Università Commerciale
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Lecture 4 – Introduction to Utility Theory under Certainty and Uncertainty

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Prep Course in Quant Methods for Finance

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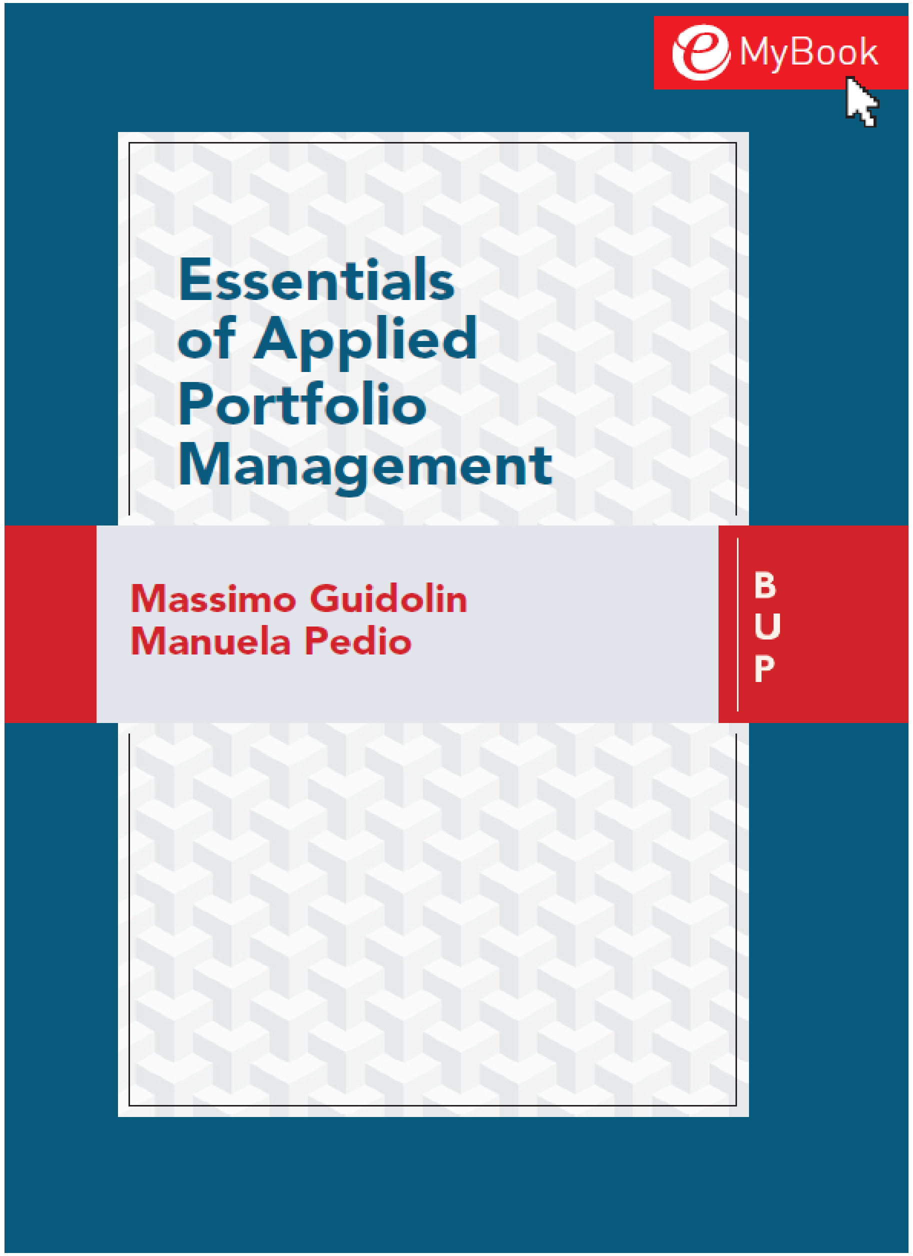
Reference Textbook (also for part I of 20135)



Outline and objectives

- Elementary choice under uncertainty: dominance
  - Guidolin-Pedio, chapter 1, sec. 2
- Elementary choice under uncertainty: mean-variance criterion
  - Guidolin-Pedio, chapter 1, sec. 2
- Preference representation theorem and its meaning
  - Guidolin-Pedio, chapter 2, sec. 1.1
- Expected utility theorem
  - Guidolin-Pedio, chapter 2, sec. 1.2
- Uniqueness of EU preferences up to monotone increasing linear transformations
  - Guidolin-Pedio, chapter 2, sec. 1.2

