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Lecture 2: Essential Concepts in Time Series Analysis

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20192– Financial Econometrics

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Overview

- Time Series: When Can We Focus on the First Two Moments Only?
- Strict vs. Weak Stationarity
- White noise processes
- The sample autocorrelation function vs. the population ACF
- The sample partial autocorrelation function vs. the population PACF
- Box-Pierce-Ljung test for sample ACF
- The sample partial autocorrelation function vs. the population PACF

Time Series

- A **time series** consists of a sequence of random variables, y_1, y_2, \dots, y_T , also known as a stochastic process $\{y_t\}_{t=1}^T$, of which we only observe the empirical realizations
 - An observed time series $\{y_t\}_{t=1}^T$ (technically, a sub-sequence because limited to a finite sample) of the realized values of a family of random variables $\{Y_t\}_{t=-\infty}^{+\infty}$ defined on an appropriate probability space
 - See difference between **sample** ($\{y_t\}_{t=1}^T$) and **population** ($\{Y_t\}_{t=-\infty}^{+\infty}$)
- A **time series model** for the observations $\{y_t\}_{t=1}^T$ is a specification of the joint distribution of the set of random variables of which the sampled data are a realization
 - We often exploit the **linearity** of the process to specify only the first- and second-order moments of the joint distribution, i.e., the mean, variances and covariances of $\{Y_t\}_{t=-\infty}^{+\infty}$

Linear Processes

Definition **(Linear process)** A time series $\{y_t\}$ is said to be a linear process if it has the representation

$$y_t = \mu + \sum_{j=-\infty}^{\infty} \phi_j z_{t-j},$$

for all t , where μ is a constant, $\{\phi_j\}$ is a sequence of constant coefficients where $\phi_0 = 1$ and $\sum_{j=-\infty}^{\infty} |\phi_j| < \infty$, and $\{z_t\}$ is a sequence of independent and identically distributed (IID) random variables with a defined distribution function. In particular, we assume that the distribution of z_t is continuous, with $E[z_t] = 0$ and $\text{var}(z_t) = \sigma_z^2$. Noticeably, if $\sigma_z^2 \sum_{i=1}^{\infty} \phi_i^2 < \infty$, then y_t is weakly stationary, with the meaning that we shall see below.

- If a time series process is linear, modelling its conditional mean and variance is sufficient in a **mean-squared error sense**

Strict Stationarity

- To use past realizations of a variable of interest to forecast its future values, it is necessary for the stochastic process that has originated the observations to be stationary
- Loosely speaking, a process is said to be stationary if its statistical properties do not change over time

Definition **(Strict stationarity)** A process is **strictly stationary** if the joint distribution of the variable associated to any sub-sequence of times t_1, t_2, \dots, t_n is the same as the joint distribution of the sequence of all times $t_{1+k}, t_{2+k}, \dots, t_{n+k}$ (where k is an arbitrary time shift). In other words, a strictly stationary time series $\{y_t\}$ has the following properties:

- the random variables y_t are identically distributed;
- the two random vectors $[y_t, y_{t+k}]'$ and $[y_1, y_{1+k}]'$ have the same joint distribution for any t and k .

Weak (Covariance) Stationarity

- In many applications, a weaker form of stationarity generally provides a useful sufficient condition

Definition **(Weak stationarity)** A stochastic process $\{y_t\}$ is **weakly stationary** (or, alternatively, **covariance stationary**) if it has time invariant first and second moments, i.e., if for any choice of $t = 1, 2, \dots, \infty$, the following conditions hold:

