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Aversion to Risk and Optimal Portfolio Selection in the Mean-Variance Framework

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20135 – Theory of Finance, Part I

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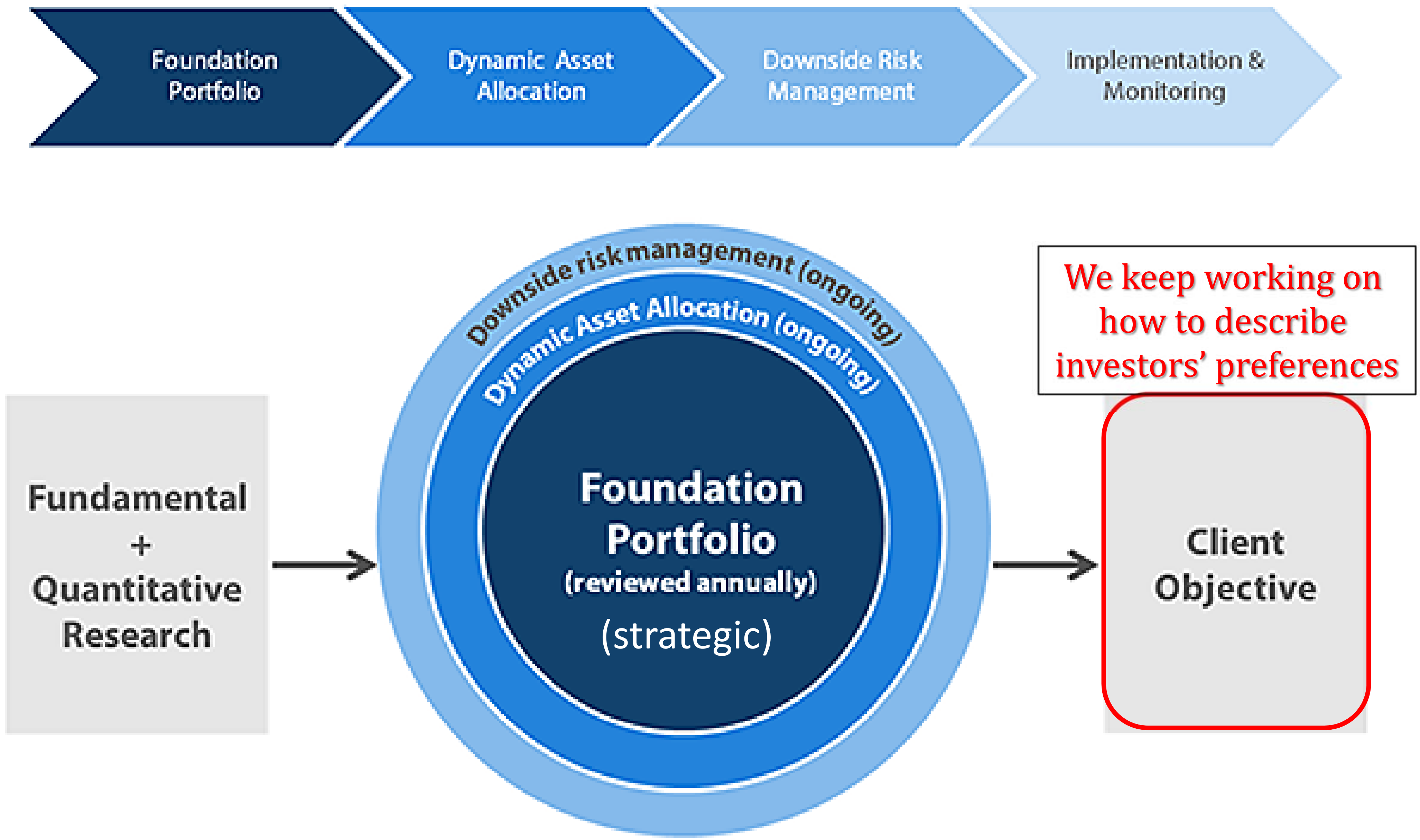
Outline and objectives

- Four alternative ways to provide foundations to mean-variance preferences
  - Guidolin-Pedio, chapter 4, sec. 2.1
- Indifference curves in mean-variance space
  - Guidolin-Pedio, chapter 4, sec. 2.2
- Optimal mean-variance portfolio selection
  - Guidolin-Pedio, chapter 4, sec. 2.3
- The separation theorem
  - Guidolin-Pedio, chapter 4, sec. 2.3

