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Supplement to Lecture 1: Fundamentals of Mean-Variance Analysis

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Portfolio Management

EMFI Angola's Sovereign Fund 2019

The Efficient Frontier with Two Risky Assets

- Assume no borrowing or lending at the risk-free rate
- Re-cap of a few basic algebraic relationships that exploit the fact that with two risky assets, $\omega_B = 1 - \omega_A$
  - See textbook for detailed derivations $\mu_P = \omega_A \mu_A + (1 - \omega_A)\mu_B$
  - Portfolio mean & variance: $\sigma_P^2 = \omega_A^2 \sigma_A^2 + (1 - \omega_A)^2 \sigma_B^2 + 2\omega_A(1 - \omega_A)\sigma_{AB}$
  - Using the definitions of correlation and of standard deviation:

$$\sigma_P = \sqrt{\omega_A^2 \sigma_A^2 + (1 - \omega_A)^2 \sigma_B^2 + 2\omega_A(1 - \omega_A)\rho_{AB}\sigma_A\sigma_B}$$

  - Solve mean equation for ω_A and plug the result into st. dev. equation ⇒ a system of 2 equations in 2 unknowns
  - The system has in general a unique solution ⇒ the opportunity set is a curve and it coincides with the mean-variance frontier (there is only one possible level of risk for a given level of return)
  - The shape of set depends on the correlation between the 2 securities
- Three possible cases: (i) ρ_{AB} = + 1; (ii) ρ_{AB} = -1; (iii) $\rho_{AB} \in (0,1)$
- Case (i): $\rho_{AB} = +1$: the expression for σ^2_P becomes a perfect square sum and this simplifies the algebra

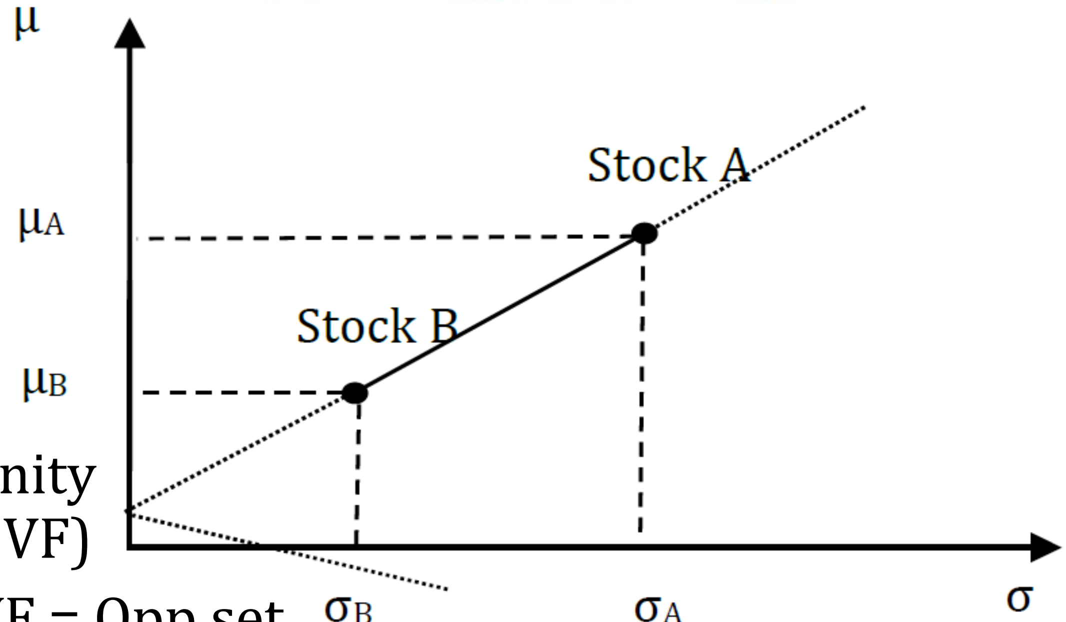
The Efficient Frontier with Two Risky Assets

- After algebra (see textbook), we have:

$$\mu_P = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\mu_A + \left(1 - \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\,(\mu_A - \mu_B)$$

the equation of a straight line, with slope $(\sigma_P - \sigma_A)/(\sigma_A - \sigma_B)$

  - In the picture, dashed lines == ptfs. require short selling
  - Without short sales, the least risky stock == GMVP
  - With short sales, the GMVP has zero risk
  - In this special case, the opportunity set = mean-variance frontier (MVF)
  - With no short sales, EffSet = MVF = Opp set



- <u>Case (ii): ρ_{AB} = -1</u>: the expression for σ^2_P becomes a perfect square difference and this simplifies the algebra (see textbook) to yield:

$$\sigma_P = \omega_A\sigma_A - (1-\omega_A)\sigma_B \quad \text{or to} \quad \sigma_P = -\omega_A\sigma_A + (1-\omega_A)\sigma_B$$

