_{n+1}) = 0$ if $n > m$. By the proof of Step 3, we have that $p(x_n) = F(p)(x_n) - F(p)(x_{n+1}) = 0$ if $n > m$. \square

Step 8: Given $m \in \mathbb{N}$, $(U - (\mathbb{R}_0^C)_+) \cap F_m = U \cap F_m - (\mathbb{R}_0^C)_+ \cap F_m$.

Proof of the Step.

The inclusion $(U - (\mathbb{R}_0^C)_+) \cap F_m \supseteq U \cap F_m - (\mathbb{R}_0^C)_+ \cap F_m$ is obvious. For the opposite inclusion, pick $F(p) \in (U - (\mathbb{R}_0^C)_+) \cap F_m$. Then, $F(p) = F(p_1) - r_1$ where $F(p_1) \in U$, $r_1 \in (\mathbb{R}_0^C)_+$, $F(p_1) - r_1 \in F_m$, $p_1 \lesssim F^{-1}(s)$. We have two cases.

- 1) $F(p_1) \in F_m$ and $r_1 \in F_m$ then $F(p) \in U \cap F_m - (\mathbb{R}_0^C)_+ \cap F_m$.
- 2) $F(p_1)$ or r_1 do not belong to F_m . In such case, since $F(p) \in F_m$, we have that

$$\begin{aligned} F(p_1)(x_n) &= r_1(x_n) \geq 0 & \forall n > m \\ F(p_1)(x_n) &= r_1(x_n) > 0 & \text{for some } n > m. \end{aligned}$$

Notice that, by definition, $p_1 \lesssim F^{-1}(s)$. If $m \geq 2$ define $p_2 \in \Delta(C)$ to be such that

$$p_2(x_n) = \begin{cases} p_1(x_n) + \sum_{k=m+1}^{\infty} p_1(x_k) & n = m \\ p_1(x_n) & n \in \{1, \dots, m-1\} \\ 0 & \text{otherwise} \end{cases}.$$

Otherwise, define p_2 to be such that

$$p_2(x_n) = \begin{cases} p_1(x_n) + \sum_{k=m+1}^{\infty} p_1(x_k) & n = m = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Since C is ordered with a maximum element and by A.5, it follows that $p_2 \lesssim p_1$. Hence, by definition and Step 7, $F(p_2) \in U \cap F_m$. Next, notice that

$$\begin{aligned} F(p_1)(x_n) &= F(p_2)(x_n) & n \in \{1, \dots, m\} \\ F(p_2)(x_n) &= 0 & \forall n > m. \end{aligned}$$

Define $r_2 \in (\mathbb{R}_0^C)_+$ to be such that

$$r_2(x_n) = \begin{cases} r_1(x_n) & n \in \{1, \dots, m\} \\ 0 & \text{otherwise} \end{cases}.$$

It is immediate to see that $r_2 \in (\mathbb{R}_0^C)_+ \cap F_m$. Therefore, it follows that

$$\begin{aligned} (F(p_2) - r_2)(x_n) &= F(p_2)(x_n) - r_2(x_n) = F(p_1)(x_n) - r_1(x_n) = F(p)(x_n) & n \in \{1, \dots, m\} \\ (F(p_2) - r_2)(x_n) &= F(p_2)(x_n) - r_2(x_n) = 0 - 0 = F(p)(x_n) & n > m. \end{aligned}$$

\square

Step 9: The set $U - (\mathbb{R}_0^C)_+$ is closed and convex.

Proof of the Step.

Fix $m \in \mathbb{N}$. Since $(\mathbb{R}_0^C)_+$ is \mathcal{E} -closed, it is immediate to see that $-(\mathbb{R}_0^C)_+$ is \mathcal{E} -closed. By [2, Corollary 2.1], it follows that $-(\mathbb{R}_0^C)_+ \cap F_m$ is closed. By Step 6 and the proof of Lemma 32, we have that $U \cap F_m$ is closed as well. We next show that $U \cap F_m$ is compact. Pick $\{r_\alpha\}_{\alpha \in A} \subseteq U \cap F_m$. By Step 6 and Step 7, there exists $\{p_\alpha\}_{\alpha \in A} \subseteq \Delta(C) \cap F_m$ such that $F(p_\alpha) = r_\alpha$ for all $\alpha \in A$. Since $\Delta(C) \cap F_m$ is compact, there exists $\{p_{\alpha_\beta}\}_{\beta \in B} \subseteq \{p_\alpha\}_{\alpha \in A}$ such that $p_{\alpha_\beta} \rightarrow p \in \Delta(C) \cap F_m$. By construction and Lemma 32, it follows that $p_{\alpha_\beta} \succsim F^{-1}(s)$ for all $\beta \in B$, hence $p \succsim F^{-1}(s)$. By Step 4 and Step 7, we can conclude that $r_{\alpha_\beta} = F(p_{\alpha_\beta}) \rightarrow F(p) \in U \cap F_m$. By Step 8, it follows that $(U - (\mathbb{R}_0^C)_+) \cap F_m = U \cap F_m - (\mathbb{R}_0^C)_+ \cap F_m$ is closed and convex, being the algebraic sum of a compact and convex set with a closed and convex set. By [2, Corollary 2.1], $U - (\mathbb{R}_0^C)_+$ is closed with respect to the inductive limit topology \mathcal{E} of Lemma 32. By [33, Theorem 3.12] and since $U - (\mathbb{R}_0^C)_+$ is convex, it follows that $U - (\mathbb{R}_0^C)_+$ is closed. \square

Step 10: If $\hat{p} \notin \{p \in \Delta(C) : p \succsim F^{-1}(s)\}$ then $F(\hat{p}) \notin U - (\mathbb{R}_0^C)_+$.

Proof of the Step.

We argue by contradiction. First, notice that for each $q \in \Delta(C)$ we have that

$$\sum_{y \succsim x_n} q(y) = 1 - \sum_{x_n \succ y} q(y) = 1 - \sum_{x_{n+1} \succ y} q(y) = 1 - F(q)(x_{n+1}) \quad \forall n \in \mathbb{N}.$$

By contradiction, if $\hat{p} \notin \{p \in \Delta(C) : p \succsim F^{-1}(s)\}$ and $F(\hat{p}) \in U - (\mathbb{R}_0^C)_+$ then there would exist $p \succsim F^{-1}(s)$ such that $F(\hat{p}) = F(p) - r$ where $r \in (\mathbb{R}_0^C)_+$. This implies that $F(\hat{p}) \leq F(p)$. Therefore, for each $n \in \mathbb{N}$

$$\sum_{y \succsim x_n} \hat{p}(y) = 1 - F(\hat{p})(x_{n+1}) \geq 1 - F(p)(x_{n+1}) = \sum_{y \succsim x_n} p(y).$$

Since \succsim satisfies A.1 and A.5, this implies that $\hat{p} \succ p \succsim F^{-1}(s)$, a contradiction. \square

Step 11: If $q \in \Delta(C)$ and $\hat{p} \notin \{p \in \Delta(C) : p \succsim q\}$ then there exists $v' \in \mathcal{V}_{inc}$ such that

$$\langle \hat{p}, v' \rangle < \langle p, v' \rangle \quad \forall p \in \{p \in \Delta(C) : p \succsim q\}.$$

Proof of the Step.

Define $s = F(q)$. By Step 9 and Step 10, it follows that $F(\hat{p}) \notin U - (\mathbb{R}_0^C)_+$ where the latter set is a closed and convex set in a locally convex topological vector space. By [33, Theorem 3.4], it follows that there exists a function $v \in \mathbb{R}^C \setminus \{0\}$ such that $\langle F(\hat{p}), v \rangle < \langle r, v \rangle$ for all $r \in U - (\mathbb{R}_0^C)_+$. Therefore, if we fix a generic $y \in C$, $\bar{r} \in U$ we have that $\langle F(\hat{p}), v \rangle < \langle \bar{r} - n\delta_y, v \rangle$ for all $n \in \mathbb{N}$. This implies that $v(y) \leq 0$ for all $y \in C$. By Step 3 and since $0 \in (\mathbb{R}_0^C)_+$, it follows that for each $p \succ q$ we have that

$$\langle \hat{p}, v' \rangle = \langle F(\hat{p}), v \rangle < \langle F(p), v \rangle = \langle p, v' \rangle.$$

By definition of v' and since C is ordered with a maximum element, it follows that v' is increasing. After a normalization, v' can be chosen to belong to \mathcal{V}_{inc} . \square

Step 12: There exists $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x_1))$ such that the function $u : \Delta(C) \rightarrow \mathbb{R}$

$$u(p) = \inf_{v \in \mathcal{V}_1(x_1)} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_{inc}} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C)$$

represents \succsim .