Outline and objectives



The Foundations of Mean-Variance Analysis

- One can show that a non-satiated investor with quadratic utility is characterized by an expected utility functional with structure:

$$E[U(W)] = E[W] - \frac{1}{2}\kappa E[W^2] = E[W] - \frac{1}{2}\kappa[Var[W] + (E[W])^2]$$
$$= E[W]\left(1 - \frac{1}{2}\kappa E[W]\right) - \frac{1}{2}\kappa Var[W]$$

  - It explicitly trades off the variance of terminal wealth with its mean because $W < 1/\kappa$ implies that $E[W] < 1/\kappa < 2/\kappa$ which is necessary and sufficient for $(1 - 1/2\kappa E[W]) > 0$
  - Quadratic utility isn't monotone increasing and may imply ARA,RRA < 0
- More generally, a MV framework is characterized by

$$E[U(W)] = \Gamma(E[W], Var[W]),$$

i.e., by dependence of the VNM functional only on mean and variance

  - If $U(\cdot)$ is quadratic, then $\Gamma(\cdot)$ will be linear in mean and variance
- A MV objective can be justified on grounds other than as the expected value of a quadratic utility function
- There are at least three additional ways of justifying a MV objective

The Foundations of Mean-Variance Analysis

A MV functional, $E[U(W)]=\Gamma(E[W], Var[W])$, can be micro-founded on: (i) quadratic utility, (ii) a Taylor expansion to any general VNM utility $U(\cdot)$, (iii) the EUT when joint return distribution is normal, (iv) directly

- First, a quadratic approximation (i.e., 2nd-order Taylor expansion), see the Appendix for details
- Second, $E[U(W)]=\Gamma(E[W], Var[W])$ may derive from an application of the EUT when the rates of return are described according to a multivariate Normal distribution
  - Normal distributions are characterized entirely by their means (expectations), variances, and covariances;
  - Linear combinations of Normal random variables are also Normal (hence, terminal wealth, or the rate of return on a portfolio of assets with Normally distributed returns, is also Normally distributed)
- Third, often a MV objective is directly assumed, on the grounds that such a criterion is plausible, without recourse to deep assumptions
- Less innocent than it seems, as it implies investors ignore features of the distribution of asset returns besides mean and variance

Mean-Variance of Terminal Wealth or Ptf. Returns?

- E.g., any skewness in the distribution would be ignored
- A less obvious feature not captured by just variance, is the thickness of the tails of a distribution; an index of this tendency is the kurtosis

- The problems with MV are not over – normally MV objectives are applied to portfolio returns, $W_{t+1} = (1 + R_{PF,t+1})W_t$ but note that:

$$E_t[W_{t+1}] = E_t[(1 + R_{PF,t+1})]W_t = (1 + E_t[R_{PF,t+1}])W_t$$

$$Var_t[W_{t+1}] = Var_t[(1 + R_{PF,t+1})]W_t^2 = Var_t[R_{PF,t+1}]W_t^2$$

- Therefore, plugging into $E_t[U(W_{t+1})] = E[W_{t+1}](1 - 0.5\kappa E[W_{t+1}]) - 0.5\kappa Var_t[W_{t+1}]$ and dropping $(1 - 0.5\kappa E[W_{t+1}])$ one has:

$$E_t[U(W_{t+1})] = (1 + E_t[R_{PF,t+1}])W_t - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]W_t^2$$

$$= \left\{(1 + E_t[R_{PF,t+1}]) - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]W_t\right\}W_t$$

$$\propto E_t[R_{PF,t+1}] - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]W_t,$$

Not the same as:
$$E_t[R_{PF,t+1}] - \frac{1}{2}\kappa Var_t[R_{PF,t+1}]$$

- We call the MV functions that depend on moments of portfolio returns $G(E_t[R_{PF,t+1}], Var_t[R_{PF,t+1}]) = G(\mu_{PF}, \sigma^2_{PF})$

MV Indifference Curves and Their Meaning

The loci in the mean-standard deviation space of the infinite combinations (μ_{PF}, σ_{PF}) that yield some fixed level of identical MV utility as measured by G(μ_{PF}, σ^2_{PF}) is called a MV indifference curve

