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One Variable Calculus: Foundations and Applications

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20550– Quantitative Methods for Finance

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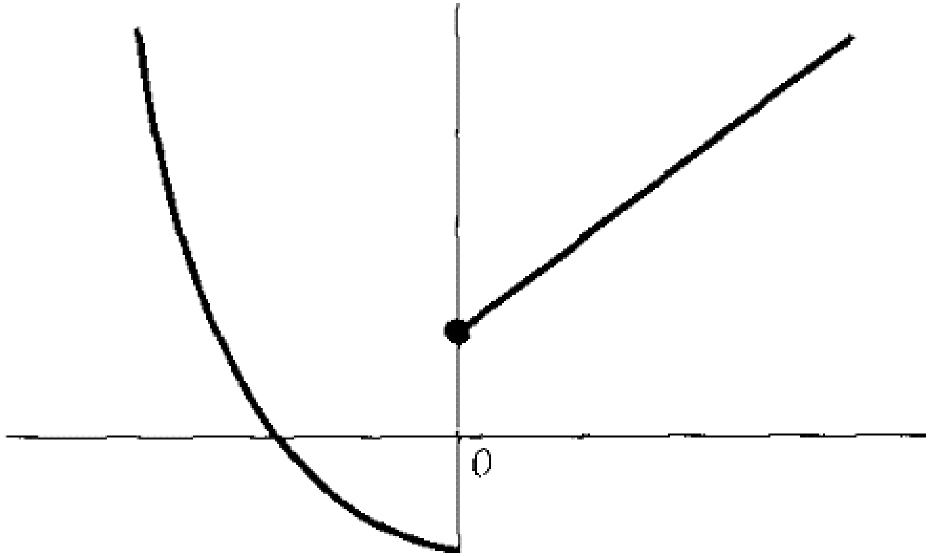
Differentiability and continuity (1/2)

A function *f(x)* is **continuous** at x_0 (a point in the domain of the function) if for any sequence $\{x_n\}$, $f(x_n)$ converges to $f(x_0)$. We say that a function is continuous if it is continuous in any point of its domain.

Continuity (at a point) is a **necessary** but **not sufficient** condition for differentiability.

Example of a function that is NOT continuous



Differentiability and continuity (2/2)

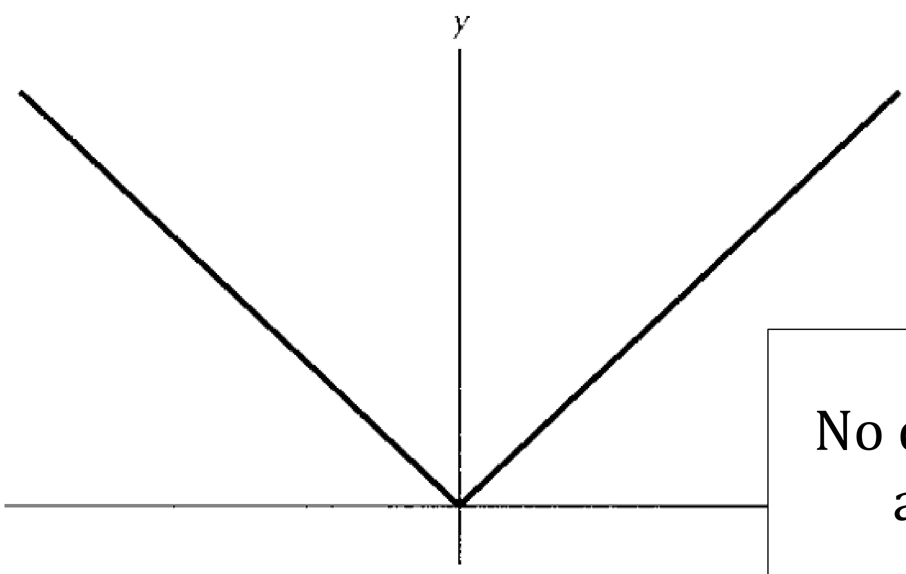
A function f is **differentiable** at x_0 if the limit

$$\lim_{h_n \to 0} \frac{f(x_0+h_n)-f(x_0)}{h_n}$$

exists and is the same for each sequence $\{h_n\}$ which converges to 0.