The Formal Set Up

- Most financial assets (securities) are risky, i.e., they can be characterized as contracts that give different (K) payoffs in different states of the world to occur at a future point in time
  - The assets of interest are said to belong to some **asset menu**
  - Only one state will occur, though investors do not know, at the outset, which one, i.e., the states are mutually exclusive
  - The description of each state is complete and exhaustive
  - The set of states, S, is given exogenously and cannot be affected by the choices of the investors
- Standard probability theory is used to capture the uncertainty on the payoffs of securities, for instance:

State	*Security A*		*Security B*		*Security C*	
	Payoff	Prob.	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15
ii	18	5/15	18	5/15	16	5/15
iii	14	4/15	10	4/15	12	4/15
iv	10	2/15	5	2/15	12	2/15
v	6	1/15	5	1/15	8	1/15

The Formal Set Up

- An investor's task is a complex one and the optimal choice will result from three distinct sets of (interacting) factors:
  1. An investor's aversion toward or tolerance for risk
  2. Some measure of the quantity of risk
  3. How risk attitudes interact with the subjective uncertainties associated with available assets to determine an investor's desired portfolio holdings (demands)
  - In the table, it is not evident why a rational investor ought to prefer security C over security A (if any)
  - An investor who pays more for security C than for A may be motivated by a desire to lower range of variation of the payoffs
  - Unclear how such inclinations against risk may be balanced off in the light of the probability distribution that characterizes different states
- The criteria of choice under uncertainty may be complete or incomplete: a complete criterion is always able to rank all securities or investment opportunities on the basis of their objective features; an incomplete criterion is not

Choice under uncertainty: (strong) dominance

A security (strongly) dominates another security (on a state-by-state basis), if the former pays as much as the latter in all states of nature, and strictly more in at least one state

- Complete criteria form a good basis for portfolio choice
  - E.g., an investor may rank all available assets and to invest in some pre-determined fraction starting from the top of the resulting ranking
- A starkly incomplete criterion is **strong dominance**
- A security (strongly) dominates another security (on a state-by-state basis), if the former pays as much as the latter in all states of nature, and strictly more in at least one state
  - All rational individuals would prefer the dominant security to the security that it dominates
  - Here rational means that the investor is non-satiated, that is, she always prefers strictly more consumption (hence, monetary outcomes that may be used to finance such consumption) to less consumption
- The following example shows that strong dominance often does not allow to rank assets or portfolios

Choice under uncertainty: (strong) dominance

State	*Security A*		*D*	*Security B*		*D*	*Security C*	
	Payoff	Prob.		Payoff	Prob.		Payoff	Prob.
i	20	3/15	>	18	3/15	=	18	3/15
ii	18	5/15	=	18	5/15	>	16	5/15
iii	14	4/15	>	10	4/15	<	12	4/15
iv	10	2/15	>	5	2/15	<	12	2/15
v	6	1/15	>	5	1/15	<	8	1/15

- For instance, security B does not dominate security C and security A does not dominate security C
- Hence, both securities A and C are not dominated by any other security, while security B is (by security A)
- A rational investor may then decide to select between assets A and C, ignoring B
- However, she cannot find an equivalently strong rule to decide to decide between security A and C, hence the criterion is incomplete

- The problem of dominance is that it escapes a definition of risk
- However, in general, a security yields payoffs that in some states are larger and in some other states are smaller than under any other

Choice under uncertainty: mean-variance (dominance)

A security MV-dominates another security if it is characterized by a higher expectation and by lower variance of payoffs than another one

- When this is the case, the best known approach at this point consists of summarizing the distributions of asset returns through their mean and variance:

$$E[R_i] = \sum_{s=1}^{S} Prob(state = s) R_i(s)$$

$$Var[R_i] = \sum_{s=1}^{S} Prob(state = s) \left[R_i(s) - E[R_i]\right]^2$$

- Under mean-variance (MV), the variance of payoffs measures risk
- MV dominance establishes that a security dominates another one in a mean variance sense, if the former is characterized by a higher expected payoff and a by lower variance of payoffs
  - The following example shows how mean and variance are used to rank different securities
  - Both securities A and C are more attractive than asset B as they have a higher mean return and a lower variance

Choice under uncertainty: mean-variance (dominance)

State	*Security A*		*Security B*		*Security C*	
	Payoff	Prob.	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15
ii	18	5/15	18	5/15	16	5/15
iii	14	4/15	10	4/15	12	4/15
iv	10	2/15	5	2/15	12	2/15
v	6	1/15	5	1/15	8	1/15
Mean	15.47		13.27		14.27	
Variance	16.78		28.46		8.46	