- How can we represent MV preference of ptf. returns?
  - Consider some small, countervailing changes in μ_{PF} and σ^2_{PF} to keep the total level of the MV satisfaction constant at some initial level:

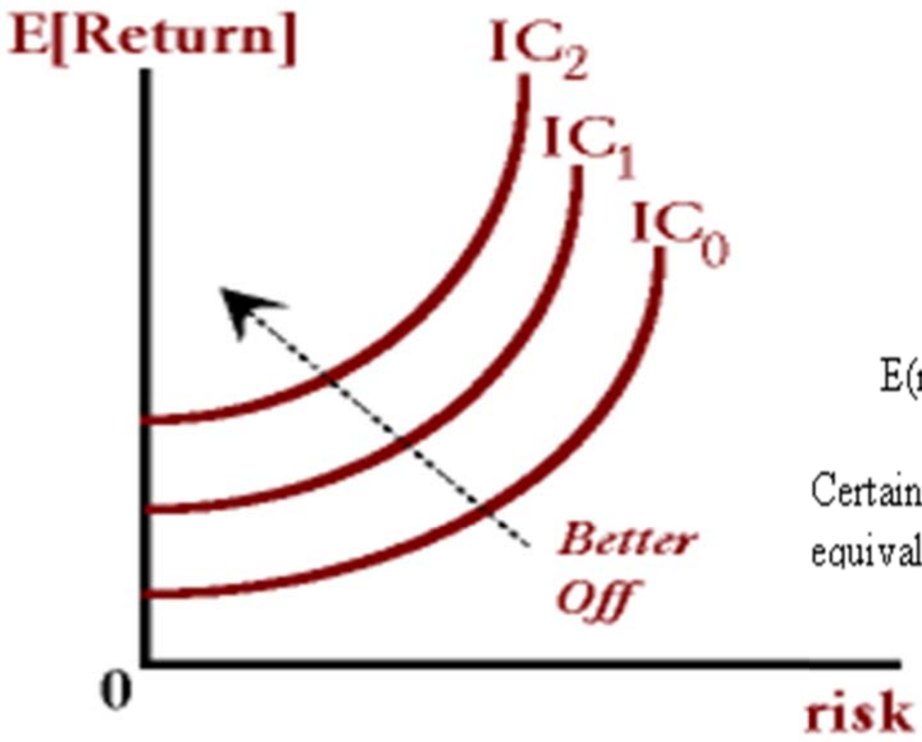
$$0 = dG(\mu_{PF}, \sigma^2_{PF}) = \left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \mu_{PF}}\right|_{G=\bar{G}} d\mu_{PF} + \left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \sigma^2_{PF}}\right|_{G=\bar{G}} d\sigma^2_{PF}$$