Proof of the Step.

Since \succsim satisfies A.5 and C is ordered with a maximum element, it follows that \succsim satisfies A.6. It is enough to consider $\{\delta_{x_k}\}_{k \in \mathbb{N}}$ and notice that, given A.5, for each $p \in \Delta(C)$ we have that there exists k and k' such that $\delta_{x_k} \succsim p \succsim \delta_{x_{k'}}$. By Lemma 32 and Corollary 29, it follows that there exists a mixture continuous, upper semicontinuous, and quasiconcave utility function $u : \Delta(C) \rightarrow \mathbb{R}$ such that $u(p) \geq u(q)$ if and only if $p \succsim q$. Since u is mixture continuous, upper semicontinuous, and quasiconcave, this implies that u is evenly quasiconcave. Choose then $U : \mathbb{R} \times \mathcal{V}_1(x_1) \rightarrow [-\infty, \infty]$ such that

$$U(t, v) = \sup \{\bar{u}(p) : \langle p, v \rangle \leq t\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x_1)$$

where \bar{u} is defined as in (37). Since u is mixture continuous, upper semicontinuous, and quasiconcave, by Lemma 24 and Theorem 25 we have that $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x_1))$ and

$$u(p) = \inf_{v \in \mathcal{V}_1(x_1)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C).$$

Fix $p \in \Delta(C)$. By a careful inspection of the proof of Theorem 25 (“Only if” part), it follows that if p is a global maximum then $u(p) = \inf_{v \in \mathcal{V}_{inc}} U(\mathbb{E}_v(p), v)$. Otherwise, by Step 11, we can choose the separating functional \bar{v} in (43) to belong to \mathcal{V}_{inc} . This proves that

$$u(p) = \inf_{v \in \mathcal{V}_{inc}} U(\mathbb{E}_v(p), v)$$

and it concludes the proof, since essential uniqueness follows from the same argument provided in the proof of Theorem 13. \square

(ii) implies (i). By Theorem 9, we have that \succsim satisfies A.1, A.2, and A.3. Next, observe that if p and q satisfy (26) then $\mathbb{E}_v(p) \geq \mathbb{E}_v(q)$ for all $v \in \mathcal{V}_{inc}$. Since $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x_1))$, this implies that $U(\mathbb{E}_v(p), v) \geq U(\mathbb{E}_v(q), v)$ for all $v \in \mathcal{V}_{inc}$. Since u satisfies (27), it follows that $p \succsim q$, proving that \succsim satisfies A.5. \blacksquare

Proof of Proposition 15.

Fix $x = 1$.

(i) implies (ii). It is immediate to see that since \succsim , among the others, satisfies A.8 and A.9 then it satisfies A.6. It is enough to consider $\{\delta_k\}_{k \in \mathbb{Z}}$ to see that for each $p \in \Delta(\mathbb{R})$ there exist k, k' such that $\delta_k \succsim p \succsim \delta_{k'}$. By Corollary 29, it follows that there exists a mixture continuous, upper semicontinuous, and quasiconcave function, $u : \Delta(\mathbb{R}) \rightarrow \mathbb{R}$, representing \succsim . By A.9, it follows that $u(\mathbb{R}) = u(\Delta(\mathbb{R}))$ where the latter set turns out to be an interval.³³ By A.8, it follows that $x \geq y$ if and only if $u(x) \geq u(y)$. Therefore, there exists a strictly increasing function $f : u(\mathbb{R}) \rightarrow \mathbb{R}$ such that $f \circ u|_C = id_{\mathbb{R}}$. Since $u(\mathbb{R}) = u(\Delta(\mathbb{R}))$, we can define $\hat{u} = f \circ u$. This implies that \hat{u} represents \succsim and that $\mathbb{R} \supseteq \hat{u}(\Delta(\mathbb{R})) \supseteq \hat{u}(\mathbb{R}) = \mathbb{R}$. Since u is mixture continuous, upper semicontinuous, and quasiconcave, this latest fact implies that \hat{u} is a mixture continuous, upper semicontinuous, and quasiconcave utility function. Moreover, by construction, it is a certainty equivalent utility function for \succsim . Without loss of generality, then we can assume that u was already chosen to be \hat{u} . Define \bar{u} as in (38). Moreover, choose $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ defined by

$$U(t, v) = \sup \{\bar{u}(p) : \langle p, v \rangle \leq t\} = U_v(t) \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x). \quad (51)$$

It is immediate to see that $U \in \mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$ and it satisfies (29). Indeed, by Theorem 25 part (i), U satisfies (29). By (51) and Lemma 24, it follows that U satisfies P.1-P.3. By (51), Theorem 25, and

³³Recall that we embed C in $\Delta(C)$ given the injective map $x \mapsto \delta_x$ and that u is mixture continuous over the connected set $\Delta(C)$.

since u is real valued and mixture continuous, it follows that U satisfies P.5. By (51) and Theorem 25 point (ii), it follows that U satisfies P.6. Finally, by (29) and since u is a certainty equivalent utility function, it follows that for each $y \in \mathbb{R}$ we have that

$$y = u(y) = \inf_{v \in \mathcal{V}_1(x)} U(\langle \delta_y, v \rangle, v) = \inf_{v \in \mathcal{V}_1(x)} U(v(y), v).$$

The essential uniqueness follows from the same argument used in Theorem 13.

(ii) implies (i). Consider $U \in \mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$ and u defined as in (29). Since $U \in \mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$, it follows that $U \in \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$. By Theorem 13, this implies that \succsim satisfies A.1, A.2, A.3, and A.7. By P.7, u is a certainty equivalent utility function for \succsim . It follows that for any two $x, y \in \mathbb{R}$ we have that

$$x \geq y \iff u(x) \geq u(y) \iff x \succsim y. \quad (52)$$

Therefore, \succsim satisfies A.8. Since u is a certainty equivalent utility function, we have that $u(\mathbb{R}) = \mathbb{R}$. This implies that for each $p \in \Delta(\mathbb{R})$ there exists $x_p \in \mathbb{R}$ such that $x_p \sim p$ and, by (52), we have that such x_p is unique.

Finally, assume that (i) or equivalently (ii) is satisfied. By the proof of (i) implies (ii), there exists a certainty equivalent utility function u that represent \succsim and is mixture continuous, upper semicontinuous, and quasiconcave. Consider U as in (51). By the previous part of the proof we have that $u = u_U$ on $\Delta(\mathbb{R})$ and $U \in \mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$. Since u is a certainty equivalent utility function, it follows that $U = U^*$. \blacksquare

Proof of Proposition 16.

Recall that $x = 1$.

\succsim_1 and \succsim_2 satisfy (i) of Proposition 15 and consider U_1^*, U_2^* as in Proposition 15. Recall that $u_1 = u_{U_1^*}$ and $u_2 = u_{U_2^*}$, defined as in (29) on $\Delta(\mathbb{R})$, are two certainty equivalent utility functions, respectively, for \succsim_1 and \succsim_2 .

(i) implies (ii). Since \succsim_1 is more risk averse than \succsim_2 , it follows that $x_p^1 \leq x_p^2$ for all $p \in \Delta(\mathbb{R})$. Therefore, if we fix $(t, v) \in \mathbb{R} \times \mathcal{V}_1(x)$, it follows that

$$\begin{aligned} U_1^*(t, v) &= \sup \{x_p^1 : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(\mathbb{R})\} \\ &\leq \sup \{x_p^2 : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(\mathbb{R})\} \\ &= U_2^*(t, v). \end{aligned}$$

(ii) implies (i). Since $U_1^* \leq U_2^*$, it follows that $U_1^*(\mathbb{E}_v(p), v) \leq U_2^*(\mathbb{E}_v(p), v)$ for all $v \in \mathcal{V}_1(x)$ and for all $p \in \Delta(\mathbb{R})$. Since u_1 and u_2 satisfy (29), respectively, for U_1^* and U_2^* , we have that $u_1(p) \leq u_2(p)$ for all $p \in \Delta(\mathbb{R})$. Finally, since u_1 and u_2 are two certainty equivalent utility functions, it follows that

$$x_p^1 = u_1(p) \leq u_2(p) = x_p^2 \quad \forall p \in \Delta(\mathbb{R}),$$

proving the statement. \blacksquare

Next lemma gives sufficient conditions for a function $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ to satisfy P.3. This will turn out to be extremely useful in proving the statements contained in Example 17, Proposition 18, and Proposition 19.