If a function is differentiable at every x_0 in its domain we say that the function is differentiable

Example of a function that is continuous but NOT differentiable



No derivative at $x_0=0$

The graph of $f(x) = |x|$.

Higher order derivatives

The derivative of a function *f* is often called the **first derivative** of *f.* If *f’* is also differentiable, we can differentiate *f’* to get the **second derivative**, *f’’.* The derivative of *f’’* (if it exists) is called the third derivative of f and so on. Typically, for our applications first and second derivatives are enough, bur higher-order derivatives may be computed

- For example, consider the function $f(x) = 2x^3 + 6x^2$
  - The first derivative is $f'(x) = 6x^2 + 12x$
  - The second derivative is the derivative of *f’,* that is $f''(x) = 12x + 12$
  - The third derivative is $f'''(x) = 12$
  - Derivatives from the fourth onwards are equal to zero

Linear approximation and differentials (1/4)

- Linear functions are very easy to manipulate: therefore it is natural to try to find a "linear approximation" to a given function

- Consider a function *f(x)* that is differentiable at $x = x_0$

- The tangent to the graph at $(x_0, f(x_0))$ follows the equation

$$y = f(x_0) + f'(x_0)(x - x_0) \text{ for } x \text{ close to } x_0$$

- Therefore, a linear approximation of the function *f* around x_0 is given by

$$f(x) \cong f(x_0) + f'(x_0)(x - x_0) \text{ for } x \text{ close to } x_0$$

or $\Delta y \cong \boxed{f'(x_0)\Delta x}$ provided that Δx is small

Differential of the function *f* at x_0

Linear approximation and differentials (2/4)

- The differential is NOT the actual increment in y if x is changed to $x + \Delta x$ but rather the change in y that would occur if y continued to change at the fixed rate $f'(x)$ as x changes to $x + \Delta x$

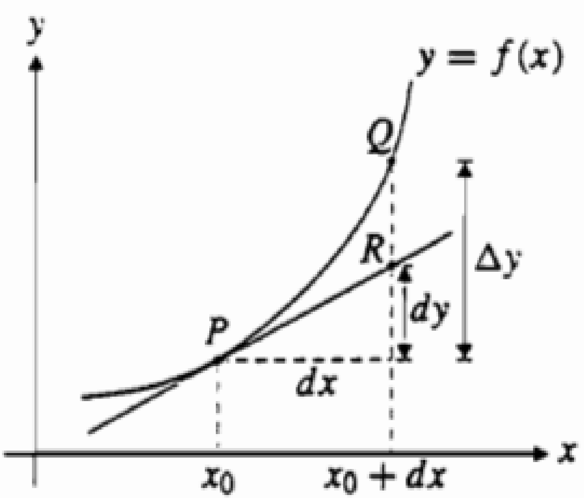


FIGURE A geometric representation of the differential.

Rules for Differentials

$$d(af + bg) = a\,df + b\,dg \qquad (a \text{ and } b \text{ are constants})$$

$$d(fg) = g\,df + f\,dg$$

$$d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2} (g \neq 0)$$

Linear approximation and differentials (3/4)

- The less is the slope of f, the more precise is the approximation; in addition, the larger is Δx, the less precise the approximation
- We shall see this point in more depth with an example: consider the function $y = \ln(x)$
- Starting from $ln(10) = 2.3$ compare the actual change in y given a certain change in x with its linear approximation that is, $\Delta y \approx \frac{1}{x_0} \Delta x$

ΔX	0.1	0.5	1	5	10	100
Actual Change	0.0100	0.0488	0.0953	0.4055	0.6931	2.3979
Linear Approximation	0.0100	0.0500	0.1000	0.5000	1.0000	10.0000
Size of the "mistake"	0.0000	0.0012	0.0047	0.0945	0.3069	7.6021

Linear approximation and differentials (4/4)

