Università Commerciale
Luigi Bocconi

One Variable Calculus: Foundations and Applications

Prof. Manuela Pedio

20550– Quantitative Methods for Finance

August 2018

About myself

- I graduated from Bocconi's MSc in Finance in 2013 (yes, I remember what it is like to be in your shoes)
- I have a four-year work experience in banking (as a derivatives sales analyst) before enrolling in a Ph.D.
- I cooperate closely with prof. Guidolin (you will meet him in the next few days) so that we will see each other again during the year … (I am also his assistant in his role as MSc Director)
- If you want to contact me look at my Bocconi's page to find out office hours and my email:

http://didattica.unibocconi.it/docenti/cv.php?rif=196456

Objectives of the Course

- The objective of this (prep)-course is to review a number of fundamental concepts from calculus and optimization with a selection of applications to economics and finance
- In my part, I listed three books in the syllabus, the first two of them are substitutes:
  - *Mathematics for Economists* by Blume and Simon (BS) (classical book to teach mathematics for economists)
  - *Fundamental Methods of Mathematical Economics* by Chiang (C) (similar to Blume and Simon)
- But dozens of other textbooks that you may have already used in your previous studies may do the job
  - *Advanced Modelling in Finance Using Excel and VBA* by Jackson and Staunton (JS) (highly enjoyable book about financial modelling in Excel)

Outline of the Course

- **Lectures 1 and 2 (3 hours, in class):**
  - Linear and non-linear functions on $\mathbb{R}$
  - Limits, continuity, differentiability, rules to compute derivatives, approximation with differentials
  - Logarithmic and exponential functions
  - Introduction to integration
- **Lecture 3 (1.5 hours, in the lab):**
  - Review of matrix algebra with applications in Excel
- Lectures 4 and 5 (3 hours, 1.5 of which in the lab):
  - Introduction to optimization: functions of one variable
  - Generalization: functions of several variables
  - Use of Excel Solver for constrained optimization

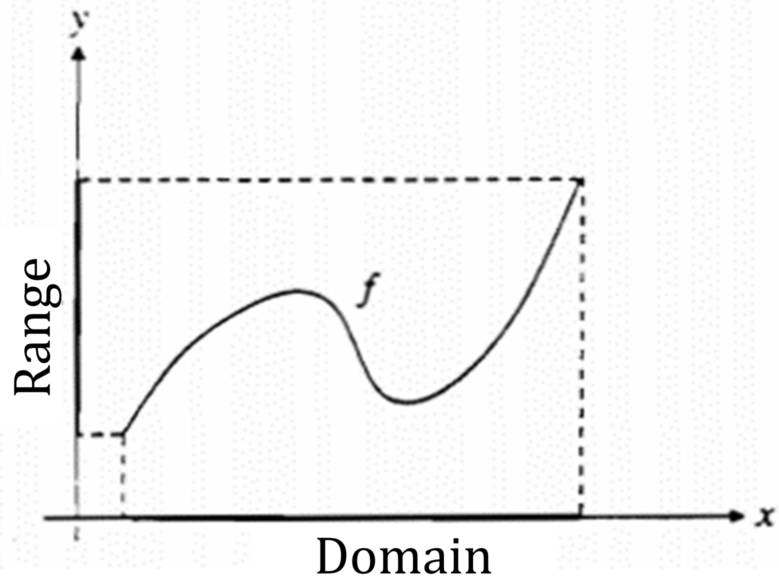
Warm up: a bit of definitions (1/2)

FUNCTIONS ON ℝ: A function of a real variable *x* with domain D is a rule that assigns a **unique** real number to each number *x* in D

- We typically use letters like *f* or *g* to denote such a rule
- For the time being, we will consider $f\colon \mathbb{R} \rightarrow I$, with $I \subset \mathbb{R}$, that is, functions from ℝ to ℝ
- Usually, $y = f(x)$ denotes the value that the function *f* assigns to the real number *x* belonging to its domain, or, in other words, the "value of *f* at *x*"
- As an example, $f(x) = x+1$ is a function that assigns to each *x* of the domain, a number that is one unit larger; for instance, $f(2) = 3$ is the value of this function at 2

Warm up: a bit of definitions (2/2)

- The domain is the set of numbers *x* at which *f(x)* is defined
- Typically, when the domain is not specified, we assume that it includes all the real numbers for which the function takes meaningful values
- For instance, for the function $f(x) = \frac{1}{x+3}$ the domain will be $(-\infty, -3) \cup (-3, \infty)$, that is, -3 is excluded from the domain
- The ***range*** (or co-domain) of a function is instead the set of the values assumed by the function
- As an example, the domain of *f*: $f(x) = |x|$ is equal to the entire $\mathbb{R}$ but the range is equal to $\mathbb{R}^+$

