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Optimal Portfolio Selection under EUT: A Few Analytical Results

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Outline and objectives

- The mathematical representation of the portfolio problem
 - Guidolin-Pedio, chapter 4, sec. 1
- Characterizing positive demands for risky assets
 - Guidolin-Pedio, chapter 4, sec. 1.1
- The percentage invested in risky assets and the behavior of ARA as a function of wealth
 - Guidolin-Pedio, chapter 4, sec. 1.2
- The elasticity of the percentage invested in risky assets and the behavior of RRA as a function of wealth
 - Guidolin-Pedio, chapter 4, sec. 1.2
- Cass and Stiglitz's theorem
 - Guidolin-Pedio, chapter 4, sec. 1.3

Key Ideas

- Even though the EU framework is far from simple, for simple **canonical problems**, it is possible to derive stark results
- Canonical problem = selection of cash vs. risky ptf. shares
- In general, a portfolio choice problem is a programming task subject to linear and non linear constraints
- In the simplest case, the problem is characterized by N necessary FOCs, whose sufficiency should also be investigated
- Computers, programming skills, and numerical optimization knowledge are often useful
- An individual who is risk averse and who strictly prefers more to less will undertake risky investments if and only if the rate return on at least one risky asset exceeds the risk-free interest
- Sensible that positive weights will be assigned to the positive risk premium assets, but also additional assets may be demanded if they play a hedging role

The Portfolio Problem

- Consider the problem of a risk averse individual who has a strictly increasing and differentiable utility function
- If the individual invests a_j euros in the j th risky asset and the remainder in the risk free asset (which yields the sure rate R^f), her uncertain end of period wealth, W , starting from W_0 is:

$$W = \sum_{j=1}^N a_j(1 + R_j) + \left(W_0 - \sum_{j=1}^N a_j \right) (1 + R^f) = W_0(1 + R^f) + \sum_{j=1}^N a_j(R_j - R^f)$$

- a_j is not a portfolio weights, but actual amount invested in an asset
- Because of EUT and a given VNM utility function, an investor solves:

$$\max_{\{a_j\}_{j=1}^N} E \left[U \left(W_0(1 + R^f) + \sum_{j=1}^N a_j(R_j - R^f) \right) \right]$$

- This is a static, one-period problem: it does not address the issues of
 - Revising decisions with the passage of time and taking the possibility of such revisions into account already at the initial time 0
 - The investor wishes to consume wealth (or add by saving non-asset income)

The Portfolio Problem

- Assume that there exists a solution to the problem
- Because $U(\cdot)$ is concave and the investor risk-averse, the necessary FOCs are also sufficient:

$$E \left[U' \left(W_0(1 + R^f) + \sum_{j=1}^N a_j(R_j - R^f) \right) (R_j - R^f) \right] = 0 \quad j = 1, 2, \dots, N$$

- Because $U(\cdot)$ is strictly increasing, the FOCs restrict the expectation of the product between a random variable (here MU) that is always positive and the random variable $(R_j - R^f)$
- This product can be zero if and only if $(R_j - R^f) < 0$ with some prob.: an interior optimum to the problem may exist only if any of the risky assets is risky, their distributions imply positive probability of losses
- The FOCs are a system of N equations in N unknowns that need to be solved to characterize the optimal portfolio
- Their solution requires knowledge of the form of the function $U'(\cdot)$ and must be dealt with on a case-by-case basis
- Resort to the computational power provided by computers, through numerical simulations, **see your textbook for worked-out examples**

Demand for Risky Assets

At least one risky asset must carry a positive risk premium for risky investments to be rational

- An individual who is risk averse and who strictly prefers more to less will undertake risky investments if and only if the rate return **on at least one risky asset** exceeds the risk-free interest
 - For some j btw. 1 and N , the risk premium is positive
 - Note that for the individual to invest nothing or even short sell the risky assets as an optimal choice, it is necessary that FOCs evaluated at no risky investments ($a_1 = a_2 = \dots = a_N = 0$) be non-positive, i.e. :
$$E[U'(W_0(1 + R^f))(R_j - R^f)] = U'(W_0(1 + R^f)) E[R_j - R^f] \leq 0 \quad \forall j$$
 - Because by assumption $MU > 0$, this is equivalent to
$$E[(R_j - R^f)] = E[R_j] - R^f \leq 0 \quad \text{for } j = 1, \dots, N$$
- Nowhere is said that the positive investment in the risky asset will be limited to the one with a positive risk premium
- Even though it is sensible that positive weights will be assigned to the positive risk premium assets, also additional assets may be demanded by a rational investor because they play a hedging role

Comparative Statics in the Canonical Problem

- We know that $E \left[U' \left(W_0(1 + R^f) + \hat{a}(R_j - R^f) \right) (R_j - R^f) \right] = 0$ for the case of a single risky asset
- Implicit differentiation of this expression makes it possible to prove that **when the risk** on the risky mutual fund **is small**, then

$$\hat{a} \cong \frac{E[R - R^f]}{\text{Var}[R] \text{ARA}(W_0)}$$

- An individual who is risk averse and who strictly prefers more to less will demand a growing (decreasing/constant) amount of the unique risky asset as her wealth increases, if and only if her ARA declines (grows/is constant) as function of initial wealth:

$$\frac{d\hat{a}}{dW_0} \lesseqgtr 0 \Leftrightarrow \frac{d\text{ARA}(W_0)}{dW_0} \gtrless 0$$