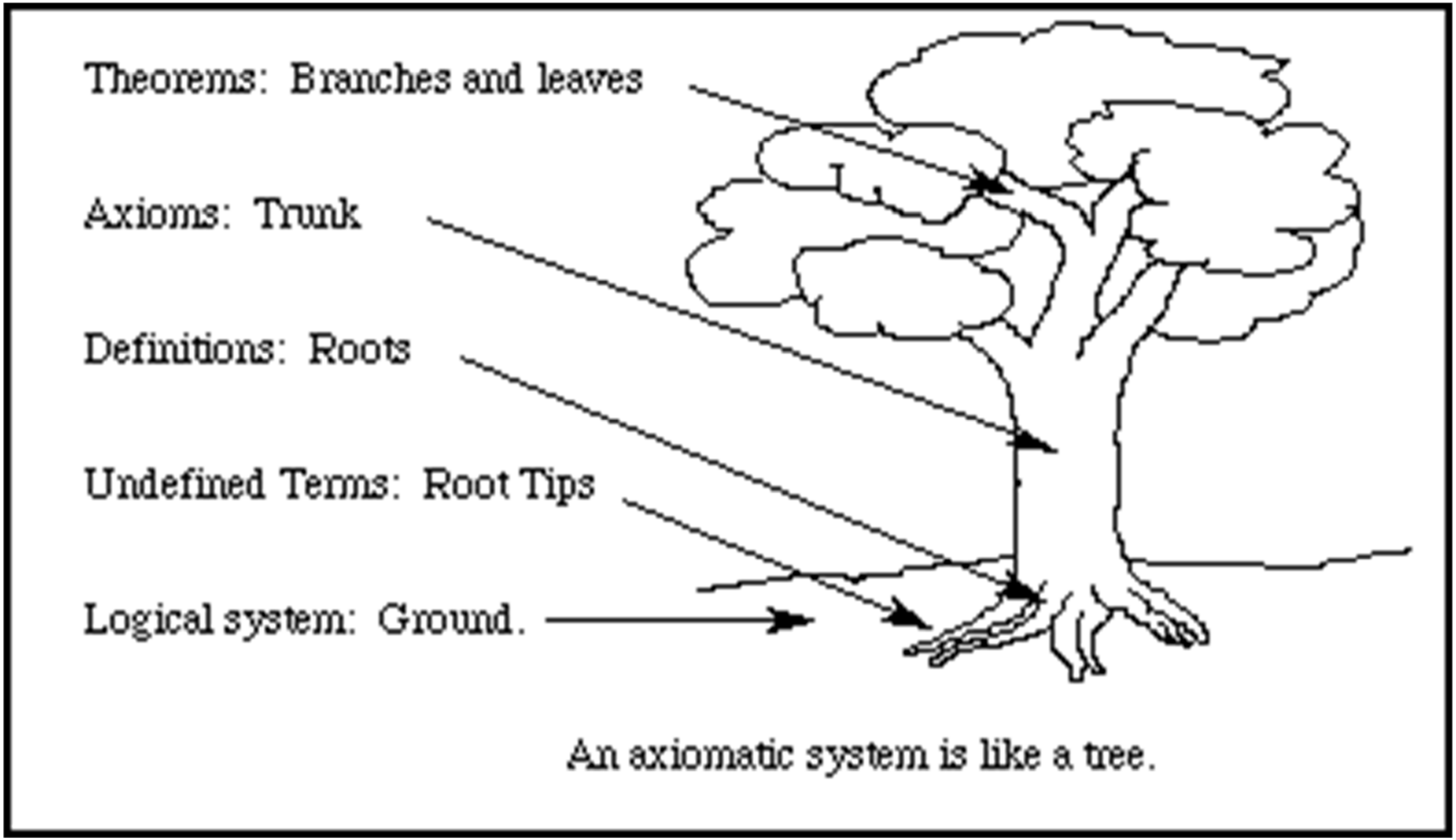
- However, security A fails to dominate security C (and vice versa) in a mean-variance sense since has higher variance.