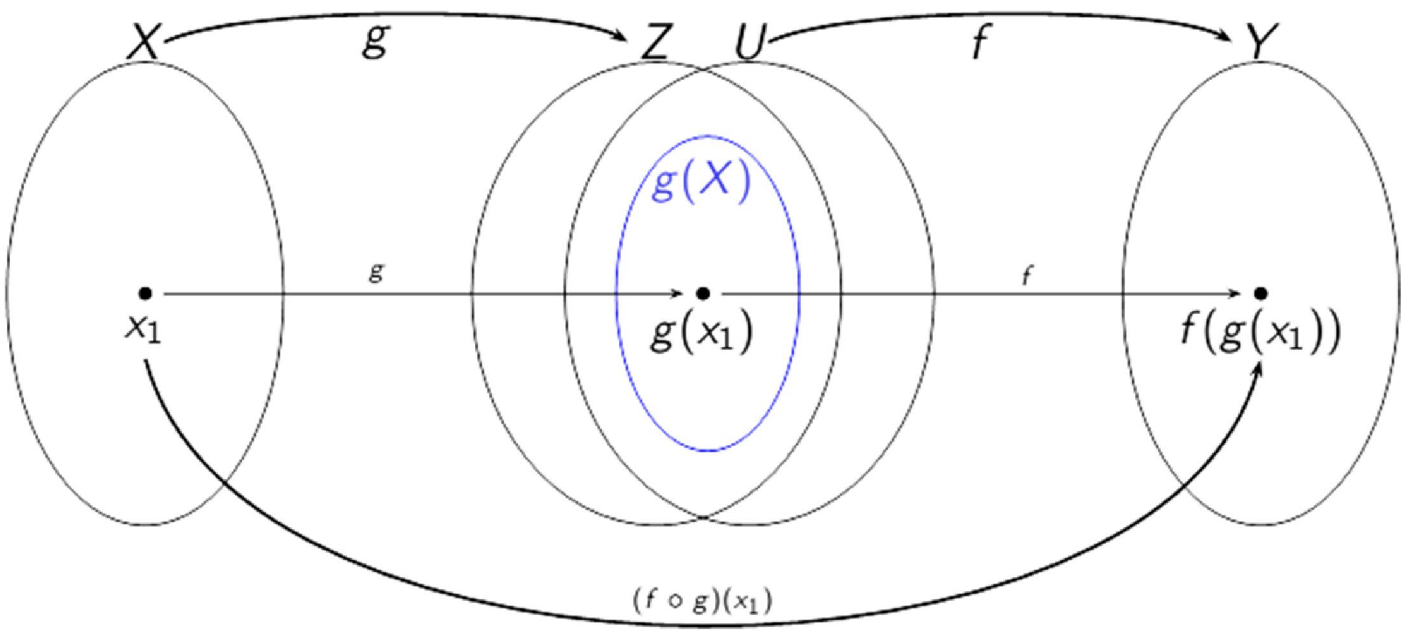


Warm up: composition of functions

- Let f: U $\rightarrow Y$, g: X $\rightarrow Z$, and $g(X) \subset U$. Then the function

$$f \circ g\colon\ X \rightarrow Y \text{ defined by}$$
$$(f \circ g)(x) := f(g(x))\ \forall\, x\ \in X$$

is called composition of f and g



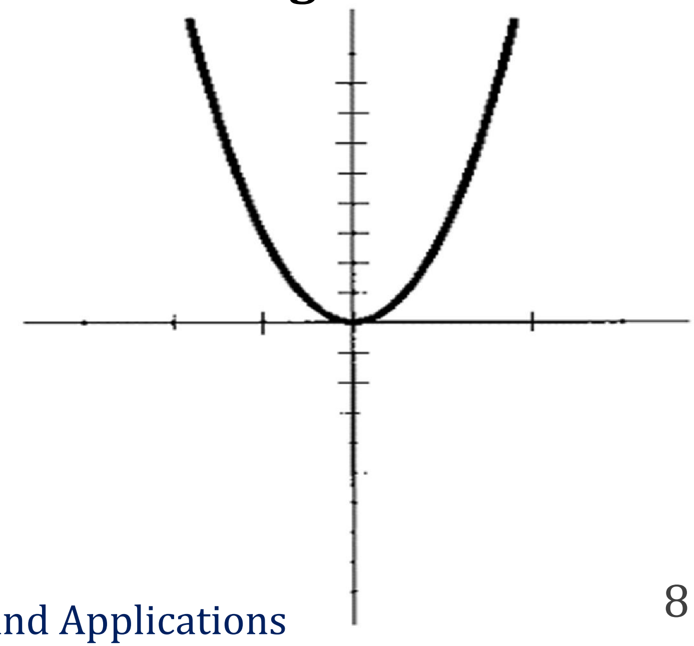
Basic geometric properties of a function (1/2)

