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# Lecture 4: Forecasting with option implied information

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Advanced Financial Econometrics III

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# Overview

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- A two-step approach
- Black-Scholes single-factor model
- Heston's two-factor square root stochastic volatility model
- Model-free implied volatility
- Option-implied correlations
- Model-free option-implied skewness and kurtosis
- Model-implied, parametric forecasts of skewness and kurtosis
- Model-free forecasts of densities
- From risk-neutral to physical forecasts

# The key point

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- Derivative prices contain information useful to forecast any twice differentiable function of the future underlying price
  - Focus on European-style options, especially equity index
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- Derivative prices contain useful **information on the conditional density of future underlying asset returns**
  - A derivative contract is an asset whose future payoff depends on the uncertain realization of the price of an underlying asset
    - Futures and forward contracts, swaps (e.g., CDS and variance swaps), collateralized debt obligations (CDOs) and basket options, European style call and put options, American style and exotic options, etc.
    - Several of these classes of derivatives exist for many different types of underlying assets, such as commodities, equities, and equity indexes
    - However, some derivative contracts such as forwards and futures are linear in the return on the underlying security, and therefore their payoffs are too simple to contain useful reliable information
    - Other securities, such as exotic options, have path-dependent payoffs, which may make information extraction cumbersome

# A two-step approach

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- Forecasting with option-implied information proceeds in two steps:
  - ① Derivative prices are used to extract a relevant aspect of the option-implied distribution of the underlying asset
    - E.g., ATM implied volatility (IV), closest to 30-day to maturity
  - ② An econometric model is used to relate this option-implied information to the forecasting object of interest
    - E.g., realized 30-day variance is regressed on IV inferred from observed option prices 30 days before
- In this lecture, brief review of the methods used in the first step
- The two most commonly used models for option valuation are the Black and Scholes (1973, JPE) and Heston (1993, RFS) models
- **Black and Scholes**  $\Rightarrow$  constant volatility geometric Brownian motion
$$dS = rSdt + \sigma Sdz$$
where  $r$  is a constant risk-free,  $\sigma$  is volatility, and  $dz$  a normal shock
- The future **log price is normally distributed** and option price for a European call with maturity  $T$  and strike price  $X$  is

# Black-Scholes, single-factor model

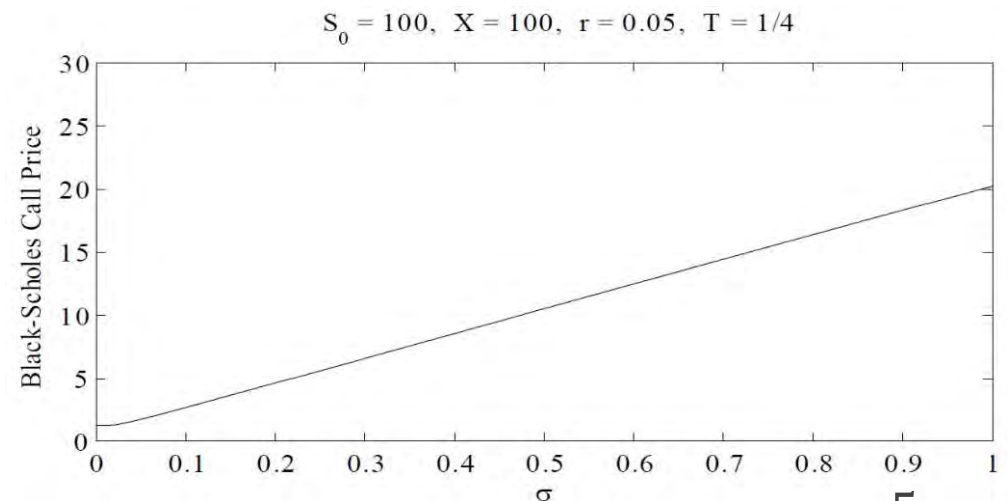
$$C^{BS}(T, X, S_0, r; \sigma) = S_0 N(d) - X \exp(-rT) N(d - \sigma\sqrt{T})$$
$$d = \frac{\ln(S_0/X) + T(r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T}} \quad P_0 + S_0 = C_0 + X \exp(-rT)$$

(put-call parity)

- The formula has just one unobserved parameter, namely volatility that can be backed out for any given option with market price as:

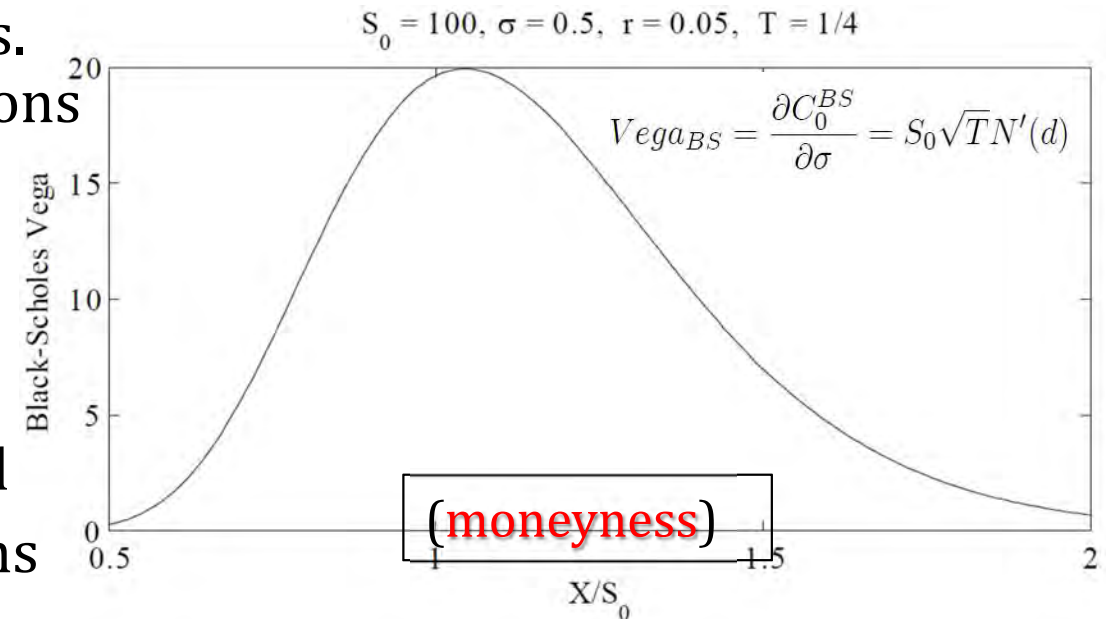
$$C_0^{Mkt} = C^{BS}(T, X, S_0, r; BSIV)$$

- The resulting option-specific volatility is called BSIV
- Although the BS formula is clearly non-linear, for ATM options, the relationship between volatility and option price is virtually linear
  - In general the relationship btw volatility and option prices is increasing and monotone
  - Solving for BSIV is quick even if it must be done numerically
  - **Vega** captures the sensitivity of the price w.r.t. changes in  $\sigma$



# Black-Scholes, single-factor model

- For equity index options, BSIV as an adjusted  $R^2$  of 62% and summarizes all other information
- The sensitivity of the price vs.  $\sigma$  is the highest for ATM options
  - In the following table we show a predictive regression of realized volatility on BSIV
  - $RV_M$ ,  $RV_W$ ,  $RV_D$  are monthly, weekly, and daily realized vol
  - RV is from intraperiod returns and C nets jumps out