Lemma 38 *Let $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$ be a function that satisfies P.1 and P.2. If U is lower semicontinuous, quasiconvex, and such that $\lim_{t \rightarrow \infty} (\inf_{v \in \mathcal{V}_1(x)} U(t, v)) = \lim_{t \rightarrow \infty} U(t, v')$ for some $v' \in \mathcal{V}_1(x)$ then U satisfies P.3.*

Proof.

First, observe that since U satisfies P.2 there exists $\beta \in [-\infty, \infty]$ such that $\beta = \lim_{t \rightarrow \infty} U(t, v)$ for all $v \in \mathcal{V}_1(x)$. Next, consider $\alpha \in \mathbb{R}$. Define $L_\alpha = \{(t, v) \in \mathbb{R} \times \mathcal{V}_1(x) : U(t, v) \leq \alpha\}$. If L_α is empty then it is vacuously \diamond -evenly convex. Since U satisfies P.1 and P.2, if $\alpha \geq \beta$ then $L_\alpha = \mathbb{R} \times \mathcal{V}_1(x)$ and again L_α is vacuously \diamond -evenly convex. Finally, suppose that $L_\alpha \neq \emptyset$ and $\alpha < \beta$. Consider $(\bar{t}, \bar{v}) \notin L_\alpha$. Since $\lim_{t \rightarrow \infty} (\inf_{v \in \mathcal{V}_1(x)} U(t, v)) = \beta$, we have that there exists $\bar{\alpha}$ such that

$$t > \bar{\alpha} \Rightarrow U(t, v) > \alpha \quad \forall v \in \mathcal{V}_1(x). \quad (53)$$

Moreover, since U is lower semicontinuous and quasiconvex, we have that L_α is nonempty, closed in the product topology, and convex.³⁴ Pick $(\bar{t}, \bar{v}) \notin L_\alpha$. By [33, Theorem 3.4], there exist $\varepsilon > 0$, $\bar{p} \in \mathbb{R}_0^C$, and $\bar{s} \in \mathbb{R}$ such that

$$\langle \bar{p}, \bar{v} \rangle + \bar{s}\bar{t} + \varepsilon < \langle \bar{p}, v \rangle + \bar{s}t \quad \forall (t, v) \in L_\alpha. \quad (54)$$

We next show that we can choose \bar{s} to be different from zero. If $\bar{s} \neq 0$ to start with then there is nothing to prove. Otherwise, (54) becomes $\langle \bar{p}, \bar{v} \rangle + \varepsilon < \langle \bar{p}, v \rangle$ for all $(t, v) \in L_\alpha$. Or equivalently, $\langle \bar{p}, v \rangle - \langle \bar{p}, \bar{v} \rangle > \varepsilon > 0$ for all $(t, v) \in L_\alpha$. Notice that, by (53), we have that $(t, v) \in L_\alpha$ only if $t \leq \bar{\alpha} < \infty$. Therefore, $\bar{t} - t \geq \bar{t} - \bar{\alpha}$ for all $(t, v) \in L_\alpha$. Choose \bar{s} such that

$$\bar{s} = \begin{cases} \frac{\varepsilon}{2(\bar{t} - \bar{\alpha} - 1)} & \text{if } \bar{t} - \bar{\alpha} = 0 \\ \frac{\varepsilon}{2(\bar{t} - \bar{\alpha})} & \text{if } \bar{t} - \bar{\alpha} < 0 \\ -\frac{\varepsilon}{2(\bar{t} - \bar{\alpha})} & \text{if } \bar{t} - \bar{\alpha} > 0 \end{cases}.$$

Notice that, in any case, $\bar{s} < 0$. Then, it follows that for each $(t, v) \in L_\alpha$

$$\bar{s}(\bar{t} - t) \leq \bar{s}(\bar{t} - \bar{\alpha}) = \begin{cases} \frac{\varepsilon}{2(\bar{t} - \bar{\alpha} - 1)}(\bar{t} - \bar{\alpha}) = 0 & \text{if } \bar{t} - \bar{\alpha} = 0 \\ \frac{\varepsilon}{2(\bar{t} - \bar{\alpha})}(\bar{t} - \bar{\alpha}) = \frac{\varepsilon}{2} & \text{if } \bar{t} - \bar{\alpha} < 0 \\ -\frac{\varepsilon}{2(\bar{t} - \bar{\alpha})}(\bar{t} - \bar{\alpha}) = -\frac{\varepsilon}{2} & \text{if } \bar{t} - \bar{\alpha} > 0 \end{cases}.$$

This implies that

$$\bar{s}(\bar{t} - t) < \varepsilon < \langle \bar{p}, v \rangle - \langle \bar{p}, \bar{v} \rangle \quad \forall (t, v) \in L_\alpha.$$

Since α and (\bar{t}, \bar{v}) were chosen to be generic, it follows that for each $\alpha \in \mathbb{R}$ and for each $(\bar{t}, \bar{v}) \notin L_\alpha$ there exist $\bar{s} \in \mathbb{R} \setminus \{0\}$ and $\bar{p} \in \mathbb{R}_0^C$ such that $\langle \bar{p}, \bar{v} \rangle + \bar{s}\bar{t} < \langle \bar{p}, v \rangle + \bar{s}t$ for all $(t, v) \in L_\alpha$. This implies that U satisfies P.3. \blacksquare

Proof of Example 17.

Recall that x was assumed to be equal to 1. Consider a binary relation \succsim' over $\Delta(\mathbb{R})$, defined by

$$p \succsim' q \Leftrightarrow \mathbb{E}_v(p) \geq \mathbb{E}_v(q) \quad \forall v \in \mathcal{W}, \quad (55)$$

where the set $\mathcal{W} \subseteq \mathcal{V}_1(x)$ is nonempty, compact, and convex, and each element $v \in \mathcal{W}$ is a strictly increasing function over the real line such that $v(\mathbb{R}) = \mathbb{R}$. Given the properties of \mathcal{W} , equivalently, we have that

$$p \succsim' q \Leftrightarrow v^{-1}(\mathbb{E}_v(p)) \geq v^{-1}(\mathbb{E}_v(q)) \quad \forall v \in \mathcal{W}.$$

Define the function $u : \Delta(\mathbb{R}) \rightarrow [-\infty, \infty)$ to be such that

$$u(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_v(p)) \quad \forall p \in \Delta(\mathbb{R}).$$

³⁴ $\{(t, v) \in \mathbb{R} \times \mathcal{V}_1(x) : U(t, v) \leq \alpha\}$ is a subset of the vector space $\mathbb{R} \times \mathbb{R}^C$. This latest set, endowed with the usual operations and the product topology, is a locally convex topological vector space.

Further, define \succsim on $\Delta(\mathbb{R})$ to be such that $p \succsim q$ if and only if $u(p) \geq u(q)$. We first show that we can replace, in the previous equation, infimum with minimum, proving that u is real valued as well. Fix $p \in \Delta(\mathbb{R})$ and consider the real valued mapping with domain \mathcal{W} such that $v \mapsto v^{-1}(\mathbb{E}_v(p))$. Notice that for each $\alpha \in \mathbb{R}$

$$\{v \in \mathcal{W} : v^{-1}(\mathbb{E}_v(p)) \leq \alpha\} = \{v \in \mathcal{W} : \mathbb{E}_v(p) \leq v(\alpha)\}.$$

Given the last equality, it is immediate to see that $\{v \in \mathcal{W} : v^{-1}(\mathbb{E}_v(p)) \leq \alpha\}$ is closed. This implies that the map $v \mapsto v^{-1}(\mathbb{E}_v(p))$ is lower semicontinuous. Since \mathcal{W} is compact, it follows that

$$u(p) = \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_v(p)) \quad \forall p \in \Delta(\mathbb{R}). \quad (56)$$

Next, fix $v \in \mathcal{W}$ and consider the real valued mapping with domain $\Delta(\mathbb{R})$ such that $p \mapsto v^{-1}(\mathbb{E}_v(p))$. Notice that for each $\alpha \in \mathbb{R}$

$$\{p \in \Delta(\mathbb{R}) : v^{-1}(\mathbb{E}_v(p)) \geq \alpha\} = \{p \in \Delta(\mathbb{R}) : \mathbb{E}_v(p) \geq v(\alpha)\}.$$