- Similarly to dominance, also MV is an incomplete criterion, i.e., pairs of securities exist that cannot be simply ranked by this criterion
- **Because of its incompleteness, the MV criterion can at best only isolate a subset of securities that are not dominated by any others**
  - E.g., security B, being dominated by both securities A and C, can be ruled out from portfolio selection
  - However, neither security A nor C can be ruled out because they belong to the set of non-dominated assets

Utility-Based Choice Under Certainty

- Formally, the starting point is a preference relation, denoted as $\succsim$
- Rationality means that you can always express a precise preference between any pair of bundles, that you should not contradict yourself when asked to express preferences over three or more bundles in successive pairs…
- Such properties are formally derived from axioms of choice
- The first step is that under such axioms, there exists a continuous, time-invariant, real-valued ordinal utility function $u(\cdot)$ that ranks bundles in the same way as $\succsim$
- That is, under rationality the ranking of bundles that you may determine on a qualitative basis using your preferences $\succsim$ corresponds to the ranking derived from the utility function u(·)

Utility-Based Choice Under Certainty



Utility-Based Choice Under Certainty

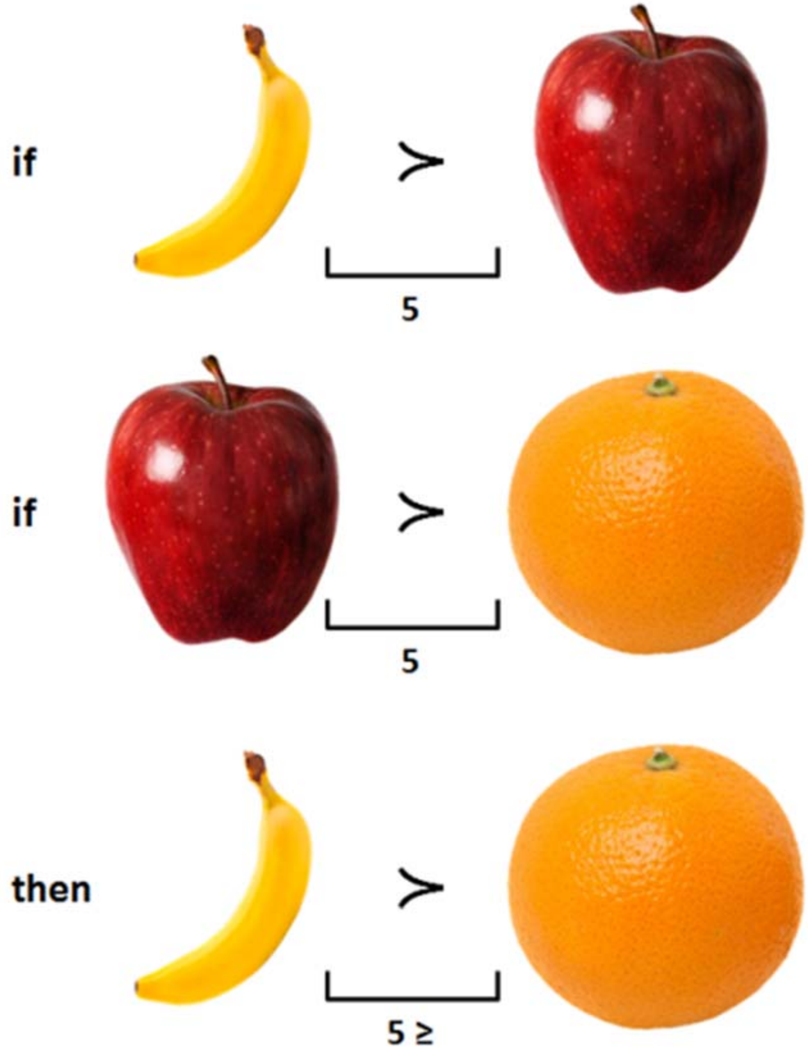
- Modern microeconomic theory describes individual behavior as the result of a process of optimization under constraints
  - The objective is determined by individual preferences
  - Constraints depend on an investor's wealth and on market prices
- To develop such a rational theory of choice under certainty, we postulate the existence of a preference relation, represented by the symbol $\succsim$
- For two bundles a and b, we can express preferences as: when $a \succsim b$, for the investor in question, bundle a is strictly preferred to bundle b, or she is indifferent between them
- Pure indifference is denoted by $a \sim b$, strict preference by $a \succ b$
- In such a framework of choice rationality derives from a set of axioms

  1 Completeness: Every investor is able to decide whether she prefers a to b, b to a, or both, in which case she is indifferent with respect to the two bundles; for any two bundles a and b, either $a \succ b$ or $b \succ a$ or both; if both conditions hold, we say that the investor is indifferent btw. the bundles