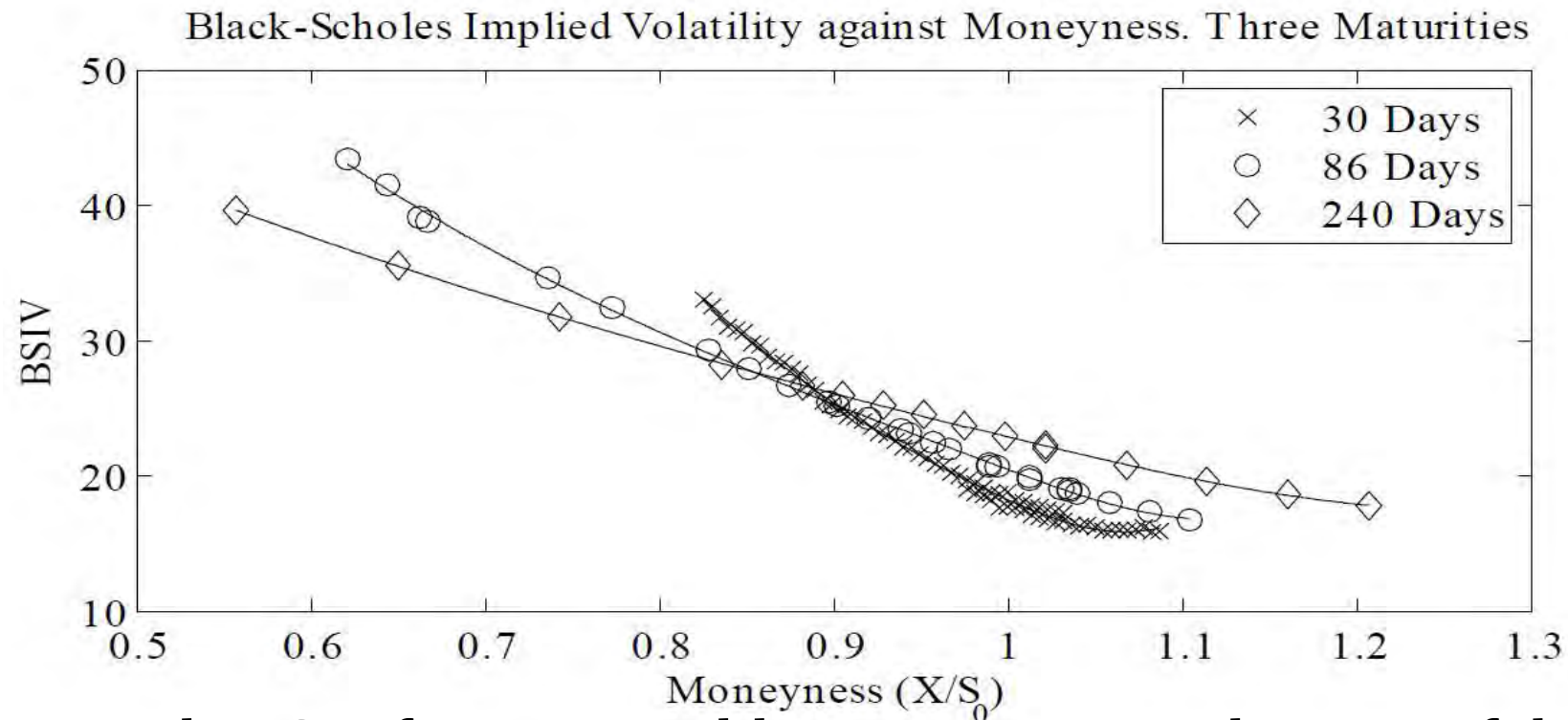
Panel B: S&P 500 data

Constant	$RV_M$	$RV_W$	$RV_D$	$C_M$	$C_W$	$C_D$	$BSIV$	Adj. $R^2$
0.0053 (0.0025)	0.6240 (0.1132)	-0.3340 (0.1039)	0.6765 (0.1007)	—	—	—	—	53.0
0.0037 (0.0023)	—	—	—	0.1568 (0.1327)	0.0407 (0.1353)	0.9646 (0.1088)	—	61.9
-0.0050 (0.0027)	—	—	—	—	—	—	1.0585 (0.0667)	62.1
-0.0052 (0.0027)	0.0378 (0.1311)	-0.1617 (0.0943)	0.3177 (0.1026)	—	—	—	0.9513 (0.1391)	64.0
-0.0051 (0.0027)	—	—	—	-0.1511 (0.1336)	0.0633 (0.1237)	0.6016 (0.1194)	0.7952 (0.1447)	68.2



# Black-Scholes, single-factor model

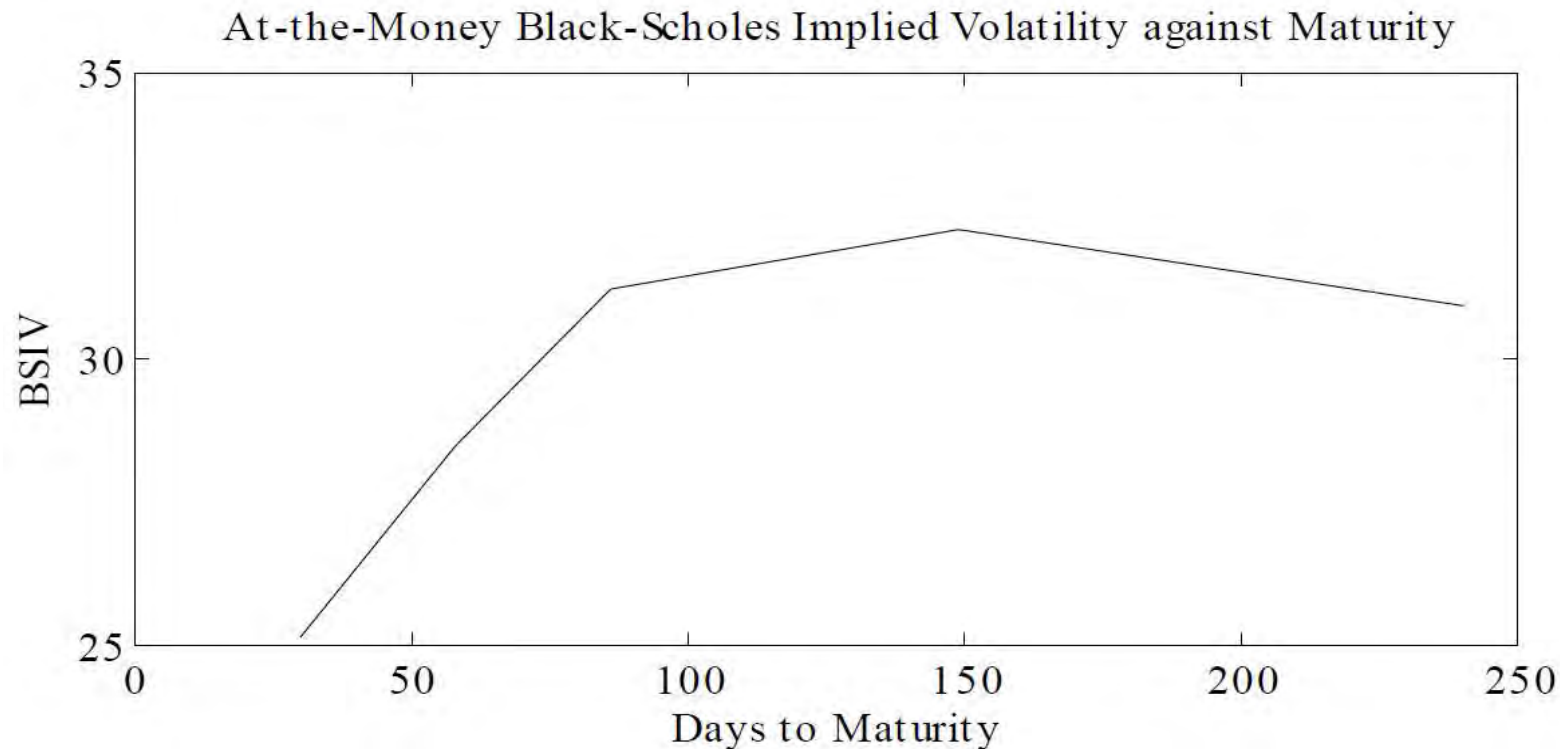
- Nonconstant patterns in BSIV vs. **moneyness**  $\Rightarrow$  misspecification



