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Choice under Uncertainty and State- Preference Approach to Portfolio Decisions

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20135 – Theory of Finance, Part I
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Outline and objectives

- Definition and characterization of risk averse behavior
 - Guidolin-Pedio, chapter 2, sec. 2.1
- Risk-loving and risk neutral investors
 - Guidolin-Pedio, chapter 2, sec. 2.1
- How to measure and compare risk aversion: ARA and RRA coefficients
 - Guidolin-Pedio, chapter 2, sec. 2.2
- Economic interpretation of ARA and RRA/1: relationship to the acceptable odds of a given, small bet
 - Guidolin-Pedio, chapter 2, sec. 2.3
- Economic interpretation of ARA and RRA/2: relationship to the size of economic risk (insurance) premia
 - Guidolin-Pedio, chapter 2, sec. 2.4
- Commonly employed utility functions of monetary wealth
 - Guidolin-Pedio, chapter 2, sec. 3

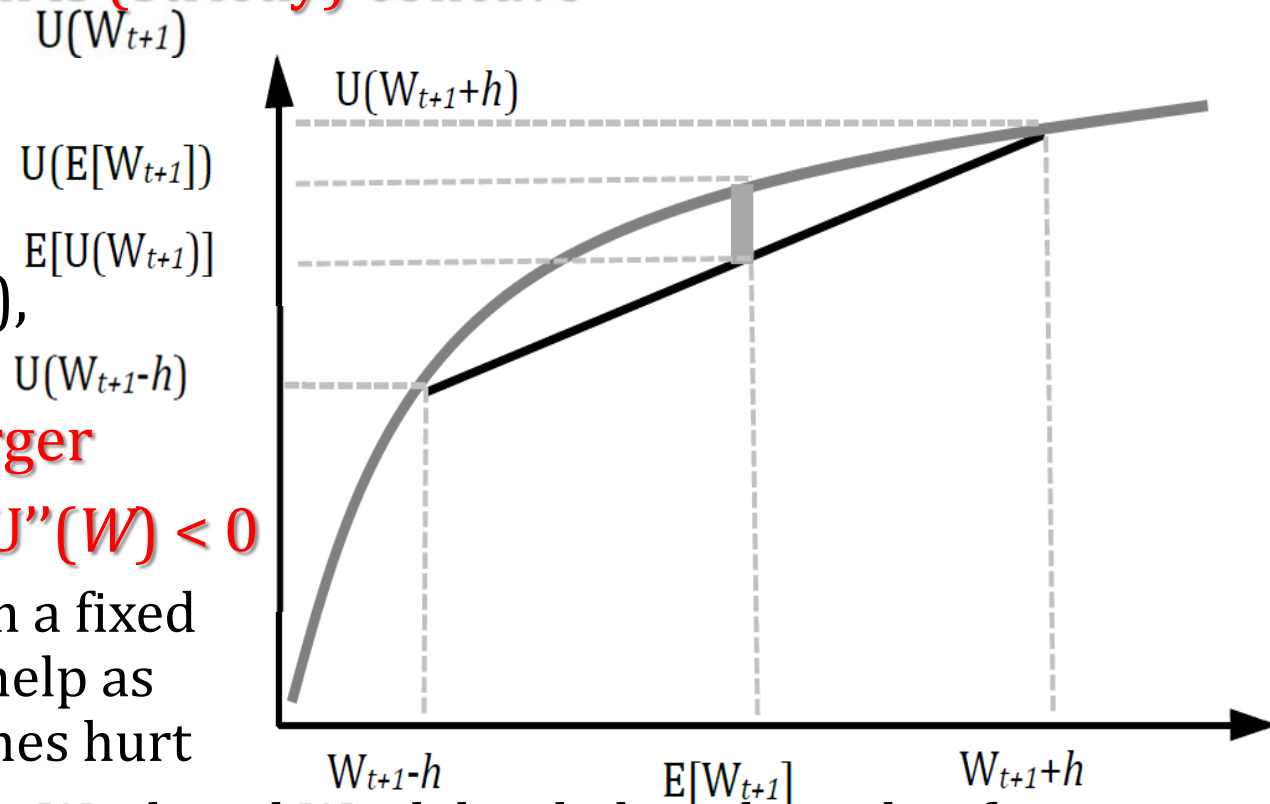
Measuring Risk Aversion

- Given a specification of probabilities, the utility function of monetary wealth $U(\cdot)$ uniquely characterizes the behavior of each investor
 - Alternative assumptions on $U(\cdot)$ identify an investor's tolerance or aversion to risk
 - If the utility of the quantities purchased and consumed of M goods, $u(x_1, x_2, \dots, x_M)$, is increasing, and all prices are strictly positive, we can show that the utility of wealth is strictly increasing in wealth W , $U'(W) > 0$
- We shall always assume **non-satiated** individuals, $U'(W) > 0$
 - Gordon Gekko's greed, <https://www.youtube.com/watch?v=VVxYOQS6ggk>
- To understand what risk aversion means, consider a bet where the investor either receives an amount h with probability $\frac{1}{2}$ or must pay an amount h with probability $\frac{1}{2}$, so that in expectation it is **fair**
- The intuitive notion of “being averse to risk” is that that for any level of wealth W , **an investor would not wish to enter in such a bet:**
$$U(W) > \frac{1}{2}U(W + h) + \frac{1}{2}U(W - h) = E[U(W + H)]$$
utility of wealth with no gamble exceeds expected utility of wealth+gamble
 - H is a zero-mean random variable that takes value h with prob. $\frac{1}{2}$ and $-h$ with prob. $\frac{1}{2}$

Defining Risk Aversion

A risk-averse investor is one who always prefers the utility of the expected value of a fair bet to the expectation of the utility of the same bet; when her VNM $U(\cdot)$ is differentiable, the $U(\cdot)$ must be **concave**

- This inequality can be satisfied for all wealth levels W if the agent's utility function has the form below
- We say **the utility function is (strictly) concave**
- Equivalently, the slope of $U(\cdot)$ decreases as the investor gets wealthier
- The marginal utility (MU), **$U'(W) \equiv d(U(W))/dW$** decreases as W grows larger
- If $U'(W)$ decreases, then **$U''(W) < 0$**
 - Positive deviations from a fixed average wealth do not help as much as the negative ones hurt
 - The segment connecting $W - h$ and $W + h$ lies below the utility function



