SDA Bocconi

Bocconi School of Management

EMFI

EXECUTIVE MASTER IN FINANCIAL INVESTMENTS

INVEST IN YOUR VALUE

Fundamentals of Mean-Variance Analysis

03.07.2019

MILANO I ITALY

Università Commerciale
Luigi Bocconi

Lecture 1: Fundamentals of Mean-Variance Analysis

Prof. Massimo Guidolin

Portfolio Management

EMFI Angola’s Sovereign Fund 2019

Overview

- Generalities on the investment management process
- How to correctly measure portfolio performance
- The characteristics of the opportunity set
- Portfolio combinations
- Measuring linear co-movements: correlations
- The effects of diversification: systematic risk
- Mean-variance and efficient frontiers: logical meaning
- The case of no borrowing and lending and two risky assets
- Generalizations to the case of N risky assets
- Two-fund separation result

Generalities

Investment (asset) management begins with an analysis of the investment objectives of the entity whose funds are managed

- The purpose of this course is to describe the activities and methodological tools/frameworks used in investment management
  - Investment management also referred to as ptf./money management
- Our immediate goals are to achieve an understanding of:

① How investment objectives are determined

② The investment vehicles to which an investor can allocate funds (not necessarily these are securitized), the asset menu

③ The investment strategies that can be employed by an investor to realize a specified investment objective

④ The best way to construct a ptf., given an investment strategy

⑤ The techniques for evaluating performance

- The allocating entities may consist of either individual investors or institutional investors

Generalities

- Institutional investors include:
  - Pension funds
  - Depository institutions (commercial banks, savings and loan associations, and credit unions)
  - Insurance companies (life companies, property and casualty companies, and health companies)
  - Regulated investment companies (mutual and closed-end funds)
  - Endowments and foundations
  - Treasury department of corporations, municipal governments, and government agencies
- We can classify institutional investors into two categories: those that must meet contractually specified liabilities vs. those that don't
- In the 1st category we have institutions with "liability-driven objectives" vs. those in the 2nd with "non-liability driven objectives"
  - An institutional investor is concerned with both the amount and timing of liabilities because its assets must produce the cash flow to meet any payments it has promised to make

Generalities

Defining clear objectives delimits with precision a sensible asset menu, the strategies and techniques of portfolio selection, and eventually the methodologies of performance assessment

- To form an investment policy, many factors are considered: client constraints, regulatory constraints, tax and accounting issues
  - Examples of client-imposed constraints are restrictions that specify the types of securities in which to invest and concentration limits on how much invested in a particular asset class or in a particular issuer
  - When a benchmark is established, there may be a restriction as to the degree to which the manager may deviate from some key characteristics of that benchmark (passive management)
  - Regulatory constraints also affect the asset classes that are permissible and impose concentration limits on investments
- In making the asset allocation decision, consideration must be given to any risk-based capital requirements
- Sometimes, certain institutional investors are exempt from income taxation if they invest in certain ways or asset classes

Generalities

Portfolio strategies may be active or passive

- Portfolio strategies can be classified as either active or passive
- An active portfolio strategy uses available information and forecasting techniques to seek a better performance than a portfolio that is simply diversified broadly
  - Essential to all active strategies are expectations about the factors that have been found to influence the performance of an asset class
- A passive portfolio strategy involves minimal expectational input, and instead relies on diversification to match the performance of some market index
  - Passive strategies assume that prices impound all information
- Which should be selected? The answer depends on (1) the client's or money manager's view of how "price-efficient" the market is; (2) the client's risk tolerance; and (3) a client's liabilities
- Once a portfolio strategy is selected, the next task is to construct the portfolio (i.e., select the specific assets to be included)

Generalities

- An efficient portfolio is one that provides the highest expected return for a given level of risk, or equivalently, the lowest risk for a given expected return
  - Three key inputs are needed: future expected return, variance of asset returns, and correlation of asset returns
  - Sometimes more complex inputs will be needed, in the limit the dynamics of the joint conditional density of the returns of the asset in the menu of selection
- Finally, performance measurement involves the calculation of the return realized by a portfolio manager over some time interval
- Performance evaluation is concerned with three issues:

① Whether the portfolio manager added value by outperforming the established benchmark

② Identifying how the manager achieved the calculated return

③ Assessing whether the portfolio manager achieved superior performance (i.e., added value) by skill or by luck

The characteristics of the opportunity set

Asset payoffs under risk are described by return distributions

- The existence of risk means that the investor can no longer associate a single number or payoff with investing in any asset
- The payoff must be described by a set of outcomes and each of their associated probability, called a frequency or **return distribution**
- However, to work with densities is extremely complex
- Often specific attributes (**moments**) are used to summarize the key features of such distribution: a measure of central tendency, called the **expected return**, and a measure of risk or dispersion around the mean, called the **standard deviation**
- A frequency distribution is a list of all possible out-comes along with the prob. of each
- Usually we do not delineate all of the possibilities