- The simple BSIV forecast is able to compete with some of the most sophisticated historical return-based forecasts
- Index-option BSIVs display a distinct downward sloping pattern commonly known as the **smirk or the skew**
- This is evidence that the BS model which relies on the normal distribution is misspecified

# Black-Scholes, single-factor model

- Nonconstant patterns in BSIV vs. **maturity**  $\Rightarrow$  misspecification



- Deep out-of-the-money put options ( $X/S_0 \ll 1$ ) are more expensive than the normal-based Black-Scholes model would suggest
- Only a distribution with a fatter left tail (that is negative skewness) would be able to generate these much higher prices for OTM puts
- BSIV for ATM ( $X/S_0 = 1$ ) tends to be larger for long-maturity than short-maturity options



# Heston, two-factor square root process

- Heston's model makes variance stochastic and square root
- For variances to change over time, we need a richer setup than the Black-Scholes models
- Most famous model that provides this result is Heston (1993, RFS), who assumes that the underlying follows a square-root process:

$$dS = rSdt + \sqrt{V}Sdz_1$$

$$dV = \kappa(\theta - V)dt + \sigma_V\sqrt{V}dz_2$$

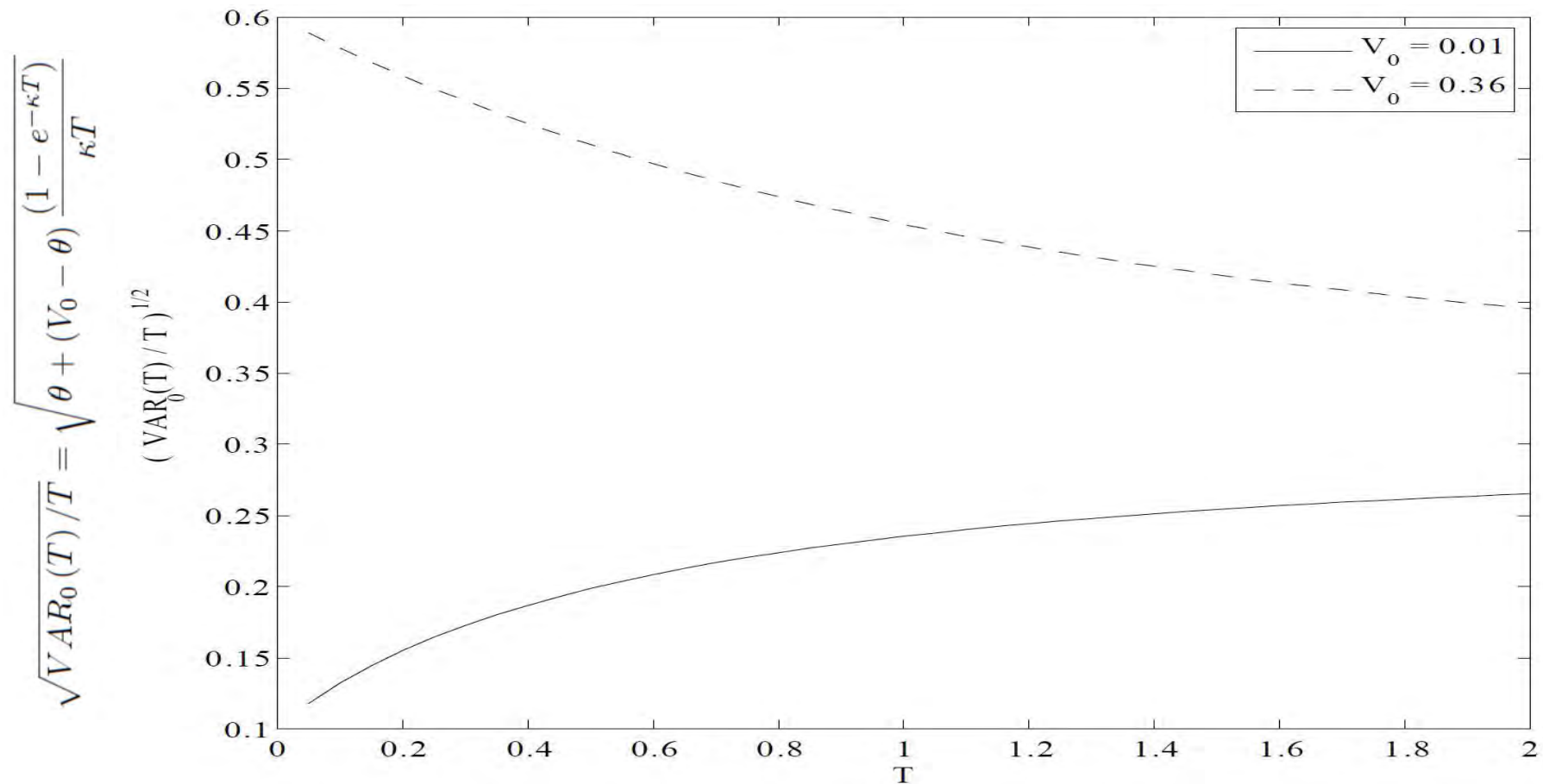
where the two innovations are correlated with parameter  $\rho$

- At time zero, the variance forecast for horizon T can be obtained as

$$VAR_0(T) \equiv E_0 \left[ \int_0^T V_t dt \right] = \theta T + (V_0 - \theta) \frac{(1 - e^{-\kappa T})}{\kappa}$$

- The mean-reversion parameter  $\kappa$  determines the extent to which the difference between current spot volatility and long run volatility,  $(V_0 - \theta)$ , affects the horizon T forecast
- Whereas the BS only has one parameter, Heston has 4 parameters

# Heston, two-factor square root process



- Bakshi, Cao, and Chen (1997, JF) re-estimate the model daily treating  $V_0$  as a fifth parameter to be estimated
- What if the model assumed to forecast volatility from option prices turns out to be misspecified?
- The answer is tragic: nothing good can be expected of the forecasts

# Model-free volatility estimation and forecasting

- Luckily a few methods to achieve **model-free volatility estimation** are possible
- When investors can trade continuously, interest rates are constant, and **the underlying futures price is a continuous semi-martingale**, Carr and Madan (1998) and Britten-Jones and Neuberger (2000, JF) show that the expected value of future realized variance is:

$$E_0 \left[ \int_0^T (dS_t/S_t)^2 dt \right] = 2 \int_0^\infty \frac{C_0^F(T, X) - \max(F_0 - X, 0)}{X^2} dX$$

- Jiang and Tian (2005, RFS) generalize this result and show that it holds even if the price process contains jumps

$$VAR_0(T) = 2 \int_0^\infty \frac{C_0(T, e^{-rT}X) - \max(S_0 - X, 0)}{X^2} dX$$

- In practice, a finite range,  $X_{\max} - X_{\min}$ , of discrete strikes are available and Jiang and Tian consider using the trapezoidal integration rule

$$VAR_0(T) \approx \sum_{i=1}^m \left\{ \frac{[C_0^F(T, X_i) - \max(F_0 - X_i, 0)]}{X_i^2} + \frac{[C_0(T, X_{i-1}) - \max(F_0 - X_{i-1}, 0)]}{X_{i-1}^2} \right\} \Delta X$$
$$\Delta X = (X_{\max} - X_{\min}) / m$$

# Empirical evidence

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- Overall, the evidence indicates that **option-implied volatility** is a biased predictor of the future volatility of the underlying asset
- Yet, most studies find that it contains useful information over traditional predictors based on historical prices
  - Option IV by itself often outperforms historical volatility
- BSIV is predictable and helps forecast volatility, but because arbitrage profits are impossible under transaction costs, predictability is consistent with EMH (see Goncalves and Guidolin, 2006, JoB)
- There is recent, strong evidence that the **variance risk premium** (VRP) can predict the equity risk premium
  - VRP is the difference between implied variance and realized variance
- Bakshi, Panayotov, and Skoulakis (2011, JFE) compute **forward variance**, the implied variance between two future dates, and find that it forecasts stocks, T-bills, and changes in real activity
- Feunou, Fontaine, Taamouti, and Tedongap (2013, RoF) find that the **term structure of IVs** can predict both equity risk and VRP