$$\mu_y \equiv E(y_t), \text{ with } |\mu_y| < \infty$$

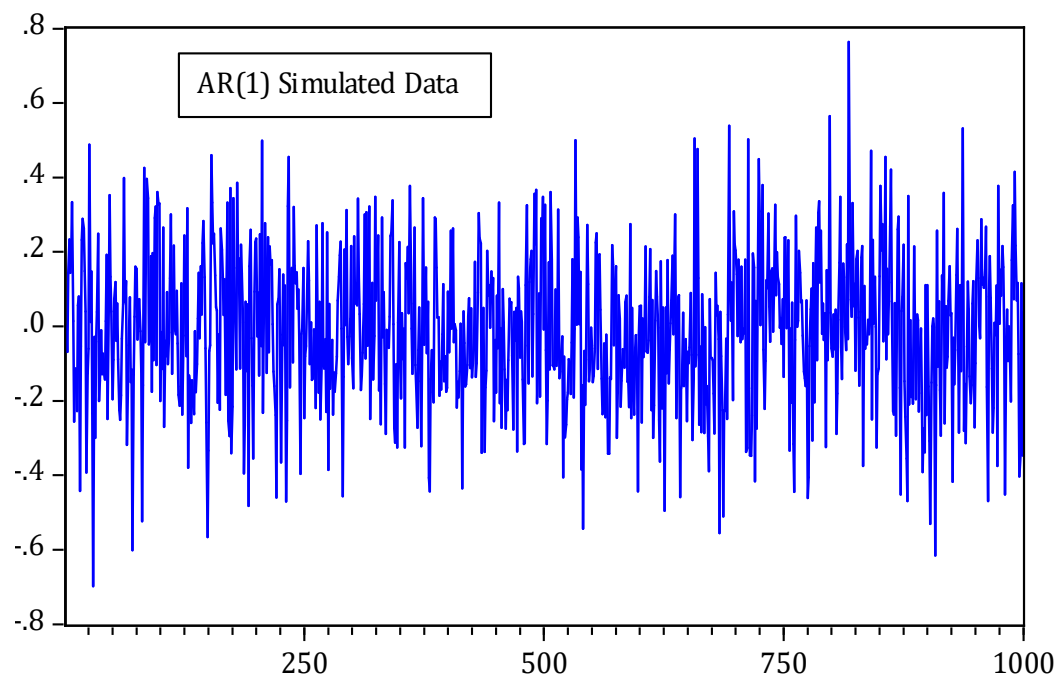
$$\sigma_y^2 \equiv E[(y_t - \mu_y)(y_t - \mu_y)] = E[(y_t - \mu_y)^2] < \infty$$

(Autocovariance function) $\gamma_h \equiv E[(y_t - \mu_y)(y_{t-h} - \mu_y)] \quad \forall h, \text{ with } |\gamma_h| < \infty.$

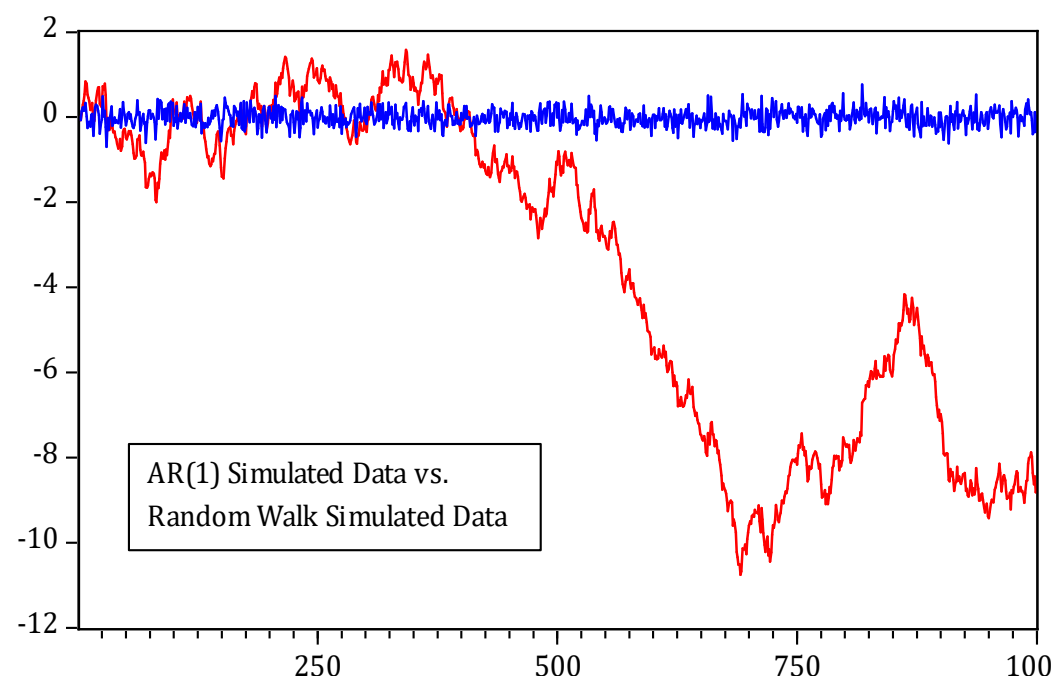
where $h = \dots, -3, -2, -1, 1, 2, 3, \dots$