Other Risk Preference Types

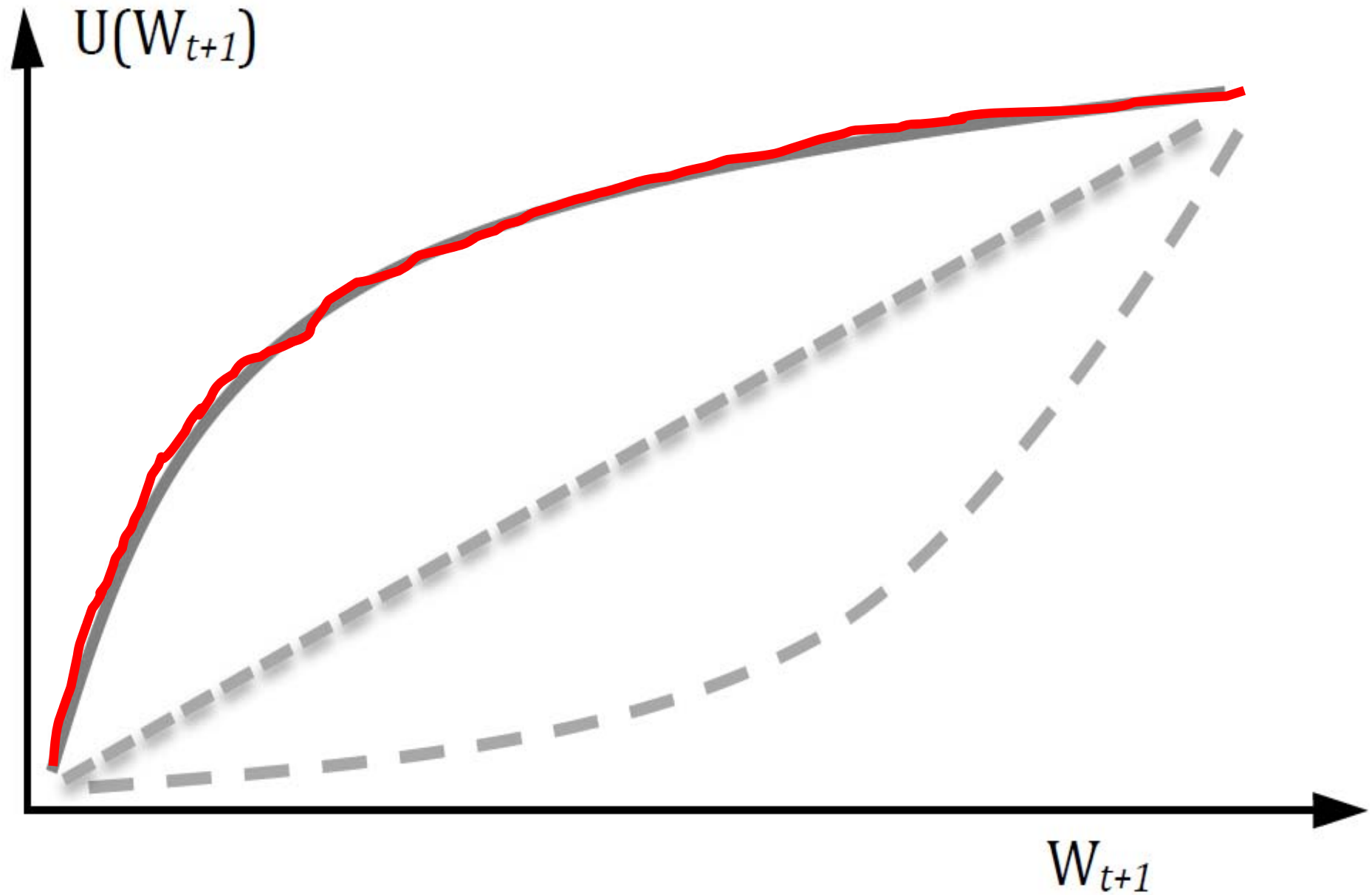
A risk-loving (neutral) investor is one who always prefers (is indifferent to) the expectation of the utility of a fair bet to the utility of the expected value of the bet; if $U(\cdot)$ is differentiable, the $U(\cdot)$ must be **convex** (**linear**)

- We obtain risk-loving behavior when

$$U(W) < \frac{1}{2}U(W + h) + \frac{1}{2}U(W - h) = E[U(W + H)]$$

- When this inequality is satisfied for all wealth levels, we say **the utility function is (strictly) convex**
- Equivalently, the slope of $U(\cdot)$ increases as the investor gets wealthier
- The marginal utility (MU), **$U'(W) \equiv d(U(W))/dW$ increases as W grows larger**
- If $U'(W)$ decreases, then **$U''(W) > 0$**
 - Positive deviations from a fixed average wealth give more happiness than the unhappiness caused by negative deviations
- The case of **risk neutral** investors obtains if $U'(W)$ is constant
 - From standard integration of the marginal utility function, it follows that $U'(W) = b \implies U(W) = a + bW$, a linear utility function

Other Risk Preference Types



Choice under Uncertainty

Absolute and Relative Risk Aversion Coefficients

- How can we manage to measure risk aversion and compare the risk aversion of different decision makers?
- Given that under mild conditions, risk aversion is equivalent to $U''(W) < 0$ for all wealth levels, one simplistic idea is to measure risk aversion on the basis of the second derivative of $U(\cdot)$
 - E.g., John is more risk averse than Mary is iff $|U_{\text{John}}''(W)| > |U_{\text{Mary}}''(W)|$
- Unfortunately, this approach leads to an inconsistency because when $U_{\text{John}}(W) = a + bU_{\text{Mary}}(W)$ with $b > 0$ and $b \neq 1$, clearly $U_{\text{John}}''(W) = bU_{\text{Mary}}''(W) \neq U_{\text{Mary}}''(W) > 0$
- But we know that by construction, John and Mary have the same preferences for risky gambles and therefore that it makes no sense to state the John is more risk averse than Mary
- Two famous measures that escape these drawbacks are the **coefficients of absolute/relative risk aversion**:

$$ARA(W) \equiv -\frac{U''(W)}{U'(W)} \quad RRA(W) \equiv -\frac{U''(W)}{U'(W)} W = ARA(W) \cdot W$$

- Because $U(W)$ is a function of wealth, $ARA(W)$ and $RRA(W)$ are too

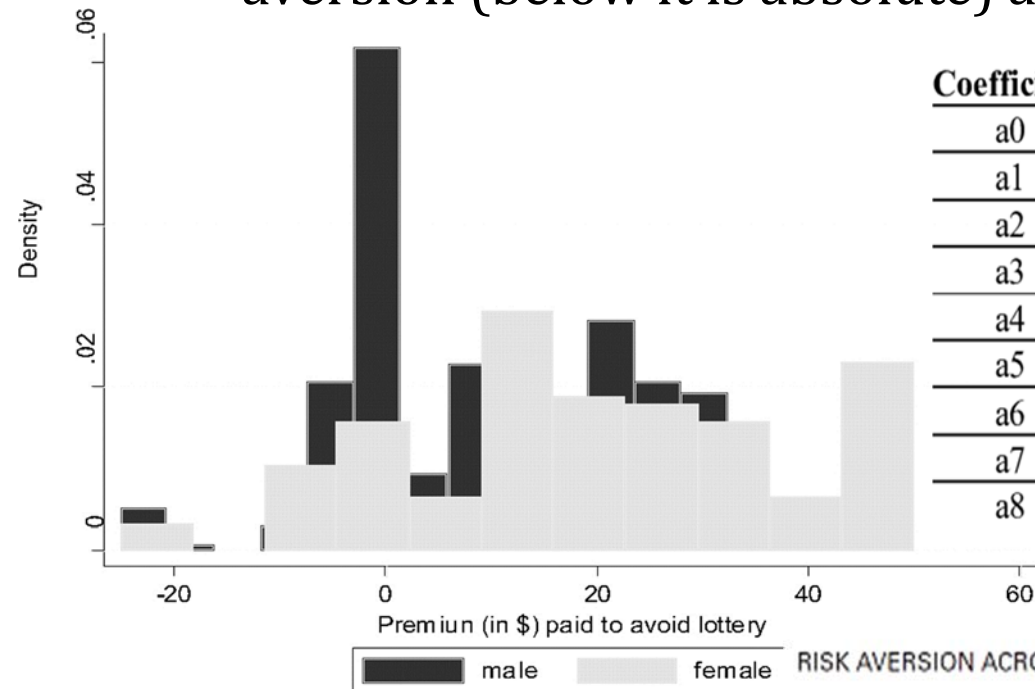
Absolute and Relative Risk Aversion Coefficients