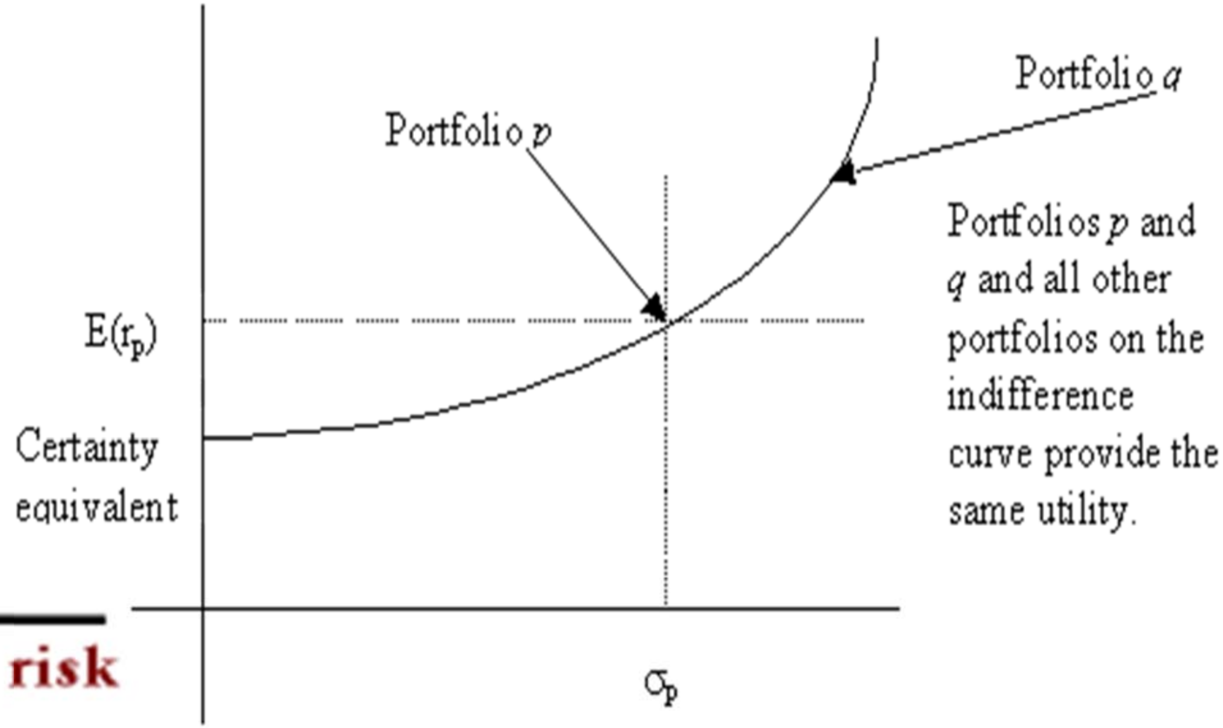
  - Solving to find the implied local slope when MV satisfaction is constant:
  - This is the slope of a MV indifference curve

$$\alpha(\bar{G}) \equiv \left.\frac{d\mu_{PF}}{d\sigma^2_{PF}}\right|_{G=\bar{G}} = -\frac{\left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \sigma^2_{PF}}\right|_{G=\bar{G}}}{\left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \mu_{PF}}\right|_{G=\bar{G}}} > 0$$

  - Because for positive σ, σ^2_{PF} is a monotone increasing function of std. dev., if the slope of the loci is positive as σ^2_{PF} increases, the same must be true of increase in standard deviation, σ_{PF}
- The issue now concerns the type of concavity of the indifference curves, because the earlier definition fails to rule out any case

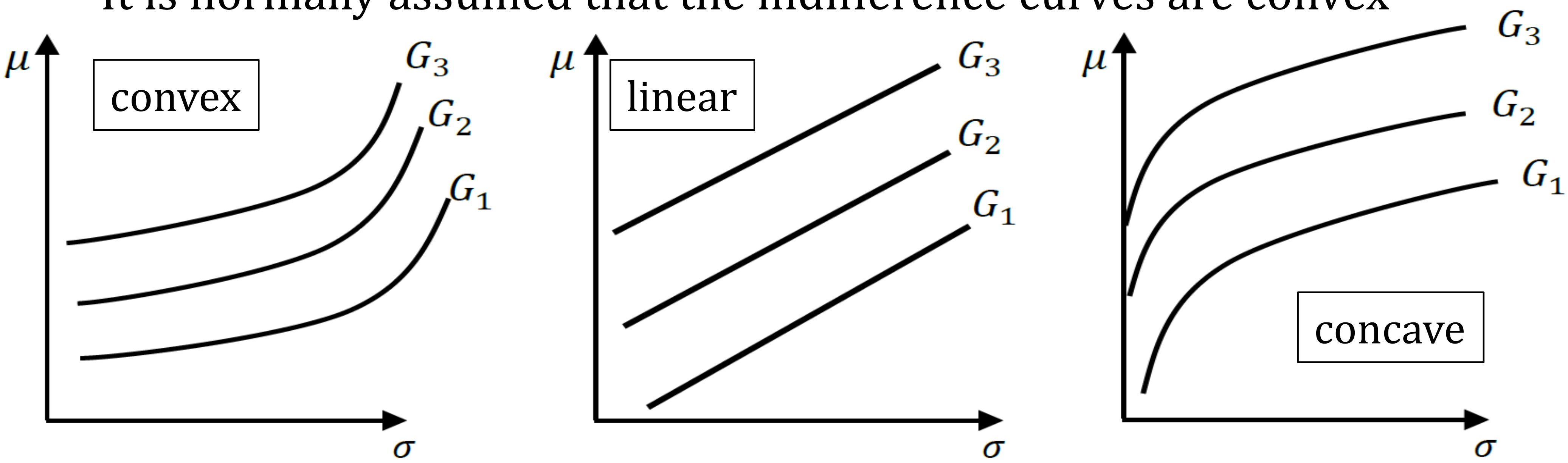
MV Indifference Curves and Their Meaning