State	*Security A*		*Security B*	
	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15
ii	18	5/15	18	5/15
iii	14	4/15	10	4/15
iv	10	2/15	5	2/15
v	6	1/15	5	1/15

The characteristics of the opportunity set

The common measure of location for returns is the expectation

- It takes at least two measures to capture the relevant information about a frequency function: one to measure the average value and one to measure the dispersion around the average value
- Using the summation notation, we have:

$$E[R_i] = \sum_{s=1}^{S} Prob(state = s) R_i(s)$$

  where Prob(*state*) is the probability of the *s*th state for asset *i*
- Certain properties of expected values are extremely useful
  - The expected value of the sum of two returns is equal to the sum of the expected value of each return, that is, $E[R_1 + R_2] = E[R_1] + E[R_2]$
  - The expected value of a constant "*C*" times a return is the constant times the expected return, that is, $E[CR_1] = CE[R_1]$
- Not only is it necessary to have a measure of the average return, it is also useful to have some measure of how much the outcomes differ from the average

The characteristics of the opportunity set

The common measure of dispersion for returns is the variance

- The average squared deviation is the variance

$$Var[R_i] = \sum_{s=1}^{S} Prob(state = s)\left[R_i(s) - E[R_i]\right]^2$$

  - Many utility functions can be expressed either exactly or approximately in terms of mean and variance
  - Furthermore, regardless of the investor's utility function, if returns are normally distributed, the mean and variance contain all relevant information about the distribution
  - The square root of the variance is called the standard deviation
  - In this case, even though security B gives a lower mean return, it is more risky than A

State	*Security A*		*Security B*	
	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15
ii	18	5/15	18	5/15
iii	14	4/15	10	4/15
iv	10	2/15	5	2/15
v	6	1/15	5	1/15
Mean	15.47		13.27	
Variance	16.78		28.46	

The characteristics of the opportunity set

Several measures of dispersion focus on downside risk only

- There are other measures of dispersion that could be used
  - The average absolute deviation
  - A semi-variance based only on deviations below the mean
  - The argument is that returns above the average return are desirable: the only returns that disturb an investor are those below average
- It is just one of a number of possible measures of downside risk
  - More generally, we can consider returns relative to other benchmarks, including a risk-free return or zero return
  - These generalized measures are referred to as lower partial moments
  - Other measure of downside risk is the so-called Value at Risk, which is widely used by banks to measure their exposure to adverse events
  - It measures the least expected loss (relative to zero, or relative to wealth) that will be expected with a certain probability
  - E.g., if 5% of the outcomes are below -30% and if the decision maker is concerned about how poor the outcomes are 5% of the time, then -30% is the 5% value at risk

Portfolio combinations

Several measures of dispersion focus on downside risk only

- Intuitively, these measures of downside risk are reasonable and some portfolio theory has been developed using them
- They are difficult to use when we move from single assets to ptfs
- In cases where the distribution of returns is symmetrical, the ordering of portfolios in mean variance space will be the same as the ordering of portfolios in mean semi-variance space or mean and any of the other measures of downside risk discussed above
    - If returns on an asset are symmetrical, the semi-variance is proportional to the variance; therefore in most of the portfolio literature the variance is used as a measure of dispersion
- Consider two assets. How can we decide which one we prefer?
- Intuitively one would think that most investors would prefer the one with the higher expected return if standard deviation was held constant
- In the following table, most investors would prefer asset 2 to 1

Portfolio combinations

The risk of a combination of assets is very different from a simple average of the risk of individual assets

Market Condition	Return[a] Asset 1	Return[a] Asset 2	Return[a] Asset 3	Return[a] Asset 5	Rainfall	Return[a] Asset 4
Good	15	16	1	16	Plentiful	16
Average	9	10	10	10	Average	10
Poor	3	4	19	4	Poor	4
Mean return	9	10	10	10		10
Variance	24	24	54	24		24
Standard deviation	4.9	4.9	7.35	4.90		4.9

[a]The alternative returns on each asset are assumed equally likely and, thus, each has a probability of $\frac{1}{3}$.

- Similarly, if expected return were held constant, investors would prefer the one with the lower variance
  - In the table, the investor would prefer asset 2 to asset 3
- However, the options open to an investor are not to simply pick between assets 1, 2, 3, 4, or 5 but also to consider combinations
- This makes sense because the risk of a combination of assets is very different from a simple average of the risk of individual assets