- The basic geometric properties of a function are whether it is increasing or decreasing and the location of its local and global minima and maxima (if any)

A function f is **increasing** if $x_1 > x_2$ implies $f(x_1) > f(x_2)$ while a function f is **decreasing** if $x_1 > x_2$ implies $f(x_1) < f(x_2)$

- The function depicted in Figure 1 is decreasing over $\mathbb{R}^-$ and increasing over $\mathbb{R}^+$
- The point where the function turns from decreasing to increasing is a (global) **minimum** for this function, in this case zero

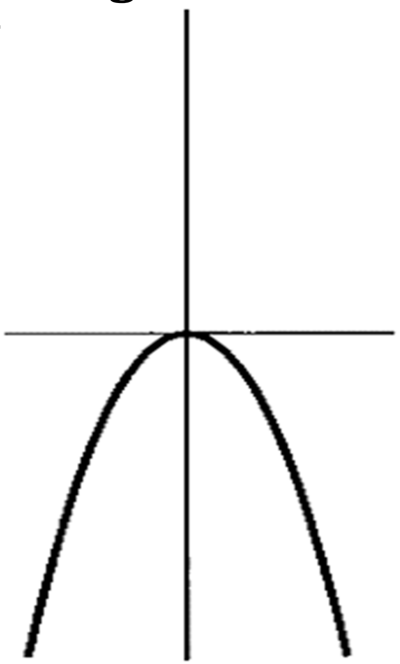
Figure 1



Basic geometric properties of a function (2/2)

- The function depicted in Figure 2 is increasing over $\mathbb{R}^-$ and decreasing over $\mathbb{R}^+$
- The point where the function turns from decreasing to increasing, is a (global) **maximum** for this function, here zero
- If a function f changes from decreasing to increasing at x_0, the point $(x_0, f(x_0))$ is a **local minimum** of the function f; if $f(x) \geq f(x_0)$ for all x then the point is a **global minimum**

Figure 2



- If a function f changes from increasing to decreasing at x_0, the point $(x_0, f(x_0))$ is a **local maximum** of the function f; if $f(x) \leq f(x_0)$ for all x then the point is a **global maximum**
- We will come back to these notions when we speak of optimization