MV Indifference Curves and Their Meaning

- It is normally assumed that the indifference curves are convex



- The justification for this type of convexity is:

  ① Plausibility, reasonable that, at higher levels of risk, the increments to expected return needed to compensate for increments in risk are larger

  ② As an implication of quadratic VNM utility: $E_t[U(W_{t+1})] = \mu_{PF} - \frac{1}{2}\kappa\, \sigma_{PF}^2$

$$\left.\frac{d\mu_{PF}}{d\sigma_{PF}}\right|_{G=\bar{G}} = \kappa\, \sigma_{PF} > 0 \qquad \left.\frac{d^2\mu_{PF}}{d(\sigma_{PF})^2}\right|_{G=\bar{G}} = \kappa > 0$$

  i.e., provided the investor is risk-averse, indifferences curves are convex

  ③ As an implication of a negative exponential VNM U(·) when returns are jointly normally distributed

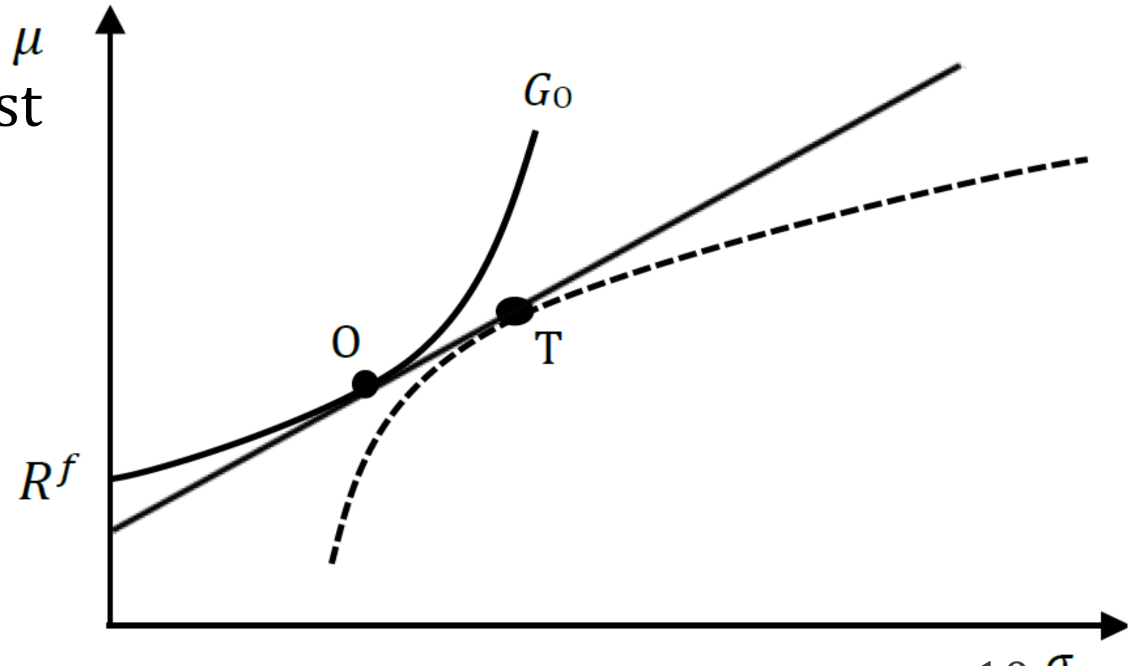
Optimal MV Portfolio Selection

For each investor, the optimal MV investor lies at the tangency btw. the highest indifference curve and the CML; as a result all investors will demand a unique, risky tangency ptf., the separation theorem