- Note: a change $\Delta x = 100$ is quite big if we are at $x_0 = 10$, leading to a rather imprecise approximation (see previous slide)
- But what if we are at $x_0 = 10000$?
- Starting from $\ln(10000) = 9.20$ compare the actual change in y given a certain change in x with its linear approximation that is, $\Delta y \approx \frac{1}{x_0}\Delta x$
- Now $\Delta x = 100$ can be considered small enough to lead to a quite accurate approximation

ΔX	0.1	0.5	1	5	10	100
Actual Change	0.0000	0.0000	0.0001	0.0005	0.0010	0.0100
Linear Approximation	0.0000	0.0001	0.0001	0.0005	0.0010	0.0100
Size of the "mistake"	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Polynomial Approximations (1/2)

- If approximations provided by linear functions are not sufficiently accurate it is natural to try quadratic approximations or approximations by polynomials of higher order
- The **quadratic approximation** to $f(x)$ about $x = x_0$ is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

- More generally, we can approximate $f(x)$ about $x = a$ by an nth polynomial (the nth-order **Taylor polynomial**)

Approximation to $f(x)$ about $x = a$:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

- However, typically linear or quadratic approximations are sufficient in most of the applications we are interested in

Polynomial Approximations (2/2)

- Consider the function $f(x) = x^5$ and see what happens when we try to approximate it around 2 with linear vs. quadratic approximations:

Δx	0.1	0.2	0.3	0.4	0.5
Actual change	8.84101	19.53632	32.36343	47.62624	65.65625
Linear Approximation	8.00000	16.00000	24.00000	32.00000	40.00000
Quadratic Approximation	8.80000	19.20000	31.20000	44.80000	60.00000

- Other example: $f(x) = \ln(x)$ to be approximated around 1 (recall that $f(1) = ln(1) = 0$)

Δx	0.1	0.2	0.3	0.4	0.5
Actual change	0.09531	0.18232	0.26236	0.33647	0.40547
Linear Approximation	0.10000	0.20000	0.30000	0.40000	0.50000
Quadratic Approximation	0.09500	0.18000	0.25500	0.32000	0.37500

Natural exponential functions (1/5)

- Consider the following function:

$$f(m) = \left(1 + \frac{1}{m}\right)^m$$

- When m increases towards infinity, $f(m)$ will converge to $2.71828\ldots \equiv e$

$$e \equiv \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m$$

Heuristic proof (if interested look at Chang, chapter 10)

m/f(m)	
1	2
2	2.25
3	2.37037
4	2.44141
5	2.48832
…	
100	2.70481
1000	2.71692
100000	2.71827
10000000	2.71828

Natural exponential functions (2/5)

- Why do we care?
- In economics and finance, the number *e* carries a special meaning, as it can be interpreted as the result of a special process of interest compounding, that is, continuous compounding
- Suppose that we invest 1 euro today and we earn an annual nominal interest of 100% (just for simplicity, we shall consider something more reasonable in a minute); clearly if the interest is compounded once a year, at the end of the year we will get 2 euros:

$$V(1) = \left(1 + \frac{1}{1}\right)^1$$

Natural exponential functions (3/5)

- Alternatively, if the interest is compounded semi-annually, we will get $$V(2) = (1 + 50\%)\,(1 + 50\%) = \left(1 + \tfrac{1}{2}\right)^2$$
- More generally, $$V(m) = \left(1 + \frac{1}{m}\right)^m$$
- Therefore, if we increase the frequency of compounding to infinity, we know from the previous slide that we will earn euro 1 x e = 2.718
- This result can be generalized in three ways:
  - (1) More years of compounding
  - (2) Principal different from 1 euro
  - (3) Nominal interest rate different from 100%

Natural exponential functions (4/5)

- (1) and (2) are trivial to implement
- Simply, the amount of money that one will have after t years with an annual rate of 100% continuously compounded is equal to Ae^{t}, where A is the capital invested at the beginning
- Suppose now that we want to feature an annual interest rate r = 5%
- We can manipulate the formula to get

$$V(m) = A\left[\left(1 + \frac{r}{m}\right)^{m/r}\right]^{rt}$$

- With continuous compounding we get

$$V(m) = Ae^{rt}$$

Natural exponential functions (5/5)