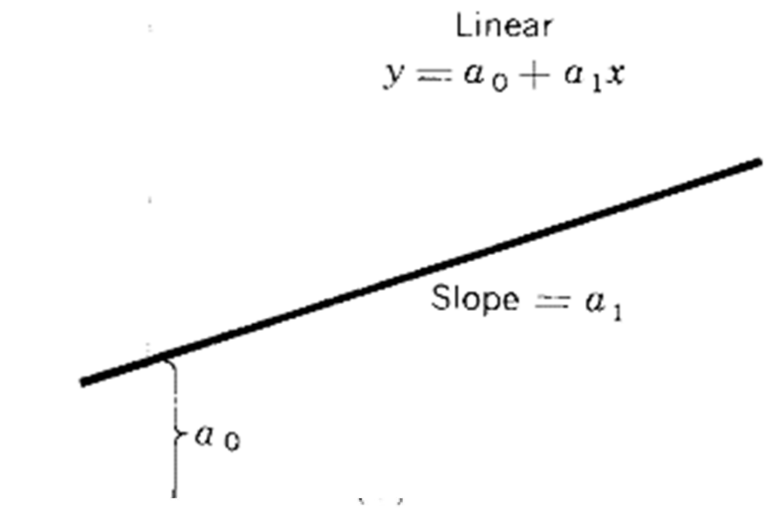
Different functions

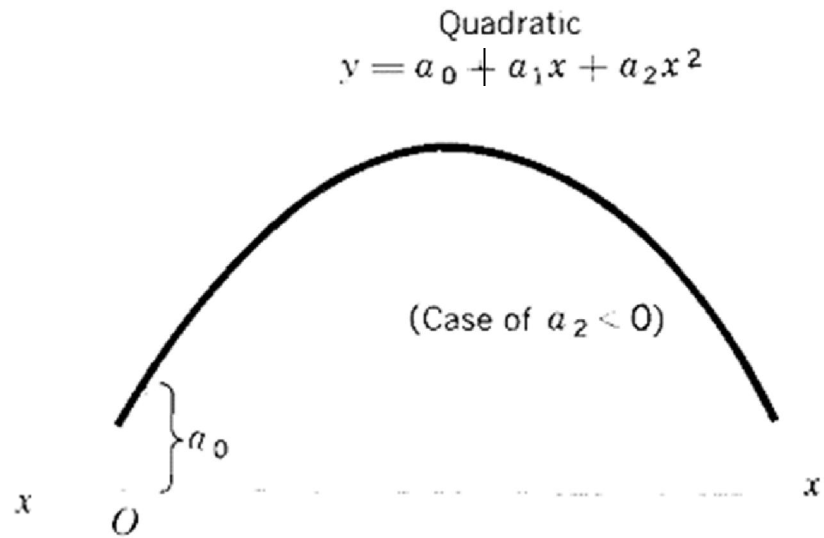
- The simplest functions are the **monomials**, those functions which can be written as $f(x) = ax^k$ for some number a (**coefficient**) and some positive integer k, where k is said to be the **degree of the monomial**
  - For instance $f(x) = 6x^2$ is a monomial of the second order
- A polynomial is a function formed by adding up different monomials; the degree of the polynomial is highest degree of any monomial that appears in the function
  - For instance $f(x) = 6x^3$+2x is a polynomial of the third order
- **Rational functions** are ratios of polynomials, e.g., $f(x) = \frac{6x^3+2x}{5x^2+2x}$
- **Non algebraic functions**: exponential functions (where x appears at the exponent), trigonometric functions, logarithmic functions, etc. (I will focus on exponential and logarithmic functions later on)

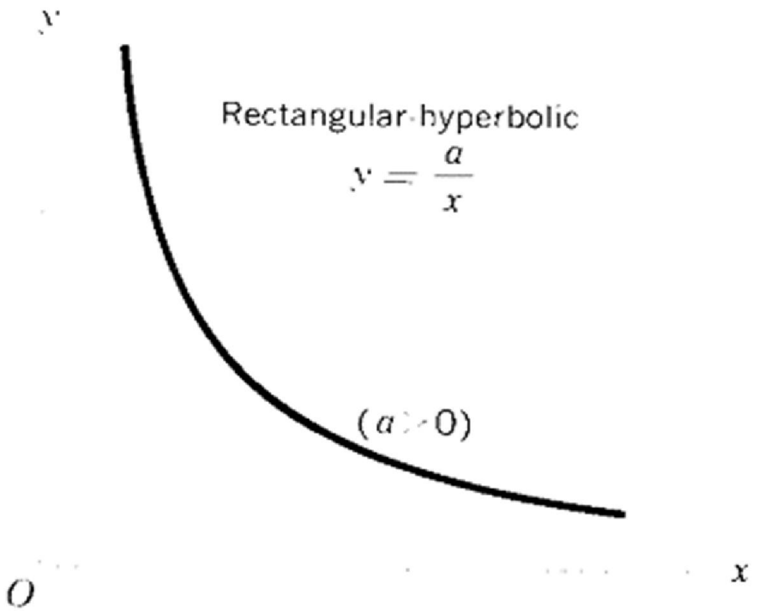
Examples of popular functions (1/2)