- There is also a relationship involving the risk premium for any given lottery H: $\frac{d\Pi(W, H)}{dW_0} \lesseqgtr 0 \Leftrightarrow \frac{d\text{ARA}(W_0)}{dW_0} \gtrless 0$

i.e., the risk premium grows/declines/is constant with wealth when the absolute risk aversion coefficient grows/declines/is constant

Comparative Statics in the Canonical Problem

- When $dARA(W_0)/dW_0 < 0$, we write about **decreasing absolute risk aversion** (DARA); when $dARA(W_0)/dW_0 = 0$, we write of **constant absolute risk aversion** (CARA); finally, when $dARA(W_0)/dW_0 > 0$, we have the case of **increasing absolute risk aversion** (IARA)
- Under negative exponential utility, $dARA(W_0)/dW_0 = 0$ and this is equivalent to $d\hat{a}/dW_0 = 0$, when an investor's wealth increases, the weight invested in the risky asset declines
 - Correspondingly, the weight invested in cash will increase
 - Riskless borrowing and lending absorb all changes in initial wealth
 - These facts cast doubts on the plausibility of CARA case
 - IARA utility functions are usually deemed rather implausible too, because they imply that as an individual gets wealthier, she will sell risky assets to hoard cash in a more-than-proportional fashion
- Only DARA utility functions enjoy adequate plausibility, e.g., power utility such that $ARA(W) = RRA(W)/W = \gamma/W$
- Arrow-Pratt's measure $RRA(W_0)$ may also reveal important information when the investor's wealth undergoes a change

Comparative Statics in the Canonical Problem

- A non-satiated individual who is risk averse will display an elasticity of the amount demanded of the risky asset vs. initial wealth that exceeds unity (is one/less than one) if and only if her relative risk aversion declines (is constant/grows) with initial wealth:

$$\hat{\eta}(W_0) \equiv \frac{d\hat{a}(W_0)}{dW_0} \frac{W_0}{\hat{a}(W_0)} \gtrless 1 \Leftrightarrow \frac{dRRA(W_0)}{dW_0} \gtrless 0$$

- The **elasticity of optimal investment the risky asset measures the % change in optimal demand of asset per unit % change in wealth**
 - Under power utility, the investor is characterized by CRRA $\Rightarrow dRRA(W_0)/dW_0 = 0$ and $\eta = 1$, i.e., when wealth grows, a power utility investor will keep her percentage holdings of the risky asset constant, and hence increase them at the same proportion as wealth grows
- This result lies at the heart of one common practice in the asset management industry: offer **identical advice to investors with very different wealth**
- Or, how come the same equity strategies inform the trades of mutual funds irrespectively of the number of shares of the funds that an investor may purchase?

Cass and Stiglitz's Theorem

- This result is interesting but entirely depends on the canonical nature of the problem, i.e., there is only one risky asset
 - Under $N > 2$, we cannot say that the wealth elasticity of the demands for risky assets are > 1 when an individual exhibits decreasing RRA
 - An investor may change his ptf. composition such that investment in one risky asset increases while investment in another asset decreases
 - Such shifts in demands may also be motivated by hedging purposes
- Only if an individual always chooses to hold the same risky portfolio and simply changes the mix between that portfolio and the riskless asset for differing levels of initial wealth, then the comparative statics for the two-asset case will be valid in a multi-asset world
- This property of optimal choices in a multi-asset world is commonly called **two fund monetary separation**
- The ability to extend a number of earlier results to the real, multi-asset world is highly attractive
- Cass and Stiglitz (1970) have proven necessary and sufficient condition on utility functions for two fund monetary separation

Cass and Stiglitz's Theorem

- An individual who is risk averse and who strictly prefers more to less will exhibit two-fund separation if and only if either

$$U'(W) = (A + BW)^C \quad \text{or} \quad U'(W) = A \exp(BW)$$

where in the former case, $B > 0$, $C < 0$, and $W \geq \max[0, -(A/B)]$, or $A > 0$, $B < 0$, $C > 0$ and $0 \leq W < -(A/B)$; $A > 0$, $B < 0$ and $W \geq 0$ in the latter

- It implies that for a number of VNM felicity functions, an investor always holds the same risky ptf. independently of her initial wealth == the composition of such a ptf. is constant

- The fact that the $U(\bullet)$ functions are of standard types comes from

$$\begin{aligned} U(W) &= \text{const} + \int U'(W) dW \\ &= \text{const} + \int (A + BW)^C dW = \text{const} + \frac{1}{B} \frac{(A + BW)^{C+1}}{C + 1} \end{aligned}$$

- This is implicitly at the heart of big portions of the modern financial architecture in which standardized investment products are offered
 - The composition of the risky portfolio is homogeneous across different investors and the latter differ in a cross-sectional dimension simply because they invest in different proportions in such a risky mutual fund

One Example

- John is characterized by negative exponential, CARA utility function
- John enjoys two-fund separation: if his wealth changed, he would keep investing the same proportions in the available risky assets
- However, because he has CARA preferences, when his wealth increases (decreases), John will not change the total amount he invests in a constant-proportion risky mutual fund

Initial wealth	Stock A		Stock B		Stock C		Cash		Tot. risky
	Total	% risky	Total	% risky	Total	% risky	Total	% total	
100	30	60	10	20	10	20	50	50	50
50	30	60	10	20	10	20	0	0	50
150	30	60	10	20	10	20	100	66.7	50

- Mary has instead power utility function

Initial wealth	Stock A		Stock B		Stock C		Cash		Tot. risky
	Total	% risky	Total	% risky	Total	% risky	Total	% total	
100	30	60	10	20	10	20	50	50	50
50	15	60	5	20	5	20	25	50	25
150	45	60	15	20	15	20	75	50	75