Both $ARA(W)$ and $RRA(W)$ are invariant to linear monotonic transforms; this occurs because both are “scaled” at the denominator $U'(W)$

- If nonzero, the reciprocal of the measure of absolute risk aversion, $T(W) \equiv 1/ARA(W)$ can be used as a measure of **risk tolerance**
- When ARA is constant, $RRA(W)$ must be a linear (increasing) function of wealth; when RRA is constant, then it must be the case that $ARA(W) = RRA/W$, a simple inverse function of wealth
- ARA and RRA are invariant to linear monotonic transformations; e.g.,
$$ARA_{John}(W) \equiv -\frac{U''_{John}(W)}{U'_{John}(W)} = -\frac{bU''_{Mary}(W)}{bU'_{Mary}(W)} = -\frac{U''_{Mary}(W)}{U'_{Mary}(W)} = ARA_{Mary}(W)$$
- To rank John and Mary's risk aversion, we need to verify whether $ARA_{John}(W) > ARA_{Mary}(W)$ (or the opposite) for all wealth levels
 - Same applies to their coefficient of relative risk aversion for all wealth
 - Possible that for some intervals of wealth it may be $(R)ARA_{John}(W) > (R)ARA_{Mary}(W)$ but for other levels/intervals the inequality be reversed
- Both measures are local as they characterize the behavior of investors only **when the risks (lotteries) considered are small**

Absolute and Relative Risk Aversion Coefficients

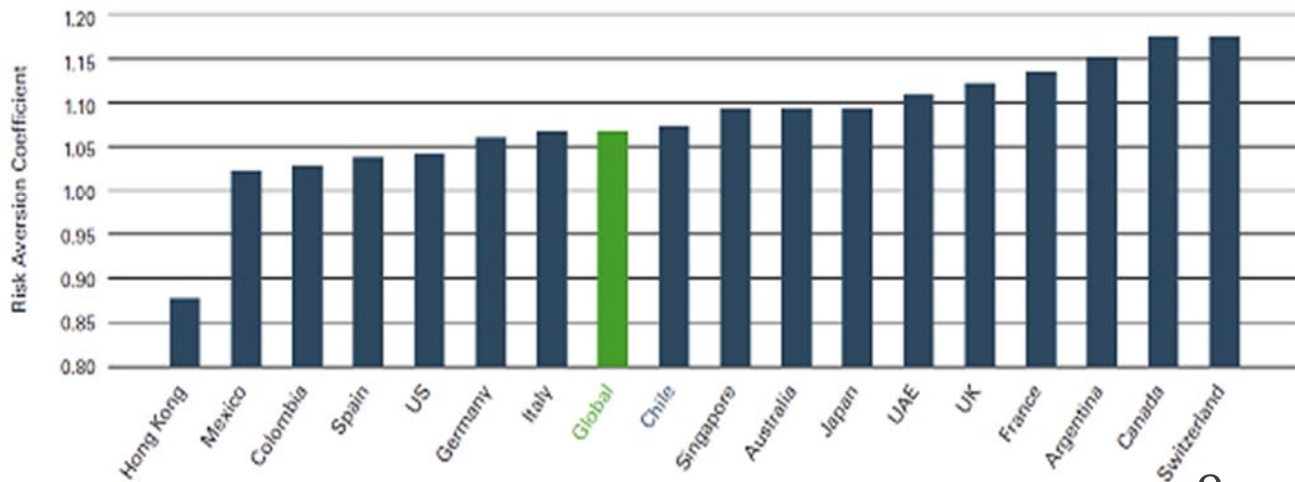
- Psychological research has documented differences in measured risk aversion (below it is absolute) across sex and age...



Coefficient	Age class	Absolute Risk Aversion
a0	18-24	0
a1	25-31	0,1
a2	32-38	0,45
a3	39-45	0,6
a4	46-52	0,7
a5	53-59	0,75
a6	60-66	0,8
a7	67-73	0,83
a8	74-80	0,85

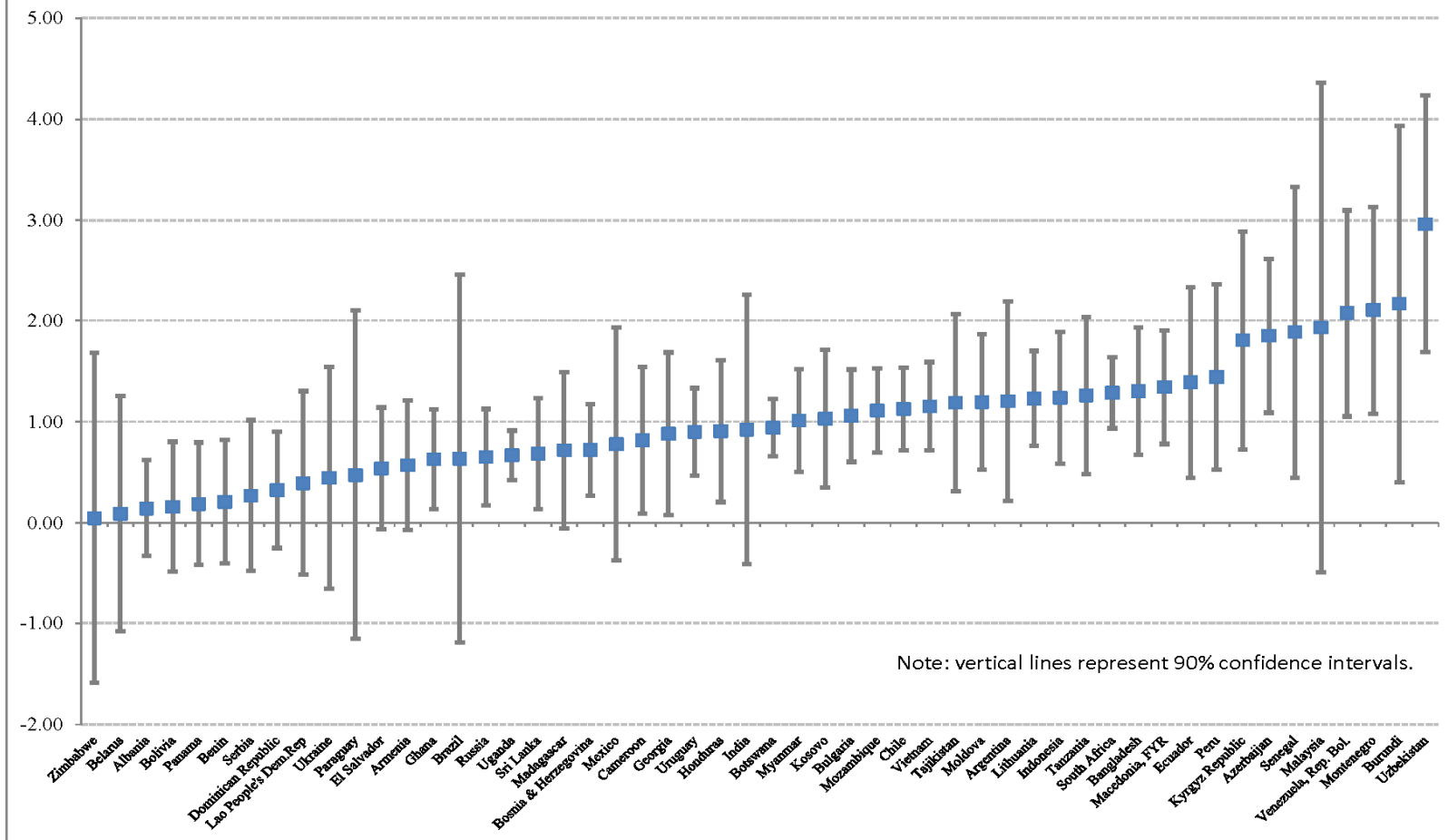
- ... and countries
- In the first picture above, what is the link between the premium to avoid a lottery and (absolute) risk aversion?