Portfolio combinations

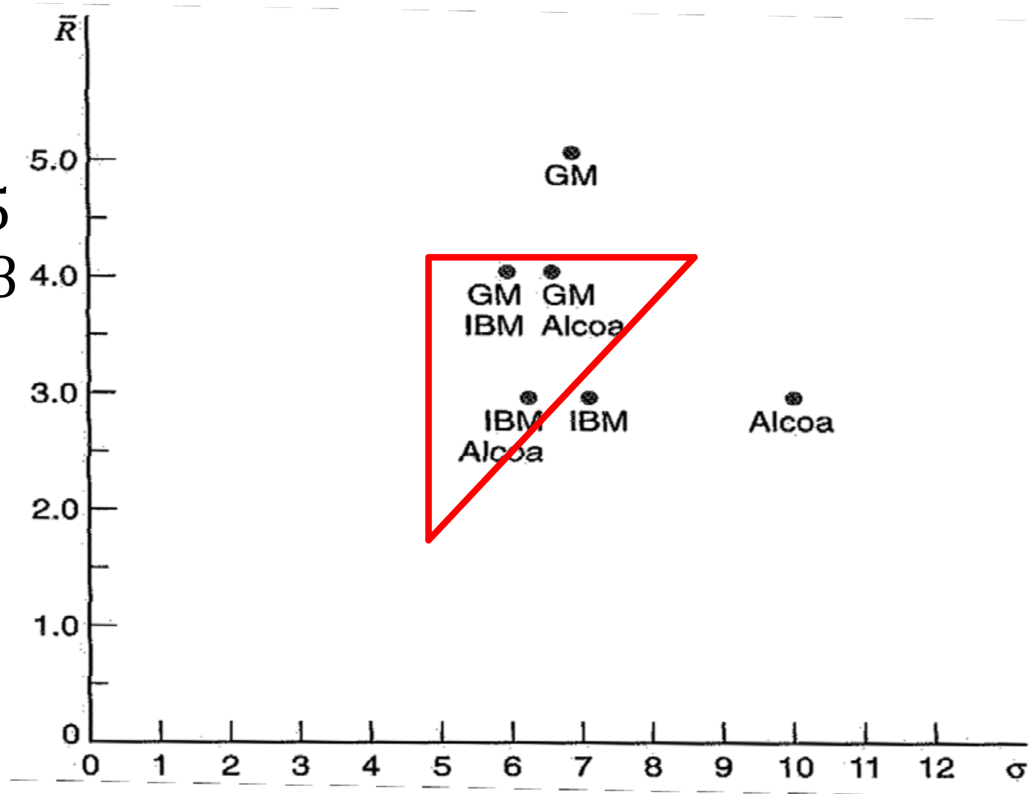
When returns of assets that are less than perfectly correlated, the resulting ptf. has less dispersion than either assets

Condition of Market	Asset 2	Asset 3	Combination of Asset 2 (60%) and Asset 3 (40%)
Good	$1.16	$1.01	$1.10
Average	1.10	1.10	1.10
Poor	1.04	1.19	1.10

- Most dramatically, the variance of a combination of two assets may be less than the variance of either of the assets themselves
  - In the table, there is a combination of asset 2 and asset 3 that is less risky than either asset 2 or 3
  - When two assets have their good and poor returns at opposite times, an investor can always find some combination of these assets that yields the same return under all market conditions
  - Even when the returns on assets are independent such as the returns on assets 2 and 4, a portfolio of such assets can have less dispersion than either asset

Portfolio combinations

- The figure shows a real-life combinations of 3 stocks
  - Correlations: IBM-Alcoa =0.05 GM-Alcoa=0.22; IBM-GM=0.48
  - 50-50 combinations shown
  - A ptf. 50% IBM and 50% Alcoa has the same return as each stock but less risk



- As mean and variance are used to summarize return distributions, we need formal ways to study how assets co-move
- The return on a portfolio of assets is simply a weighted average of the return on the individual assets: $$R_{Pj} = \sum_{i=1}^{N} \left(X_i R_{ij} \right)$$

  X_i is the fraction of the investor's funds invested in the *i*th asset
- Given properties of expectations, $\bar{R}_P = E(R_P) = E\left(\sum_{i=1}^{N} X_i R_{ij} \right) = \sum_{i=1}^{N} \left(X_i \bar{R}_i \right)$

Portfolio combinations: return covariances

- The variance on a portfolio is a little more difficult to determine than the expected return; we start out with a two-asset example:

$$\sigma_P^2 = E\left(R_P - \overline{R}_P\right)^2 = E\left[X_1 R_{1j} + X_2 R_{2j} - \left(X_1 \overline{R}_1 + X_2 \overline{R}_2\right)\right]^2$$

$$= E\left[X_1\left(R_{1j} - \overline{R}_1\right) + X_2\left(R_{2j} - \overline{R}_2\right)\right]^2$$

$$= E\left[X_1^2\left(R_{1j} - \overline{R}_1\right)^2 + 2X_1X_2\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right) + X_2^2\left(R_{2j} - \overline{R}_2\right)^2\right]$$

- Applying our two rules on expected values, we have

$$\sigma_P^2 = X_1^2 E\left[\left(R_{1j} - \overline{R}_1\right)^2\right] + 2X_1X_2 E\left[\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right)\right] + X_2^2 E\left[\left(R_{2j} - \overline{R}_2\right)^2\right]$$

$$= X_1^2\sigma_1^2 + 2X_1X_2 E\left[\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right)\right] + X_2^2\sigma_2^2$$

