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Linear Algebra, Vectors and Matrices

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Systems of linear equations (1/2)

- In general, an equation is said to be linear if it has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

- The coefficients (letters) a_1, a_2, ..., b are fixed numbers and therefore they are called **parameters**

- x_1, x_2, ..., x_n stand for variables

- A system of linear equations consists of a set of linear equations that must be solved simultaneously

- For instance, the one below is a system of two linear equations

$$6x_1 + x_2 = 3$$
$$2x_1 + 5x_2 = 4$$

Systems of linear equations (2/2)

- Matrix algebra provides a compact way to write system of equations (even large ones)
- For instance the previous system of equations can be written as follows

$$\begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

VECTOR OF THE CONSTANTS

MATRIX OF THE COEFFICIENTS

VECTOR OF THE VARIABLES

- It leads to a way of testing the existence of a solution by the analysis of the **determinant** of the matrix of coefficients
- It gives a method for finding the solution (if it exists)
- In compact format we can write the system as $\boldsymbol{Ax} = \boldsymbol{b}$

Matrix Operations (1/6)

- The **size** of a matrix is *n x m* where *n* is the number of rows and *m* is the number of columns; for example, a 4 *x* 3 is a matrix with 4 rows and 3 columns; a *n x n* matrix is called square matrix (same number of rows and columns)
- In order to perform algebraic operations, matrices must meet some requirements about their size
- We say that the matrices must be **conformable** for a given operation
- Multiplication by a scalar can always be performed (the size of the matrix does not matter)
- It consists in multiplying every element of the matrix for a given scalar

$$4\begin{bmatrix} -2 & 1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ 20 & -12 \end{bmatrix}$$

Matrix Operations (2/6)

- In order to be conformable for addition two matrices must have the same size

ADDITION OF TWO MATRICES

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,m} \\ \dots & a_{j,j} & \dots \\ a_{n,1} & \dots & a_{n,m} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ \dots & b_{j,j} & \dots \\ b_{n,1} & \dots & b_{n,m} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} + b_{1,1} & \dots & a_{1,m} + b_{1,m} \\ \dots & a_{j,j} + b_{j,j} & \dots \\ a_{n,1} + b_{n,1} & \dots & a_{n,m} + b_{n,m} \end{bmatrix}$$

For instance,

$$\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 3 & 0 \end{bmatrix}$$

Matrix Operations (3/6)

- In order to be conformable for subtraction, two matrices must have the same size

SUBTRACTION OF TWO MATRICES

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,m} \\ \dots & a_{j,j} & \dots \\ a_{n,1} & \dots & a_{n,m} \end{bmatrix} - \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ \dots & b_{j,j} & \dots \\ b_{n,1} & \dots & b_{n,m} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} - b_{1,1} & \dots & a_{1,m} - b_{1,m} \\ \dots & a_{j,j} - b_{j,j} & \dots \\ a_{n,1} - b_{n,1} & \dots & a_{n,m} - b_{n,m} \end{bmatrix}$$

For instance

$$\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 2 \end{bmatrix}$$

Matrix Operations (4/6)

- In order to be conformable for multiplication, if the size of the first matrix is *n x m*, the size of the second matrix must be *m x h;* the result of the multiplication has size *n x h*

MULTIPLICATION OF MATRICES

$$\boxed{\begin{bmatrix} a_{1,1} & \dots & a_{1,m} \\ \dots & a_{j,j} & \dots \\ a_{n,1} & \dots & a_{n,m} \end{bmatrix}}_{\text{LEAD MATRIX}} \underbrace{\begin{bmatrix} b_{1,1} & \dots & b_{1,h} \\ \dots & b_{j,j} & \dots \\ b_{m,1} & \dots & b_{m,h} \end{bmatrix}}_{\text{LAG MATRIX}} = \begin{bmatrix} c_{1,1} & & c_{1,h} \\ & c_{j,j} & \\ c_{n,1} & & c_{n,h} \end{bmatrix}$$