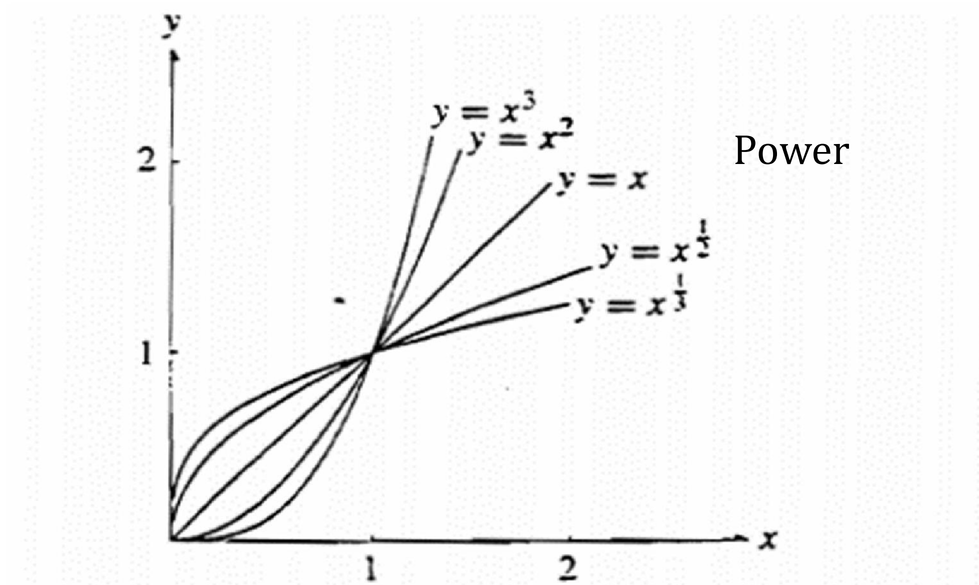
Type	Description	Example
Constant: $y = a_0$	Constant function (polynomial of degree zero)	y = 5
Straight line: $y = a_1 x + a_0$	Linear function (polynomial of degree one)	y = 5x + 3
Parabola: $y = a_1 x + a_2 x^2 + a_0$	Quadratic function (polynomial of degree two)	$y = 2x^2 + 3x + 2$
Hyperbola: $y = a/x$	Rational function	$y = a/x = ax^{-1}$
Power function: $y = x^k$	Monomial of degree k	$y = x^{1/2} = \sqrt{x}$

Examples of popular functions (2/2)









Limits: a short detour (1/4)

- To get a first intuition of the concepts of limit consider that a function f is defined for all x near a but not necessarily at $x = a$
- We say that the function $f(x)$ has the number A as its limit as x tends to a if $f(x)$ tends to A when x tends to a
- We write

$$\lim_{x \to a} f(x) = A$$

- It is possible that the value of $f(x)$ does not tend to any fixed number as x tends to a; then we say that $f(x)$ does not have a limit as x tends to a

Writing $\lim_{x \to a} f(x) = A$ means that we can make $f(x)$ as close to A as we want for all x sufficiently close to (but not equal to) a.

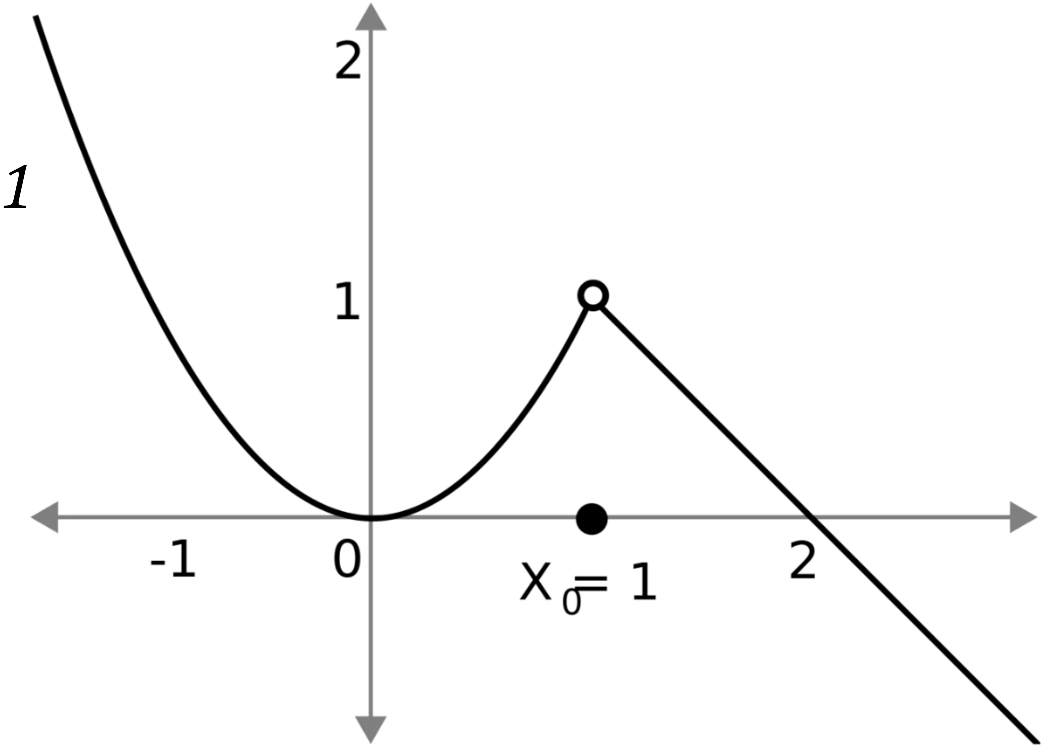
Limits: a short detour (2/4)