- $E[(R_{1i} - \overline{R}_1)(R_{2i} - \overline{R}_2)]$ has a special name: the covariance, σ_{12}
- The covariance is the expected value of the product of two deviations: the deviations of the returns on security 1 from its mean and the deviations of security 2 from its mean
- As such it can be positive or negative

Portfolio combinations: return covariances

Covariance and correlations measure linear co-movements

- It will be large when the good outcomes for each stock occur together and when the bad outcomes for each stock occur together
- The covariance is a measure of how returns move together
- For many purposes it is useful to standardize the covariance: dividing the covariance between two assets by the product of the standard deviation of each asset produces a variable with the same properties as the covariance but with a range of -1 to + 1
- This measure is the correlation coefficient: $\varrho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$
- Covariances and correlations are traditionally collected in symmetric matrices
  - Note that when ρ_{12} = -1 a ptf. may be built with no risk
  - In general ptf. risk is less than individual securitities when ρ_{12} = 0

	1	2	3	4	5
1		24 (+1)	−36 (−1)	0 (0)	24 (+1)
2			−36 (−1)	0 (0)	24 (+1)
3				0 (0)	−36 (−1)
4					0 (0)
5					

Aggregate portfolio variance

A ptf. of N uncorrelated assets has a zero variance as $N \rightarrow \infty$

- The formula for variance of a portfolio can be generalized to any generic number N of uncorrelated assets
  - The variance of each asset is multiplied by the square of the proportion invested in it: $\sum_{j=1}^{N}\left(X_j^2\sigma_j^2\right)$
  - With N assets the proportion invested in each asset is l/N, applying our formula yields:
$$\sigma_P^2 = \sum_{j=1}^{N}(1/N)^2\sigma_j^2 = 1/N\left[\sum_{j=1}^{N}\frac{\sigma_j^2}{N}\right]$$
  - Thus the formula reduces to $\sigma_P^2 = 1/N\bar{\sigma}_j^2$, where "sigma-bar" represents the average variance of the stocks in the portfolio
  - As N gets larger and larger, the variance of the portfolio gets smaller and smaller and the variance of the portfolio approaches zero
  - This is a general result: if we have enough uncorrelated assets, the variance of a portfolio of these assets approaches zero
  - In general, we are not so fortunate as in most markets the correlation coefficient and the covariance between assets is positive
  - In these markets the risk on the portfolio cannot be made to go to zero

Aggregate portfolio variance

While as N $\to \infty$ the contribution to variance goes to zero, the contribution to covariance has a finite limit

- The general formula is: $$\sigma_P^2 = \sum_{j=1}^{N}\left(X_j^2\sigma_j^2\right) + \sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\left(X_j X_k \sigma_{jk}\right)$$
  - Once again, consider equal investment in N assets:
  $$\sigma_P^2 = \sum_{j=1}^{N}(1/N)^2\sigma_j^2 + \sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}(1/N)(1/N)\sigma_{jk}$$
  - Factoring out l/N from the first summation and (N - l)/N from the second yields: $$\sigma_P^2 = (1/N)\sum_{j=1}^{N}\left[\frac{\sigma_j^2}{N}\right] + \frac{(N-1)}{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\left[\frac{\sigma_{jk}}{N(N-1)}\right]$$
  - Both of the terms in the brackets are averages: the second term is the summation of covariances divided by the number of covariances
  - Replacing the summations by averages: $\sigma_P^2 = \frac{1}{N}\bar{\sigma}_j^2 + \frac{N-1}{N}\bar{\sigma}_{jk}$
  - The contribution to the portfolio variance of the variance of the individual securities goes to zero as N gets very large

The effects of diversification: systematic risk

The contribution to total risk caused by covariance is systematic

- The contribution of the covariance terms approaches the average covariance as N→∞
- The individual risk of securities can be diversified away, but the contribution to total risk caused by the covariance terms cannot be diversified away
- The table illustrates how this relationship looks when dealing with U.S. equities
  - The average variance and average covariance were calculated using monthly data for all stocks listed on the New York Stock Exchange
- As more and more securities are added, the average variance on the portfolio declines until it approaches the average covariance
  - The figure illustrate this same relationship for common equities in a number of countries