- $c_{1,1}$ is the result of the multiplication of the first row of the lead matrix "with" the first column of the lag matrix
- …
- $c_{n,h}$ is the result of the multiplication of the *n* row of the lead matrix "with" the *h-th* column of the lag matrix

Matrix Operations (5/6)

- As an example of multiplication of matrices try

$$\begin{bmatrix} 3 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Important: unlike multiplication of scalars the multiplication of matrices is **not commutative**!
- Note that, for instance, if A is 2 x 3 and B is a 3 x 4, not only AB and BA are not the same product, but BA is non conformable
- Unlike scalars, matrices cannot be divided one by the other (although it is possible to divide every element of a matrix by a scalar)
- In order to perform division we need to introduce the concept of inverse of a matrix (but before we need other definitions)

Matrix Operations (6/6)

- The **transpose** of a matrix **A** denoted by **A'** or $\mathbf{A}^T$ is obtained by interchanging the rows and the columns of the matrix, e.g.,

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & -4 \\ -2 & 9 & 2 \end{bmatrix} \text{ and } \mathbf{A}' = \begin{bmatrix} 6 & -2 \\ 0 & 9 \\ -4 & 2 \end{bmatrix}$$

- Note that if **A** is *n x m* then **A'** is *m x n*
- If **A=A'** then the matrix is said to be **symmetric**

Properties of Transposes

The following properties characterize transposes:

(4.9) $(A')' = A$

(4.10) $(A + B)' = A' + B'$

(4.11) $(AB)' = B'A'$

The use of matrices in Excel

- A lot of operations with vectors and matrices can be performed easily in Excel
- When dealing with matrices in Excel you need to remember two things:
  - you always need to know the size of the results and to highlight the entire area where you want to put it, this is why knowing theory of operators is important
  - you need to press **CTRL + SHIFT + ENTER**
- The only exception to the two requirements above concerns the function **SUMPRODUCT [MATR.SOMMA.PRODOTTO in ITA]**
- This function allows to multiply two vectors

Operations with matrices in Excel

- Suppose that **A** and **B** are two 2-by-2 matrix and that you want to perform their multiplication
  - **MMULT (C5:D6,C9:D10)**

	A	B	C	D	E	F	G	H	I
1									
2									
3									
4									
5		**A=**	5	8		**AB=**	181	57	
6			7	3			114	47	
7									
8									
9		**B=**	9	5					
10			17	4					
11									

  - You need to select the result area G5:H6 (matrix is 2 x 2 so the result will be 2 x 2 too) and press CTRL + SHIFT + ENTER
  - In case you want to **transpose** the matrix **A** you need to use **TRANSPOSE (C5:D6)** (again, you need to select the result area and press CTRL + SHIFT + ENTER)

Important definitions (1/5)

- An **identity matrix** (typically denoted by **I**) is a square matrix with 1 in is principal diagonal and 0 everywhere else, e.g.,

$$\mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PRINCIPAL DIAGONAL

- The identity matrix is the matrix counterparty of the number 1, because $\mathbf{AI}_{\boldsymbol{m}}$ **=A**, and $\mathbf{I}_{\boldsymbol{n}}$ **A = A** where **A** is a generic *n* x *m* matrix

- A **null matrix** is the counterparty of the number zero; it is a matrix whose all elements are zeros, e.g.,

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Important definitions (2/5)

- A **diagonal** matrix is a square matrix in which all non-diagonal entries are zero, e.g.,

$$\mathbf{D} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- An **upper** (**lower**) **triangular** matrix is a square matrix in which all the entries below (above) the diagonal are zeros, e.g.,

$$\mathbf{U} = \begin{bmatrix} -3 & -1 & 9 \\ 0 & 5 & -2 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} -3 & 0 & 0 \\ -1 & 5 & 0 \\ 5 & -3 & 0 \end{bmatrix}$$

UPPER LOWER

Important definitions (3/5)

- **Euclidean norm :** the Euclidean norm of a vector **x** is denoted by ||**x**|| is defined as follows: ||**x**|| = $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