- Note: we do not need the function to be defined at x_0 in order for the limit $\lim_{x \to x_0} f(x)$ to exist

- Example: the function plotted in the figure is not defined at $x_0 = 1$

- However, the limit at $x_0=1$ exists

$$\lim_{x \to 1} f(x) = 1$$



Limits: a short detour (3/4)

Rules for Limits

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, then

- **i.** $\lim_{x \to a} [f(x) + g(x)] = A + B$
- **ii.** $\lim_{x \to a} [f(x) - g(x)] = A - B$
- **iii.** $\lim_{x \to a} [f(x)g(x)] = A \cdot B$
- **iv.** $\lim_{x \to a} [f(x)/g(x)] = A/B$ (provided $B \neq 0$)
- **v.** $\lim_{x \to a} [f(x)]^{p/q} = A^{p/q}$ (if $A^{p/q}$ is defined)

- Exercise: compute

$$\lim_{x \to -2} x^2 + 5x$$

Limits: a short detour (4/4)

$$\lim_{x \to -2} x^2 + 5x$$

Numerical solution (with excel)

x	-1.800	-1.900	-1.990	-2.000	-2.001	-2.010	-2.100	-2.200
f(x)	-5.76	-5.89	-5.99		-6.001	-6.01	-6.09	-6.16

Using the rules

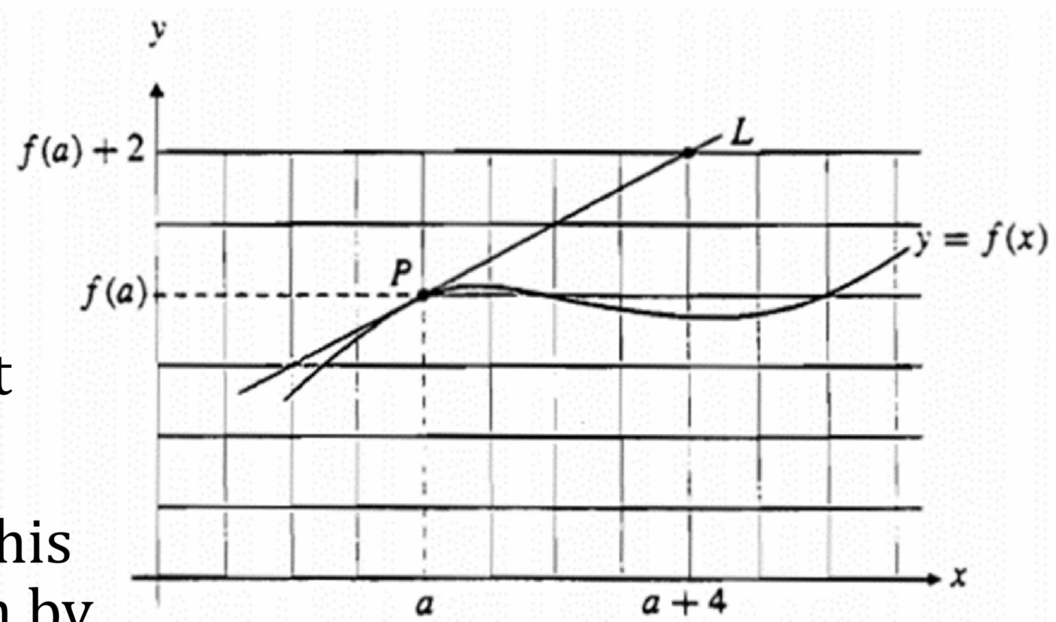
$$\lim_{x \to -2}(x^2 + 5x) = \lim_{x \to -2}(x \cdot x) + \lim_{x \to -2}(5 \cdot x)$$

$$= \left(\lim_{x \to -2} x\right)\left(\lim_{x \to -2} x\right) + \left(\lim_{x \to -2} 5\right)\left(\lim_{x \to -2} x\right)$$

$$= (-2)(-2) + 5 \cdot (-2) = -6$$

The slope of functions (1/4)

- Start from a geometric interpretation: when we study the graph of a function we would like to have a measure of the steepness of the graph at a point or several points
- If the function is linear, this is easy: the slope is given by the coefficient that multiplies *x*