Given the last equality, it is immediate to see that $\{p \in \Delta(\mathbb{R}) : v^{-1}(\mathbb{E}_v(p)) \geq \alpha\}$ is closed and convex. By construction, this implies that u is the lower envelope of a family of upper semicontinuous and quasiconcave functions. It follows that u is upper semicontinuous and quasiconcave as well. Next, observe that for each $x \in \mathbb{R}$ and for each $v \in \mathcal{W}$ we have that $v^{-1}(\mathbb{E}_v(\delta_x)) = x$. This implies that $u(\delta_x) = x$ for all $x \in \mathbb{R}$. By definition, this implies that u is a certainty equivalent utility function and that u is not constant. Next, consider $p, q \in \Delta(\mathbb{R})$, $\gamma \in \mathbb{R}$, and a converging sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ to $\bar{\lambda}$ contained in

$$\begin{aligned} \{\lambda \in [0, 1] : \gamma \geq u(\lambda p + (1 - \lambda) q)\} &= \\ &= \bigcup_{v \in \mathcal{W}} \{\lambda \in [0, 1] : \gamma \geq v^{-1}(\mathbb{E}_v(\lambda p + (1 - \lambda) q))\} \\ &= \bigcup_{v \in \mathcal{W}} \{\lambda \in [0, 1] : v(\gamma) \geq \mathbb{E}_v(\lambda p + (1 - \lambda) q)\}. \end{aligned}$$

By (56), it follows that for each $n \in \mathbb{N}$ there exists $v_n \in \mathcal{W}$ such that $v_n(\gamma) \geq \lambda_n \mathbb{E}_{v_n}(p) + (1 - \lambda_n) \mathbb{E}_{v_n}(q)$. Since \mathcal{W} is compact, it follows that there exists a converging subnet $\{v_{n_\alpha}\}_{\alpha \in A} \subseteq \{v_n\}_{n \in \mathbb{N}}$ such that $v_{n_\alpha} \rightarrow v \in \mathcal{W}$. We can then conclude that

$$\begin{aligned} v(\gamma) &= \lim_{\alpha} v_{n_\alpha}(\gamma) \geq \lim_{\alpha} \lambda_{n_\alpha} \mathbb{E}_{v_{n_\alpha}}(p) + \lim_{\alpha} (1 - \lambda_{n_\alpha}) \mathbb{E}_{v_{n_\alpha}}(q) \\ &= \bar{\lambda} \mathbb{E}_v(p) + (1 - \bar{\lambda}) \mathbb{E}_v(q) = \mathbb{E}_v(\bar{\lambda} p + (1 - \bar{\lambda}) q). \end{aligned}$$

This implies that $\bar{\lambda} \in \bigcup_{v \in \mathcal{W}} \{\lambda \in [0, 1] : v(\gamma) \geq \mathbb{E}_v(\lambda p + (1 - \lambda) q)\}$. Hence, it follows that

$$\{\lambda \in [0, 1] : \gamma \geq u(\lambda p + (1 - \lambda) q)\}$$

is closed. Since u is upper semicontinuous, we have that $\{\lambda \in [0, 1] : u(\lambda p + (1 - \lambda) q) \geq \gamma\}$ is closed as well. It follows that u is a mixture continuous, upper semicontinuous, and quasiconcave (certainty equivalent) utility function for \succsim . By Corollary 29, this implies that \succsim satisfies A.1, A.2, A.3, A.7. Since u is a certainty equivalent utility function for \succsim , it follows that \succsim satisfies A.8 and A.9.

Finally, we show that $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$U(t, v) = \begin{cases} v^{-1}(t) & v \in \mathcal{W} \\ \infty & \text{otherwise} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x),$$

represents \succsim as in (ii) of Proposition 15. Notice that each function $v \in \mathcal{W}$ is strictly increasing and surjective. This implies that v is continuous and it follows that v^{-1} is strictly increasing, surjective, and continuous. It follows that U is well defined. By definition, it is immediate to see that $u(p) = \min_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v)$ for all $p \in \Delta(\mathbb{R})$. We next show that $U \in \mathcal{U}_n^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$. Since each v in \mathcal{W} is strictly increasing so is v^{-1} . This implies that $U(\cdot, v)$ is increasing for each $v \in \mathcal{V}_1(x)$. That is, U satisfies P.1. Since each function in \mathcal{W} is strictly increasing and such that $v(\mathbb{R}) = \mathbb{R}$ so is v^{-1} , this implies that $\lim_{t \rightarrow \infty} v^{-1}(t) = \infty$ for all $v \in \mathcal{W}$. In turn, this implies that U satisfies P.2. Next, consider $\alpha \in \mathbb{R}$ and the lower contour set $L_\alpha = \{(t, v) \in \mathbb{R} \times \mathcal{V}_1(x) : U(t, v) \leq \alpha\}$. It is immediate to see that

$$\begin{aligned} L_\alpha &= \{(t, v) \in \mathbb{R} \times \mathcal{W} : v^{-1}(t) \leq \alpha\} \\ &= \{(t, v) \in \mathbb{R} \times \mathcal{W} : t \leq v(\alpha)\}. \end{aligned}$$

This implies that L_α is a closed set with respect to the product topology and that L_α is convex. Therefore, U satisfies P.1, P.2, it is lower semicontinuous and quasiconvex. Moreover, it is not hard to check that

$$\lim_{t \rightarrow \infty} \left(\inf_{v \in \mathcal{V}_1(x)} U(t, v) \right) = \infty = \lim_{t \rightarrow \infty} U(t, v) \quad \forall v \in \mathcal{V}_1(x).$$

By Lemma 38, this implies that U satisfies P.3. Since u is mixture continuous and real valued, it follows that U satisfies P.5. Since v^{-1} is continuous for all $v \in \mathcal{W}$, it follows that $U^+ = U$, hence U satisfies P.6. P.7 is easily verified to be satisfied by U . \blacksquare

Proof of Proposition 18.

(ii) implies (i). It follows from a routine argument.

(i) implies (ii). Since \succsim satisfies A.1, A.2, and A.10, there exists $\bar{v} \in \mathbb{R}^C$ such that the function $u : \Delta(C) \rightarrow \mathbb{R}$, defined by

$$u(p) = \mathbb{E}_{\bar{v}}(p) \quad \forall p \in \Delta(C),$$

represents \succsim . Without loss of generality, we can assume that $\bar{v} \in \mathcal{V}_1(x)$. Consider $U : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$U(t, v) = \begin{cases} t & v = \bar{v} \\ \infty & v \neq \bar{v} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x).$$

It is immediate to check that U satisfies P.1 as well as P.2. Next, consider $\alpha \in \mathbb{R}$ and the lower contour set $L_\alpha = \{(t, v) \in \mathbb{R} \times \mathcal{V}_1(x) : U(t, v) \leq \alpha\}$. It follows that $L_\alpha = (-\infty, \alpha] \times \{\bar{v}\}$. This implies that L_α is a closed set with respect to the product topology and that L_α is convex. Therefore, U satisfies P.1, P.2, it is lower semicontinuous and quasiconvex. Moreover, it is immediate to see that

$$\lim_{t \rightarrow \infty} \left(\inf_{v \in \mathcal{V}_1(x)} U(t, v) \right) = \infty = \lim_{t \rightarrow \infty} U(t, v) \quad \forall v \in \mathcal{V}_1(x).$$

By Lemma 38, this implies that U satisfies P.3. Since u is real valued, continuous (resp., mixture continuous), and such that $u = u_U$ on $\Delta(C)$, it follows that U satisfies P.4 (resp., P.5). It is immediate to check that $U = U^+$, hence U satisfies P.6. Given these facts, we have that $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x)) \cap \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(x))$ and U represents \succsim as in (28) of Theorem 13. \blacksquare

Proof of Proposition 19.

(ii) implies (i). It follows from a routine argument.

(i) implies (ii). Since \succsim satisfies A.1, A.3, A.4, A.11, A.12, by Lemma 31 and [27, Theorem 1], there exists a closed and convex set $\mathcal{W} \subseteq \mathbb{R}^C$ such that $v(\bar{x}) = v'(\bar{x})$ for all $v, v' \in \mathcal{W}$ and the function $u : \Delta(C) \rightarrow \mathbb{R}$, defined by

$$u(p) = \min_{v \in \mathcal{W}} \mathbb{E}_v(p) \quad \forall p \in \Delta(C),$$

is continuous and represents \succsim . Without loss of generality, we can assume that $\mathcal{W} \subseteq \mathcal{V}_1(\bar{x})$. Then, consider $U : \mathbb{R} \times \mathcal{V}_1(\bar{x}) \rightarrow [-\infty, \infty]$, defined by

$$U(t, v) = \begin{cases} t & v \in \mathcal{W} \\ \infty & v \notin \mathcal{W} \end{cases} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(\bar{x}).$$

It is immediate to check that U satisfies P.1 as well as P.2. Next, consider $\alpha \in \mathbb{R}$ and the lower contour set $L_\alpha = \{(t, v) \in \mathbb{R} \times \mathcal{V}_1(\bar{x}) : U(t, v) \leq \alpha\}$. It follows that $L_\alpha = (-\infty, \alpha] \times \mathcal{W}$. This implies that L_α is a closed set with respect to the product topology and that L_α is convex. Therefore, U satisfies P.1, P.2, it is lower semicontinuous and quasiconvex. Moreover, it is immediate to see that

$$\lim_{t \rightarrow \infty} \left(\inf_{v \in \mathcal{V}_1(\bar{x})} U(t, v) \right) = \infty = \lim_{t \rightarrow \infty} U(t, v) \quad \forall v \in \mathcal{V}_1(\bar{x}).$$

By Lemma 38, this implies that U satisfies P.3. Since u is real valued, continuous (resp., mixture continuous), and such that $u = u_U$ on $\Delta(C)$, it follows that U satisfies P.4 (resp., P.5). It is immediate to check that $U = U^+$, hence U satisfies P.6. Given these facts, we have that $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(\bar{x})) \cap \mathcal{U}^{mc}(\mathbb{R} \times \mathcal{V}_1(\bar{x}))$ and U represents \succsim as in (28) of Theorem 13. \blacksquare

B.1 Proofs of Subsection 3.2

In this subsection, we prove Theorem 3, Proposition 4, and Corollary 41. We proceed by steps.