RISK AVERSION ACROSS COUNTRIES (ABSOLUTE)



Absolute and Relative Risk Aversion Coefficients

Figure 2. Relative Risk Aversion among Developing Countries



Choice under Uncertainty

ARA and RRA and the Odds of Accepting a Bet

- What is the economic interpretation of ARA and RRA coefficients?
- A first interpretation is that ARA and RRA are related to **the odds that a risk-averse agent may accept a bet**
 - Consider an investor with wealth W who is offered at no charge, a bet involving winning or losing an amount h , with probabilities π and $1 - \pi$
 - Any investor will accept such a bet if π is high enough and reject it if π is small enough (surely if $\pi = 0$, because the bet is a lump-sum tax of h)
 - Such a bet is defined as a fair bet when $\pi = \frac{1}{2}$ because it costs nothing and its expected payoff is $(\frac{1}{2})h + (\frac{1}{2})(-h) = 0$
 - When π differs from $\frac{1}{2}$, the bet is not fair and when $\pi > \frac{1}{2}$ the bet is clearly tilted in favor of the investor
 - An investor's willingness to accept the bet may depend on her wealth W
- Let $\pi = \pi(W; h)$ be that probability at which the agent is indifferent between accepting or rejecting the gamble, i.e., such that:

$$U(W) = \pi(W; h)U(W + h) + [1 - \pi(W; h)]U(W - h)$$

i.e., the sure-thing utility she derives in the absence of the bet equals its expected utility

ARA and RRA and the Odds of Accepting a Bet

As the ARA coefficient of an investor grows, her probability required to enter a bet grows, at least locally (for small bets)

- Your textbook (**please see it**) show that by applying a Taylor's expansion to the previous equation, one can show that for a small bet, there is a link btw. $ARA(W)$ and the minimum odds required to enter in the bet:

$$\pi(W; h) \cong \frac{1}{2} + \frac{1}{4} ARA(W)h$$

- The higher is ARA, the larger is the difference $\pi(W; h) - 1/2 > 0$, i.e., the “mark-up” in the odds of the bet that the investor requires to tolerate it
- The expression for $\pi(W; h)$ depends on the size of the bet, h , in a very simple way, i.e., linearly, although this is due only on the fact that we are considering a second-order approximation that applies for $h \rightarrow 0$
- If one accepts a characterization in which John is more risk averse than Mary if and only if $\pi_{\text{John}}(W; h) > \pi_{\text{Mary}}(W; h)$, we know that as a first approximation this is equivalent to stating that $ARA_{\text{John}}(W) > ARA_{\text{Mary}}(W)$ for all wealth levels
- Exploiting $ARA(W) \equiv RRA(W)/W$, we can re-write this result as:

$$\pi(W; \varpi) \cong \frac{1}{2} + \frac{1}{4} RRA(W)\varpi$$

Relative size of the bet

Two Examples

- John is characterized by VNM function $U_{John}(W) = 1 - e^{-\theta W}$ with $\theta > 0$
- Therefore $U'(W) = \theta e^{-\theta W} > 0$, $U''(W) = -\theta^2 e^{-\theta W} < 0$ so that

$$ARA_{John}(W) = -\frac{-\theta^2 e^{-\theta W}}{\theta e^{-\theta W}} = \theta$$

which is clearly constant

- As a result, in the face of a two-outcome symmetric bet with size h , we have:

$$\pi_{John}(W; h) = \pi(h) \cong \frac{1}{2} + \frac{1}{4}\theta h$$

- An increase in either absolute risk aversion and in the size of the bet have identical effects
- The minimal odds $\pi(W; h)$ turns out to be independent of wealth
- Mary is instead characterized by a VNM power utility function:

$$U_{Mary}(W) = \frac{W^{1-\gamma}}{1-\gamma} \quad \text{with } \gamma > 0$$

- Therefore $U'(W) = W^{-\gamma} > 0$, $U''(W) = -\gamma W^{-\gamma-1} < 0$ so that