- Yet, each of the equations only holds when the RHS is positive

The Efficient Frontier with Two Risky Assets

- The opportunity set is a straight line, but its slope depends on which of the equations above holds
- If the first equation applies, the opportunity set is equal to:

$$\mu_P = \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\mu_A + \left(1 - \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}(\mu_A - \mu_B)$$

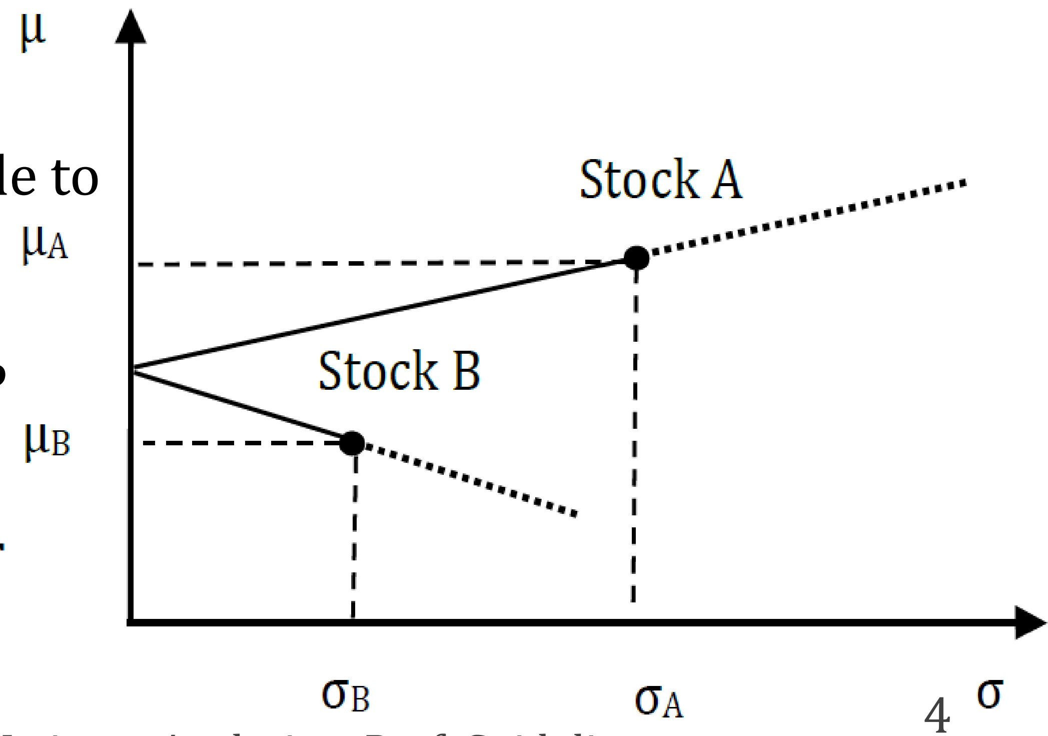
while if the second equation holds, the opportunity set is equal to:

$$\mu_P = \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\mu_A + \left(1 - \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}(\mu_A - \mu_B)$$

  - In the picture, dashed lines == ptfs. require short selling
  - Even without short sales, possible to find a combination that has zero variance, i.e., it is risk-free
  - Such a riskless portfolio is GMVP
  - The expression for such a ptf. is:

$$\omega_A^{GMVP}\sigma_A - (1 - \omega_A^{GMVP})\sigma_B = 0 \text{ or}$$

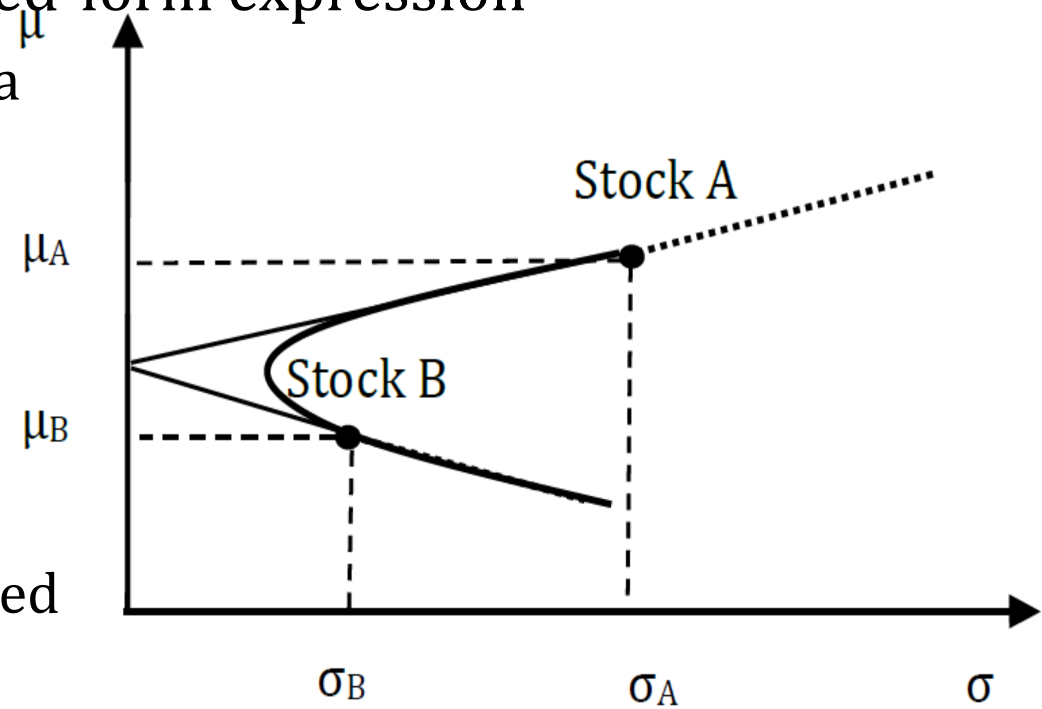
$$-\omega_A^{GMVP}\sigma_A + (1 - \omega_A^{GMVP})\sigma_B = 0$$