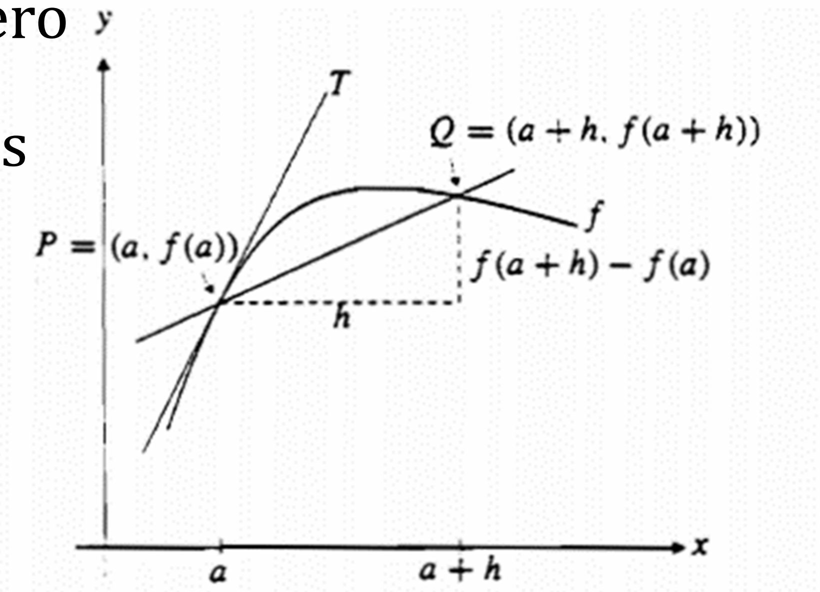
- However, this is less trivial for a non-linear function as the one depicted above
- We can define the steepness of a curve at a particular point as the slope of the straight line that just touches the curve at that point, i.e., that is tangent to the curve at that point (point P in the figure)

The slope of curves (2/4)

The slope of the tangent to the graph at point *P*, with coordinates (a, $f(a)$), is called **derivative** of *f* at *P* and is denoted by $f'(a)$

- How can we find the slope of the tangent of *f* at the point P, with coordinates (a, $f(a)$)?
- Consider a point Q that is also on the graph of *f* and is close to P, i.e., suppose that the *x-coordinate* of Q is $a+h$ where h is a small number different from zero
- Therefore the *y-coordinate* of Q is $f(a+h)$
- The slope of the secant PQ is:

$$m_{PQ} = \frac{f(a+h)-f(a)}{h}$$

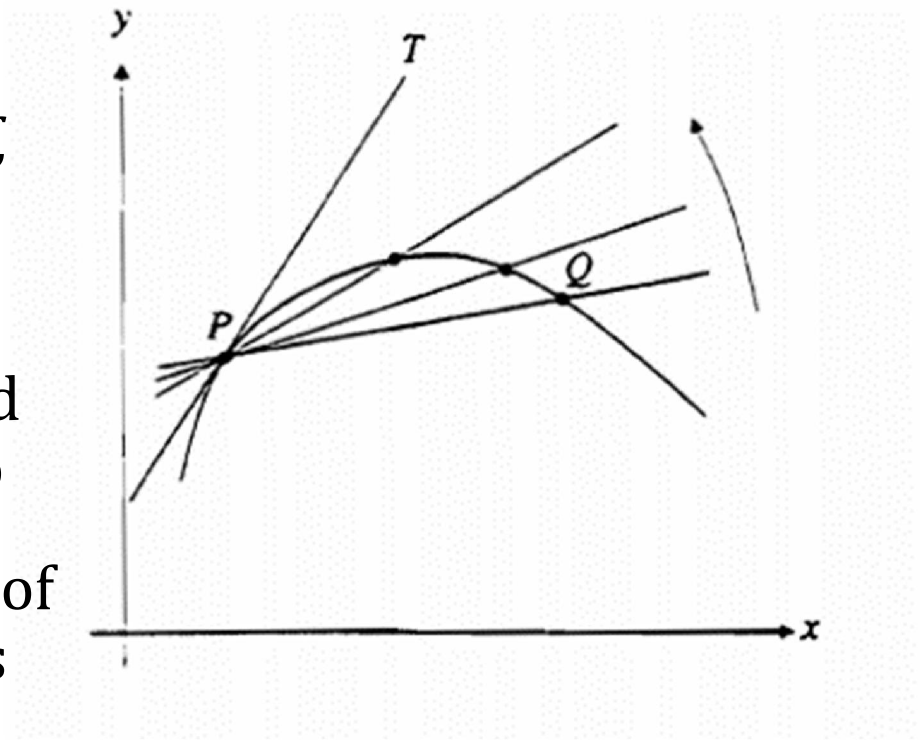


The slope of curves (3/4)

- When Q moves towards P (Q tends to P) along the graph of *f*, the secant PQ tends to the tangent to the graph at P