# Option-implied correlations

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- BSIV is useful in forecasting the volatility of **individual stocks**
- Implied volatility has also been used to predict **future stock returns**
- Information in options leads **analyst recommendation changes**
- The **VIX is a priced risk factor with a negative price of risk**, so that stocks with higher sensitivities to the innovation in VIX exhibit on average future lower returns
  - VIX is a weighted average of BSIVs
  - The CBOE computes VIX using OTM and ATM call and put options
  - It calculates the volatility for the two available maturities that are the nearest and second-nearest to 30 days.
- Certain derivatives contain very rich information on correlations between financial time series
- E.g., in currency markets  $S_{\$/\pounds} = S_{\$/\yen} S_{\yen/\pounds} \Rightarrow R_{\$/\pounds} = R_{\$/\yen} + R_{\yen/\pounds}$  where R denotes a continuously compounded return
- Therefore  $VAR_{\$/\pounds} = VAR_{\$/\yen} + VAR_{\yen/\pounds} + 2COV(R_{\$/\yen}, R_{\yen/\pounds})$



# Option-implied correlations

- While implied correlations for currencies are derived from the **triangular equality**, in the case of stocks only an **implied average correlation** may be estimated
- The implied correlation is:
 
$$CORR(R_{\$/\yen}, R_{\yen/\pounds}) = \frac{(VAR_{\$/\pounds} - VAR_{\$/\yen} - VAR_{\yen/\pounds})}{2VAR_{\$/\yen}^{1/2}VAR_{\yen/\pounds}^{1/2}}$$
- Provided we have option-implied variance forecasts for 3 currencies, we can use this to get an implied correlation forecast
- Option-implied exchange rate correlations for the DM/GBP pair and the DM/JPY, and USD/DM/JPY pairs predict significantly better than historical correlations between the pairs
- There is a measure of **average** option-implied correlation between the stocks in an index, I,
 
$$\rho_{ICI} = \frac{VAR_I - \sum_{j=1}^n w_j^2 VAR_j}{2 \sum_{j=1}^{n-1} \sum_{i>j}^n w_i w_j VAR_i^{1/2} VAR_j^{1/2}}$$

Weight of stock j
- Skintzi and Refenes (2005, JFM) use options on the DJIA index



# Option-implied correlations

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- Smirks (asymmetric smiles) in IVs indicate left-skewness in the density of underlying returns, while symmetric smiles point to excess kurtosis
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- Implied correlation index is biased upward, but is a better predictor of future correlation than historical correlation
  - Implied correlations may be used to estimate betas and the literature finds that option-implied betas predict realized betas well
  - However, using option-implied information in portfolio allocation does not improve the Sharpe ratio or CER of the optimal portfolio
  - We saw earlier that BS is unlikely to be correctly specified: the very option prices (IVs) contain robust evidence of asymmetries and fat tails in the predictive density of underlying asset returns
  - Can we extract **option-implied skewness and kurtosis**?
  - It is sensible to proceed with a model-free approach, called option replication approach, see Bakshi and Madan (2000, JFE)

# Model-free option-implied skewness and kurtosis

- For any twice differentiable function of the future underlying price, there is a **spanning portfolio** made of bonds, stock, and European call and put options
- Bakshi, Carr, and Madan show that **any twice continuously differentiable fnct,  $H(S_T)$ , of terminal price  $S_T$ , can be replicated (spanned) by a unique position in the risk-free, stocks and European options**

$$H(S_T) = \underbrace{[H(S_0) - H'(S_0)S_0]}_{\text{Units of risk-free bond}} + \underbrace{H'(S_0)S_T}_{\text{Units of underlying}} + \int_0^{S_0} H''(X) \max(X - S_T, 0) dX + \int_{S_0}^{\infty} H''(X) \max(S_T - X, 0) dX$$

- $H''(X)dX$  are units of OTM call and put options with strike price  $X$
- From a forecasting perspective, for any  $H(\bullet)$ , there is a portfolio of risk-free bonds, stocks, and options whose current aggregate market value provides an option-implied forecast of  $H(S_T)$

$$E_0 [e^{-rT} H(S_T)] = e^{-rT} [H(S_0) - H'(S_0)S_0] + S_0 H'(S_0) + \int_0^{S_0} H''(X) P_0(T, X) dX + \int_{S_0}^{\infty} H''(X) C_0(T, X) dX$$

# Model-free option-implied skewness and kurtosis

- Under mild assumptions, the prices of OTM puts and calls can be used to infer risk-neutral volatility, skewness, and kurtosis

- Consider now higher moments of simple returns:

$$H(S_T) = \left(\frac{S_T - S_0}{S_0}\right)^2, \quad H(S_T) = \left(\frac{S_T - S_0}{S_0}\right)^3, \quad \text{and} \quad H(S_T) = \left(\frac{S_T - S_0}{S_0}\right)^4$$

- We can use OTM European call and put prices to derive the **quadratic, cubic, and quartic contracts** as

$$M_{0,2}(T) \equiv E_0 \left[ e^{-rT} \left( \frac{S_T - S_0}{S_0} \right)^2 \right] = \frac{2}{S_0^2} \left[ \int_0^{S_0} P_0(T, X) dX + \int_{S_0}^{\infty} C_0(T, X) dX \right]$$

$$M_{0,3}(T) \equiv E_0 \left[ e^{-rT} \left( \frac{S_T - S_0}{S_0} \right)^3 \right] = \frac{6}{S_0^2} \left[ \int_0^{S_0} \left( \frac{X - S_0}{S_0} \right) P_0(T, X) dX + \int_{S_0}^{\infty} \left( \frac{X - S_0}{S_0} \right) C_0(T, X) dX \right]$$

$$M_{0,4}(T) \equiv E_0 \left[ e^{-rT} \left( \frac{S_T - S_0}{S_0} \right)^4 \right] = \frac{12}{S_0^2} \left[ \int_0^{S_0} \left( \frac{X - S_0}{S_0} \right)^2 P_0(T, X) dX + \int_{S_0}^{\infty} \left( \frac{X - S_0}{S_0} \right)^2 C_0(T, X) dX \right]$$

- High option prices imply high volatility
- High OTM put and low OTM call prices  $\Rightarrow$  negative skewness
- High call and put prices at extreme moneyness  $\Rightarrow$  high kurtosis

# Model-free option-implied skewness and kurtosis

- Now compute option-implied volatility, skewness, and kurtosis:

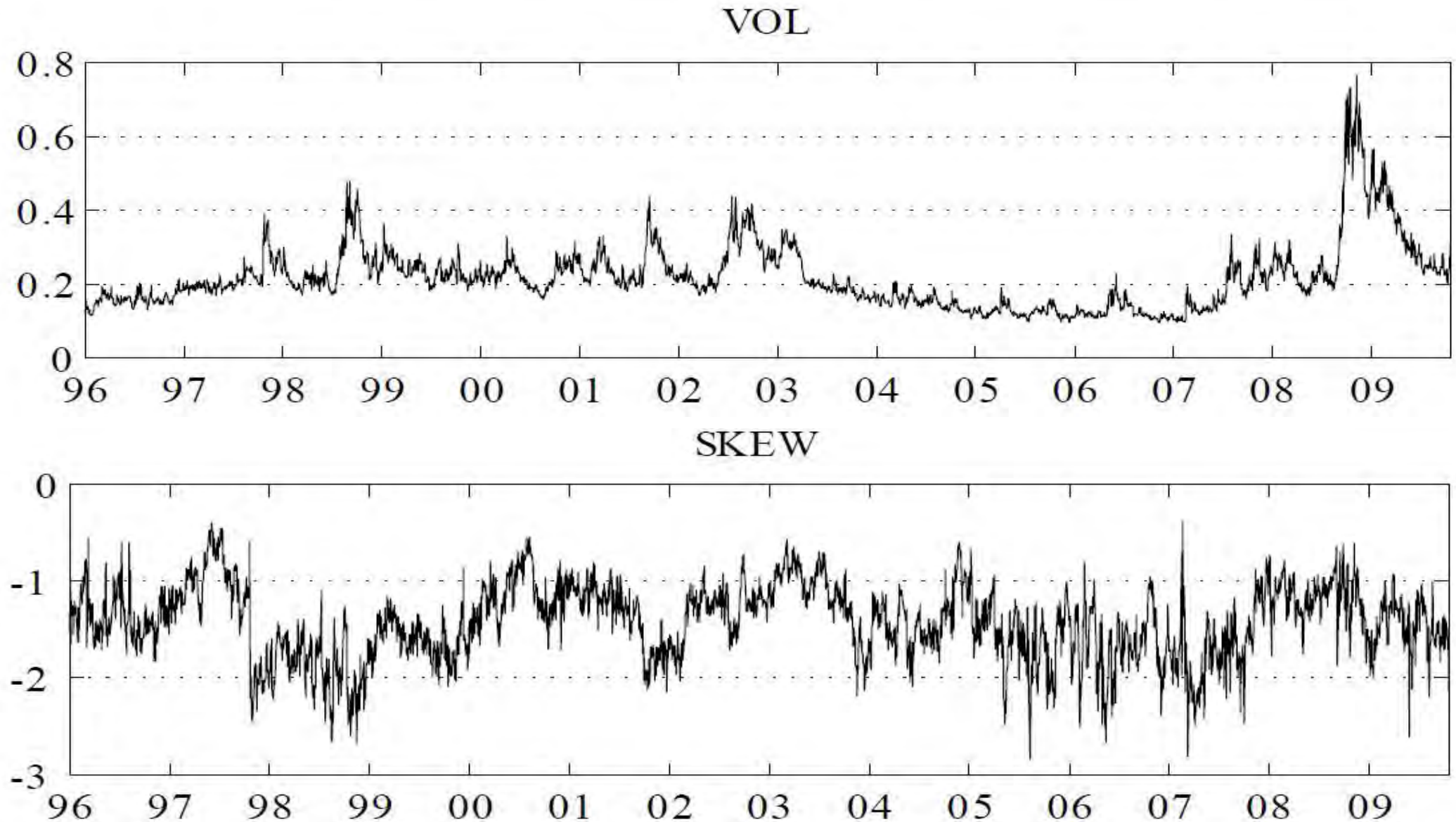
$$\begin{aligned}
 VOL_0(T) &\equiv [VAR_0(T)]^{1/2} = [e^{rT} M_{0,2} - M_{0,1}^2]^{1/2} \\
 SKEW_0(T) &= \frac{e^{rT} M_{0,3} - 3M_{0,1}e^{rT} M_{0,2} + 2M_{0,1}^3}{[e^{rT} M_{0,2} - M_{0,1}^2]^{\frac{3}{2}}} \\
 KURT_0(T) &= \frac{e^{rT} M_{0,4} - 4M_{0,1}e^{rT} M_{0,3} + 6e^{rT} M_{0,1}^2 M_{0,2} - 3M_{0,1}^4}{[e^{rT} M_{0,2} - M_{0,1}^2]^2}
 \end{aligned}$$

$M_{0,1} \equiv E_0 \left[ \left( \frac{S_T - S_0}{S_0} \right) \right] = e^{rT} - 1$

- Using S&P 500 index options over January 1996 - September 2009 we plot higher moments of log returns for the one-month horizon
  - The volatility series is very highly correlated with the VIX index, with a correlation of 0.997
- The estimate of skewness is negative for every day in the sample
- The estimate of kurtosis is always higher than 3
- Both skewness and kurtosis do not show significant or persistent alterations during the 2008-2009 Great Financial Crisis



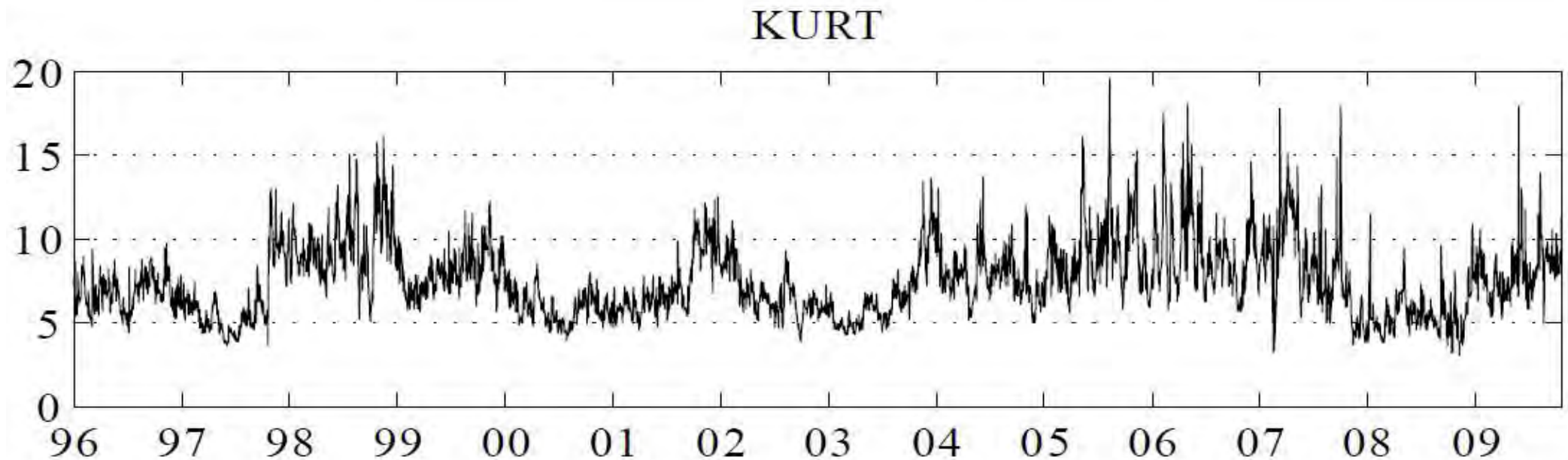
# Model-free option-implied skewness and kurtosis



- In February 2011, the CBOE began publishing the CBOE S&P 500 Skew Index computed according to this methodology

# Model-based option-implied skewness and kurtosis

- Jarrow and Rudd have proposed an option price approximation based of **Edgeworth expansions of BS**,  $C_0^{BS}(T, X)$



- There are also parametric, model-based methods to extract skewness and kurtosis from option prices
- Some are based on **functional «expansions»/approximations of BS**
- Most famous approach is due to Jarrow and Rudd (1982, JFE) who propose a method where the density of the security price at option maturity,  $T$ , is approximated using an Edgeworth series expansion



# Model-based option-implied skewness and kurtosis

- Jarrow and Rudd's pricing formula is:

$$C_0^{JR}(T, X) \approx C_0^{BS}(T, X) - e^{-rT} \frac{(K_3 - K_3(\Psi))}{3!} \frac{d\psi(T, X)}{dX} + e^{-rT} \frac{(K_4 - K_4(\Psi))}{4!} \frac{d^2\psi(T, X)}{dX^2}$$