The Efficient Frontier with Two Risky Assets

$$\Longrightarrow \quad \omega_A^{GMVP} = \frac{\sigma_B}{\sigma_A + \sigma_B}$$

- Case (iii): $\rho_{AB} \in (0,1)$: In this case, although tricks exist to trace it out, the MVF does not have a closed-form expression
  - The MVF is non-linear, a parabola (i.e., a quadratic function) in the variance-mean space
  - Or a a (branch of) hyperbola in standard deviation-mean space
  - In such a space, the MVF is not a function, it is just a «correspondence», a "right-rotated hyperbola"

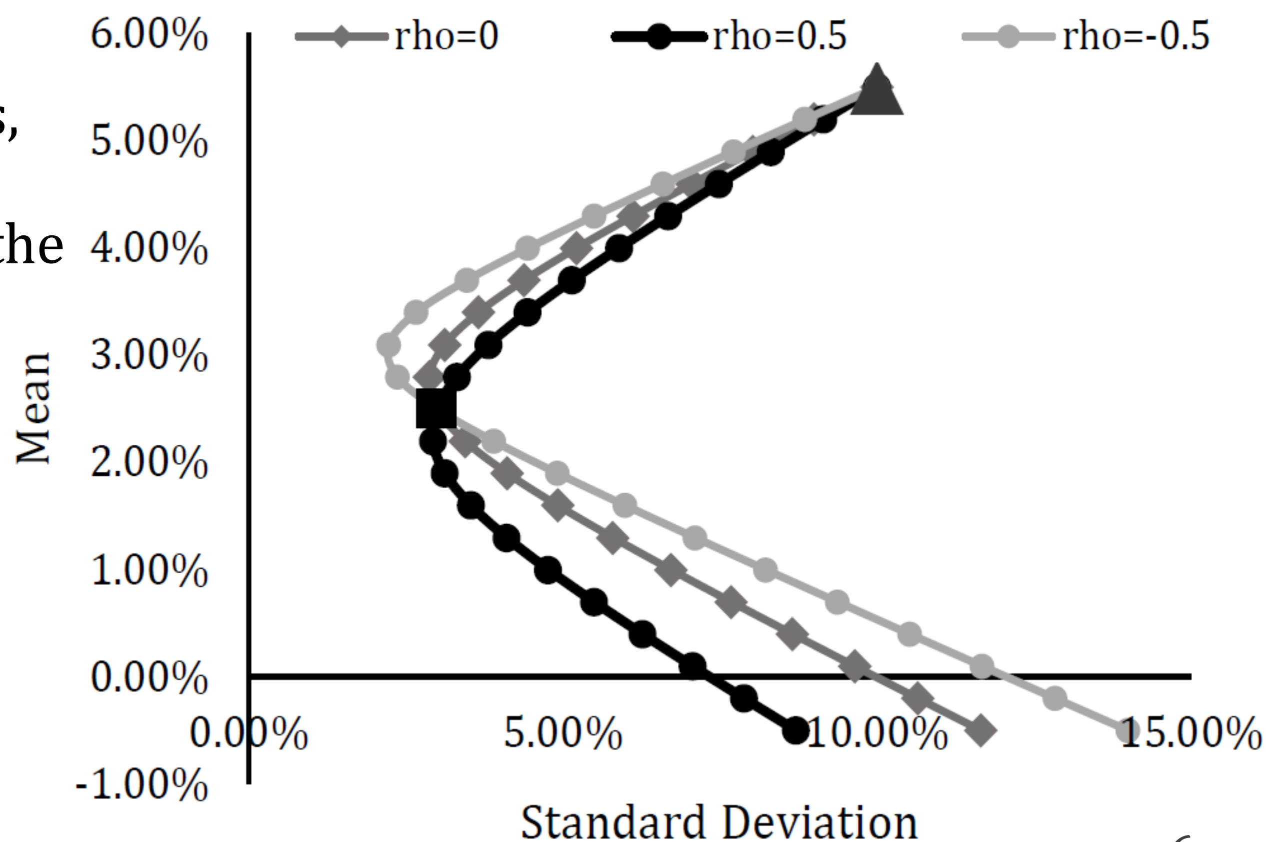


  - The efficient set == a portion of the MVF, the branch of the "rotated hyperbola" that lies above (and includes) the GMVP
  - To distinguish the efficient set from the MVF we have to find the GMVP:

$$\frac{\partial \sigma_P^2}{\partial \omega_A} = 2\omega_A\sigma_A^2 - 2(1-\omega_A)\sigma_B^2 + 2(1-2\omega_A)\rho_{AB}\sigma_A\sigma_B \Longrightarrow \omega_A^{GMVP} = \frac{\sigma_B^2 - \rho_{A,B}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{A,B}\sigma_A\sigma_B}$$

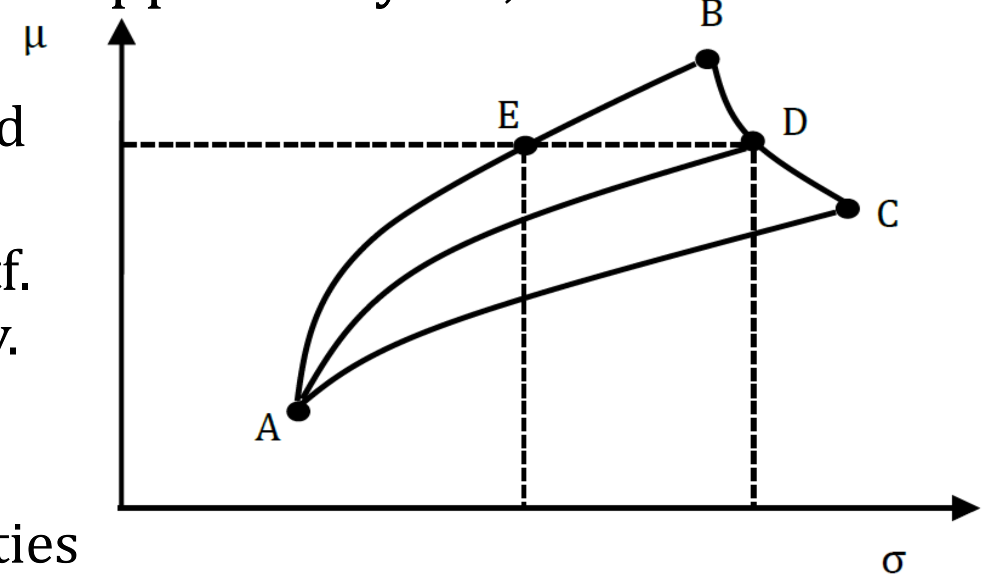
One Numerical Example

- For instance, consider stock A with $\mu_A = 5.5\%$ and $\sigma_A = 10\%$, and stock B with $\mu_B = 2.5\%$ and $\sigma_B = 3\%$
- Draw MVF in Excel for $\rho_{A,B} = 0$, $\rho_{A,B} = 0.5$, and $\rho_{A,B} = -0.5$
- See textbook for calculations and details and book's website for exercises in Excel related to this case
  - When the $\rho_{A,B} < 0$, it is possible to form ptfs. that have a lower risk than each of the 2 assets
  - Clearly, as $\rho_{A,B}$ declines, risk characterizing the GMVP moves towards the left, inward
  - The entire MVF rotates upward, less risk may be borne for identical expected ptf. return
  - Note tha the GMVP often needs to include short positions



The Case of N Risky Assets

- Usually investors choose among a large number of risky securities
  - E.g., allocation among the 500 stocks in the S&P 500
- Extend our framework to the general case, with *N* risky assets
- The MVF no longer coincides with the opportunity set, which now becomes **a region** and not a line



  - Ptf. D, a combination of assets B and C, is not MV efficient
  - It gives the same mean return as ptf. E but implies a higher standard dev. and a risk-averse investor would never hold portfolio D
  - To exclude all the inefficient securities and ptfs., as first step the investor needs to trace out the MVF, i.e., select ptfs. with minimum variance (std. dev.) for each level of μ
  - Only interested in the upper bound of the feasible region
- We solve the following **quadratic programming problem**:

$$\min_{\{\omega\}} \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \quad Subject\ to \quad \boldsymbol{\iota}'\boldsymbol{\omega} = 1, \quad \boldsymbol{\mu}'\boldsymbol{\omega} = \bar{\mu}$$

Nx1 vector of 1s ($\boldsymbol{\iota}$)

Target mean ($\bar{\mu}$)