A Survey of the Properties of e^x

The natural exponential function

$$f(x) = e^x \qquad (e = 2.71828\ldots)$$

is differentiable and strictly increasing for all real numbers x. In fact,

$$f(x) = e^x \Longrightarrow f'(x) = f(x) = e^x$$

The following properties hold for all exponents s and t:

(a) $e^s e^t = e^{s+t}$ (b) $e^s/e^t = e^{s-t}$ (c) $(e^s)^t = e^{st}$

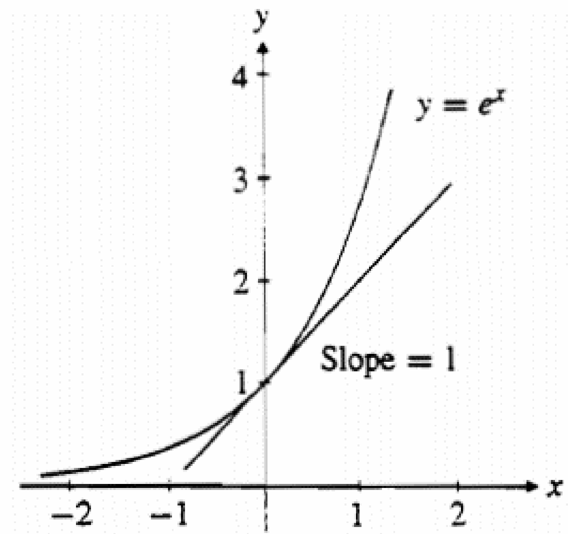


FIGURE 8.3 The natural exponential function.

Natural logarithmic functions (1/2)

- The inverse of the natural exponential function is the **natural logarithmic function**, $y = lnx$

$$\ln x = y \iff e^y = x; \qquad e^{\ln x} = x \text{ and } \ln e^x = x.$$

Useful Rules for ln

$$\ln(xy) = \ln x + \ln y \qquad (x \text{ and } y \text{ are positive}) \qquad \text{(a)}$$

(The logarithm of a product is equal to the *sum* of the logarithms of each of the factors.)

$$\ln \frac{x}{y} = \ln x - \ln y \qquad (x \text{ and } y \text{ are positive}) \qquad \text{(b)}$$

(The logarithm of a quotient is equal to the *difference* between the logarithms of its numerator and denominator.)

$$\ln x^p = p \ln x \qquad (x \text{ is positive}) \qquad \text{(c)}$$

(The logarithm of a power is equal to the exponent multiplied by the logarithm of the base.)

$$\ln 1 = 0, \qquad \ln e = 1, \qquad x = e^{\ln x} \qquad \text{and} \qquad \ln e^x = x \qquad \text{(d)}$$

Natural logarithmic functions (2/2)

- Sometimes economists prefer to represent (and study) a function $y = f(x)$ in log-log terms
- This means that they apply a change to the variables such that Y= $\ln(y)$ and X $= \ln(x)$; therefore $x = e^X$ and $\frac{dx}{dX} = e^X = x$
- Hence, in XY-coordinates, f becomes $Y = \ln f(x) = lnf(e^X) \equiv F(X)$
- The slope of the graph in the log-log terms is

$$= \frac{d}{dx}(\ln f(x)) \frac{dx}{dX} = \frac{f'(x)}{f(x)} x.$$

$$\frac{df(x)}{dx} \frac{x}{f(x)} \approx \frac{\Delta f}{\Delta x} \cdot \frac{x}{f(x)} = \boxed{\frac{\Delta f}{f(x)} \Big/ \frac{\Delta x}{x}},$$

% change of a function f relative to a % change of x

elasticity

A primer on integration (1/7)

- Integration is the "inverse operator" vs. differentiation
- If differentiation of a given **primitive** function *F(x)* yields the derivative *f(x)* then we can integrate *f(x)* to find *F(x)*
- The function *F(x)* is referred to as the integral of the function *f(x)*
- Importantly, while any primitive function has a unique derivative, the reverse is not true: if *F(x)* is an integral of *f(x)*, then also *F(x)* plus any constant is an integral of *f(x)* (the derivative of a constant is zero!)
- Standard notation for the integration of *f(x)* with respect to *x* is $\int \boxed{f(x)}\, dx$

$$\frac{d}{dx}F(x) = f(x) \quad \Rightarrow \quad \int f(x)\, dx = F(x) + c$$

INTEGRAND

A primer on integration (2/7)