- $K_j$  is the  $j$ th cumulant of the actual density,  $K_j(\Psi)$  is the cumulant of the lognormal density and other quantities are reported in Appendix A
- If one approximates around a log-normal density, then

$$VAR(X) = \exp \left( 2 \left( \ln(S_0) + \left( r - \frac{1}{2} \sigma^2 \right) T \right) + \sigma^2 T \right) (\exp(\sigma^2 T) - 1)$$

$$SKEW(X) = (\exp(\sigma^2 T) + 2) \sqrt{\exp(\sigma^2 T) - 1}$$

$$KURT(X) = \exp(4\sigma^2 T) + 2 \exp(3\sigma^2 T) + 3 \exp(2\sigma^2 T) - 3$$

- The model now has 3 parameters to estimate, VAR, SKEW and KURT
- As an alternative to the Edgeworth expansion, Corrado and Su (1996, JFR) consider a **Gram-Charlier series expansion**:

$$C_0^{CS}(T, X) = C_0^{BS}(T, X) + \frac{1}{3!} S_0 \sigma \sqrt{T} \left( (2\sigma\sqrt{T} - d) N'(d) + \sigma^2 T N(d) \right) SKEW$$

# Model-based option-implied skewness and kurtosis

- Also parametric **jump-diffusion models** may be used to forecast skewness and kurtosis from option prices

$$+ \frac{1}{4!} S_0 \sigma \sqrt{T} \left( \left( d^2 - 1 - 3\sigma \sqrt{T} (d - \sigma \sqrt{T}) \right) N'(d) + \sigma^3 T^{3/2} N(d) \right) (KURT - 3)$$

- Additional parametric models have become popular in the literature to infer skewness and kurtosis from option prices
- E.g., in Bates (2000, JoE), the futures price  $F$  is assumed to follow a jump-diffusion:

Correlated with  
coefficient  $\rho$

$$\begin{aligned} dF/F &= -\lambda \bar{k} dt + \sqrt{V} dz_1 + k dq, \\ dV &= \kappa (\theta - V) dt + \sigma_V \sqrt{V} dz_2 \end{aligned}$$

- $q$  is a Poisson counter with instantaneous intensity  $\lambda$ , and  $k$  is a log-normal return jump,  $\ln(1+k) \sim N[\ln(1+\bar{k}) - \delta^2/2, \delta^2]$
- Higher-order moments can now be computed as a function of the unknown parameters, to be estimated
- Options can also be used to **forecast the density of underlying asset returns**

# Model-free risk-neutral density forecasts

- The option-implied conditional density for the underlying at maturity  $T$  is the **forward second derivative of an ATM call**
- Breeden and Litzenberger (1978, JBus) and Banz and Miller (1978, JBus) show that the option-implied density can be extracted from a set of European option prices with a continuum of strike prices
  - This result is a special case of Carr and Madan's result reviewed above
- The value of a European call,  $C_0$ , is the discounted expected value of payoff on expiry date  $T$ , i.e., under the implied measure,  $f_0(S_T)$ :

$$C_0(T, X) = e^{-rT} \int_0^{\infty} \max\{S_T - X, 0\} f_0(S_T) dS_T = e^{-rT} \int_X^{\infty} (S_T - X) f_0(S_T) dS_T$$

- Take the partial derivative of  $C_0$  with respect to the strike price  $X$ :

$$\frac{\partial C_0(T, X)}{\partial X} = -e^{-rT} [1 - \tilde{F}_0(X)] \Rightarrow \tilde{F}_0(S_T) = 1 + e^{rT} \frac{\partial C_0(T, X)}{\partial X} \Big|_{X=S_T}$$

- The conditional density function (PDF) denoted by  $f_0(X)$  is obtained as:

$$f_0(S_T) = e^{rT} \frac{\partial^2 C_0(T, X)}{\partial X^2} \Big|_{X=S_T}$$

# Model-free risk-neutral density forecasts

- A discrete strike approximation of the conditional PDF in terms of calls is:

$$f_0(X_n) \approx e^{rT} \frac{C_0(T, X_{n+1}) - 2C_0(T, X_n) + C_0(T, X_{n-1}))}{(\Delta X)^2}$$

- Because of put-call parity,  $S_0 + P_0 = C_0 + Xe^{-rT}$  can use puts instead:

$$f_0(S_T) = e^{rT} \frac{\partial^2 P_0(T, X)}{\partial X^2} \Big|_{X=S_T}$$

- In practice, we can obtain an approximation to the CDF using finite differences of call or put prices observed at discrete strike prices:

$$\tilde{F}_0(X_n) \approx 1 + e^{rT} \left( \frac{C_0(T, X_{n+1}) - C_0(T, X_{n-1}))}{X_{n+1} - X_{n-1}} \right)$$

$$\tilde{F}_0(X_n) \approx e^{rT} \left( \frac{P_0(T, X_{n+1}) - P_0(T, X_{n-1}))}{X_{n+1} - X_{n-1}} \right)$$

- In terms of the log return, the CDF and PDF are

$$\tilde{F}_{0,R_T}(x) = F_0(e^{x+\ln S_0}) \quad \text{and} \quad f_{0,R_T}(x) = e^{x+\ln S_0} f_0(e^{x+\ln S_0})$$



# Parametric and approximated RN density forecasts

- The key issue in implementing this method is that typically only a limited number of options are traded, with a handful of strikes
- Tricks exist: e.g., the simple but flexible ad-hoc BS (AHBS) model constructs the density forecast off a **BS implied volatility curve**
- In a first step, estimate a second-order polynomial or other well-fitting function for implied BS volatility as a function of strike and maturity, to obtain fitted BSIV values:

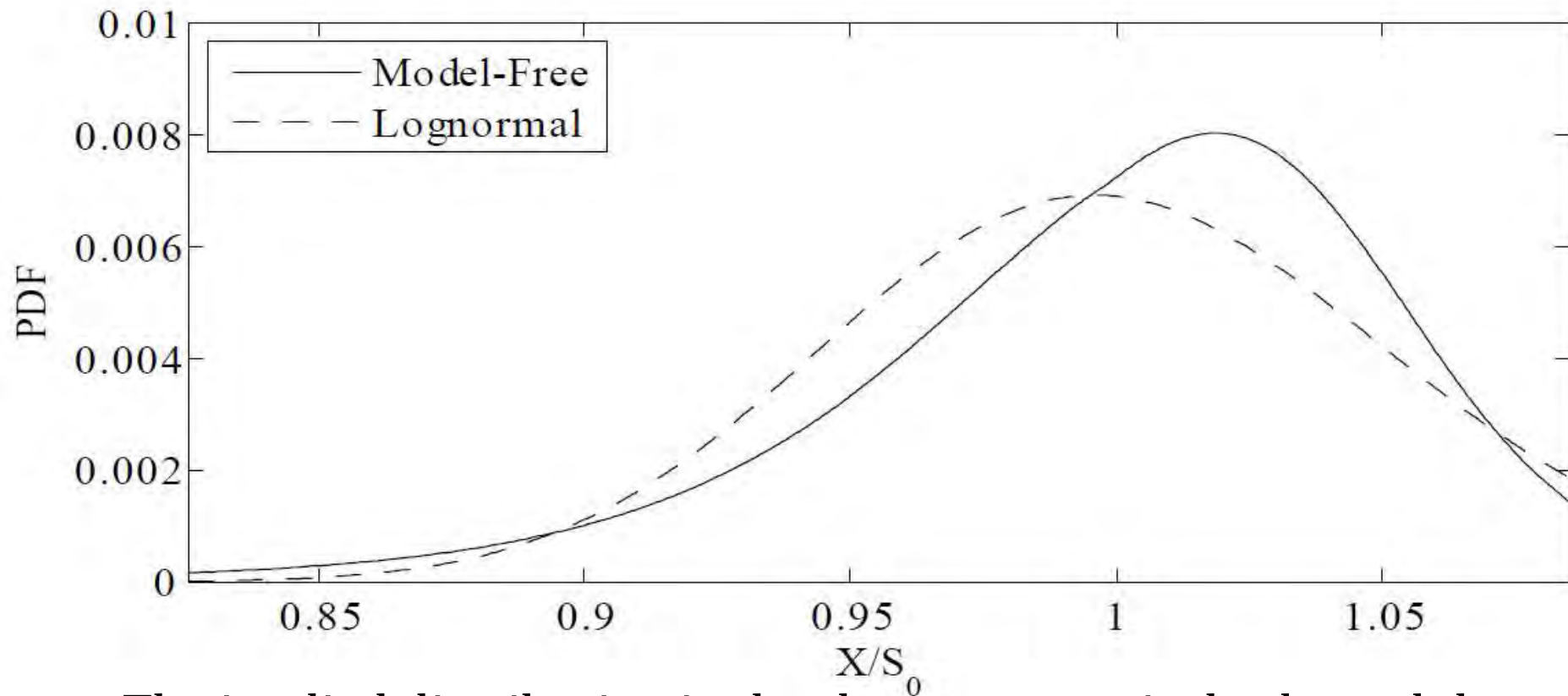
$$BSIV(X, T) = a_0 + a_1X + a_2X^2 + a_3T + a_4T^2 + a_5XT$$