- $\rho_h \equiv \gamma_h / \gamma_0$ (where γ_0 is the variance) is called **autocorrelation function (ACF)**, for $h = \dots, -2, -1, 1, 2, \dots$
 - Often more meaningful than ACVF because it is expressed as pure numbers that fall in $[-1, 1]$

An Example of Stationary Series



Panel (a)



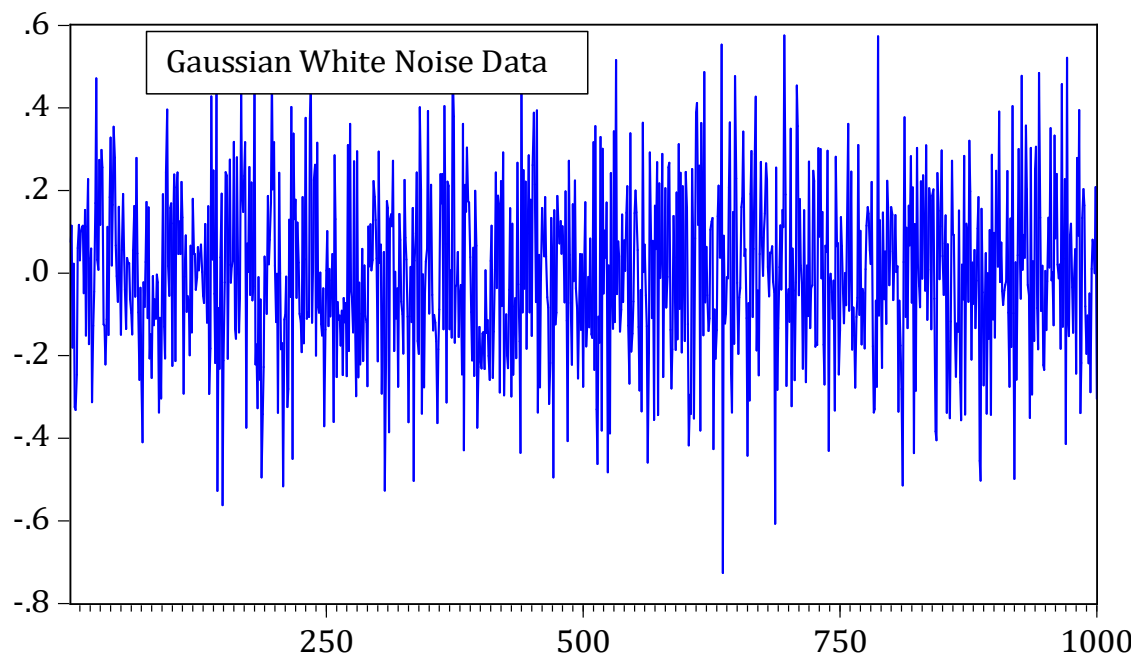
Panel (b)

- A time series generated by a stationary process fluctuates around a constant mean, because its memory of past shocks decays over time
 - The data plotted in panel (a) are 1,000 realizations of a **first-order autoregressive** (henceforth, AR) process of the type $y_t = \phi_0 + \phi_1 y_{t-1} + \epsilon_t$, with $\phi_0 = 0$ and $\phi_1 = 0.2$
 - In panel (b) we have a nonstationary **random walk**, $y_t = y_{t-1} + \epsilon_t$
 - We shall describe these models later

White Noise Process

- A fundamental class of stationary processes is the fundamental building block of all (covariance) stationary processes: white noise

Definition (White Noise) A white noise (WN) process is a sequence of random variables $\{z_t\}$ with mean equal to zero, constant variance equal to σ^2 , and zero autocovariances (and autocorrelations) except at lag zero. If $\{z_t\}$ is normally distributed, we shall speak of a Gaussian white noise.



