SDA Bocconi

Bocconi School of Management

EMFI

EXECUTIVE MASTER IN FINANCIAL INVESTMENTS

INVEST IN YOUR VALUE

Utility-Based Portfolio Choice

04.07.2019

MILANO I ITALY

Università Commerciale
Luigi Bocconi

Lecture 2: Utility-Based Portfolio Choice

Prof. Massimo Guidolin

Portfolio Management

EMFI Angola's Sovereign Fund 2019

Outline and objectives

- Choice under uncertainty: dominance
- Choice under uncertainty: mean-variance criterion
- Axioms of choice under certainty
- Preference representation theorem and its meaning
- Expected utility theorem
- Definition and characterization of risk averse behavior
- Risk-loving and risk neutral investors
- How to measure and compare risk aversion: ARA and RRA coefficients
- Commonly employed utility functions of monetary wealth

Key Concepts/1

- Most financial assets (securities) are risky, i.e., they can be characterized as contracts that give different payoffs in different, mutually exclusive states of the world
- Assume that investors are able to quantify such uncertainty on future states using standard probability distributions, e.g.,

State	*Security A*		*Security B*		*Security C*	
	Payoff	Prob.	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15
ii	18	5/15	18	5/15	16	5/15
iii	14	4/15	10	4/15	12	4/15
iv	10	2/15	5	2/15	12	2/15
v	6	1/15	5	1/15	8	1/15

- Criteria of choice under uncertainty may be complete or incomplete: a complete criterion is always able to rank all securities or investment opportunities on the basis of their objective features; an incomplete criterion is not

Key Concepts/2

- A starkly incomplete criterion is strong dominance: A security (strongly) dominates another security (on a state-by-state basis), if the former pays as much as the latter in all states of nature, and strictly more in at least one state
- We assume that rational (== non-satiated, who prefer more to less) investors, always prefer dominant securities over dominated ones
- Dominance escapes a definition of risk
- Because in general, a security yields payoffs that in some states are larger and in some other states are smaller than under any other, the best known approach consists of summarizing the distributions of asset returns through their mean and variance
- This is the logical foundation of mean-variance dominance, that however remains incomplete
- Under MV, risk is identified with the variance of returns/payoffs

Key Concepts/3

- However, also MV dominance is incomplete:

State	*Security A*		*Security B*		*Security C*	
	Payoff	Prob.	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15
ii	18	5/15	18	5/15	16	5/15
iii	14	4/15	10	4/15	12	4/15
iv	10	2/15	5	2/15	12	2/15
v	6	1/15	5	1/15	8	1/15
Mean	15.47		13.27		14.27	
Variance	16.78		28.46		8.46	

- Because of its incompleteness, the MV criterion can at best only isolate a subset of securities that are not dominated by any other
- This will be later called the MV efficient set
- How can we overcome the pervasive incompleteness that the two criteria imply?
- We develop a theory of utility-based portfolio decisions in conditions of uncertainty

Key Concepts/4

- Formally, the starting point is preference relation, denoted as $\succsim$
- Rationality means that you can always express a precise preference between any pair of bundles, that you should not contradict yourself when asked to express preferences over three or more bundles in successive pairs…
- … and some additional technical conditions that prevent the possibility that by considering long sequences of converging bundles you may express equivocal choices
- Such properties are formally derived from axioms of choice
- The first step is that under such axioms, there exists a continuous, time-invariant, real-valued ordinal utility function $u(\cdot)$ that ranks bundles in the same way as $\succsim$
- Under rationality the ranking of bundles that you may determine on a qualitative basis using your preferences $\succsim$ corresponds to the ranking derived from the utility function $u(\cdot)$

Key Concepts/5

- $u(\cdot)$ is an ordinal function, i.e., its precise values have no economic meaning; it is just used to rank bundles/perspectives
- These concepts generalize to case of choice under uncertainty
- Under certainty, the choice is among consumption baskets with known characteristics; under uncertainty, the objects of choice are vectors of state-contingent monetary payoffs
- Disentangling preferences from probabilities is a complex problem that simplifies to a maximization under assumptions
- Such a problem admits a straightforward, indeed linear, solution under special assumptions, called the expected utility theorem
- Under the EUT, there exists a cardinal, continuous, time-invariant, real-valued Von Neumann-Morgenstern (VNM) felicity function of money $U(\cdot)$, such that for any two lotteries/gambles /securities (i.e., probability distributions of monetary payoffs) x and y, $x \succsim y$ if and only if $E[U(x)] \geq E[U(y)]$