Lemma 39 *Let C be a finite set and let \succsim be a binary relation on \mathcal{M} . If \succsim satisfies B.1-B.6 then there exists a continuous and quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ that represents \succsim restricted to $\Delta(C)$. Furthermore, given any continuous and quasiconcave utility function u for \succsim , there exists an essentially unique and upper semicontinuous $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that*

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C). \quad (57)$$

Moreover, $U^* : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x), \quad (58)$$

is upper semicontinuous, belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$, and represents u as in (57).

Proof.

Since \succsim satisfies B.1 and B.6, it follows that \succsim restricted to $\Delta(C)$ satisfies A.1 and A.3. Since \succsim satisfies B.2 and B.3, we can conclude that \succsim restricted to $\Delta(C)$ satisfies A.4. Since C is finite and by Lemma 33 part (i), it follows that \succsim satisfies A.6. By Corollary 29 and Remark 30, it follows that there exists a continuous and quasiconcave utility function, $u : \Delta(C) \rightarrow \mathbb{R}$, that represents \succsim restricted to $\Delta(C)$. Recall that x is fixed and define $U = U^*$, by Lemma 24, Theorem 25 part (i), and Lemma 27, it follows that U belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$, it is essentially unique, and it is such that

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C).$$

Next, we show that U^* is upper semicontinuous, proving that U can be chosen to be upper semicontinuous. Consider a sequence $\{(t_n, v_n)\}_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathcal{V}_1(x)$ such that $(t_n, v_n) \rightarrow (t, v)$. This implies that

$t_n \rightarrow t$ and $v_n \rightarrow v$ and that $(t, v) \in \mathbb{R} \times \mathcal{V}_1(x)$. Next, consider the subsequence $\{(t_{n_k}, v_{n_k})\}_{k \in \mathbb{N}}$ such that

$$\limsup_n U^*(t_n, v_n) = \lim_k U^*(t_{n_k}, v_{n_k}).$$

We have two cases:

- for k large enough the set $\{p \in \Delta(C) : \mathbb{E}_{v_{n_k}}(p) \leq t_{n_k}\} = \emptyset$;
- there exists $\{(t_{n_{k(l)}}, v_{n_{k(l)}})\}_{l \in \mathbb{N}}$ such that $\{p \in \Delta(C) : \mathbb{E}_{v_{n_{k(l)}}}(p) \leq t_{n_{k(l)}}\} \neq \emptyset$.

In the first case, we have that

$$\limsup_n U^*(t_n, v_n) = \lim_k U^*(t_{n_k}, v_{n_k}) = -\infty \leq U^*(t, v). \quad (59)$$

In the second case, since $\Delta(C)$ is compact and u is continuous, we have that there exists $p_l \in \Delta(C)$ such that $\mathbb{E}_{v_{n_{k(l)}}}(p_l) \leq t_{n_{k(l)}}$ and $u(p_l) = U^*(t_{n_{k(l)}}, v_{n_{k(l)}})$ for all $l \in \mathbb{N}$. Since $\{p_l\}_{l \in \mathbb{N}}$ is a sequence in a compact set, without loss of generality, we can assume that $p_l \rightarrow p \in \Delta(C)$. This implies that $\mathbb{E}_v(p) = \lim_l \mathbb{E}_{v_{n_{k(l)}}}(p_l) \leq \lim_l t_{n_{k(l)}} = t$. Since u is continuous, we can conclude that

$$\begin{aligned} \limsup_n U^*(t_n, v_n) &= \lim_k U^*(t_{n_k}, v_{n_k}) = \lim_l U^*(t_{n_{k(l)}}, v_{n_{k(l)}}) \\ &= \lim_l u(p_l) = u(p) \leq U^*(t, v). \end{aligned}$$

By this last inequality, (59), and [1, Lemma 2.42], it follows that U^* is upper semicontinuous. \blacksquare

Next, Lemma 40 shows that there exists a utility function $V : \mathcal{M} \rightarrow \mathbb{R}$ that represents \succsim . The proof relies by first deriving V by “extending” u to finite menus and then, by upper semicontinuity, to all closed menus. This is based on the results developed by Gul and Pesendorfer [23]. The proof is available upon request.

Lemma 40 (Gul-Pesendorfer, 2001, Lemma 2 and Lemma 8) *Let C be a finite set and let \succsim be a binary relation on \mathcal{M} . The following are equivalent facts:*

- (i) \succsim satisfies B.1-B.5 and B.7;
- (ii) \succsim satisfies B.1-B.6;
- (iii) there exist a continuous and quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ (as in Lemma 39) that represents \succsim restricted to $\Delta(C)$ and a function $V : \mathcal{M} \rightarrow \mathbb{R}$ that represents \succsim such that

$$V(P) = \max_{p \in P} u(p) \quad \forall P \in \mathcal{M}. \quad (60)$$

Proof of Theorem 3.

(i) implies (ii). It follows trivially.

(ii) implies (iii). It follows by applying Lemma 39 and Lemma 40.

(iii) implies (i). By construction, the function $u = u_U$ is real valued, continuous, quasiconcave on $\Delta(C)$, and it represents \succsim . Moreover, $V(P) = \max_{p \in P} u(p)$. By Lemma 40, the statement follows.

Assume (i) or, equivalently, (ii) or (iii) and define $u : \Delta(C) \rightarrow \mathbb{R}$ to be such that $p \mapsto V(p)$. It follows that u is real valued, continuous, quasiconcave, and it represents \succsim on $\Delta(C)$. Moreover,

$V(P) = \max_{p \in P} u(p)$ for all $P \in \mathcal{M}$. By Lemma 39, it follows that U^* is upper semicontinuous, belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$, and represents \succsim as in (15). \blacksquare

Proof of Proposition 4.

Assume that \succsim is a binary relation on \mathcal{M} that satisfies B.1-B.6. Furthermore, assume that U and V are as in Theorem 3. Fix a convex menu P .

Claim. Let U belong to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ and let U be upper semicontinuous. The function $\Gamma : P \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$\Gamma(p, v) = U(\mathbb{E}_v(p), v) \quad \forall (p, v) \in P \times \mathcal{V}_1(x),$$

is quasiconvex with respect to v , quasiconcave with respect to p , and upper semicontinuous.

Proof of the Claim.

We first prove that Γ is quasiconvex with respect to v . Fix $p \in P$. Consider $v_1, v_2 \in \mathcal{V}_1(x)$ and $\lambda \in (0, 1)$. Define $v_\lambda = \lambda v_1 + (1 - \lambda) v_2$. It follows that $(\mathbb{E}_{v_\lambda}(p), v_\lambda) = \lambda(\mathbb{E}_{v_1}(p), v_1) + (1 - \lambda)(\mathbb{E}_{v_2}(p), v_2)$. Since U satisfies P.3, it follows that

$$\Gamma(p, v_\lambda) = U(\mathbb{E}_{v_\lambda}(p), v_\lambda) \leq \max\{U(\mathbb{E}_{v_1}(p), v_1), U(\mathbb{E}_{v_2}(p), v_2)\} = \max\{\Gamma(p, v_1), \Gamma(p, v_2)\}.$$