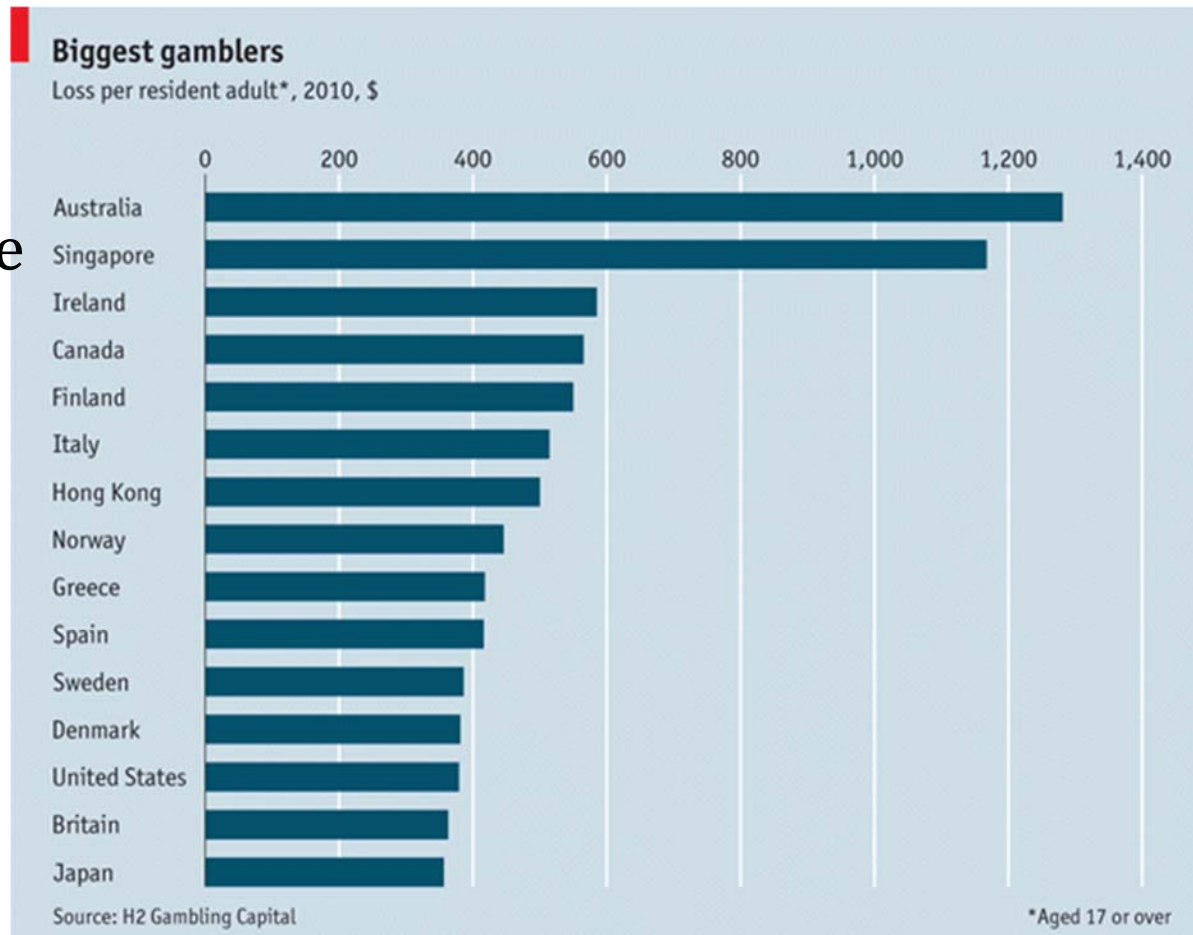
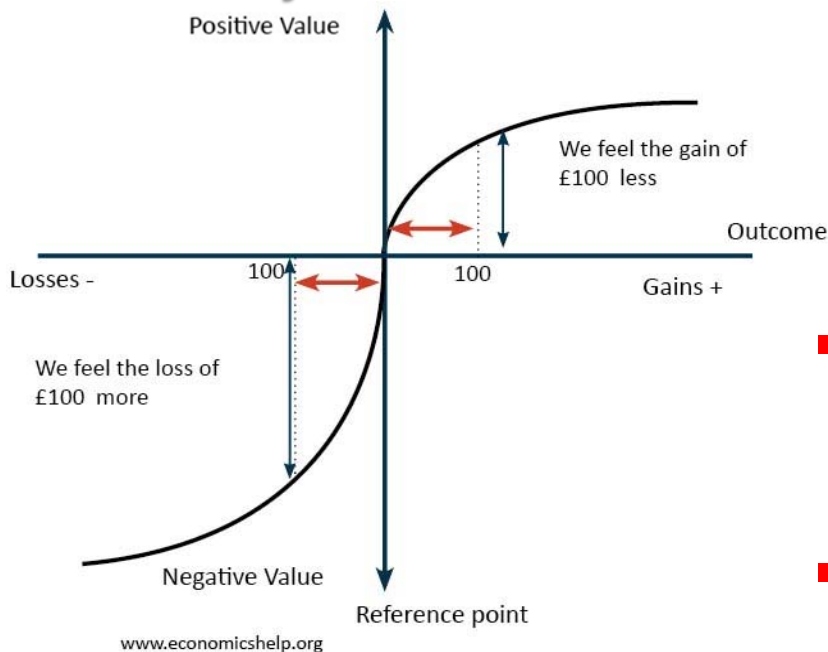
$$RRA_{Mary}(W) = -\frac{-\gamma W^{-\gamma-1}}{W^{-\gamma}} W = \gamma \implies \pi_{Mary}(W; \varpi) = \pi(h) \cong \frac{1}{2} + \frac{1}{4}\gamma \varpi$$

Applications to Real-Life Examples

- Because casinos are for-profit companies and hence they «rig» their chance games in their favor, their winning odds are structurally below $\frac{1}{2}$ and as such $< \pi(W; h)$ for all h implied by the gambles
- Therefore no risk-averse agent should ever walk into a casino, ever!
- However, not all risk-loving agents will walk in: even though for a risk-lover, θ and γ are both negative and therefore $\pi(W; h) < \frac{1}{2}$, it still takes a sufficiently large $\pi(W; h)$ to accept the risky, unfair gambles
- One easy way to spot a constant ARA agent is the following: if the agent has 1 euro in her pockets and she rejects a 1-cent gamble, she will still reject it after she inherits 999,999 euros! (That is odd, true)
- One way to spot a constant RRA agent is the following: if the agent has 1 euro in her pockets and she rejects a 1-cent gamble, she will still reject a 10,000 euros gamble after she inherits 999,999 euros!
- However she may (does not have to) accept gambles of less than 1% of 1 million euros
- In short, constant ARA agents care for the absolute size (h) of gambles, while constant RRA care for their relative size (ϖ)

Applications to Real-Life Examples

- Talking about gambling...
- In fact, there is empirical and experimental evidence that investors would be risk-averse over gains but risk-seekers over losses
- This is called **prospect theory**



- With prospect theory, we enter the domain of so-called behavioral economics/finance
- The reason is that there are no obvious axioms of rational choice supporting it

ARA and RRA and the Risk Premium

The certainty equivalent of a risky bet is the (maximum) amount of money one is willing to pay for the risky bet, less than its expected value

- The other interpretation of ARA and RRA is that they relate to size of the **risk premium** characterizing a gamble/lottery/security

- This derives from the very definition of risk aversion and it is simply an application of the standard Jensen's inequality:

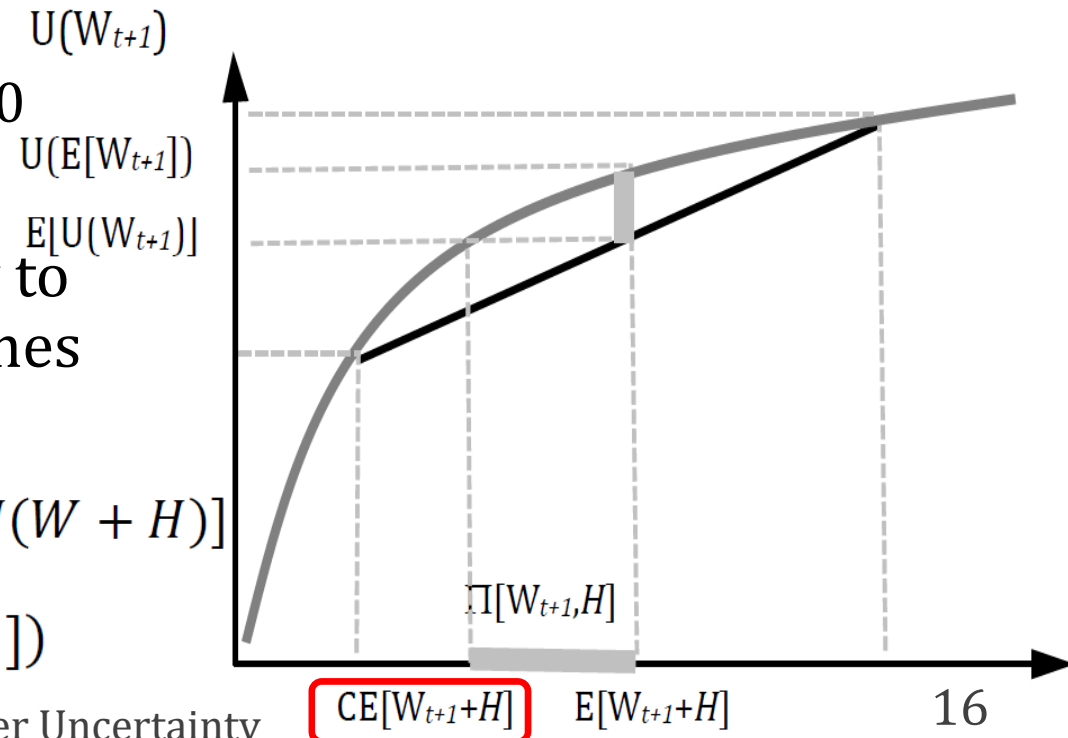
$$U(E[W + H]) = U\left(W + \sum_{s=1}^S \pi_s h_s\right) > \sum_{s=1}^S \pi_s U(W + h_s) = E[U(W + H)]$$