- Given that there are precise rules of differentiation, we can also develop a set of rules of integration

Rule I (the power rule)

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c \qquad (n \neq -1)$$

- For instance, $\int x^3 dx = \frac{1}{4} x^4 + c$

Rule II (the exponential rule)

$$\int e^x \, dx = e^x + c$$

Rule III (the logarithmic rule)

$$\int \frac{1}{x} dx = \ln x + c \qquad (x > 0)$$

Rule IIa

$$\int f'(x) e^{f(x)} \, dx = e^{f(x)} + c$$

Rule IIIa

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c \qquad [f(x) > 0]$$

A primer on integration (3/7)

Rule IV (the integral of a sum) The integral of the sum of a finite number of functions is the sum of the integrals of those functions. For the two-function case, this means that

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

- For instance, $\int (x^3 + x + 1)dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + x + c$

Rule V (the integral of a multiple) The integral of k times an integrand (k being a constant) is k times the integral of that integrand. In symbols,

$$\int kf(x)\, dx = k\int f(x)\, dx$$

- For instance, $\int 2x^2 dx = 2\frac{1}{3}x^3 + c$
- For instance, $\int 3x^2 dx = 3\frac{1}{3}x^3 + c = x^3 + c$

A primer on integration (4/7)

Rule VI (the substitution rule) The integral of $f(u)(du/dx)$ with respect to the variable x is the integral of $f(u)$ with respect to the variable u:

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + c$$

where the operation $\int du$ has been substituted for the operation $\int dx$.

- This is the integral calculus counterpart for the chain rule
- For example, find $\int 8e^{2x+3}dx$
- Let $u = 2x + 3$; then $\frac{du}{dx} = 2$ or $dx = \frac{du}{2}$

$$\int 8e^{2x+3}\,dx = \int 8e^{u}\frac{du}{2} = 4\int e^{u}\,du = 4e^{u} + c = 4e^{2x+3} + c$$

A primer on integration (5/7)

Rule VII (integration by parts) The integral of v with respect to u is equal to uv less the integral of u with respect to v:

$$\int v\,du = uv - \int u\,dv$$

- For instance, find $\int lnx\,dx$
- Note that here we cannot use the logarithmic rule! Indeed, that rule applies to integrand $\frac{1}{x}$
- Hence, let $v = lnx$, implying $dv = \frac{1}{x}dx$ and also let u=x

$$\int \ln x\,dx = \int v\,du = uv - \int u\,dv$$

$$= x\ln x - \int dx = x\ln x - x + c = x(\ln x - 1) + c$$

A primer on integration (6/7)

- Till now we have discussed **indefinite integrals**: they yield no definite numerical result
- For a given indefinite integral of a continuous function *f(x)*,

$$\int f(x)\,dx = F(x) + c$$

if we choose two values of *x* in the domain *a*, and *b, a*<*b*, then

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

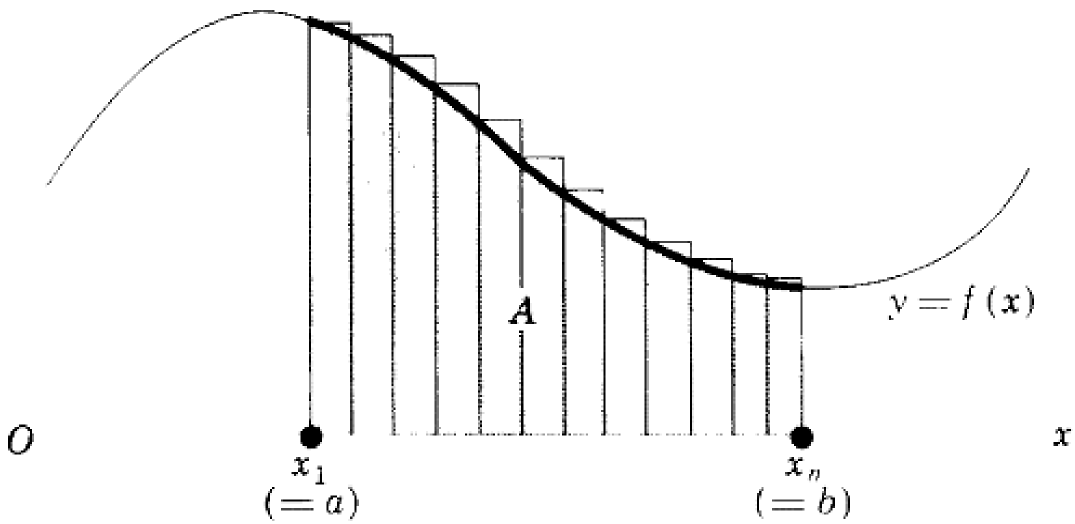
- This is called definite integral of *f(x)* from *a* to *b; a* and *b* are the lower and upper limit of integration, respectively
- We write $\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$