Next, we show that Γ is quasiconcave with respect to p . Fix $v \in \mathcal{V}_1(x)$. Consider $p_1, p_2 \in P$ and $\lambda \in (0, 1)$. Define $p_\lambda = \lambda p_1 + (1 - \lambda) p_2$. Without loss of generality, assume that $\mathbb{E}_v(p_1) \geq \mathbb{E}_v(p_2)$. Since U satisfies P.1, it follows that $\mathbb{E}_v(p_1) \geq \mathbb{E}_v(p_\lambda) \geq \mathbb{E}_v(p_2)$ and $U(\mathbb{E}_v(p_1), v) \geq U(\mathbb{E}_v(p_\lambda), v) \geq U(\mathbb{E}_v(p_2), v)$. We can conclude that

$$\begin{aligned} \Gamma(p_\lambda, v) &= U(\mathbb{E}_v(p_\lambda), v) \geq U(\mathbb{E}_v(p_2), v) \\ &= \min\{U(\mathbb{E}_v(p_1), v), U(\mathbb{E}_v(p_2), v)\} = \min\{\Gamma(p_1, v), \Gamma(p_2, v)\}. \end{aligned}$$

Finally, we show that Γ is upper semicontinuous. Consider a sequence $\{(p_n, v_n)\}_{n \in \mathbb{N}} \subseteq P \times \mathcal{V}_1(x)$ such that $(p_n, v_n) \rightarrow (p, v) \in P \times \mathcal{V}_1(x)$. Particularly, it follows that $\mathbb{E}_{v_n}(p_n) \rightarrow \mathbb{E}_v(p)$ and $v_n \rightarrow v$. Since U is upper semicontinuous, it follows that

$$\limsup_n \Gamma(p_n, v_n) = \limsup_n U(\mathbb{E}_{v_n}(p_n), v_n) \leq U(\mathbb{E}_v(p), v) = \Gamma(p, v),$$

proving the statement. \square

By previous Claim, [35, Corollary 2] and since U satisfies P.1., it follows that

$$\begin{aligned} V(P) &= \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} \Gamma(p, v) = \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} \Gamma(p, v) \\ &= \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right). \end{aligned}$$

\blacksquare

Corollary 41 Let C be a finite set and let \succsim be a binary relation on \mathcal{M} . The following are equivalent facts:

- (i) \succsim satisfies B.1-B.6 and B.8;
- (ii) there exists an essentially unique and upper semicontinuous $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that the function $V : \mathcal{M} \rightarrow \mathbb{R}$, defined by

$$V(P) = \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right) \quad \forall P \in \mathcal{M}, \quad (61)$$

represents \succsim .

Proof of Corollary 41.

Before starting observe that $\max_{p \in P} \mathbb{E}_v(p) = \max_{p \in \text{co}(P)} \mathbb{E}_v(p)$ for all $P \in \mathcal{M}$ and for all $v \in \mathcal{V}_1(x)$.

(i) implies (ii). By Theorem 3 and since \succsim satisfies B.1-B.6, it follows that there exists an essentially unique and upper semicontinuous function $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that the function $V : \mathcal{M} \rightarrow \mathbb{R}$, defined by

$$V(P) = \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall P \in \mathcal{M},$$

represents \succsim . Fix $P \in \mathcal{M}$. By B.8 and by Proposition 4, it follows that

$$\begin{aligned} V(P) &= V(\text{co}(P)) = \max_{p \in \text{co}(P)} \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_1(x)} \max_{p \in \text{co}(P)} U(\mathbb{E}_v(p), v) \\ &= \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in \text{co}(P)} \mathbb{E}_v(p), v\right) = \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right). \end{aligned}$$

(ii) implies (i). By assumption there exists an essentially unique and upper semicontinuous function $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that the function $V : \mathcal{M} \rightarrow \mathbb{R}$, defined by,

$$V(P) = \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right) \quad \forall P \in \mathcal{M}, \quad (62)$$

represents \succsim . Given (62), this implies that

$$\begin{aligned} V(P) &= \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right) \\ &= \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in \text{co}(P)} \mathbb{E}_v(p), v\right) = V(\text{co}(P)) \quad \forall P \in \mathcal{M}, \end{aligned}$$

that is, \succsim satisfies B.8. On the other hand, by assumption, we have that

$$V(P) = \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall P \in \mathcal{M}.$$

By Theorem 3, it follows that \succsim satisfies B.1-B.6, proving the statement. ■

B.2 Proofs of Subsubsection 3.2.1

In this subsection, we prove Proposition 5. We proceed by steps. Recall that we denote

$$\mathcal{C} = \{P \subseteq \Delta(C) : \emptyset \neq P \text{ closed and convex}\}.$$

Recall that the Hausdorff metric $d_h : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is such that

$$d_h(P, Q) = \max \left\{ \sup_{p \in P} \inf_{q \in Q} d(p, q), \sup_{q \in Q} \inf_{p \in P} d(q, p) \right\}$$

where we consider d to be the distance induced by the supnorm.

Lemma 42 *Let C be a finite set and let \succsim be a binary relation on C . If \succsim satisfies B.1-B.6 then there exists a continuous and quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ that represents \succsim restricted to $\Delta(C)$. Furthermore, given any continuous and quasiconcave utility function u for \succsim , there exists an essentially unique and upper semicontinuous $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that*

$$u(p) = \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall p \in \Delta(C). \quad (63)$$

Moreover, $U^* : \mathbb{R} \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$U^*(t, v) = \sup \{u(p) : \mathbb{E}_v(p) \leq t \text{ and } p \in \Delta(C)\} \quad \forall (t, v) \in \mathbb{R} \times \mathcal{V}_1(x), \quad (64)$$

is upper semicontinuous, belongs to $\mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$, and represents u as in (63).

Proof.

Since \succsim satisfies B.1 and B.6, it follows that \succsim restricted to $\Delta(C)$ satisfies A.1 and A.3. Since \succsim satisfies B.2 and B.3, we can conclude that \succsim restricted to $\Delta(C)$ satisfies A.4. Since C is finite and by Lemma 33 part (i), it follows that \succsim restricted to $\Delta(C)$ satisfies A.6. By Corollary 29 and Remark 30, it follows that there exists a continuous and quasiconcave utility function, $u : \Delta(C) \rightarrow \mathbb{R}$, that represents \succsim restricted to $\Delta(C)$. The statement then follows by using the same arguments contained in the proof of Lemma 39. \blacksquare

Lemma 43 *Let C be a finite set and let \succsim be a binary relation on C that satisfies B.1-B.6. If $P \in \mathcal{C}$ then there exists $\bar{p} \in P$ such that $\bar{p} \succsim P$.*

Proof.

We proceed by Steps. Before starting we have to introduce quite a bit of notation. We denote with p an element of \mathbb{R}_0^C . Since C is finite, we can identify it with a vector of $|C|$ components. We define p with a superscript, p^i or \bar{p}^i , to be a real number and we define with $p(i)$ the i -th component of a vector p . Consider $l \in \mathbb{N}$ where l is the cardinality of C . We define $B = [0, 1]^l \subseteq \mathbb{R}_0^C$ and fix $k \in \mathbb{N}$. Finally, we define

$$\mathcal{K} = \left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\right\} \text{ and } \mathcal{K}_l = \{p \in B : p(i) \in \mathcal{K} \text{ for all } i \in \{1, \dots, l\}\}.$$

If $l' \in \mathbb{N}$ is such that $1 \leq l' \leq l$ and $\{\bar{p}^1, \dots, \bar{p}^{l'}\} \subseteq \mathcal{K}$ then we define

$$B_k(\bar{p}^1, \dots, \bar{p}^{l'}) = \begin{cases} \times_{i=1}^{l'} \left[\bar{p}^i - \frac{1}{k}, \bar{p}^i + \frac{1}{k}\right] & \text{if } l' = l \\ \bigcup_{p^{l'+1}, \dots, p^l \in \mathcal{K}} B_k(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1}, \dots, p^l) & \text{if } l' < l \end{cases}.$$

Notice that if $l' = l$ then $B_k(\bar{p}^1, \dots, \bar{p}^{l'})$ is nothing else than the closed ball of radius $\frac{1}{k}$ and center $(\bar{p}^1, \dots, \bar{p}^{l'}) = \bar{p} \in \mathcal{K}_l$, with respect to the supnorm. Therefore, equivalently, we denote $B_k(\bar{p}^1, \dots, \bar{p}^{l'}) = B_k(\bar{p})$. If $l' < l$ then

$$B_k(\bar{p}^1, \dots, \bar{p}^{l'}) = \left[\bar{p}^1 - \frac{1}{k}, \bar{p}^1 + \frac{1}{k}\right] \times \dots \times \left[\bar{p}^{l'} - \frac{1}{k}, \bar{p}^{l'} + \frac{1}{k}\right] \times \times_{i=l'+1}^l \left[-\frac{1}{k}, 1 + \frac{1}{k}\right]. \quad (65)$$

Next, fix $P \in \mathcal{C}$ and define

$$\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}) = B_k(\bar{p}^1, \dots, \bar{p}^{l'}) \cap P.$$

Given $l' \in \mathbb{N}_0$ such that $0 \leq l' < l$, a subset $H \subseteq \mathbb{N}$, and if $l' > 0$ a subset $\{\bar{p}^1, \dots, \bar{p}^{l'}\} \subseteq \mathcal{K}$, we say that a family $\{\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_h^{l'+1})\}_{h \in H}$ is a family of adjacent sets if and only if each of them is nonempty and given $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{h_1}^{l'+1})$ and $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{h_2}^{l'+1})$ in such family with $p_{h_1}^{l'+1} < p_{h_2}^{l'+1}$, it follows that for each $p^{l'+1} \in \mathcal{K}$ such that $p_{h_1}^{l'+1} < p^{l'+1} < p_{h_2}^{l'+1}$ the set $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1})$ belongs to $\{\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_h^{l'+1})\}_{h \in H}$.