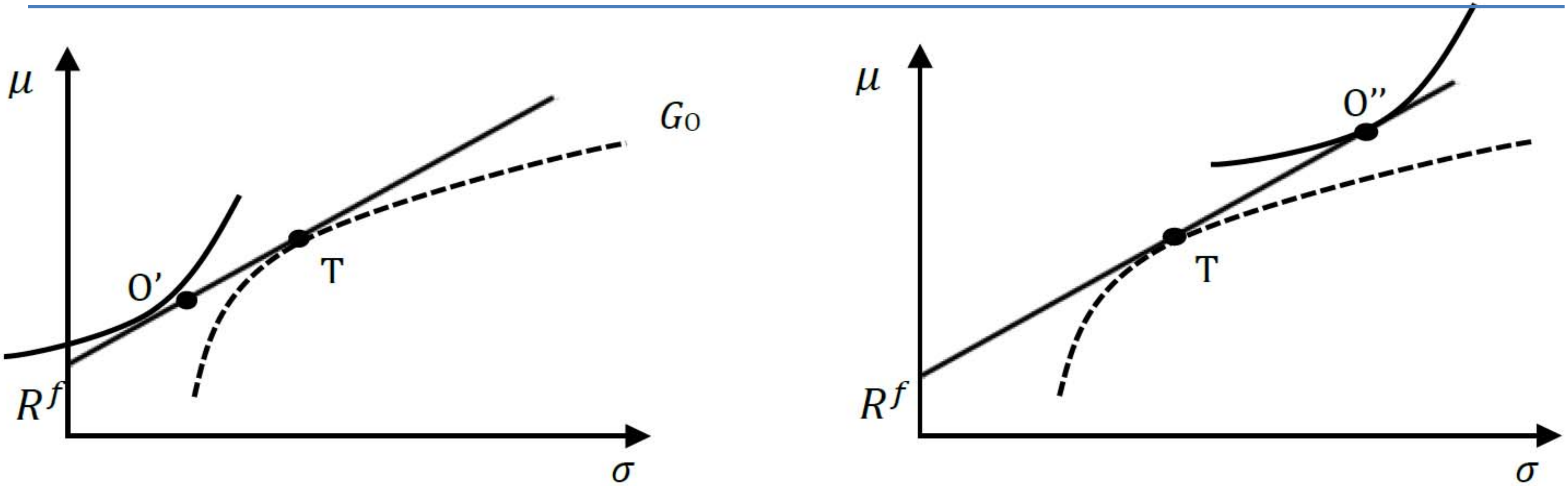
④ Because linear or concave indifference curves would otherwise lead to predictions that are inconsistent with observed behavior

- Ready to assemble all the MV machinery:
  - The minimum-variance frontier and the efficient set
  - Indifference curves describing MV-type preferences
- The optimal MV ptf. for one investor lies then on the highest indifference curve attainable s.t. being feasible == on or below CML
- The tangency condition gives that at the optimum it must be