A primer on integration (7/7)

- Every definite integral gives a definite numerical result
- This value can be interpreted to be the area under the graph of continuous function $y = f(x)$ over the $[a, b]$ interval inside its domain

$$A^* = \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

APPROXIMATION OF THE AREA UNDER *f(x)*



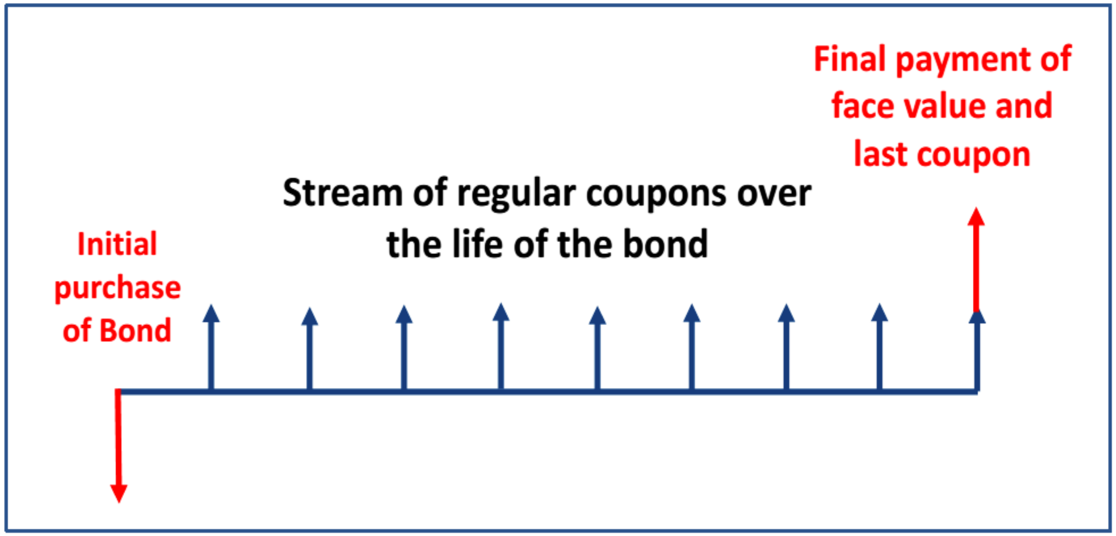
(b)

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \lim_{n \to \infty} A^* = \text{area } A$$

CONTINUOUS COUNTERPART OF SUMMATION

Appendix: Bond Mathematics (1/8)

- A bond is a security issued with a fixed “face value” (redemption value)
- Bonds feature a maturity date at which point the principal is repaid to the holder of the security
- Bonds pay fixed periodic amounts of interest (coupons) (but ZCB, zero coupon bonds, are instead purchased at discount vs. face value)
- Their market value differs from their face value depending on the coupon rate and how it compares with current market rates



Appendix: Bond Mathematics (2/8)