Utility-Based Choice Under Certainty

Under the axioms of choice, a continuous, time-invariant, real-valued ordinal utility function $u(\cdot)$ that ranks bundles in the same way as $\succsim$

② Transitivity: For bundles a, b, and c, if $a \succsim b$ and $b \succsim c$, then $a \succsim$ c



Utility-Based Choice Under Certainty

Under the axioms of choice, a continuous, time-invariant, real-valued ordinal utility function $u(\cdot)$ that ranks bundles in the same way as $\succsim$

② **Transitivity**: For bundles a, b, and c, if $a \succsim b$ and $b \succsim c$, then $a \succsim c$

③ **Continuity**: Let $\{x_n\}$ and $\{y_n\}$ be two sequences of consumption bundles such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. The preference relation $\succsim$ is continuous if and only if $x_n \succsim y_n$ for all n, then the same relationship is preserved in the limit, $x \succsim y$

- Completeness, transitivity, and continuity are **sufficient** to guarantee the existence of a **continuous**, time-invariant, real-valued **ordinal** utility function $u(\cdot)$, such that for any two objects of choice a and b,

$$a \succsim b \text{ if and only if } u(a) \geq u(b)$$

- Equivalently, a decision-maker, instead of optimizing by searching and choosing the best possible bundle of goods and services, may simply maximize the utility function $u(\cdot)$ (possibly, subject to constraints)
  - Because of the continuity axiom, $u(\cdot)$ is a continuous function
  - Because $u(\cdot)$ is an **ordinal function**, no special meaning may be attached to its values, i.e., the exact size of the difference $u(a) - u(b) \geq 0$ is not

Utility-Based Choice Under Certainty

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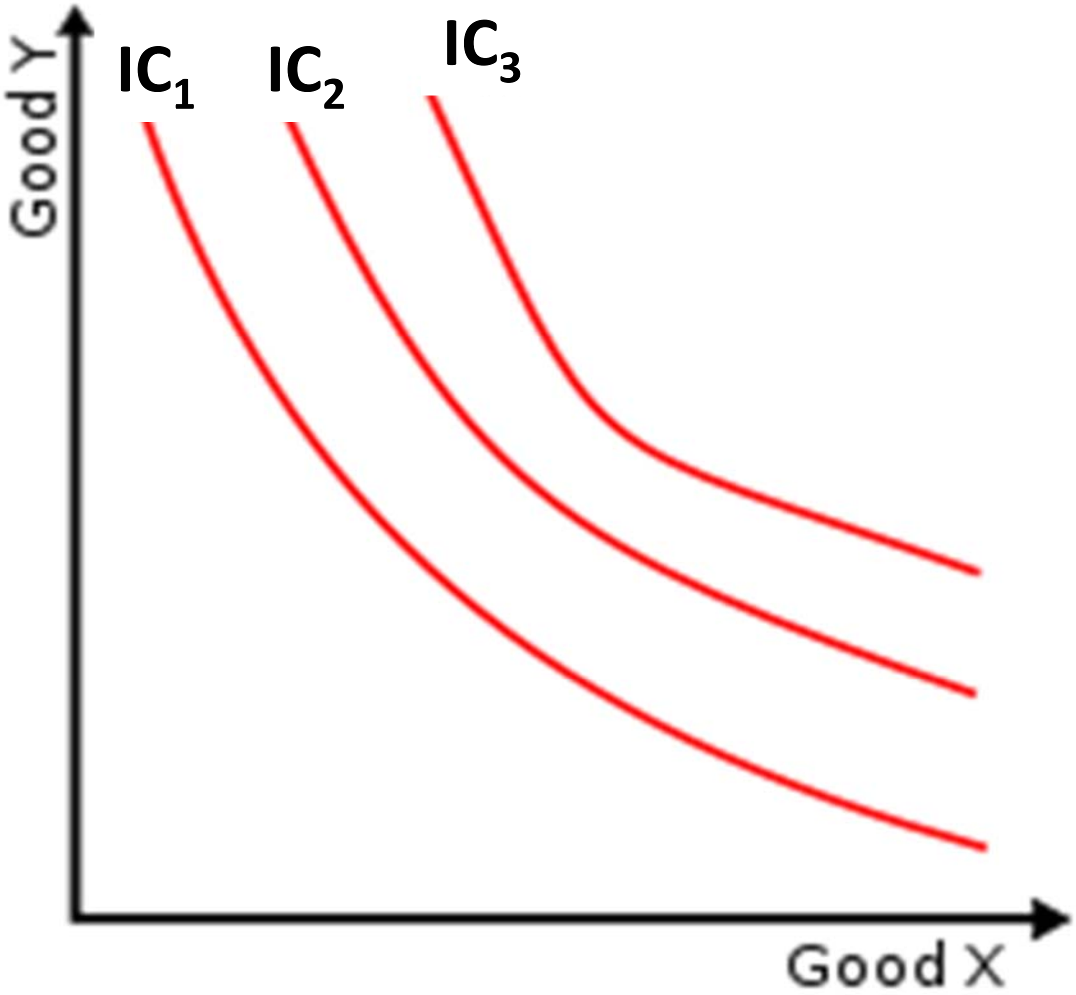