$$\alpha(G_O) = \frac{\mu_T - R^f}{\sigma_T} = SR_T$$



The Separation Theorem

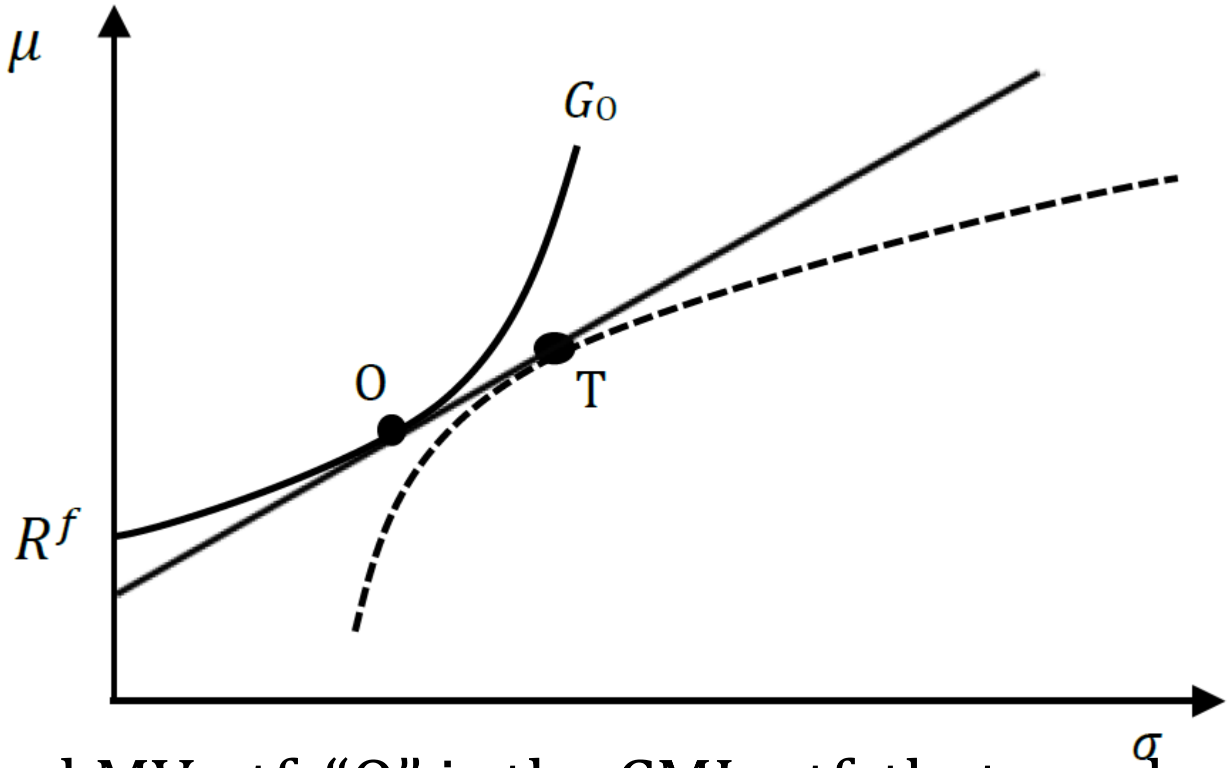


- On the left, the case of an extremely risk-averse, cautious investor who leaves almost all of her wealth in cash
- On the right, an aggressive investor who levers her initial wealth by borrowing to invest more than 100% in the tangency portfolio
- In the case of quadratic utility or when expected utility may be approximated by a MV objective, the optimal share is the solution to:

$$\max_{\omega_t}(1 + R^f) + E\left[\left(R_{T,t+1} - R^f\right)\omega_t\right] - \frac{1}{2}\kappa Var\left[\left(R_{T,t+1} - R^f\right)\omega_t\right]$$

$$\Leftrightarrow \max_{\omega_t} E\left[\left(R_{T,t+1} - R^f\right)\right]\omega_t - \frac{1}{2}\kappa\omega_t^2\sigma^2$$

The Separation Theorem



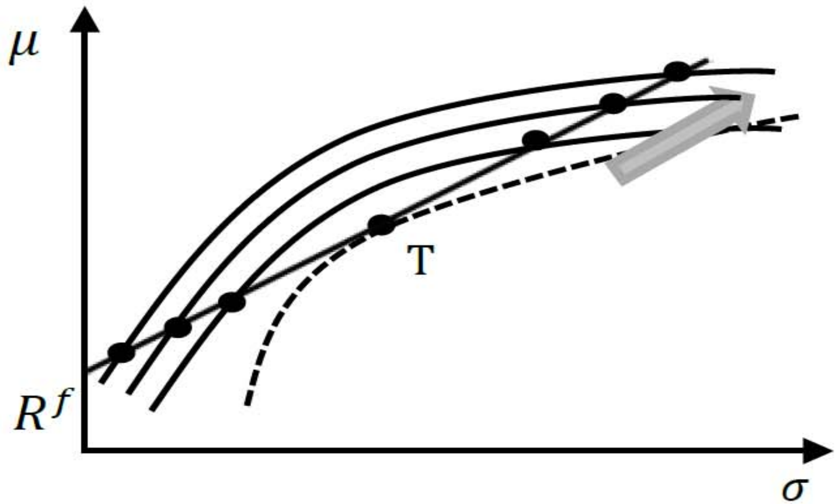
- The optimal MV ptf. "O" is the CML ptf. that reaches the highest possible indifference curve
- «O» is by construction a linear combination btw. the tangency ptf. «T» and the risk-free asset, because it is on the CML
- Investors with different indifference curves will then require different mixes of risk-free asset and tangency ptf.