- The yield-to-maturity of a bond is the internal rate of return implied by a certain coupon, i.e., the rate of discount that equates the discounted value of coupons and final payment to today's market price
- The yield is inversely related to the price
- Suppose that you buy a bond with face value 100 Eur and coupon of 5% paid each year, which matures in 3 years from now and whose price today is 96 Euro (for simplicity we assume that last coupon has been just paid)

$$96 = \frac{5}{1 + yield} + \frac{5}{(1 + yield)^2} + \frac{105}{(1 + yield)^3}$$

YIELD= 6.5%

- The yield-to-maturity is the effective rate of return of a bond under the assumption that coupons are reinvested at the same yield-to-maturity

Appendix: Bond Mathematics (3/8)

- It is easy to see that the relationship between yield and price is an inverse one
  - If price increases to 102 Euro, yield decreases to 4.27%

Relationship	Description	Price
Coupon Rate = Yield to Maturity	At Par	Price = 100
Coupon Rate > Yield to Maturity	At a Premium 'Above Par'	Price > 100
Coupon Rate < Yield to Maturity	At a Discount 'Below Par'	Price < 100

Appendix: Bond Mathematics (4/8)

- For simplicity earlier we have assumed that the coupon has been just paid
- If this is not the case, i.e., when we buy the bond between two coupon dates, "clean" and "dirty" price are different, because the seller is entitled to receive part of the coupon
- The dirty price will be the clean price plus the "accrued interest", that is calculated as

$$\text{Face Value} \times \begin{array}{c}\text{Coupon Rate}\\ \text{Per Period}\end{array} \times \boxed{\text{Day Count Fraction}}$$

$$\text{Day Count Fraction} = \frac{\text{The number of days assumed since the previous coupon date}}{\text{The total number of days assumed between coupon payments}}$$

Appendix: Bond Mathematics (5/8)

- In order to understand the risk of a bond, we need to measure how sensitive the bond price is to changes in yield
- The **Macaulay duration** of a bond is the sum of the present values of each cash flow divided by the **dirty price** of the bond, weighted by the time when it occurs
- It represents a measure of the average life of the bond

$$D = \frac{\sum_{k=1}^{n} t_k \frac{F_{t_k}}{(1+y)^{t_k}}}{P}$$

Cash flows

y is the yield-to-maturity

- The **modified duration** is instead equal to

$$MD = \frac{D}{1+y}$$

Appendix: Bond Mathematics (6/8)

- Look at the mathematical meaning of the two:
- $P = \sum_{k=1}^{n} \frac{F_t}{(1+y)^{t_k}}$ so that its derivative is

$$\sum_{k=1}^{n} -t_k \frac{F_t}{(1+y)^{t_k+1}} = -\sum_{k=1}^{n} t_k \frac{F_t}{(1+y)^{t_k+1}} = -MD \times P$$

- Therefore, MD can be used to approximate the (%) change in the price for a certain (small) change in the yield exactly as we discussed in the section about differentials
- Exploiting the fact that we know that for a function *f(x),* the linear approximation is $\Delta f(x) \approx f'(x)\Delta x$, we get that

$$dP \approx \boldsymbol{-MDxP} x\Delta y$$

$$\boxed{\frac{dP}{P} \approx -MDx\Delta y}$$

Appendix: Bond Mathematics (7/8)

- Because the relationship between yield and price is non-linear, we may find it convenient to approximate it with a quadratic Taylor expansion
- **Convexity** measures the degree of the curvature in the relationship between prices and yields

$$C = \sum_{k=1}^{n} (t_k + t_k{}^2) \frac{\frac{F_t}{(1+y)^{t_k}}}{P}$$

- We can then approximate the % change of price for a small change in yield by using

$$\boxed{\frac{dP}{P} \approx -MDx\Delta y + \frac{1}{2}\frac{C}{(1+y)^2}\Delta y^2}$$

Appendix: Bond Mathematics (8/8)

- Play with the excel sheet provided on the course web page to understand how duration and convexity interact with time-to-maturity, coupon rate, coupon frequency, etc.
- Some useful excel functions for bonds are:
  - PRICE (to get the bond clean price of a security paying periodic interest) [PREZZO in ITA]
  - YIELD (to compute the yield to maturity of a security paying periodic interest) [REND in ITA]
  - ACCRINT (compute accrued interest) [INT.MATURATO.PER in ITA]