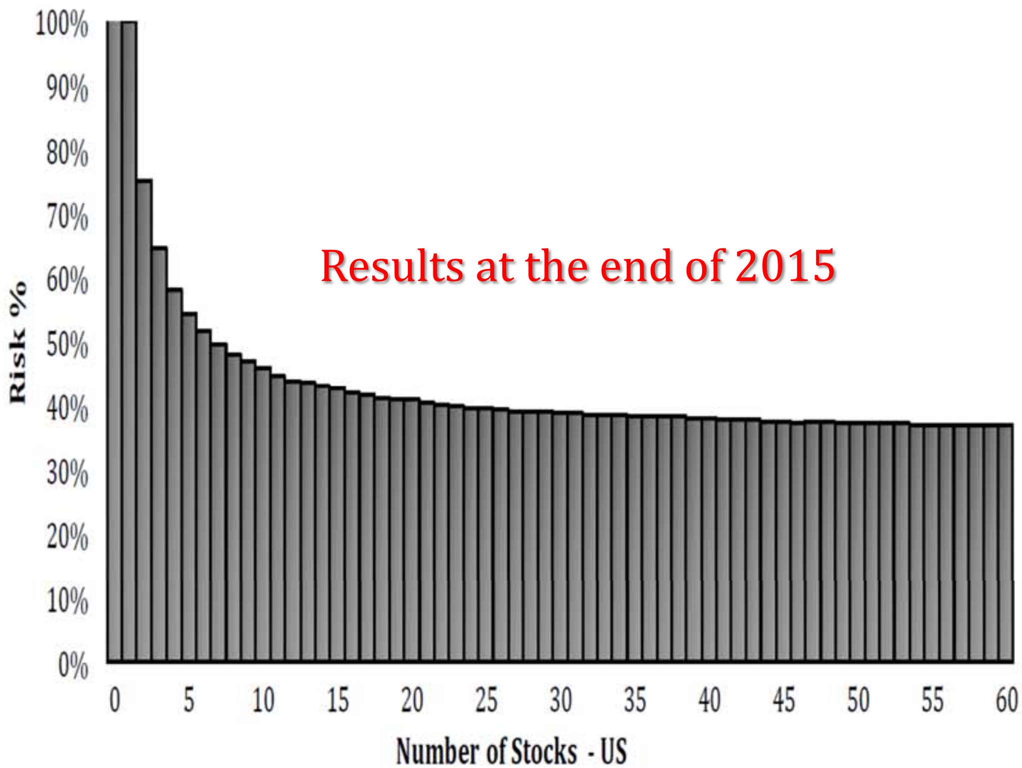
Number of Securities	Expected Portfolio Varianc
1	46.619
2	26.839
4	16.948
6	13.651
8	12.003
10	11.014
12	10.354
14	9.883
16	9.530
18	9.256
20	9.036
25	8.640
30	8.376
35	8.188
40	8.047
45	7.937
50	7.849
75	7.585
100	7.453
125	7.374
150	7.321
175	7.284
200	7.255
250	7.216
300	7.190
350	7.171
400	7.157
450	7.146
500	7.137
600	7.124
700	7.114
800	7.107
900	7.102
1000	7.097
Infinity	7.058

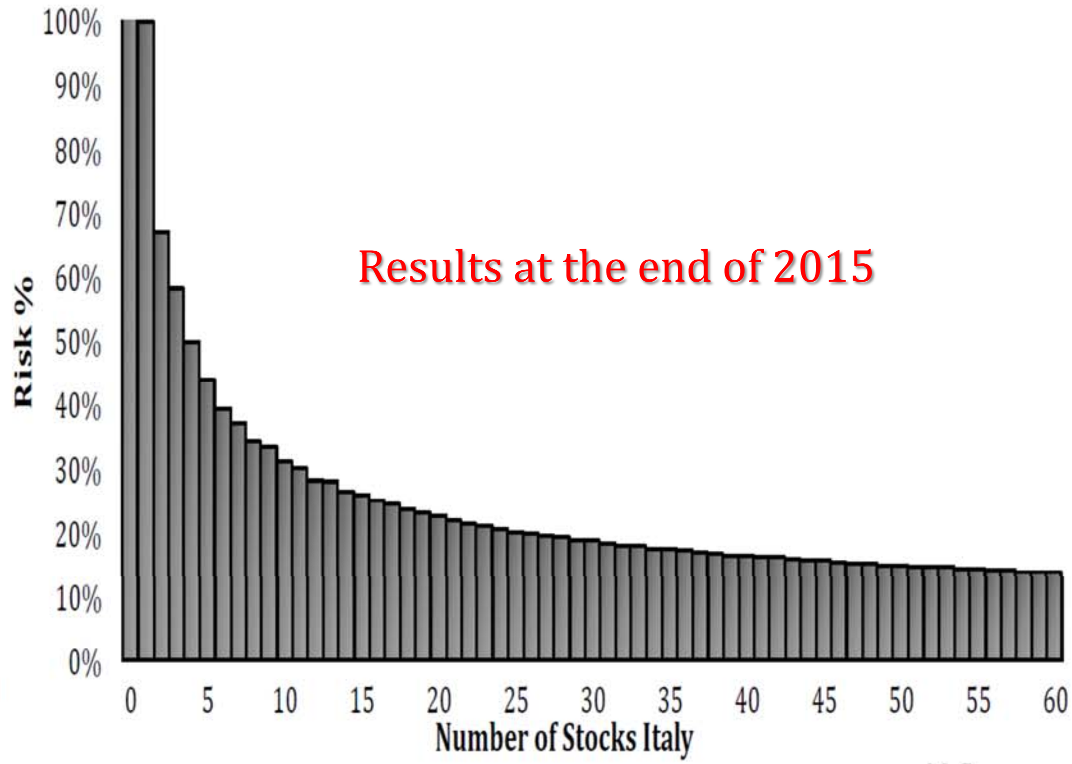
The effects of diversification: systematic risk