- Second, using this estimated polynomial, we generate a set of fixed maturity IVs across a grid of strikes
- Call prices can then be obtained using the BS formula:

$$C_0^{AHBS}(X, T) = C_0^{BS}(T, X, S_0, r; BSIV(X, T))$$

- Option-implied density can be obtained using the second derivative
- The figure shows the CDF and PDF obtained when applying a smoothing cubic spline using BSIV data on 30-day OTM calls and puts on the S&P 500 index on October 22, 2009 vs. the lognormal

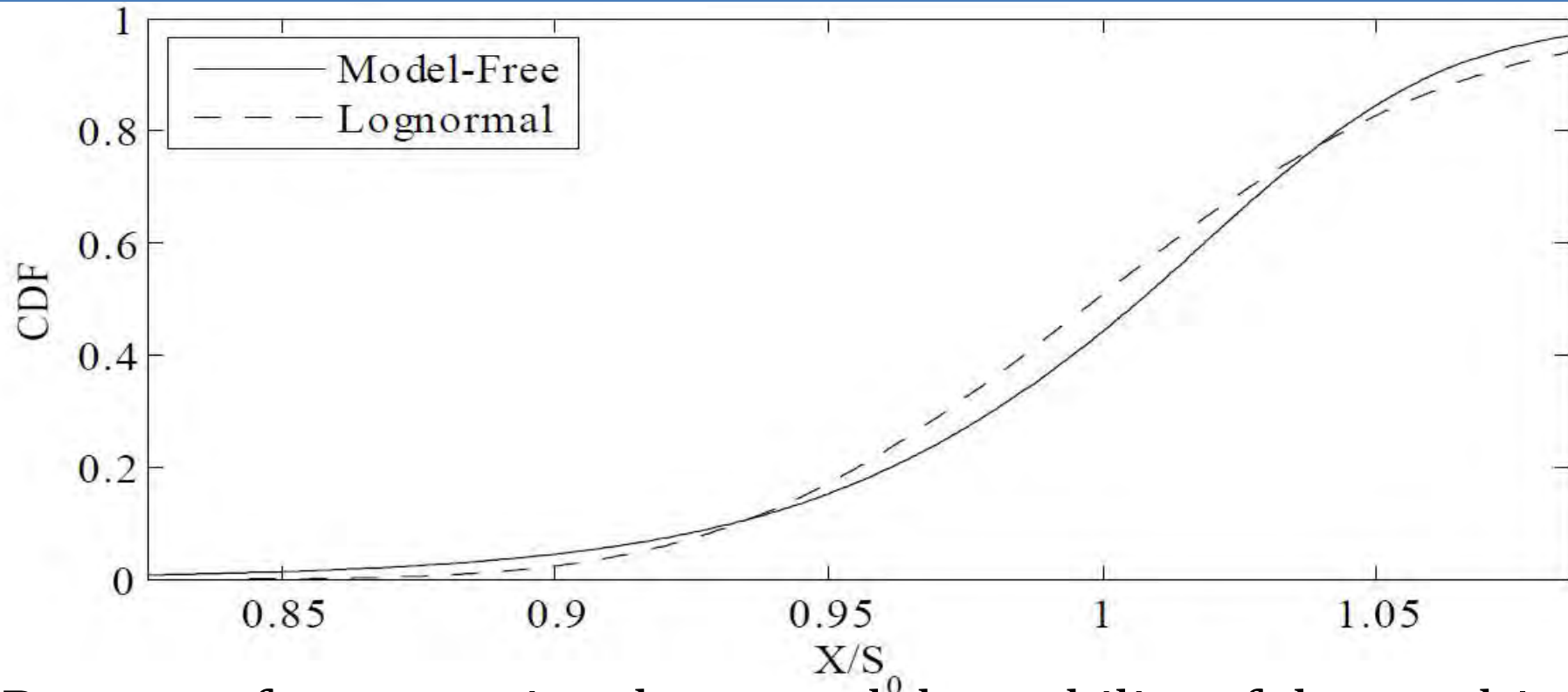
# Parametric and approximated RN density forecasts



- The implied distribution is clearly more negatively skewed than the lognormal distribution
- The limited empirical evidence on alternative methods fails to reach clear conclusions as to which method is to be preferred
- Because the resulting densities are often not markedly different from each other using different estimation methods, it makes sense to use methods that are computationally easy



# Parametric and approximated RN density forecasts



- Because of computational ease and the stability of the resulting parameter estimates, the smoothed implied volatility function method is a good choice for many purposes
- So far we have constructed forecasting objects using the so-called risk-neutral measure implied from options
- When forecasting properties of the underlying asset we ideally want to use the physical measure

# From risk-neutral to physical forecasts

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- Because forecasting occurs in the physical measure space, it is often useful to know the mapping between Q and P
- Knowing the mapping btw. the two measures is therefore required
  - Use superscript Q to describe the option-implied density used above and we use superscript P to denote the physical density
- Black and Scholes (1973) assume the physical stock price process:

$$dS = (r + \mu) S dt + \sigma S dz$$

where  $\mu$  is the equity risk premium

- In the **complete markets**, BS world the option is a redundant asset perfectly replicated by trading the stock and a risk-free bond
- The option price is independent of the degree of risk-aversion of investors because they can replicate the option using a dynamic trading strategy in the underlying asset
- **Principle of risk-neutral valuation**, all derivatives can be valued using the risk-neutral expected pay-off discounted at the risk free rate:  
$$C_0(X, T) = \exp(-rT) E_0^Q [\max\{S_T - X, 0\}]$$

# From risk-neutral to physical forecasts

- Using Ito's lemma implies that log returns are normally distributed

$$f_0^P(\ln(S_T)) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(\frac{1}{2\sigma^2 T} \left(\ln(S_T) - \ln(S_0) + \left(r + \mu - \frac{\sigma^2}{2}\right) T\right)^2\right)$$

which shows that the the only difference in drift is represented by the equity risk premium

- In a BS world, the option-implied RN density forecast will therefore have the correct volatility and functional form but a mean biased downward because of the equity premium
- Because the RN mean of the asset return is the risk-free rate, **the option price has no predictive content for the mean return**
- In the **incomplete markets** case we can still assume a pricing relationship of the form

$$\begin{aligned} C_0(X, T) &= \exp(-rT) E_0^Q[\max\{S_T - X, 0\}] \\ &= \exp(-rT) \int_X^\infty \max\{\exp(\ln(S_T)) - X, 0\} f_0^Q(\ln(S_T)) dS_T \end{aligned}$$

- But the link between the Q and P distributions is not unique and a pricing kernel  $M_T$  must be assumed to link the two distributions

# From risk-neutral to physical forecasts

$$M_T = \exp(-rT) \frac{f_0^Q(\ln(S_T))}{f_0^P(\ln(S_T))} \Rightarrow C_0(X, T) = \exp(-rT) E_0^Q[\max\{S_T - X, 0\}]$$
$$= E_0^P[M_T \max\{S_T - X, 0\}]$$