- H is a random variable with S outcomes, each with prob. $\pi_s \geq 0$

- The (maximum) certain sum of money a person is willing to pay to acquire a risky opportunity defines his **certainty equivalent** (CE):

$$U(CE(W, H)) = \sum_{s=1}^S \pi_s U(W + h_s) = E[U(W + H)]$$

$$\text{or } CE(W, H) = U^{-1}(E[U(W + H)])$$



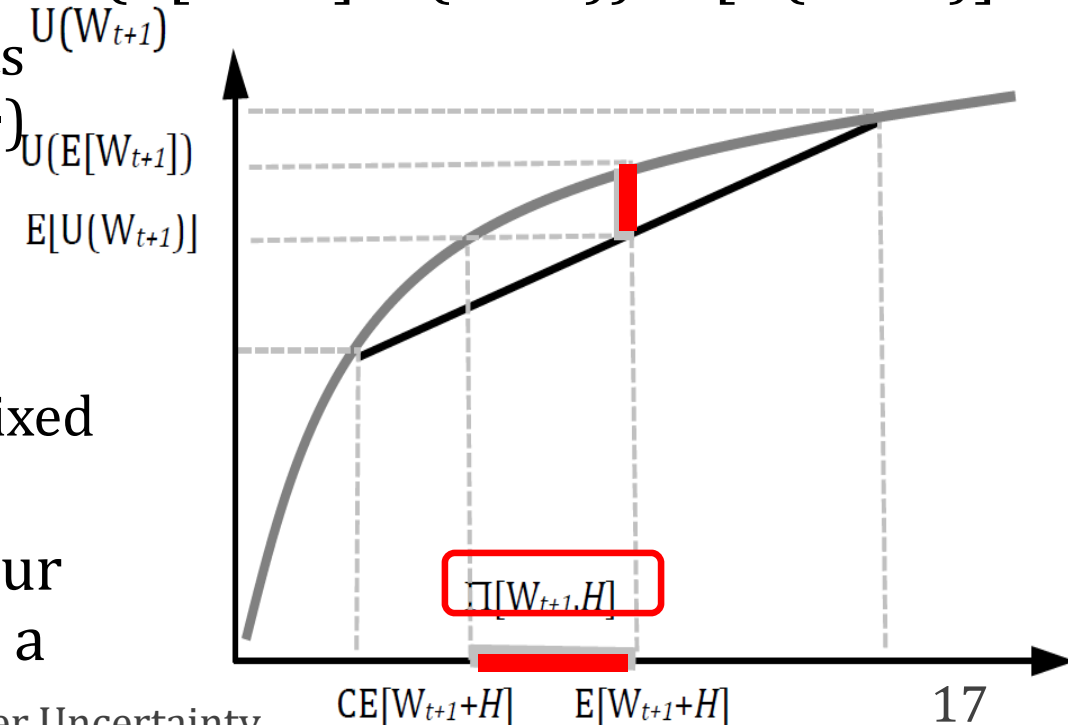
ARA and RRA and the Risk Premium

The risk premium measures the difference between the expected value of a bet and the certainty equivalent an investor is willing to pay for it

- The difference between the expected value of a risky prospect and its CE is a measure of the risky payoff's risk premium, $\Pi(W, H)$:

$$\Pi(W, H) \equiv E[W+H] - CE(W, H)$$

- It represents the maximum amount the agent would be willing to pay to avoid the gamble implied by the risky asset
- Equivalently, $\Pi(W, H)$ must be s.t.: $U(E[W+H] - \Pi(W, H)) = E[U(W+H)]$
 - The length of both red segments depends on the concavity of $U(\cdot)$
 - If one were to make it “more concave”, the size of both segments would increase
 - The same would occur if—for fixed $U(\cdot)$ —one were to increase h
- Using Taylor approximations, your textbook shows that when $H \rightarrow 0$ a result follows