Step 1. Let $l' \in \mathbb{N}_0$ be such that $0 \leq l' < l$ and if $l' > 0$ let $\{\bar{p}^1, \dots, \bar{p}^{l'}\} \subseteq \mathcal{K}$. Then,

$$\left\{ \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) : p^{l'+1} \in \mathcal{K} \text{ and } \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) \neq \emptyset \right\}$$

if nonempty is a family of adjacent sets.

Proof of the Step.

If the family $\left\{ \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) : p^{l'+1} \in \mathcal{K} \text{ and } \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) \neq \emptyset \right\}$ contains at most two elements then the statement is vacuously true. Otherwise, pick a finite index set H and pick two points in \mathcal{K} , $p_{h_1}^{l'+1}$ and $p_{h_2}^{l'+1}$ such that $p_{h_1}^{l'+1} < p_{h_2}^{l'+1}$ and

$$\hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p_{h_1}^{l'+1} \right) \neq \emptyset \neq \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p_{h_2}^{l'+1} \right).$$

Consider $p^{l'+1} \in \mathcal{K}$ such that $p_{h_1}^{l'+1} < p^{l'+1} < p_{h_2}^{l'+1}$ and define $p_1, p_2 \in P$ to be such that $p_1 \in \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p_{h_1}^{l'+1} \right)$ and $p_2 \in \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p_{h_2}^{l'+1} \right)$. By (65), it follows that:

- For $i \leq l'$ and for each $\lambda \in [0, 1]$ we have that³⁵

$$\lambda p_1(i) + (1 - \lambda) p_2(i) \in \left[\bar{p}^i - \frac{1}{k}, \bar{p}^i + \frac{1}{k} \right].$$

- For $i > l' + 1$ and for each $\lambda \in [0, 1]$ we have that³⁶

$$\lambda p_1(i) + (1 - \lambda) p_2(i) \in \left[-\frac{1}{k}, 1 + \frac{1}{k} \right].$$

- For $i = l' + 1$ we have that

$$p_1(i) \leq p_{h_1}^{l'+1} + \frac{1}{k} \leq p^{l'+1} \leq p_{h_2}^{l'+1} - \frac{1}{k} \leq p_2(i).$$

This implies that there exists a $\bar{\lambda} \in [0, 1]$ such that

$$\bar{\lambda} p_1(i) + (1 - \bar{\lambda}) p_2(i) = p^{l'+1}.$$

Since P is convex, this implies that $\bar{\lambda} p_1 + (1 - \bar{\lambda}) p_2 \in \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right)$. Therefore, $\hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right)$ is nonempty, hence it belongs to the collection. \square

Consider $l' \in \mathbb{N}_0$ such that $0 \leq l' < l$ and if $l' > 0$ consider $\{\bar{p}^1, \dots, \bar{p}^{l'}\} \subseteq \mathcal{K}$. If the class

$$\left\{ \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) : p^{l'+1} \in \mathcal{K} \text{ and } \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) \neq \emptyset \right\}$$

is nonempty then we denote $\left\{ p_h^{l'+1} \right\}_{h \in H} \subseteq \mathcal{K}$ the ordered maximal family such that $\hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p_h^{l'+1} \right) \neq \emptyset$ and $H \subseteq \mathbb{N}$. In light of Step 1, without loss of generality, we assume that $\left\{ p_h^{l'+1} \right\}_{h \in H}$ is ordered, that is, $p_h^{l'+1} < p_{h+1}^{l'+1}$. We define \bar{h} as the maximum element and h the minimum element of H .

Step 2. Let $l' \in \mathbb{N}_0$ be such that $0 \leq l' < l$ and if $l' > 0$ let $\{\bar{p}^1, \dots, \bar{p}^{l'}\} \subseteq \mathcal{K}$. If the class

$$\left\{ \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) : p^{l'+1} \in \mathcal{K} \text{ and } \hat{B}_k \left(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1} \right) \neq \emptyset \right\}$$

³⁵If $l' = 0$ this case is vacuous.

³⁶If $l' = l - 1$ this case is vacuous.

is nonempty, $\hat{h}, \tilde{h} \in H$, and $\tilde{h} \geq \hat{h}$ then $\bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1}) \in \mathcal{C}$ and is equal to

$$\left(\left[\bar{p}^1 - \frac{1}{k}, \bar{p}^1 + \frac{1}{k} \right] \times \dots \times \left[\bar{p}^{l'} - \frac{1}{k}, \bar{p}^{l'} + \frac{1}{k} \right] \times \left[p_{\hat{h}}^{l'+1} - \frac{1}{k}, p_{\hat{h}}^{l'+1} + \frac{1}{k} \right] \times \times_{i=l'+2}^l \left[-\frac{1}{k}, 1 + \frac{1}{k} \right] \right) \cap P.$$

Proof of the Step.

The proof follows from standard arguments and Step 1. \square

Step 3. Let $l' \in \mathbb{N}$ be such that $1 \leq l' < l$ and $\{\bar{p}^1, \dots, \bar{p}^{l'}\} \subseteq \mathcal{K}$. If $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}) \neq \emptyset$ then there exists $\hat{p} \in \mathcal{K}_l$ such that $\hat{B}_k(\hat{p}) \succ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$ and \hat{p} does not depend on $\{\bar{p}^1, \dots, \bar{p}^{l'}\}$.

Proof of the Step.

We prove the statement by induction. First, we consider the family of sets such that $\hat{B}_k(p) \neq \emptyset$ where $p \in \mathcal{K}_l$. Since P is nonempty, this family is nonempty and finite. We consider the best element, according to the ranking, \succ . Call such element $\hat{B}_k(\hat{p})$. We argue by backward induction.

Assume $l' = l - 1$. Since $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$ is nonempty, the family

$$\left\{ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1}) : p^{l'+1} \in \mathcal{K} \text{ and } \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1}) \neq \emptyset \right\}$$

is nonempty and finite. Call $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1})$ the best element with respect to \succ of such family. By Step 2, B.1, B.5, and induction it follows that $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1}) \sim \bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1})$

for each $\tilde{h} \in H$ such that $\tilde{h} \geq \hat{h}$. Therefore, $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1}) \sim \bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1})$.

By the same argument, it follows that $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1}) \succ \bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1}) =$

$\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$. By construction, we have that $\hat{B}_k(\hat{p}) \succ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1})$, this implies that $\hat{B}_k(\hat{p}) \succ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$.

Assume that $1 \leq l' < l - 1$ and that the statement is true for each l'' strictly greater than l' and smaller than l . Since $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$ is nonempty, the family

$$\left\{ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1}) : p^{l'+1} \in \mathcal{K} \text{ and } \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p^{l'+1}) \neq \emptyset \right\}$$

is nonempty. Call $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1})$ the best element with respect to \succ of such family. By Step 2, B.1, B.5, and induction it follows that $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1}) \sim \bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1})$ for

each $\tilde{h} \in H$ such that $\tilde{h} \geq \hat{h}$. Therefore, $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1}) \sim \bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1})$.

By the same argument, it follows that $\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1}) \succ \bigcup_{j \in H: \tilde{h} \geq j \geq \hat{h}} \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_j^{l'+1}) =$

$\hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$. By inductive hypothesis, we have that $\hat{B}_k(\hat{p}) \succ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'}, p_{\tilde{h}}^{l'+1})$. This implies that $\hat{B}_k(\hat{p}) \succ \hat{B}_k(\bar{p}^1, \dots, \bar{p}^{l'})$. \square

Step 4. For each $k \in \mathbb{N}$ there exists $\hat{p}_k \in \mathcal{K}_l$ such that $\hat{B}_k(\hat{p}_k) \succ P$.

Proof of the Step.