Percentage of the Risk on an Individual Security that Can Be Eliminated by Holding a Random Portfolio of Stocks within Selected National Markets and among National Markets [13]

Results from the 1970s

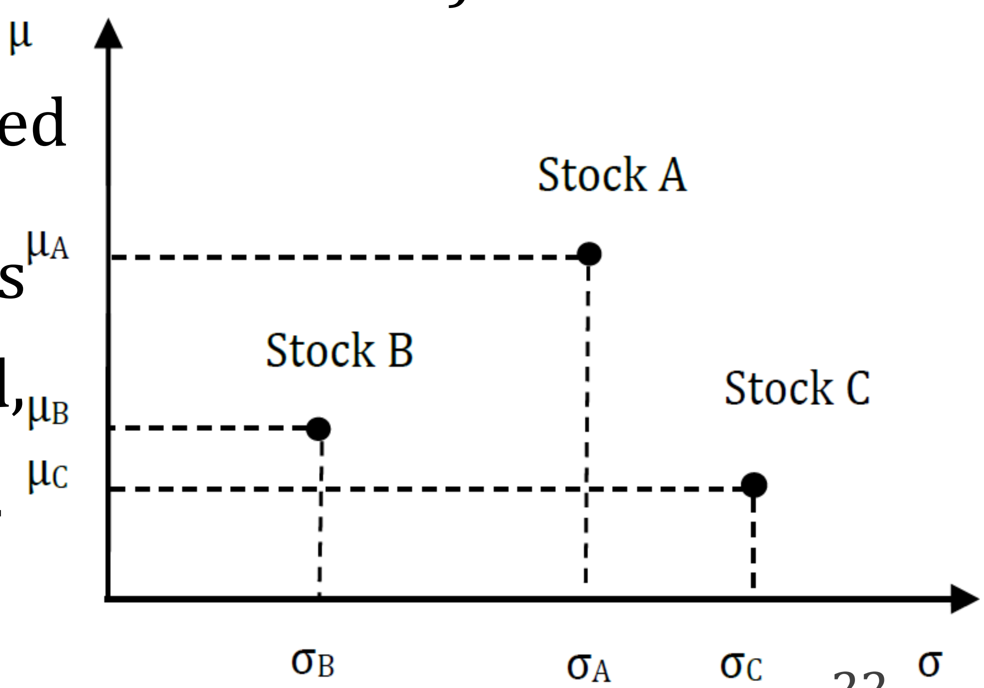
United States	73
U.K.	65.5
France	67.3
Germany	56.2
Italy	60.0
Belgium	80.0
Switzerland	56.0
Netherlands	76.1
International stocks	89.3





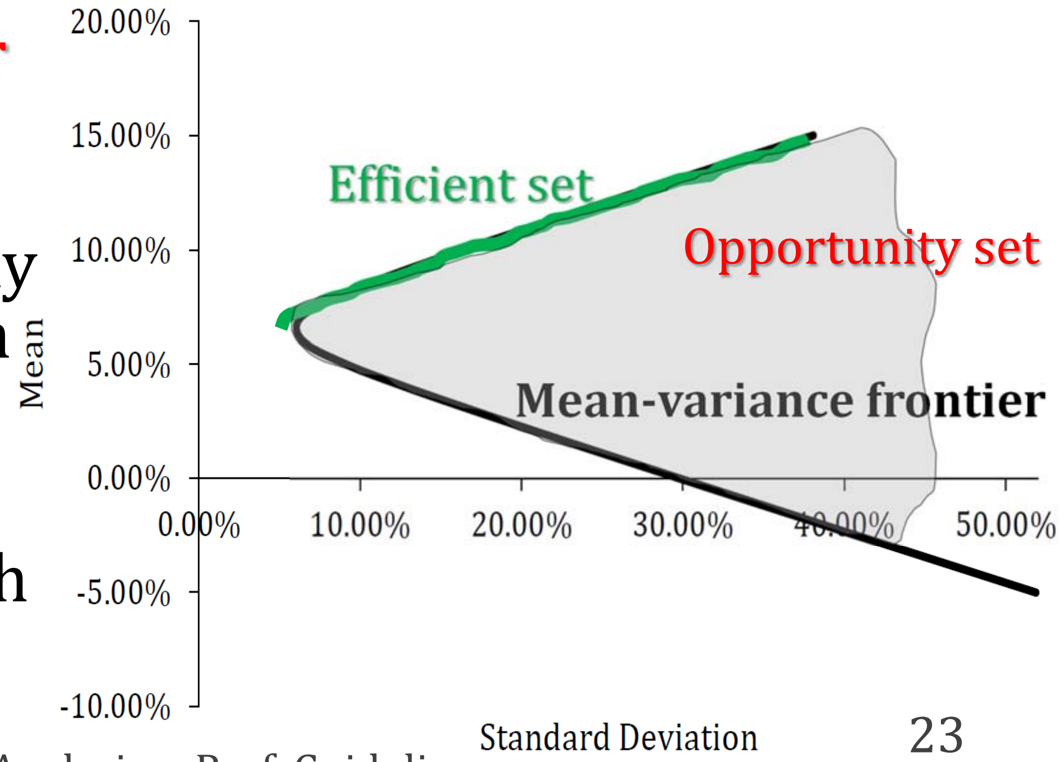
Key Concepts of Mean-Variance Analysis/1