- The pricing kernel (or stochastic discount factor) describes how in equilibrium investors trade off the current (known) option price versus the future (stochastic) pay-off
- For instance, Heston's model allows for stochastic volatility implying that the option, which depends on volatility, cannot be perfectly replicated by the stock and bond
- Heston (1993) assumes that the price of an asset follows

$$dS = (r + \mu V) S dt + \sqrt{V} S dz_1$$

$$dV = \kappa^P (\theta^P - V) dt + \sigma_V \sqrt{V} dz_2$$

where the two innovations are correlated with parameter  $\rho$

- The mapping between the P and Q-parameters is given by

$$\kappa = \kappa^P + \lambda, \quad \theta = \theta^P \frac{\kappa^P}{\kappa}$$

# From risk-neutral to physical forecasts

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- The P and Q processes imply a pricing kernel of the form

$$M_T = M_0 \left( \frac{S_T}{S_0} \right)^\gamma \exp \left( \delta T + \eta \int_0^T V(s) ds + \xi(V_T - V_0) \right)$$

where  $\xi$  is a variance preference parameter

- The risk premia  $\mu$  and  $\lambda$  are related to the preference parameters by

$$\mu = -\gamma - \xi \sigma_V \rho \quad \lambda = -\rho \sigma_V \gamma - \sigma_V^2 \xi$$

- In order to appreciate the differences btw. P- and Q-forecasts of variance, let's examine the role played by  $\xi$ :

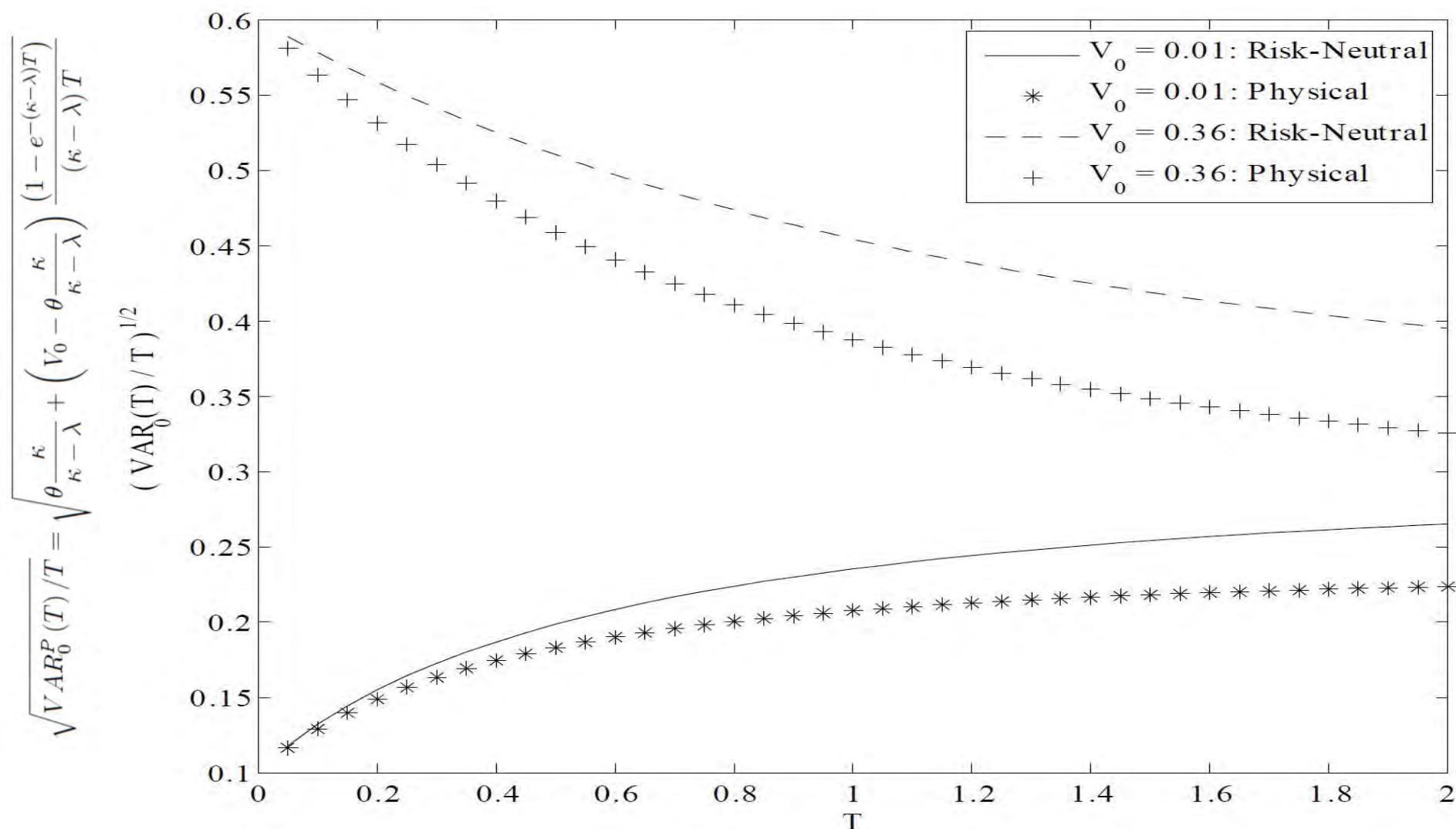
$$\begin{aligned} VAR_0^P(T) &= \theta^P T + (V_0 - \theta^P) \frac{(1 - e^{-\kappa^P T})}{\kappa^P} \\ &= \theta \frac{\kappa}{\kappa - \lambda} T + \left( V_0 - \theta \frac{\kappa}{\kappa - \lambda} \right) \frac{(1 - e^{-(\kappa - \lambda)T})}{\kappa - \lambda} \end{aligned}$$

- Under P-measure the expected variance in Heston's model differs from the RN forecast

$$VAR_0(T) \equiv E_0 \left[ \int_0^T V_t dt \right] = \theta T + (V_0 - \theta) \frac{(1 - e^{-\kappa T})}{\kappa}$$



# From risk-neutral to physical forecasts



- For short horizons and when the current volatility is low then the effect of the volatility risk premium is relatively small
- However for long-horizons the effect is much larger.



# Conclusion

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- The literature contains a large body of evidence supporting the use of option-implied information to predict physical objects of interest
- It is certainly not mandatory that the option-implied information is mapped into the physical measure to generate forecasts
- However, some empirical studies have found that transforming option-implied to physical information improves forecasting performance in certain situations
- We would expect the option-implied distribution or moments to be biased predictors of their physical counterpart
- Yet this bias may be small, and attempting to remove it can create problems of its own, for instance because based on imposing restrictions on investor preferences
- More generally, the existence of a bias does not prevent the option-implied information from being a useful predictor of the future object of interest

## Appendix A: Meaning of coefficients in Jarrow-Rudd's formula

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$$C_0^{JR}(T, X) \approx C_0^{BS}(T, X) - e^{-rT} \frac{(K_3 - K_3(\Psi))}{3!} \frac{d\psi(T, X)}{dX} + \\ + e^{-rT} \frac{(K_4 - K_4(\Psi))}{4!} \frac{d^2\psi(T, X)}{dX^2}$$

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$$\psi(T, X) = \left(X\sigma\sqrt{T2\pi}\right)^{-1} \exp\left\{-\frac{1}{2}\left(d - \sigma\sqrt{T}\right)^2\right\}$$

$$\frac{d\psi(T, X)}{dX} = \frac{\psi(T, X) \left(d - 2\sigma\sqrt{T}\right)}{X\sigma\sqrt{T}}$$

$$\frac{d^2\psi(T, X)}{dX^2} = \frac{\psi(T, X)}{X^2\sigma^2T} \left[\left(d - 2\sigma\sqrt{T}\right)^2 - \sigma\sqrt{T} \left(d - 2\sigma\sqrt{T}\right) - 1\right]$$

$$d = \frac{\ln(S_0/X) + T\left(r + \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}$$