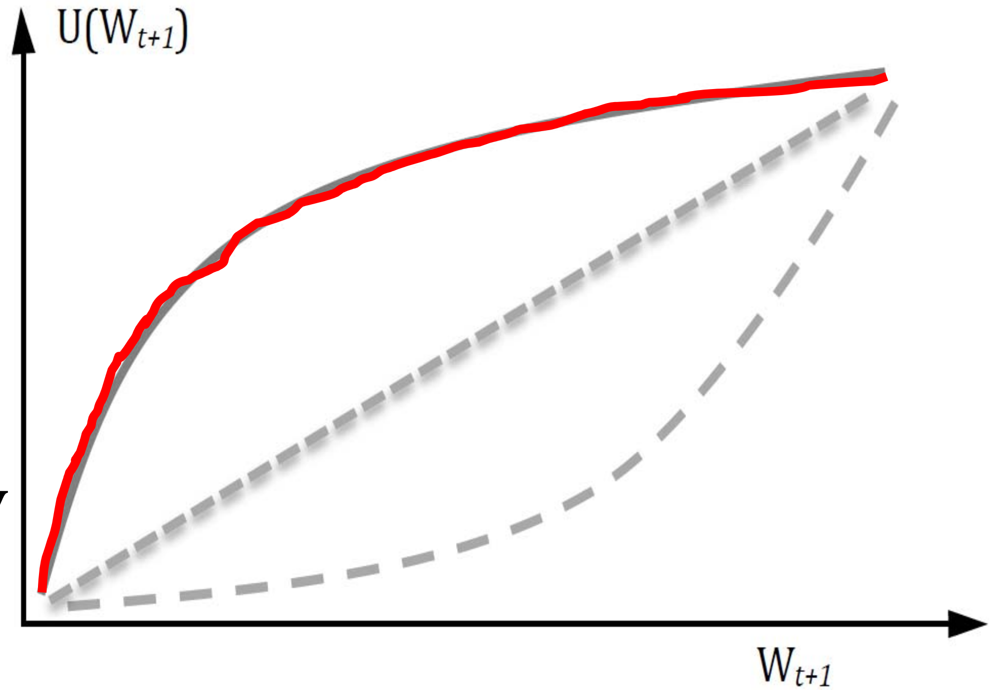
Key Concepts/6

State	*Security A*		*Security B*		*Security C*		*Security D*	
	Pay-off	Prob.	Pay-off	Prob.	Pay-off	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15	5	3/15
ii	18	5/15	18	5/15	16	5/15	14	4/15
iii	14	4/15	10	4/15	12	4/15	14	4/15
iv	10	2/15	5	2/15	12	2/15	18	2/15
v	6	1/15	5	1/15	8	1/15	18	1/15
$E[R_i]$	15.47		13.27		14.27		13.00	
$Stdev[R_i]$	4.10		5.33		2.91		4.29	
$E[lnR_i]$	2.693		2.477		2.635		2.483	

- EUT implies an enormous simplification: instead of combining probabilities and preferences over possible state-contingent payoffs in complicated ways, the probabilities are used to take the expectation of an index of preferences applied to payoffs
- Although VNM utility is cardinal, its unit of measure is unclear
- EU-based criteria are complete, they always rank all assets

Key Concepts/7

- There are precise links between the shapes and mathematical properties of (VNM) utility functions (of wealth) U(·) and the preferences/behavior of investors
- We shall always assume non-satiated individuals, $U'(W) > 0$, i.e., investors prefer more wealth (consumption power) to less
- Given non-satiation, risk-averse investors would always reject a fair bet
- When U(·) is differentiable, risk aversion implies concavity
- Equivalently, the marginal utility of the investor declines as her wealth increases



- When U(·) is convex, or marginal utility of wealth increases with wealth, then the investor is a risk-lover

Key Concepts/8

- One issue that is of interest both conceptually and practically is how to measure risk aversion and compare it across investors
- Because we know that linear affine, monotone increasing transforms of U(·) represent identical preferences, we cannot simply use the (sign and) magnitude of the second derivative of U(·) to measure and compare risk aversion
- We need a measure of risk aversion that is invariant to linear transformations
- Two widely used measures of this sort have been proposed by Pratt (1964) and Arrow (1971), the coefficients of absolute and relative risk aversion, respectively

$$ARA(W) \equiv -\frac{U''(W)}{U'(W)} \qquad RRA(W) \equiv -\frac{U''(W)}{U'(W)}W = ARA(W) \cdot W$$

- These are functions of investors' wealth, i.e., besides being positive for risk-averse investors, they may change with W

Key Concepts/9

- Because they are unique up to monotone increasing, linear affine transforms, ARA and RRA can be used to rank individuals
- Both measures are local indices of risk aversion
- The four most common VNM felicity functions are