- We review the development of the celebrated mean-variance framework introduced by Markowitz in the 1950s
- Initially at least, risky assets only, no borrowing or lending
- Assume that for some reasons, the joint distribution of asset returns is completely characterized by their means, variances, and covariances
- We represent each asset (and portfolio of assets) in a two-dimensional diagram, where expected portfolio return is plotted on the vertical axis and standard deviation is on the horizontal axis
- Not all securities may be selected, e.g., stock C is dominated by the remaining two stocks in terms of MV dominance



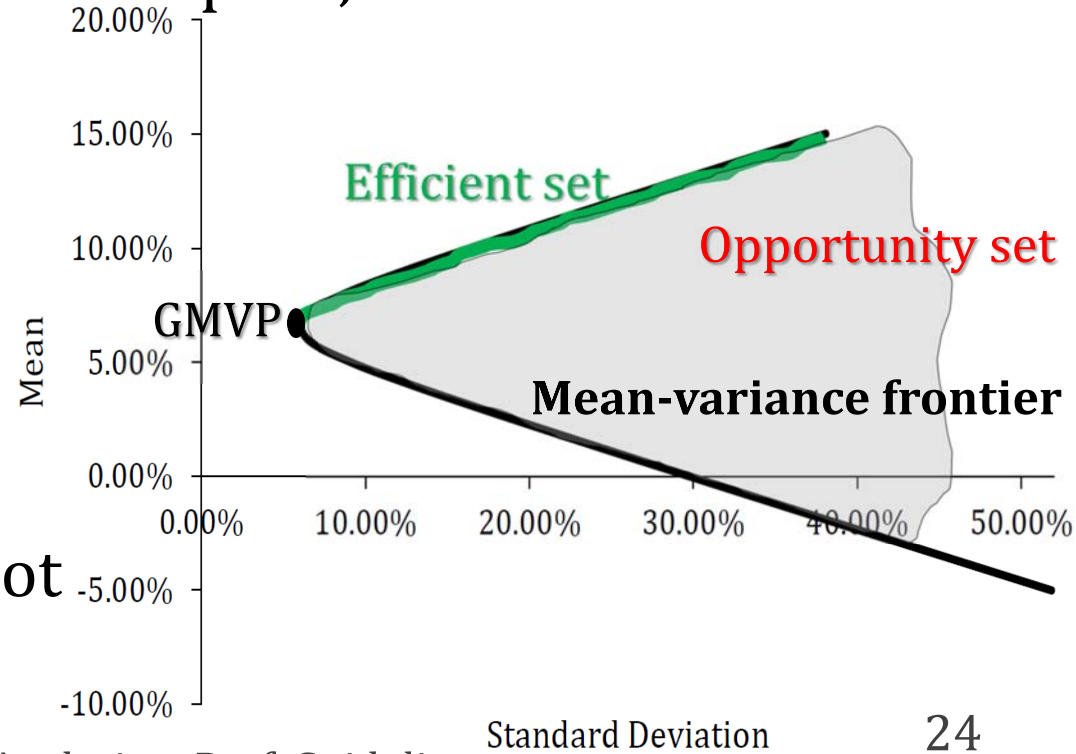
Key Concepts of Mean-Variance Analysis/2

- According to MV criterion a portfolio is efficient if and only if there is no other portfolio that allows the investor to achieve the same expected return with a lower level of risk or a higher level of expected return with the same level of risk
- Three key notions: i) the opportunity set (feasible region), which includes all the portfolios (both efficient and inefficient) that the investor is able to build given securities in the asset menu
- (ii) the mean-variance frontier (aka minimum variance frontier, MVF) subset of the opportunity set containing only the portfolio(s) with minimum variance for any target level of expected returns
- (iii) the efficient frontier, which only includes efficient ptfs



Key Concepts of Mean-Variance Analysis/3

- Because it is possible that a portfolio exists which has a higher return than another portfolio with the same level of risk, only portfolios that have a higher expected return than the global minimum variance portfolio (GMVP) are efficient
- The preferences of the investor(s) for risk are not relevant to the determination of the efficient frontier
- In the classical standard dev.-mean space, the MVF is not a function, but a «right-rotated hyperbola”
- The GMPV is Global Minimum Variance Ptf. and it is of high interest because «separates» the efficient set from the MVF
- The structure of GMVP does not depend on expected returns