Sample Autocorrelation Function

- Stationary AR and white noise processes may sometimes be hard to tell apart – what tools are available to identify them?
- The **sample ACF** reflects important information about the linear dependence of a series at different times

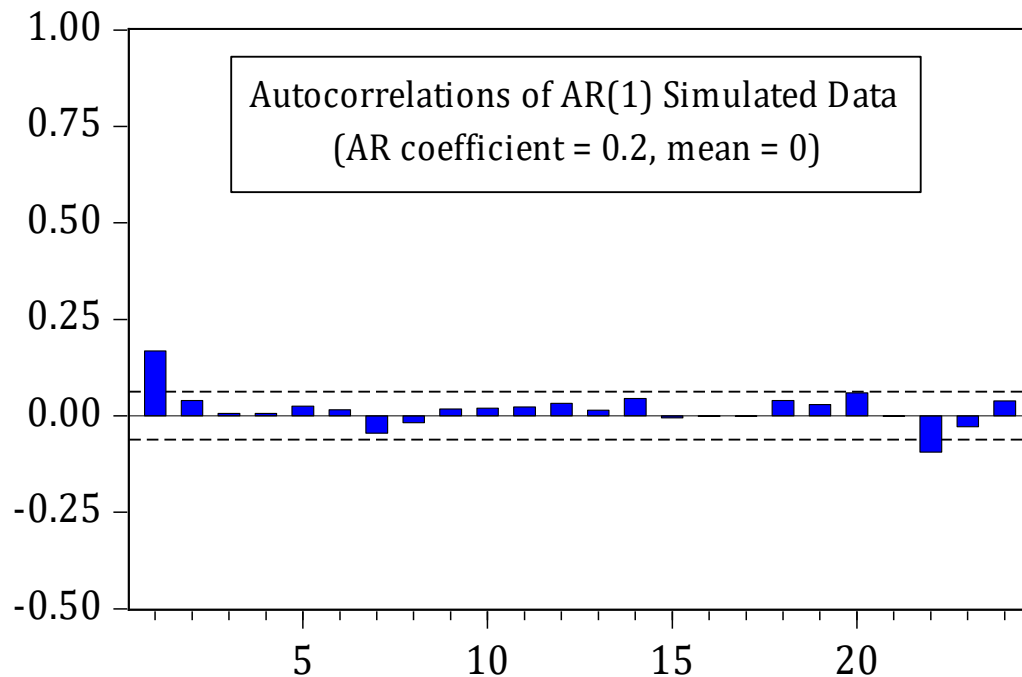
Definition **(Sample autocorrelations)** Given a sample of T observations of the variable $y_t, y_1, y_2, \dots, y_T$, the estimated or sample autocorrelation function $\hat{\rho}_h$ (where h is a positive integer) is computed as

$$\hat{\rho}_h = \frac{\sum_{t=h+1}^T (y_t - \hat{\mu})(y_{t-h} - \hat{\mu})}{\sum_{t=1}^T (y_t - \hat{\mu})^2} = \frac{\hat{\gamma}_h}{\hat{\gamma}_0},$$

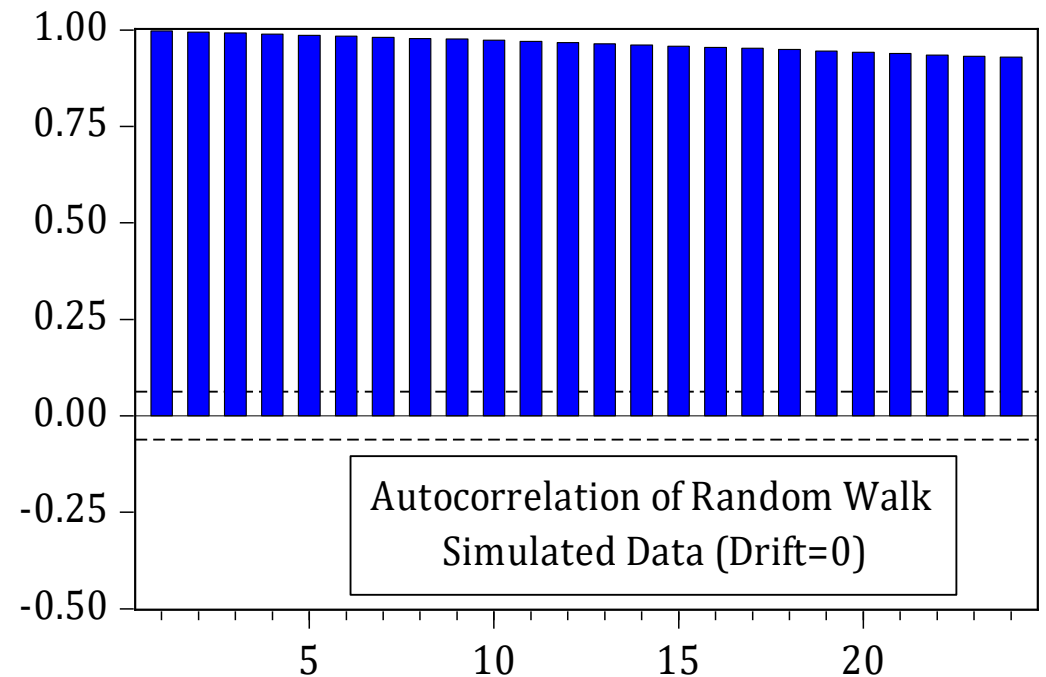
where $\hat{\mu}$ is the sample mean computed as $\hat{\mu} = T^{-1} \sum_{t=1}^T y_t$.