- The *x-coordinate a+h* must tend to *a* so that *h* must tend to zero

- Therefore, we define the slope of the tangent to the graph at P as the number to which m_{PQ} approaches when h goes to zero



- Hence, the derivative of a function f at a point a of its domain is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The slope of curves (4/4)

- Let's get the intuition for it: imagine that we want to compute the derivative of $f(x^2)$ at $x_0 = 3$
- First of all, we know that $f(3) = 3^2 = 9$
- Now compute $\frac{f(a+h)-f(a)}{h}$ for smaller and smaller values of h
- The derivative of $f(x^2)$ at $x_0 = 3$ is 6 !

h	x_0+h	f(x_0+h)	(f(x_0+h)-f(x_0))/h
0.1	3.1	9.61	6.1
0.01	3.01	9.0601	6.01
0.001	3.001	9.006001	6.001
0.0001	3.0001	9.00060001	6.0001
0.00001	3.00001	9.00006	6.00001
0.0000001	3.0000001	9.0000006	6.000000088

Interpretation of derivatives

- In economics, other (non geometric) interpretations of the derivative are often more useful
- It is a **rate of change**!
- If $C(x)$ is a function that expresses the cost of producing x units of a good, then we interpret $C'(x)$ as the marginal cost at x

$$C'(x) = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h}$$

→ Cost that we face when we make a "small" increment to the production level

Note: Do not forget that the slope of a non-linear function is NOT CONSTANT! The derivative depends on the point where you compute **it** (meaning that if the cost function is non linear, increasing the production from 10,000 to 10,001 units is not the same as increasing the production from 50,000 to 50,001

Rules for computing derivatives (1/2)

Theorem (for a proof see BS, chapter 2):
For any positive integer k, the derivative of $f(x) = x^k$ at x_0 is
$$f'(x_0) = kx_0^{k-1}$$

- The derivatives of other commonly used functions are:
  - The derivative of a constant is zero
  - The derivative of the logarithm is
    - $f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x}$
    - $f(x) = log_a(x) \Rightarrow f'(x) = \frac{1}{xln(a)}$
  - The derivative of exponential and power functions is
    - $f(x) = e^x \Rightarrow f'(x) = e^x$
    - $f(x) = a^x \Rightarrow f'(x) = \ln(a)\, a^x$

Rules for computing derivatives (2/2)

Theorem Suppose that k is an arbitrary constant and that f and g are differentiable functions at $x = x_0$. Then,

$$a)\ (f \pm g)'(x_0) = f'(x_0) \pm g'(x_0),$$

$$b)\ (kf)'(x_0) = k(f'(x_0)),$$

$$c)\ (f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0),$$

$$d)\ \left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2},$$

$$e)\ ((f(x))^n)' = n(f(x))^{n-1} \cdot f'(x),$$

$$f)\ (x^k)' = kx^{k-1}.$$

CHAIN RULE: $f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$

Exercises

- Take the derivatives of the following functions:
  - $-7x^4 + \sqrt{2x} + 1 \Rightarrow -28x^3 + \frac{1}{\sqrt{x}}$
  - $\frac{x^2}{1+x} \Rightarrow \frac{2x(1+x)-x^2}{(1+x)^2}$
  - $\ln(3x + 2) \Rightarrow \frac{1}{3x+2}3$
  - $\ln(x)5x \Rightarrow \frac{1}{x}5x + \ln(x)\,5$
  - $\frac{3e^x}{x+2} \Rightarrow \frac{3e^x(x+2)-3e^x}{(x+2)^2}$
  - $\frac{x^2+3x}{x+2} \Rightarrow \frac{(2x+3)(x+2)-(x^2+3x)}{(x+2)^2}$
  - $\ln(x^3) = \frac{1}{x}3$

Exercise (Solutions)

- Take the derivatives of the following functions:
  - $-7x^4 + \sqrt{2x} + 1 \Rightarrow -28x^3 + \frac{2}{\sqrt{x}}$
  - $\frac{x^2}{1+x} \Rightarrow \frac{2x(1+x)-x^2}{(1+x)^2}$
  - $\ln(3x+2) \Rightarrow \frac{3}{3x+2}$
  - $\ln(x)5x \Rightarrow \frac{1}{x}5x + \ln(x)\,5$
  - $\frac{3e^x}{x+2} \Rightarrow \frac{3e^x(x+2)-3e^x}{(x+2)^2}$
  - $\frac{x^2+3x}{x+2} \Rightarrow \frac{(2x+3)(x+2)-(x^2+3x)}{(x+2)^2}$
  - $\ln(x^3) = \frac{3}{x}$