The Case of N Risky Assets

- For the time being, no short-sale restrictions have been imposed
  - To solve the program, assume that no pair or general combination of asset returns are linearly dependent
  - $\Rightarrow \Sigma$ is nonsingular and invertible; in fact, Σ is (semi-)positive definite
- Under these conditions, it is a constrained minimization problem that can be solved through the use of Lagrangian multiplier method
- See your textbook for algebra and details
- If one defines $A \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$, $B \equiv \boldsymbol{\mu}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $C \equiv (\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})$, and $D = BC - A^2$ then the unique solution to the problem, $\boldsymbol{\omega}^*$, is:

$$\boldsymbol{\omega}^* = \mathbf{g} + \boldsymbol{h}\,\bar{\mu} \qquad \mathbf{g} = \frac{1}{D}[B(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})],\ \mathbf{h} = \frac{1}{D}[C(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\iota})]$$

i.e., any combination of MVF ptf. weights gives another MVF ptf.

  - Consider two MVF ptfs. P_1 and P_2 with mean μ_{P1} and μ_{P2}, and assume that P_3 is a generic portfolio on the MVF: always possible to find a quantity x such that $\mu_{P_3} = x\mu_{P_1} + (1-x)\mu_{P_2}$
  - Other MVF ptf: $\boldsymbol{\omega}_{P_3} = x\omega_{P_1} + (1-x)\omega_{P_2} = x(\mathbf{g} + \mathbf{h}\mu_{P_1}) + (1-x)(\mathbf{g} + \mathbf{h}\mu_{P_2})$
$$= \mathbf{g} + \mathbf{h}(x\mu_{P_1} + (1-x)\mu_{P_2}) = \mathbf{g} + \mathbf{h}\mu_{P_3},$$

Two-Fund Separation

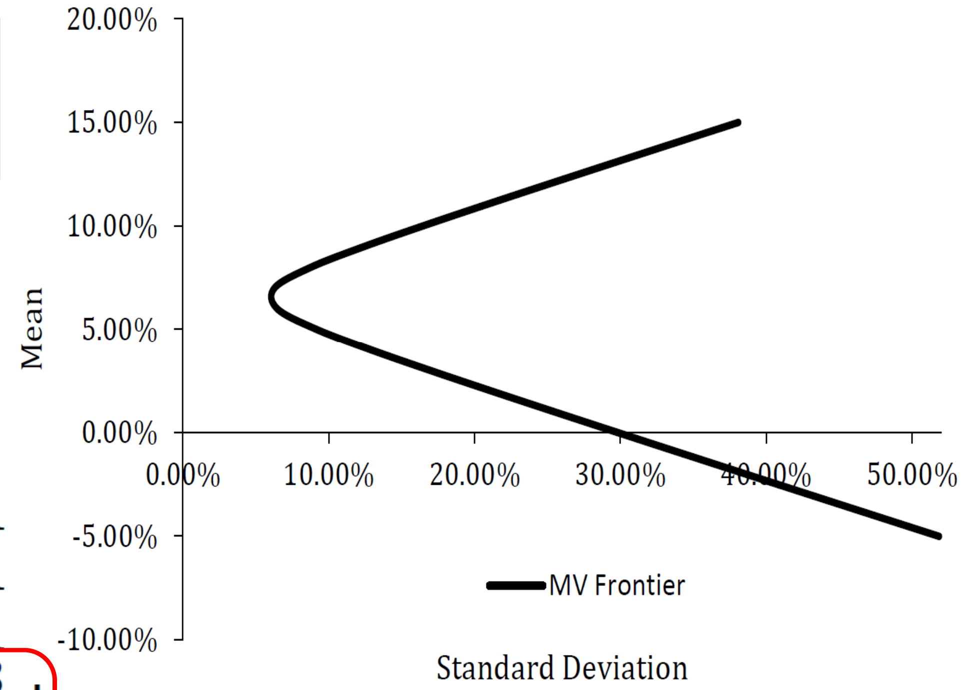
It is sufficient to know two points (portfolios) on the mean-variance frontier to generate all the others

- All MV-optimizers are satisfied by holding a combination of two mutual funds (provided they are MV efficient), regardless of preferences
- Their heterogeneous preferences will only impact the way in which they combine the two funds that they choose to hold
- In equilibrium, if all investors are rational MV optimizers, the market portfolio, being a convex combination of the optimal portfolios of all the investor, has to be an efficient set portfolio
- As for the shape of MVF when *N* assets are available, this is a rotated hyperbola as in case of 2 assets: $$\sigma_P^2 = \frac{1}{D}[C(\mu_P)^2 - 2A\mu_P + B]$$
  - Equation of a parabola with vertex in $((1/C)^{1/2}, A/C)$, which also represents the global minimum variance portfolio
  - The textbook shows that GMV weights are: $\boldsymbol{\omega}_{GMVP} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{C} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}$,