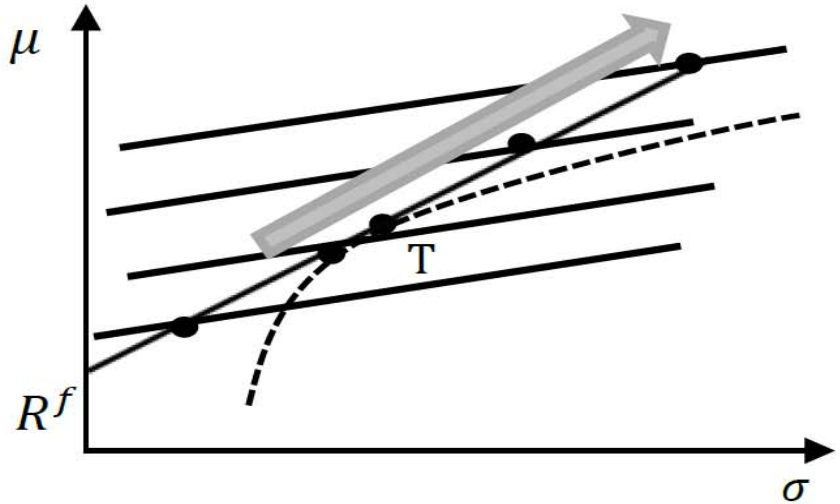
Optimal MV Portfolio Selection

- The FOC leads to the expression:

$$\widehat{\omega}_t = \frac{E\left[\left(R_{T,t+1} - R^f\right)\right]}{\kappa\sigma^2}$$

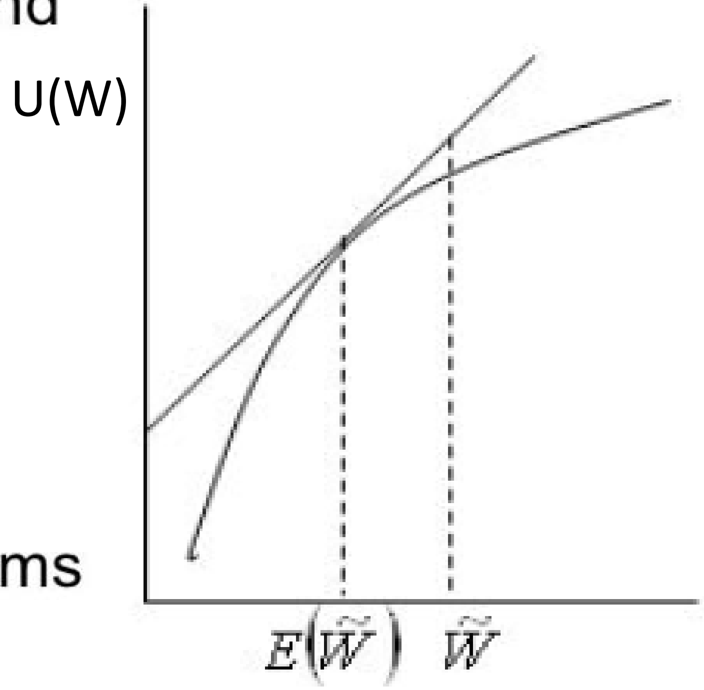
- The greater the excess expected return (risk premium), the larger the holding of the risky portfolio
- The riskier the portfolio T, the lower the holding of the risky asset
- The greater the risk tolerance (i.e. the smaller is κ), the higher the holding of the risky portfolio
- If one were not to assume convex indifference curves, an investor will borrow an infinite amount of cash at the riskless rate and to invest a share that diverges to infinity in T, which makes little sense





Appendix

- Denote the level of wealth by $\tilde{W}$. Taking a Taylor's series expansion of utility around expected wealth



$$U(\tilde{W}) = U(E[\tilde{W}]) + U'(E[\tilde{W}])[\tilde{W} - E[\tilde{W}]]$$

$$+ \frac{1}{2}U''(E[\tilde{W}])[\tilde{W} - E[\tilde{W}]]^2 + R_3$$

- Here R_3 is the error that depends on terms involving $[\tilde{W} - E[\tilde{W}]]^3$ and higher

- Taking the expectation of the expansion

$$E[U(\tilde{W})] = U(E[\tilde{W}]) + \frac{1}{2}U''(E[\tilde{W}])\sigma(\tilde{W})^2 + E[R_3]$$

- The expected error is

$$E[R_3] = \sum_{n=3}^{\infty} \frac{1}{n!} U^{(n)}(E[\tilde{W}]) m^n(\tilde{W})$$