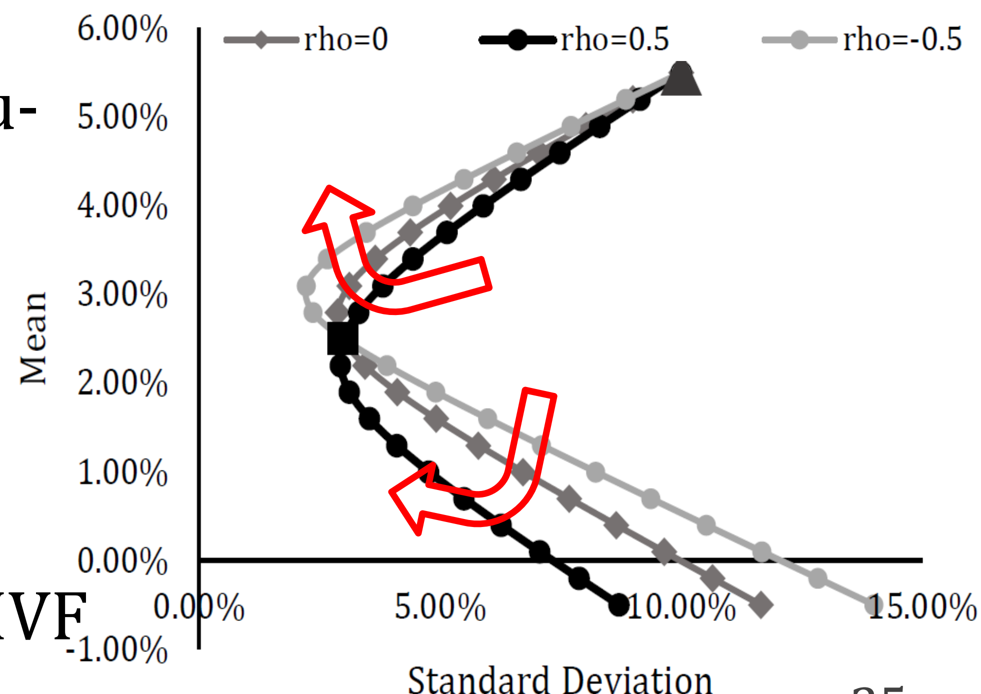
Key Concepts of Mean-Variance Analysis/4

- The GMVP and the entire MVF depend strongly on the correlation structure of security returns: the lower are the correlations (on average), the more the efficient set moves up and to the left, improving the risk-expected return trade-off
- The position and shape of the MVF reflects the diversification opportunities that a given asset menu offers
- Even though, MVF ptfs are solutions of a complex quadratic programming program, in the absence of constraints, their stru-cture is relatively simple:

$$\boldsymbol{\omega}^* = \mathbf{g} + \boldsymbol{h}\,\bar{\mu}$$

$$\mathbf{g} = \frac{1}{D}[B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})]$$

$$\mathbf{h} = \frac{1}{D}[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})]$$



- Combinations of MVF ptfs. are MVF

Key Concepts of Mean-Variance Analysis/5

- It is sufficient to know two points (portfolios) on the mean-variance frontier to generate all the others
- A two-fund separation result holds: all MV-optimizers can be satisfied by holding a combination of only two mutual funds (provided these are MV efficient), regardless of their preferences
- Their heterogeneous preferences will only impact the way in which they combine the two funds that they choose to hold
- As an implication, when there are N > 2 securities, the primitive assets need not to lie on the MVF
- Arguably, also risk-free assets exist, securities with zero variance of their returns and zero correlation with other assets
- Resorting to unlimited borrowing and lending and the risk-free rate changes the locus on which a rational investor performs her portfolio decisions

Key Concepts of Mean-Variance Analysis/6

- The presence of riskless assets creates capital transformation lines (CTL) and investors select efficient pfs. on the MVF such that they end up selecting their optimum on the steepest CTL
- When investors have homogeneous beliefs on means, variances, and correlations, in the absence of frictions, all investors will hold an identical tangency portfolio that maximizes the Sharpe ratio of the steepest CTL
- Such steepest CTL is called the Capital Market Line
- While the share of wealth an investor lends or borrows at the risk-free rate depends on the investor's preference for risk, the risky portfolio should be the same for all the investors
- This is special case of two-fund separation result stated above
- When lending and borrowing is only possible at different rates, it is no longer possible to determinate a tangency portfolio and the efficient set fails to be linear, the steepest CLT

Summary

- We have seen that there are precise ways, based on moments, to summarize the distribution of future asset returns
- Our next task is to show how the characteristics on combinations of securities can be used to define the opportunity set of investments from which the investor must make a choice
- In the process, we have discovered the principle of diversification
- By mixing a large number of assets, we can push their individual variance-driven contribution to the total portfolio variance to zero
- However, even as the number of assets $N \rightarrow \infty$, their average contribution to total risk (ptf. variance) cannot be pushed to zero if the assets are correlated on average
- The portion of risk that cannot be simply diversified away by increasing the number of assets in portfolio is called systematic or, indeed non-diversifiable, risk
- One has to bear in mind that in the process choices were made; e.g., risk has been measured as total variance and not as downside risk

SDA Bocconi

EMFI

EXECUTIVE MASTER IN FINANCIAL INVESTMENTS

INVEST IN YOUR VALUE