One Strategic Asset Allocation Example

- Consider three assets – U.S. Treasury, corporate bonds, and equity – characterized by the mean vector and the variance-covariance matrix:

$$\mu = \begin{bmatrix} 6.00 \\ 7.50 \\ 9.00 \end{bmatrix} \quad \mathbf{\Sigma} = \begin{bmatrix} 0.005 & 0.004 & -0.002 \\ 0.004 & 0.008 & 0.003 \\ -0.002 & 0.003 & 0.025 \end{bmatrix}$$



- The textbook guides you to perform calculations of A, B, C, D using Excel:

$$\mathbf{g} = \frac{1}{D}[B(\mathbf{\Sigma}^{-1}\boldsymbol{\iota}) - A(\mathbf{\Sigma}^{-1}\boldsymbol{\mu})]$$

$$= \frac{1}{14.21}\left\{1.26 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 6,00 \\ 7,50 \\ 9,00 \end{bmatrix}\right\} = \begin{bmatrix} 4.63 \\ -3.27 \\ -0.37 \end{bmatrix}$$

$$\mathbf{h} = \frac{1}{D}[C(\mathbf{\Sigma}^{-1}\boldsymbol{\mu}) - A(\mathbf{\Sigma}^{-1}\boldsymbol{\iota})] = \frac{1}{14.21}\left\{282.10 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \begin{bmatrix} 6.00 \\ 7.50 \\ 9.00 \end{bmatrix} - 18.5 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \begin{bmatrix} -57.78 \\ 48.89 \\ 8.89 \end{bmatrix}$$

Unlimited, Riskless Borrowing and Lending

- So far, we have ignored the existence of a risk-free asset == a security with return R^f known with certainty and zero variance and zero covariance with all risky assets
  - Buying such a riskless asset == lending at a risk-free rate to issuer
  - Assume investor is able to leverage at riskless rate
  - There is no limit to the amount that the investor can borrow or lend at the riskless rate (we shall remove this assumption later)
- Fictional experiment in which the possibility to borrow and lend at R^f is offered to investor who already allocated among N risky assets
- X is the fraction of wealth in an efficient frontier, risky portfolio (A) characterized μ_A and σ_A, respectively; a share 1 - X is invested in the riskless asset, to obtain mean and standard deviation:

$$\mu_P = X\mu_A + (1-X)R^f = R^f + X(\mu_A - R^f)$$

$$\sigma_P = \sqrt{\sigma_P{}^2} = \sqrt{X^2\sigma_A{}^2} = X\sigma_A$$

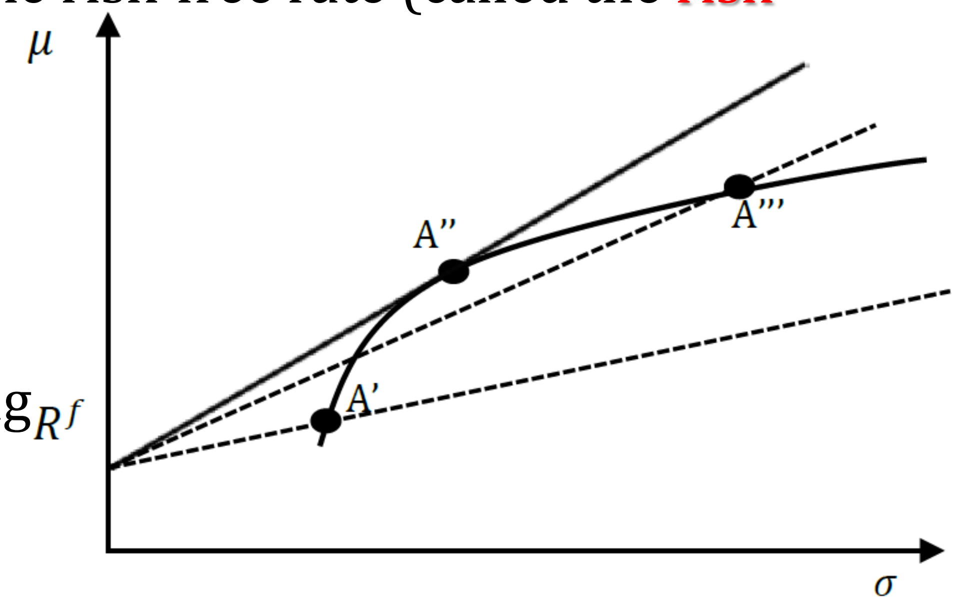
- Solving from X in the first equation and plugging into the second:

$$\mu_P = R_F + \frac{\sigma_P}{\sigma_A}(\mu_A - R^f) = \quad R^f + \frac{(\mu_A - R^f)}{\sigma_A}\sigma_P$$

Unlimited, Riskless Borrowing and Lending

The capital transformation line measures at what rate unit risk (st. dev.) can be transformed into average excess return (risk premium)