Utility-Based Choice Under Certainty

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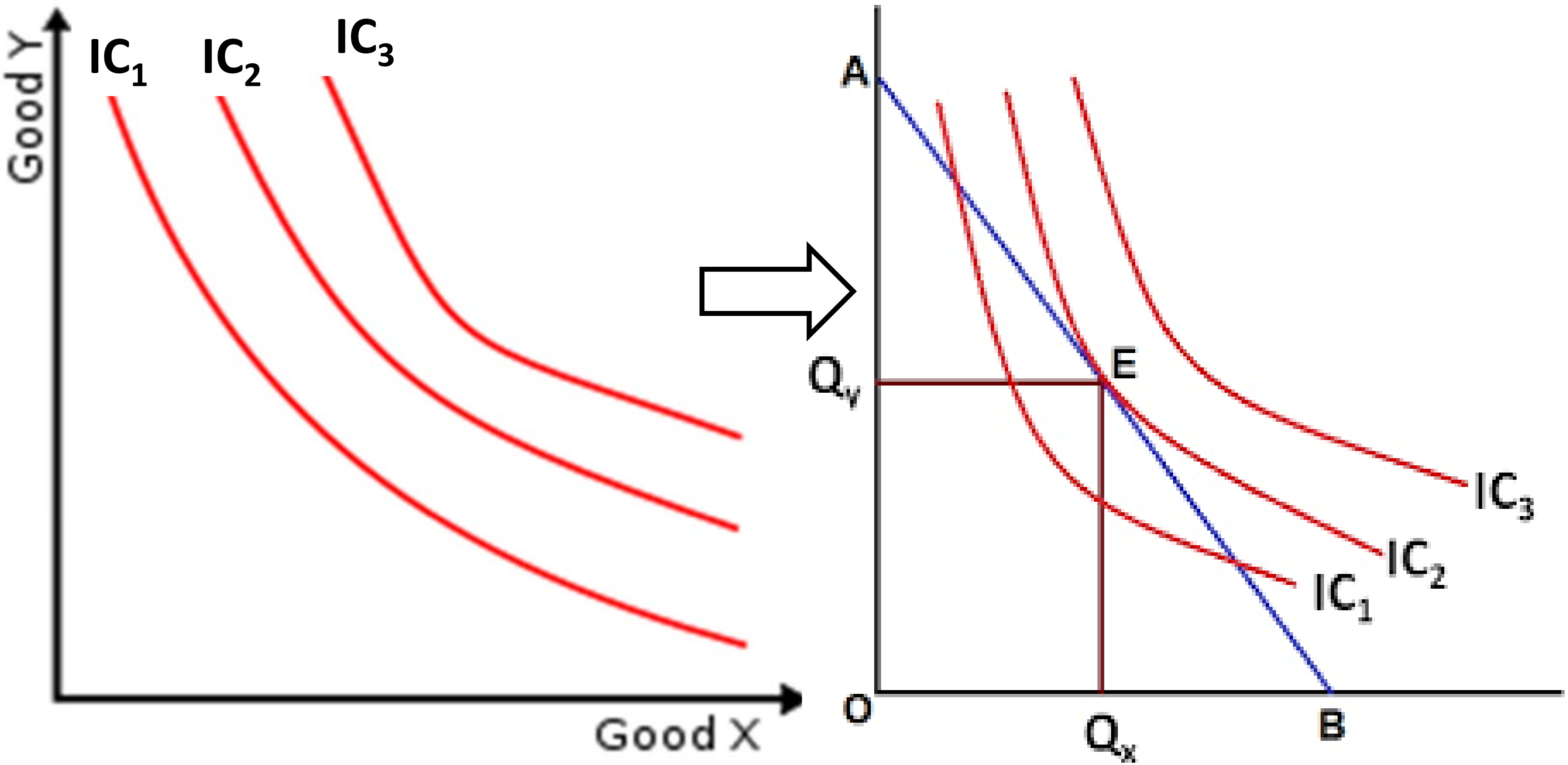