Fix $k \in \mathbb{N}$. The family $\left\{ \hat{B}_k(p^1) : p^1 \in \mathcal{K} \text{ and } \hat{B}_k(p^1) \neq \emptyset \right\}$ is obviously nonempty. By Step 3, there exists $\hat{p}_k \in \mathcal{K}_l$ such that $\hat{B}_k(\hat{p}_k) \succsim \hat{B}_k(p^1)$ for each element in the previous family. By Step 2, Step 3, B.1, B.5, and induction, it follows that $\hat{B}_k(\hat{p}_k) \succsim \bigcup_{j \in H: \bar{h} \geq j \geq h} \hat{B}_k(p_j^1)$. This implies that

$\hat{B}_k(\hat{p}_k) \succsim \bigcup_{\bar{h} \geq j \geq h} \hat{B}_k(p_j^1) = P$. Since k was chosen to be generic, the statement follows. \square

Step 5. There exists $\bar{p} \in P$ such that $\bar{p} \succsim P$.

Proof of the Step.

By Step 4, there exists a sequence $\left\{ \hat{B}_k(\hat{p}_k) \right\}_{k \in \mathbb{N}}$ such that $\hat{B}_k(\hat{p}_k) \succsim P$ and $\hat{p}_k \in \mathcal{K}_l$ for all $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ there exists $p_k \in \hat{B}_k(\hat{p}_k) \cap P$. Since P is compact there exists a converging subsequence $\{p_{k_n}\}_{n \in \mathbb{N}}$ such that $p_{k_n} \rightarrow \bar{p} \in P$. Notice that for each $n \in \mathbb{N}$ and for each $p \in \hat{B}_{k_n}(\hat{p}_{k_n})$ we have that

$$d(\bar{p}, p) \leq d(\bar{p}, p_{k_n}) + d(p_{k_n}, p) \leq d(\bar{p}, p_{k_n}) + \frac{2}{k_n}.$$

This implies that

$$\begin{aligned} d_h(\{\bar{p}\}, \hat{B}_{k_n}(\hat{p}_{k_n})) &= \max \left\{ \sup_{p \in \hat{B}_{k_n}(\hat{p}_{k_n})} d(\bar{p}, p), \inf_{p \in \hat{B}_{k_n}(\hat{p}_{k_n})} d(\bar{p}, p) \right\} \leq \sup_{p \in \hat{B}_{k_n}(\hat{p}_{k_n})} d(\bar{p}, p) \\ &\leq d(\bar{p}, p_{k_n}) + \frac{2}{k_n} \rightarrow 0. \end{aligned}$$

Since for each $n \in \mathbb{N}$ we have that $\hat{B}_{k_n}(\hat{p}_{k_n}) \succsim P$, by B.2, it follows that $\bar{p} \succsim P$. \blacksquare

Lemma 44 *Let C be a finite set and let \succsim be a binary relation on C . The following are equivalent facts:*

(i) \succsim satisfies B.1-B.6;

(ii) there exists a continuous and quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ (as in Lemma 42) that represents \succsim restricted to $\Delta(C)$ and such that the function $V : C \rightarrow \mathbb{R}$, defined by

$$V(P) = \max_{p \in P} u(p) \quad \forall P \in C, \tag{66}$$

represents \succsim .

Proof.

(i) implies (ii). Since \succsim satisfies B.1-B.6, by Lemma 42, it follows that there exists a utility function $u : \Delta(C) \rightarrow \mathbb{R}$ with the aforementioned properties. Fix $P \in C$. By Lemma 43, we have that there exists $\bar{p} \in P$ such that $\bar{p} \succsim P$. By B.1 and B.4, it follows that $\bar{p} \sim P$. By B.1 and B.4, this latter fact implies that $\bar{p} \succsim q$ for all $q \in P$. Since u represents \succsim restricted to $\Delta(C)$, if we define V as in (66) then it follows that V is well defined and such that $V(P) = u(\bar{p})$. We can conclude that

$$P_1 \succsim P_2 \iff \bar{p}_1 \succsim \bar{p}_2 \iff u(\bar{p}_1) \geq u(\bar{p}_2) \iff V(P_1) \geq V(P_2),$$

proving that V represents \succsim on C .

(ii) implies (i). Assume there exists a continuous and quasiconcave function $u : \Delta(C) \rightarrow \mathbb{R}$ that represents \succsim restricted to $\Delta(C)$. Moreover, assume there exists a function $V : C \rightarrow \mathbb{R}$ as in (66) that represents \succsim on C . Since V represents \succsim , it follows that B.1 is satisfied. Pick a sequence $\{P_n\}_{n \in \mathbb{N}} \subseteq C$ such that $P_n \rightarrow P \in C$ and consider the subsequence $\{P_{n_k}\}_{k \in \mathbb{N}}$ such that $\lim_k V(P_{n_k}) =$

$\limsup_n V(P_n)$. Since u is continuous, there exists a sequence $\{\bar{p}_{n_k}\}_{k \in \mathbb{N}} \subseteq \Delta(C)$ such that $u(\bar{p}_{n_k}) = V(P_{n_k})$ and $\bar{p}_{n_k} \in P_{n_k}$ for all $k \in \mathbb{N}$. Since C is finite, we have that $\Delta(C)$ is compact. Hence, there exists a subsequence $\{\bar{p}_{n_{k(l)}}\}_{l \in \mathbb{N}}$ of $\{\bar{p}_{n_k}\}_{k \in \mathbb{N}}$ such that $\bar{p}_{n_{k(l)}} \rightarrow \bar{p}$. It is easy to see that $\bar{p} \in P$. Then, since u is continuous, it follows that

$$\begin{aligned} \limsup_n V(P_n) &= \lim_k V(P_{n_k}) = \lim_l V(P_{n_{k(l)}}) \\ &= \lim_l u(\bar{p}_{n_{k(l)}}) = u(\bar{p}) \leq \max_{p \in P} u(P) = V(P). \end{aligned}$$

By previous inequality, [1, Lemma 2.42], and since V represents \succsim , it follows that \succsim satisfies B.2. By [1, Lemma 3.78] and since u is continuous, B.3 is satisfied. Since u is quasiconcave, B.6 is satisfied. Finally, since V satisfies (60), it follows immediately that B.4 and B.5 are satisfied. \blacksquare

Proof of Proposition 5.

(i) implies (ii). By applying Lemma 42 and then Lemma 44, it follows that there exists an essentially unique and upper semicontinuous function $U \in \mathcal{U}^c(\mathbb{R} \times \mathcal{V}_1(x))$ such that the function $V : \mathcal{C} \rightarrow \mathbb{R}$, defined by

$$V(P) = \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v) \quad \forall P \in \mathcal{C},$$

represents \succsim . Fix $P \in \mathcal{C}$. By the Claim contained in the proof of Proposition 4, the function $\Gamma : P \times \mathcal{V}_1(x) \rightarrow [-\infty, \infty]$, defined by

$$\Gamma(p, v) = U(\mathbb{E}_v(p), v) \quad \forall (p, v) \in P \times \mathcal{V}_1(x),$$

is quasiconvex with respect to v , quasiconcave with respect to p , upper semicontinuous, and such that

$$V(P) = \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} \Gamma(p, v).$$

By [35, Corollary 2] and since U satisfies P.1, it follows that

$$\begin{aligned} V(P) &= \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} \Gamma(p, v) = \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} \Gamma(p, v) \\ &= \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_1(x)} U\left(\max_{p \in P} \mathbb{E}_v(p), v\right). \end{aligned}$$

(ii) implies (i). By construction, the function $u = u_U$ is real valued, continuous, and quasiconcave on $\Delta(C)$ and it represents \succsim on $\Delta(C)$. Moreover, since U satisfies P.1, we have that

$$V(P) = \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} U(\mathbb{E}_v(p), v) \quad \forall P \in \mathcal{C}.$$

By using the same notation of previous part, fix $P \in \mathcal{C}$. By the Claim contained in the proof of Proposition 4 and [35, Corollary 2], it follows that

$$\begin{aligned} V(P) &= \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} U(\mathbb{E}_v(p), v) = \inf_{v \in \mathcal{V}_1(x)} \max_{p \in P} \Gamma(p, v) \\ &= \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} \Gamma(p, v) = \max_{p \in P} \inf_{v \in \mathcal{V}_1(x)} U(\mathbb{E}_v(p), v). \end{aligned}$$

Since P was chosen to be generic, it follows that the previous equality holds for each P in \mathcal{C} . This implies that $V(P) = \max_{p \in P} u(p)$. By Lemma 44, it follows that \succsim satisfies B.1-B.6.

Finally, assume that equivalently (i) or (ii) are satisfied, by Lemma 42, Lemma 44, and (i) implies (ii), it follows that U can be chosen to be U^* . \blacksquare

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