- If $\{Y_t\}_{t=-\infty}^{+\infty}$ is an i.i.d. process with finite variance, then for a large sample, the estimated autocorrelations $\hat{\rho}_h$ will be asymptotically normally distributed with mean ρ_h and variance **$1/T$**

Sample Autocorrelation Function



Panel (a)



Panel (b)

- The dashed lines correspond to approximate (asymptotic) 95% confidence intervals built as $\pm 1.96/\sqrt{T}$
- The SACF in panel (a) shows that a stationary process quickly “forgets” information from a distant past
- The theoretical ACF for a random walk process shall be exactly one at all lags but because SACF is a downward biased estimates of the true and unobserved ACF, the sample coefficients are less than 1

Ljung-Box Test for SACF

- It is also possible to jointly test whether several (say, M) consecutive autocorrelation coefficients are equal to zero:

$$H_o: \rho_1 = \rho_2 = \dots = \rho_M = 0 \quad vs. H_a: \exists \text{ some } j \text{ s.t. } \rho_j \neq 0$$

- Box and Pierce (1970) and Ljung and Box (1978) developed a well-known port-manteau test based on the Q- or LB-statistic

$$Q^*(M) = T(T+2) \sum_{h=1}^M \frac{\hat{\rho}_h^2}{T-h} \sim \chi_M^2$$

Serial Correlation Structure of Simulated AR(1) Data

Serial Correlation Structure of Simulated White Noise Data

| Prob | Serial Correlation Structure of Simulated AR(1) Data | | | | | | Serial Correlation Structure of Simulated White Noise Data | | | | | | Prob | |
|-------|--|---------------------|----|--------|--------|--------|--|---------------------|----|-----|--------|--------|--------|-------|
| | Autocorrelation | Partial Correlation | AC | PAC | Q-Stat | Prob | Autocorrelation | Partial Correlation | AC | PAC | Q-Stat | Prob | | |
| 0.000 | | | 1 | 0.168 | 0.168 | 28.433 | 0.000 | | | 1 | 0.022 | 0.022 | 0.4673 | 0.494 |
| 0.000 | | | 2 | 0.039 | 0.010 | 29.925 | 0.000 | | | 2 | -0.010 | -0.010 | 0.5598 | 0.756 |
| 0.000 | | | 3 | 0.006 | -0.002 | 29.961 | 0.000 | | | 3 | 0.051 | 0.051 | 3.1776 | 0.365 |
| 0.000 | | | 4 | 0.006 | 0.005 | 29.997 | 0.000 | | | 4 | 0.027 | 0.025 | 3.9294 | 0.416 |
| 0.000 | | | 5 | 0.025 | 0.024 | 30.646 | 0.000 | | | 5 | -0.056 | -0.056 | 7.1017 | 0.213 |
| 0.000 | | | 6 | 0.015 | 0.007 | 30.873 | 0.000 | | | 6 | 0.033 | 0.034 | 8.2260 | 0.222 |
| 0.000 | | | 7 | -0.045 | -0.052 | 32.956 | 0.000 | | | 7 | 0.004 | -0.001 | 8.2428 | 0.312 |
| 0.000 | | | 8 | -0.018 | -0.003 | 33.282 | 0.000 | | | 8 | 0.006 | 0.011 | 8.2735 | 0.407 |
| 0.000 | | | 9 | 0.018 | 0.024 | 33.594 | 0.000 | | | 9 | -0.011 | -0.012 | 8.4026 | 0.494 |
| 0.000 | | | 10 | 0.020 | 0.013 | 34.001 | 0.000 | | | 10 | 0.028 | 0.024 | 9.2163 | 0.512 |
| 0.000 | | | 11 | 0.023 | 0.016 | 34.531 | 0.000 | | | 11 | -0.033 | -0.032 | 10.308 | 0.503 |