- For instance, given the vector

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix}$$

the norm is $\left|\left|\mathbf{x}\right|\right| = \sqrt{-2^2 + 1^2 + 4^2 + (-3)^2} = \sqrt{30} = 5.5$

- Although I did not introduce the geometric interpretation of vectors, it is useful to know that the Euclidean norm is the "length" of a vector

Important definitions (4/5)

- In a n-dimensional space ($\mathbb{R}^n$) there exist a maximum of n linearly independent vectors; for instance, in a 3-dimensional space there exist at maximum of three linearly independent vectors
- For example, in the case of three-dimensional space, any other vector can be rewritten as a linear combination of

$$\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The vectors that contain one element equal to 1 and the other elements equal to zero are called **unit vectors**
- The **rank** of a matrix tells us the maximum number of linearly independent rows or columns of the matrix (if the matrix is *n x m* the maximum rank is *min(n, m)*

Important definitions (5/5)

- The **inverse** of a matrix, denoted as $\mathbf{A}^{-1}$ exists only if the matrix is a **square matrix**
- The inverse of matrix satisfies the conditions $\mathbf{AA}^{-1}\mathbf{=I}$ and $\mathbf{A}^{-1}\mathbf{A=I}$
- Not every square matrix has an inverse (being a square matrix is a necessary but <u>not</u> sufficient condition)
- When a matrix does have an inverse it is said to be **non-singular** while if it does not have an inverse it is said to be **singular**
- A *n x n* matrix is nonsingular if it has *n* linearly independent rows (columns) (we can also say that the matrix has **full rank**)

Check singularity of a matrix (with Excel)

- In order to check non-singularity of a square matrix (or alternatively, that the matrix has a full rank) one has to use the **determinant** of a matrix
- If the determinant of the square matrix is different from zero then the matrix is non-singular (thus invertible)
- Computing the determinant (with exception of 2x2 matrices) is typically tedious and computationally intense
- In case of a 2 x 2 matrix, the formula for determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \quad [= \text{a scalar}]$$

- Excel can help us with more tedious computations
  - Check whether $\begin{bmatrix} -1 & -2 & 5 \\ 3 & 6 & 4 \\ 6 & 12 & -1 \end{bmatrix}$ is singular using the excel function **MDETERM**

The inverse matrix in Excel

- We do not want to go into the technicalities of matrix inversion (the interested Reader may read Chang, Chapter 5)
- However, again Excel can help us with the computation
- The relevant function is **MINVERSE**
  - **MINVERSE (C5:E7)**

	A	B	C	D	E	F	G	H	I	J
1										
2										
3										
4										
5			8	5	11			-0,00241	0,231325	-0,040964
6		**A=**	6	2	1		**A^(-1)=**	-0,050602	-0,142169	0,139759
7			9	11	5			0,115663	-0,103614	-0,033735
8										

  - You need to select the result area H5:J7 (matrix is 3 x 3 so the result will be 3 x 3 too) and press CTRL + SHIFT + ENTER

Solution of a linear system

- Consider again the system $\mathbf{Ax} = \mathbf{d}$
- We now know that in order to perform division we have to employ the concept of inverse
- The solution to the system is then simply $\mathrm{x} = \mathrm{A}^{-1}\mathrm{d}$
- Therefore, we are left with the possibilities in the scheme below

Table Solution outcomes for a linear-equation system $Ax = d$

Determinant $\lvert A \rvert$ \ Vector d		$d \neq 0$ (nonhomogeneous system)	$d = 0$ (homogeneous system)
$\lvert A \rvert \neq 0$ (matrix A nonsingular)		There exists a unique, nontrivial solution $\bar{x} \neq 0$	There exists a unique, trivial solution $\bar{x} = 0$
$\lvert A \rvert = 0$ (matrix A singular)	Equations dependent	There exist an infinite number of solutions (not including the trivial one)	There exist an infinite number of solutions (including the trivial one)
	Equations inconsistent	No solution exists	[Not applicable]