- The equation of a straight line with intercept R^f and slope $(\mu_A - R^f)/\sigma_A$
- This line is sometimes referred to as **capital transformation line**
- The term $(\mu_A - R^f)/\sigma_A$ is called **Sharpe ratio** (SR), the total reward for taking a certain amount of risk, represented by the st. dev.
  - SR is the mean return in excess of the risk-free rate (called the **risk premium**) per unit of volatility
  - The plot shows 3 transformation lines for 3 choices of the risky benchmark A (A', A'', and A''') on the efficient frontier
  - Points to the left of A involve lending at the risk-free rate while the ones to the right involve borrowing



  - As investors prefer more to less, they will welcome a "rotation" of the straight line passing through R^f as far as possible in a counterclockwise direction, **until tangency**

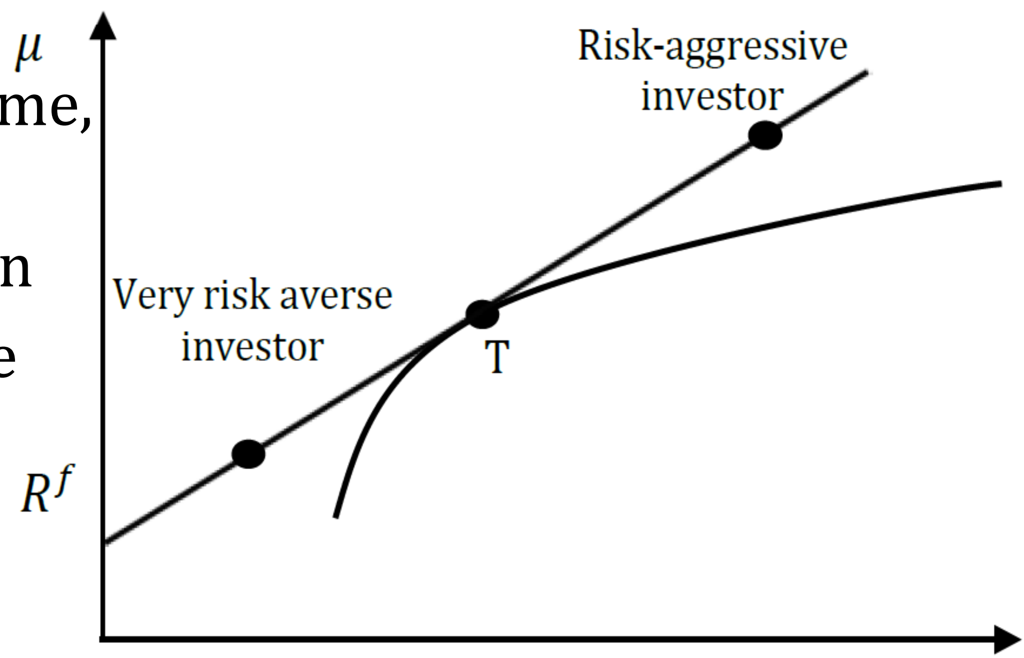
The Tangency Portfolio and the Capital Market Line

Under no frictions and homogeneous beliefs, there exists a tangency ptf. that maximizes the slope of the transformation line

- Assuming beliefs are homogeneous, there are no frictions or taxes, and that individuals face the same R^f and identical asset menus, all rational, non-satiated investors hold the same tangency portfolio
- It is combined with a certain share of risk-free lending or borrowing
- While the share of wealth an investor lends or borrows at R^f depends on the investor's preference for risk, the risky portfolio should be the same for all the investors
- The steepest CTL gets a special name, the Capital Market Line (CML)
- Special case of two-fund separation
- To determine the tangency ptf. one needs to solve:

$$\max_{\{\omega\}} \frac{(\boldsymbol{\omega}'\boldsymbol{\mu} - R^f)}{(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega})^{\frac{1}{2}}}$$

$$\textit{subject to} \quad \boldsymbol{\omega}'\boldsymbol{\iota} = 1$$



The Tangency Portfolio and the Capital Market Line

- The textbook explains how the problem may be written as a simple unconstrained max problem that we can solved by solving the FOCs: $\max_{\{\omega\}} \frac{\boldsymbol{\omega}'(\boldsymbol{\mu} - R^f \boldsymbol{\iota})}{(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega})^{1/2}}$

- The resulting vector of optimal ptf. weights is: $\boldsymbol{\omega}_T = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - R^f \boldsymbol{\iota})}{A - CR^f}$
- Using the same data as in the strategic asset allocation example on three assets – U.S. Treasury, corporate bonds, and equity – we have:

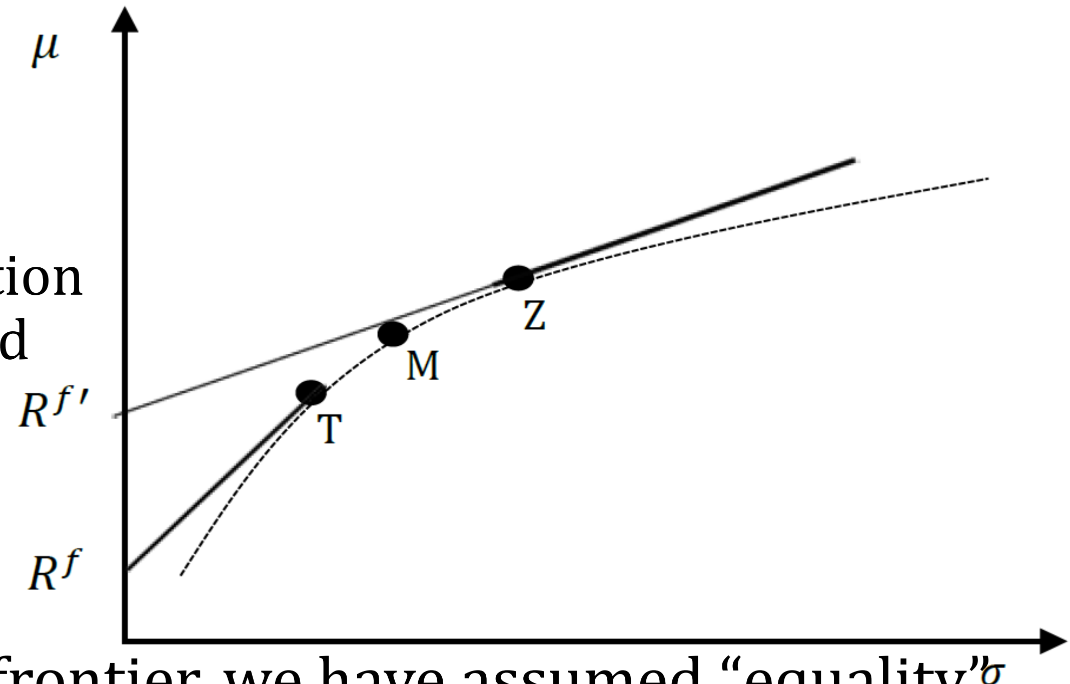
$$\boldsymbol{\omega}_T = \frac{1}{18.5 - 282.11 \cdot 2.5\%} \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \begin{bmatrix} 3.50\% \\ 5.00\% \\ 6.50\% \end{bmatrix} = \begin{bmatrix} 0.59 \\ 0.16 \\ 0.25 \end{bmatrix}$$

- Textbook gives indications on how to use Microsoft Excel'sl Solver®
  - The Solver will iteratively change the values of the cells that contain the weights until the value of the Sharpe ratio is maximized
  - We shall analyze the use of the Solver soon and in your homeworks
- Up to this point, we have assumed that the investor can borrow money at the same riskless rate at which she can lend
- More reasonable assumption: the investor is able to borrow money, but at a higher rate than the one of the risk free (long) investment

Unlimited, Riskless Borrowing and Lending

When lending and borrowing is possible at different rates, it is no longer possible to determine a single tangency portfolio

- The figure shows how the CML is modified when borrowing is only possible at a rate $R^{f\prime} > R^f$
- There are now two CTLs, both tangent to the efficient frontier
  - All the points falling on the portion of the efficient frontier delimited by T (below) and Z (above) will be efficient even though these do not fall on the straight, CML-type line



- While constructing the efficient frontier, we have assumed "equality" constraints (e.g., ptf weights summing to one), but no "inequality" constraints (e.g., positive portfolio weights)
- Inequality constraints complicate the solution techniques
- However, unlimited short-selling assumption is often unrealistic (see margin accounts)

Short-Selling Constraints

- When short-selling is not allowed, portfolio weights should be positive, i.e. the constraint $\boldsymbol{\omega} \geq 0$ (to be interpreted in an element-by-element basis) has to be imposed
  - When $\boldsymbol{\omega}$ has to be positive, the unconstrained maximum may be at a value of that is not feasible
  - Therefore, it is necessary to impose the Kuhn-Tucker conditions
  - The textbook gives a heuristic introduction to what these are
  - Fortunately, Microsoft's Excel Solver® offers the possibility to solve the problem numerically, by-passing these complex analytical details
- Consider again our earlier strategic asset allocation example and let's set $\bar{\mu} = 9\%$
  - In the absence of constraints, the solution is $\boldsymbol{\omega}_T = \begin{bmatrix} -56.66\% \\ 113.33\% \\ 43.33\% \end{bmatrix}$
  - This makes sense because the second asset is characterized by a large Sharpe ratio and hence must be exploited to yield a high mean return by leveraging the first security
  - Selling -57% of the first security is a major hurdle
  - Under nonnegativity constraints we obtain: $\boldsymbol{\omega}_T^{constrain} = \begin{bmatrix} 0\% \\ 0\% \\ 100\% \end{bmatrix}$