| 0.000 | | | 12 | 0.032 | 0.028 | 35.566 | 0.000 | | | 12 | -0.051 | -0.050 | 12.961 | 0.372 |
| 0.000 | | | 13 | 0.014 | 0.005 | 35.764 | 0.001 | | | 13 | -0.031 | -0.031 | 13.937 | 0.378 |
| 0.000 | | | 14 | 0.045 | 0.038 | 37.802 | 0.001 | | | 14 | 0.008 | 0.008 | 13.996 | 0.450 |
| 0.000 | | | 15 | -0.005 | -0.022 | 37.828 | 0.001 | | | 15 | 0.059 | 0.069 | 17.483 | 0.291 |
| 0.000 | | | 16 | -0.001 | 0.002 | 37.829 | 0.002 | | | 16 | 0.010 | 0.008 | 17.584 | 0.349 |
| 0.000 | | | 17 | -0.001 | -0.000 | 37.829 | 0.003 | | | 17 | -0.013 | -0.015 | 17.752 | 0.405 |
| 0.000 | | | 18 | 0.040 | 0.042 | 39.502 | 0.002 | | | 18 | 0.020 | 0.013 | 18.143 | 0.446 |
| 0.000 | | | 19 | 0.029 | 0.016 | 40.336 | 0.003 | | | 19 | 0.014 | 0.013 | 18.333 | 0.500 |
| 0.000 | | | 20 | 0.059 | 0.051 | 43.884 | 0.002 | | | 20 | 0.051 | 0.059 | 21.024 | 0.396 |
| 0.000 | | | 21 | -0.001 | -0.018 | 43.884 | 0.002 | | | 21 | -0.009 | -0.014 | 21.100 | 0.453 |
| 0.000 | | | 22 | -0.094 | -0.099 | 52.881 | 0.000 | | | 22 | -0.037 | -0.041 | 22.539 | 0.428 |
| 0.000 | | | 23 | -0.028 | -0.001 | 53.672 | 0.000 | | | 23 | -0.033 | -0.039 | 23.667 | 0.422 |
| 0.000 | | | 24 | 0.038 | 0.046 | 55.161 | 0.000 | | | 24 | -0.037 | -0.042 | 25.038 | 0.404 |

0.494

0.756

0.365

0.416

0.213

0.222

0.312

0.407

0.494

11

Panel (a)

Panel (b)

Sample Partial Autocorrelation Function

- The partial autocorrelation between y_t and y_{t-h} is the autocorrelation between the two random variables in the time series, conditional on $y_{t-1}, y_{t-2}, \dots, y_{t-h+1}$
- Or, the ACF measured **after netting out the portion of the variability linearly explained already by the lags between y_{t-1} and y_{t-h+1}**
 - The sample estimate of the partial autocorrelation at lag h is obtained as the ordinary least square estimator of ϕ_h in an autoregressive model:
$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} \dots + \phi_h y_{t-h} + \epsilon_t$$

| Serial Correlation Structure of Simulated AR(1) Data | | | | | | Serial Correlation Structure of Simulated White Noise Data | | | | | |
|--|---------------------|-----------|--------|--------|-------|--|---------------------|-----------|--------|--------|-------|
| Autocorrelation | Partial Correlation | AC | PAC | Q-Stat | Prob | Autocorrelation | Partial Correlation | AC | PAC | Q-Stat | Prob |
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| | | 20 0.059 | 0.051 | 43.884 | 0.002 | | | 20 0.051 | 0.059 | 21.024 | 0.396 |
| | | 21 -0.001 | -0.018 | 43.884 | 0.002 | | | 21 -0.009 | -0.014 | 21.100 | 0.453 |
| | | 22 -0.094 | -0.099 | 52.881 | 0.000 | | | 22 -0.037 | -0.041 | 22.539 | 0.428 |
| | | 23 -0.028 | -0.001 | 53.672 | 0.000 | | | 23 -0.033 | -0.039 | 23.667 | 0.422 |
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Panel (a)

Panel (b)