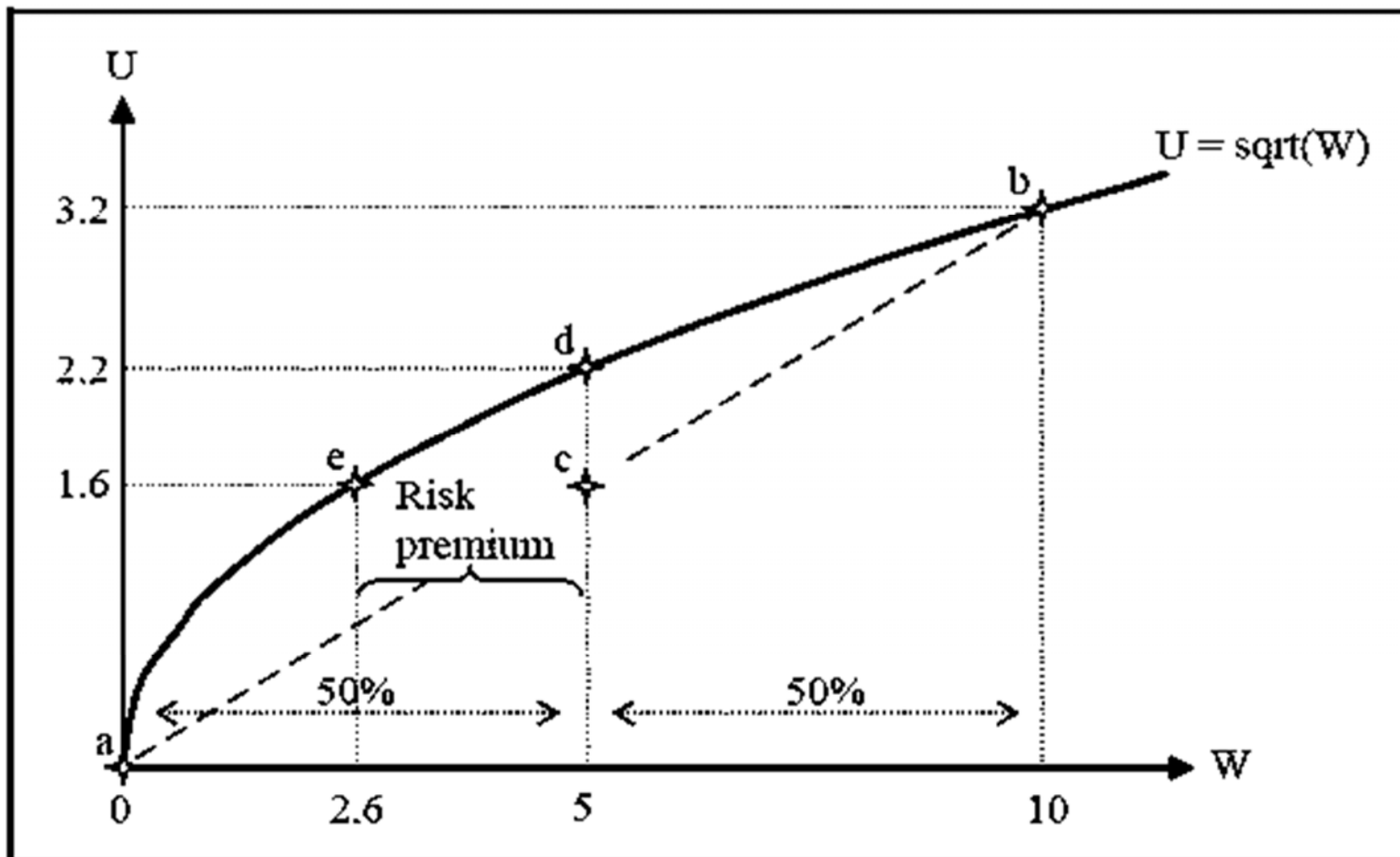


ARA and RRA and the Risk Premium

For small risks, ARA and RRA are proportional to the risk premium but are interacted with variance, i.e., the perceived quantity of risk

$$\Pi(W, H) \cong \frac{1}{2} ARA(W, H) Var[H]$$

- Time for a simple, “visual” numerical example:



ARA and RRA and the Risk Premium

$$\Pi(W, H) \cong \frac{1}{2} ARA(W, H) Var[H]$$

- Risk premium \propto (Subjective risk aversion) \times (Quantity of risk)
 - As before, because $ARA(W) \equiv RRA(W)/W$, we can re-write the result as:
$$\frac{\Pi(W, H)}{W} \cong \frac{1}{2} RRA(W, H) \frac{Var[H]}{W^2} = \frac{1}{2} RRA(W, H) Var[\varpi]$$
 - Consider a two-outcome symmetric bet with size h (i.e., the possible outcomes are h and $-h$ with fixed, objective probabilities π and $1-\pi$, respectively), we have that $Var[H] = h^2 = \pi h^2 + (1-\pi)(-h)^2$
 - E.g., if John is characterized by VNM function $U_{John}(W) = 1 - e^{-\theta W}$ then
$$\Pi_{John}(W; h) \cong \frac{1}{2} \theta h^2 \quad (\text{independent of wealth})$$
 - If $\theta = 0.1$, $W = 100$ euros and $h = 10$ euros with equally likely outcomes, then $\Pi_{John}(W; h) \cong (0.5)(0.1)(10)^2 = 5$ euros, and $CE = 95$
 - Let's check what the definition yields:

$$1 - e^{-0.1 \times CE} = 0.5(1 - e^{-(100-10)0.1}) + 0.5(1 - e^{-(100+10)0.1}) \Rightarrow CE = 95.6623$$

A Different Definition of Risk Premium

- Possible to convert these ideas into the classical definition of a percentage risk premium to be added to asset returns to compensate a decision-maker for the risk she runs
- Any risky gamble H , generates a gross return $1+(H/W) = 1 + R^H$ so that if CER is the riskless, certainty equivalent rate of return, then:

$$U((1 + CER)W) = E[U((1 + \tilde{R}^H)W)] \Rightarrow CER = \frac{U^{-1}\left(E\left[U\left((1 + \tilde{R}^H)W\right)\right]\right)}{W} - 1$$

- This equation defines CER implicitly
- The difference $E[R^H] - CER$ is often interpreted as a percentage risk premium associated to the risky asset/gamble H
- It is the percentage extra return that an investor requires to accept the risky gamble instead of settling for the riskless CER
- Consider again Mary, characterized by a power utility function of wealth
- Because $U = \frac{v^{1-\gamma}}{1-\gamma} \Rightarrow (1-\gamma)U = v^{1-\gamma} \Rightarrow v = [(1-\gamma)U]^{\frac{1}{1-\gamma}}$

One can show (see textbook) that

$$CER = \frac{\left[(1-\gamma)E\left[U\left((1 + \tilde{R}^H)W\right)\right]\right]^{\frac{1}{1-\gamma}}}{W} - 1$$

Introducing a Few Common Utility of Wealth Functions

- Our earlier examples have featured a few VNM utility functions, here we simply collect ideas on their functional form and properties
- Given an initial level of wealth W_0 , a utility of money function, which relative to the starting point has the property $U(W)/U(W_0) = h(W - W_0)$, so that **utility reacts only to the absolute difference in wealth**, is of the **absolute risk aversion** type
- Only (non-satiated) function meeting this requirement is the **(negative) exponential**, where response to changes in $W - W_0$ is constant:

$$U(W) = 1 - e^{-\theta W} \quad \text{with } \theta > 0$$
 - The textbook shows that this implies a constant ARA, and because of that the utility function is also referred to as CARA
 - As $ARA(W) = \theta$, $RRA(W) = ARA(W)W = \theta W$, a linear function of wealth
 - $RRA(W)$ depends on initial wealth level, relative quantities such as the percentage risk premium depend on initial wealth, which is problematic
- A **power, CRRA** utility function is

$$U(W) = \begin{cases} \frac{W^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \ln W & \gamma = 1 \end{cases}$$
 - The **textbook proves** that in this case $RRA(W) = \gamma$

Introducing a Few Common Utility of Wealth Functions

- As $ARA(W) = RRA(W)/W = \gamma/W$, an inverse function of wealth
- The textbook reports numerical examples that emphasize that different utility functions (even within the same power family) imply—for the same bet—rather different estimates of CE and hence risk premia
- A very popular class of utility functions is the **quadratic** one:
$$U(W) = W - \frac{1}{2}\kappa W^2 \quad \text{with } \kappa > 0$$
- Because $U'(W) = 1 - \kappa W$, $U''(W) = -\kappa$, this implies:
$$ARA(W) = -\frac{-\kappa}{1 - \kappa W} = \frac{\kappa}{1 - \kappa W} = \frac{\kappa}{W[(1/W) - \kappa]}$$
$$RRA(W) = -\frac{-\kappa W}{1 - \kappa W} = \frac{\kappa}{[(1/W) - \kappa]}$$
 - A quadratic utility investor is not always risk averse: $ARA(W)$ and $RRA(W)$ are positive if and only if $\kappa < 1/W$, or if $W < W^* = 1/\kappa =$ **bliss point**
 - In fact, $W < W^* = 1/\kappa$ is also necessary and sufficient for the investor to be non-satiated, i.e., for the utility function to be monotone increasing
- One final VNM utility function is the **linear** one: $U(W) = a + bW$, $b > 0$
- $U'(W) = b$ and $U''(W) = 0$, imply that $ARA(W) = RRA(W) = 0$