Utility-Based Choice Under Certainty

Given $u(\cdot)$ and a monotone increasing transformation $v(\cdot)$, the function $v(u(\cdot))$ represents the same preferences as the original $u(\cdot)$

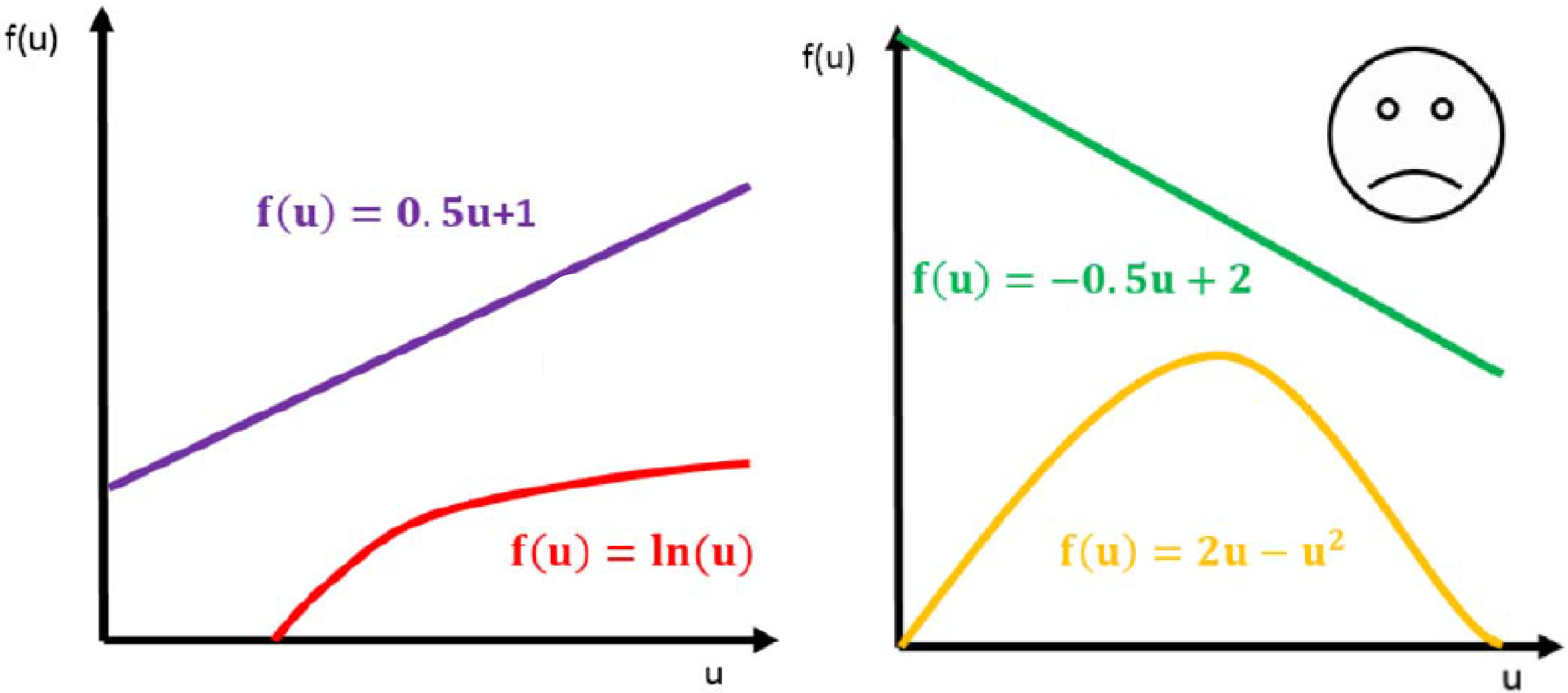
- Different investors will be characterized by heterogeneous preferences and as such will express different utility functions, as identified by heterogeneous shapes and features of their $u(\cdot)$ functions
- However, because $a \succsim b$ if and only if $u(a) \geq u(b)$, any monotone increasing transformation $v(\cdot)$ will be such that $v(u(a)) \geq v(u(b))$, or, assuming $v(\cdot)$ monotone increasing cannot change the ranking

- Given a utility function $u(\cdot)$ and a generic monotone increasing transformation $v(\cdot)$, the function $v(u(\cdot))$ represents the same preferences as the original utility function $u(\cdot)$
  - E.g., if $u(a) \geq u(b)$, $(u(a))^3 \geq (u(b))^3$ ... guys, any guess?