Negative exponential, CARA $U(W) = 1 - e^{-\theta W}$ with $\theta > 0$

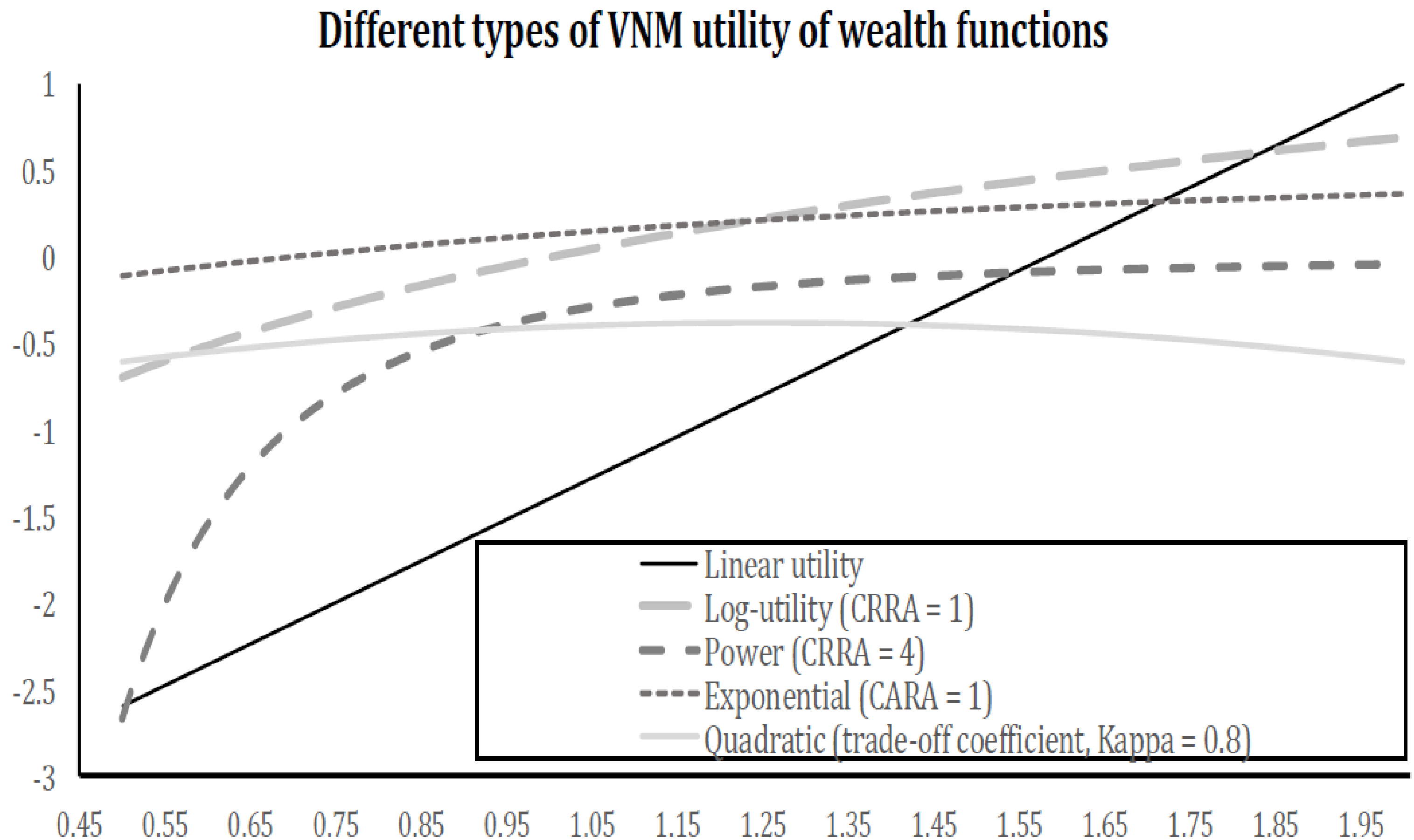
Power, CRRA $U(W) = \begin{cases} \frac{W^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \ln W & \gamma = 1 \end{cases}$

Quadratic, IARA $U(W) = W - \frac{1}{2}\kappa W^2$ with $\kappa > 0$

Linear, risk-neutral $U(W) = a + bW$ with $b > 0$

- Quadratic utility poses a few problems: e.g., the investor is not nonsatiated for all wealth levels; she is satiated below the bliss
- These functions are called linear risk tolerance (LRT) utility functions (alternatively, HARA, hyperbolic absolute risk aversion, because their ARA(W) defines a hyperbola)

Key Concepts/10



- All functions, apart from the linear, risk-neutral function, are concave
- No special meaning (or lack thereof) ought to be attached to the fact that all utility functions are negative for some wealth levels (in fact, a few are always negative for all wealth levels)

SDA Bocconi

EMFI

EXECUTIVE MASTER IN FINANCIAL INVESTMENTS

INVEST IN YOUR VALUE