Introducing a Few Common Utility of Wealth Functions

- All these utility functions are strictly increasing and concave, have risk tolerance $T(W)$ that depends of wealth in a linear affine fashion:

$$T_{exp}(W) = \frac{1}{\theta} \quad T_{power}(W) = \frac{1}{\gamma} W \quad T_{quadr}(W) = \frac{1}{\kappa} - W$$

- These functions are called **linear risk tolerance** (LRT) utility functions (alternatively, HARA utility functions, where HARA stands for hyperbolic absolute risk aversion, since $ARA(W)$ defines a hyperbola)
- LRT utility functions have many attractive properties:

$$U(W) = \frac{\gamma}{1-\gamma} \left(\frac{\theta W}{\gamma} + \beta \right)^{1-\gamma} \quad \text{with } \gamma \neq 1, \frac{\theta W}{\gamma} + \beta > 0, \theta > 0$$

- It is possible to check that

$$ARA(W) = \left(\frac{W}{\gamma} + \frac{\beta}{\theta} \right)^{-1}$$

- When $\gamma \rightarrow +\infty$ and $\beta=1$, $ARA(W) \rightarrow \theta$ (the CARA case), and when $\beta=0$, $ARA(W)=\gamma/W$ (the CRRA case)
- Correspondingly, the risk tolerance function is $T_{HARA}(W) = \frac{\beta}{\theta} + \frac{W}{\gamma}$
- It is clearly linear affine and increasing in wealth
- This nests all cases reported above

Summary: Common Utility of Wealth Functions

- Under such a definition, the risk premium is the percentage extra return that an investor requires to accept the risky gamble instead of settling for the riskless CER

- The four most common VNM felicity functions are

Negative exponential, CARA $U(W) = 1 - e^{-\theta W}$ with $\theta > 0$

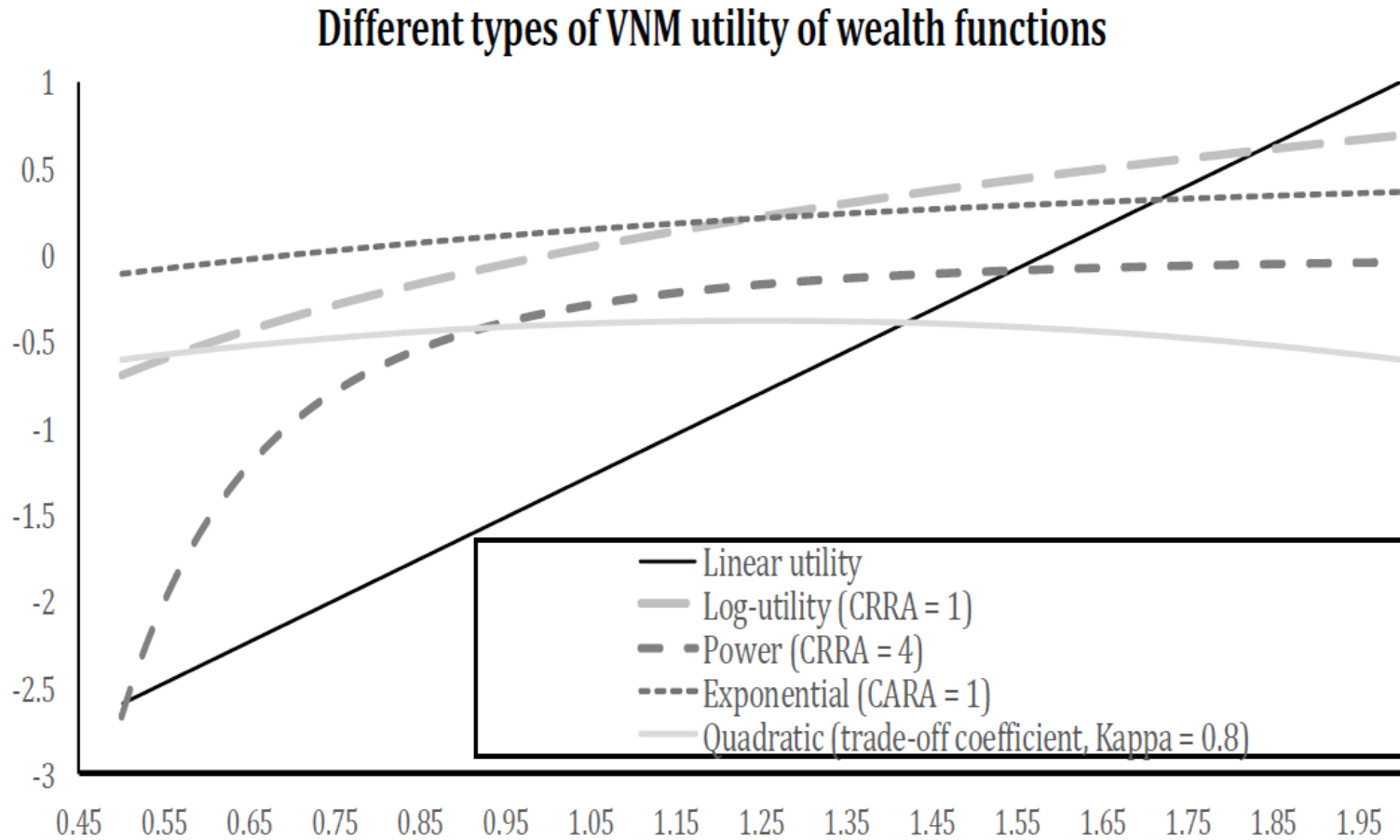
Power, CRRA
$$U(W) = \begin{cases} \frac{W^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \ln W & \gamma = 1 \end{cases}$$

Quadratic, IARA $U(W) = W - \frac{1}{2}\kappa W^2$ with $\kappa > 0$

Linear, risk-neutral $U(W) = a + bW$ with $b > 0$

- Quadratic utility poses a few problems: e.g., the investor is not nonsatiated for all wealth levels; she is satiated below the bliss
- These functions are called linear risk tolerance (LRT) utility functions (alternatively, HARA, **hyperbolic absolute risk aversion**, because their $ARA(W)$ defines a hyperbola)

Common Utility of Wealth Functions



- All functions, apart from the linear, risk-neutral function, are concave
- No special meaning (or lack therefore) ought to be attached to the fact that all utility function are negative for some wealth levels (in fact, a few are always negative for all wealth levels)

From the Density of Wealth to the Density of $U(W)$

- Risk aversion is captured by the concavity of $U(\cdot)$
- It changes the perception of the problem for an investor
- On the horizontal axis, where wealth is measured, we plot the density function of portfolio outcomes
- This does not have to be, but could be a symmetric Gaussian density
- We map the probability distribution of wealth into a probability density function for the corresponding utility index, $f(U(W))$
- The concavity of the utility function makes for one asymmetric, fat tailed distribution that certainly deviates from a (Gaussian) benchmark
- May have important effects on investors' optimal portfolios