- These concepts and the use of utility functions can be generalized to the case of choice under uncertainty concerning securities and random payoffs

Utility-Based Choice Under Certainty

Given $u(\cdot)$ and a monotone increasing transformation $v(\cdot)$, the function $v(u(\cdot))$ represents the same preferences as the original $u(\cdot)$



- These concepts and the use of utility functions can be generalized to the case of choice under uncertainty concerning securities and random payoffs

Re-Cap + What About Uncertainty?

- $u(\cdot)$ is an ordinal function, i.e., its precise values have no economic meaning; it is just used to rank bundles/perspectives
- It is not correct to state that because $l(a) = 2u(a)$, the investor with utility function $l(\cdot)$ values the bundle a twice as much the investor characterized by $u(\cdot)$
- Because it is ordinal and its precise values do not matter, any monotone increasing transformation of $u(\cdot)$, $v(u(\cdot))$, will preserve the rankings of bundles and hence represent the same preferences of $u(\cdot)$
- These concepts generalize to case of choice under uncertainty
- Under certainty, the choice is among consumption baskets with known characteristics; under uncertainty, the objects of choice are vectors of state-contingent monetary payoffs
- Disentangling preferences from probabilities is a complex problem that simplifies to a manageable maximization under assumptions

Utility-Based Choice Under Uncertainty

- Ranking vectors of monetary payoffs involves more than pure elements of taste or preferences
- E.g., when selecting between some stock A that pays out well during recessions and poorly during expansions and some stock B that pays out according to an opposite pattern, it is essential to forecasts the probabilities of recessions and expansions

- Disentangling pure preferences from probability assessments is a complex problem that simplifies to a manageable maximization problem only under special assumptions, that is when the expected utility theorem (EUT) applies
- Under the EUT, an investor's ranking over assets with uncertain monetary payoffs may be represented by an index combining, in the most elementary way (i.e., linearly):

  1. a preference ordering on the state-specific payoffs
  2. the state probabilities associated to these payoffs

- The EUT simplifies the complex interaction between probabilities and preferences over payoffs in a linear way, i.e., by a simple sum of products

The Expected Utility Theorem

Under the assumptions of the EUT, one ranks assets/securities on the basis of the expectation of the utility of their payoffs across states

- Under the six axioms specified below, there exists a cardinal, continuous, time-invariant, real-valued Von Neumann-Morgenstern (VNM) felicity function of money U(·), such that for any two lotteries/gambles/securities (i.e., probability distributions of monetary payoffs) x and y,

$$x \succsim y \quad \text{if and only} \quad \text{if } E[U(x)] \geq E[U(y)]$$

where for a generic lottery z (e.g., one that pays out either x or y),

$$\mathbb{U}(z) \equiv E[U(z)] = \sum_{s=1}^{S} Prob(\,state = s)U(z(s))$$

- The perceived, cardinal happiness of a complex and risky menu of options, is given by the weighted average of the satisfaction derived from each such individual option, weighted by the probabilities
  - In the following example we use $U(z) = \ln(z)$
  - The ranking by the EU criterion differs from MV: while according the latter only securities B and D are dominated (by A and C), and hence A and C cannot be ranked, according to EU, security A ranks above security C (and B and D)

The Expected Utility Theorem: Supporting Axioms

State	*Security A*		*Security B*		*Security C*		*Security D*	
	Pay-off	Prob.	Pay-off	Prob.	Pay-off	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15	5	3/15
ii	18	5/15	18	5/15	16	5/15	14	4/15
iii	14	4/15	10	4/15	12	4/15	14	4/15
iv	10	2/15	5	2/15	12	2/15	18	2/15
v	6	1/15	5	1/15	8	1/15	18	1/15
$E[R_i]$	15.47		13.27		14.27		13.00	
$Stdev[R_i]$	4.10		5.33		2.91		4.29	
$E[lnR_i]$	$\boxed{2.693}$		2.477		2.635		2.483	

- This example shows one fundamental advantage of EUT-based criteria over dominance and MV criteria: its completeness
- The ranking by the EU criterion differs from MV: while according the latter only securities B and D are dominated (by A and C), and hence A and C cannot be ranked, according to EU, security A ranks above security C (and B and D)

The Expected Utility Theorem: Supporting Axioms

State	*Security A*		*Security B*		*Security C*		*Security D*	
	Pay-off	Prob.	Pay-off	Prob.	Pay-off	Prob.	Payoff	Prob.
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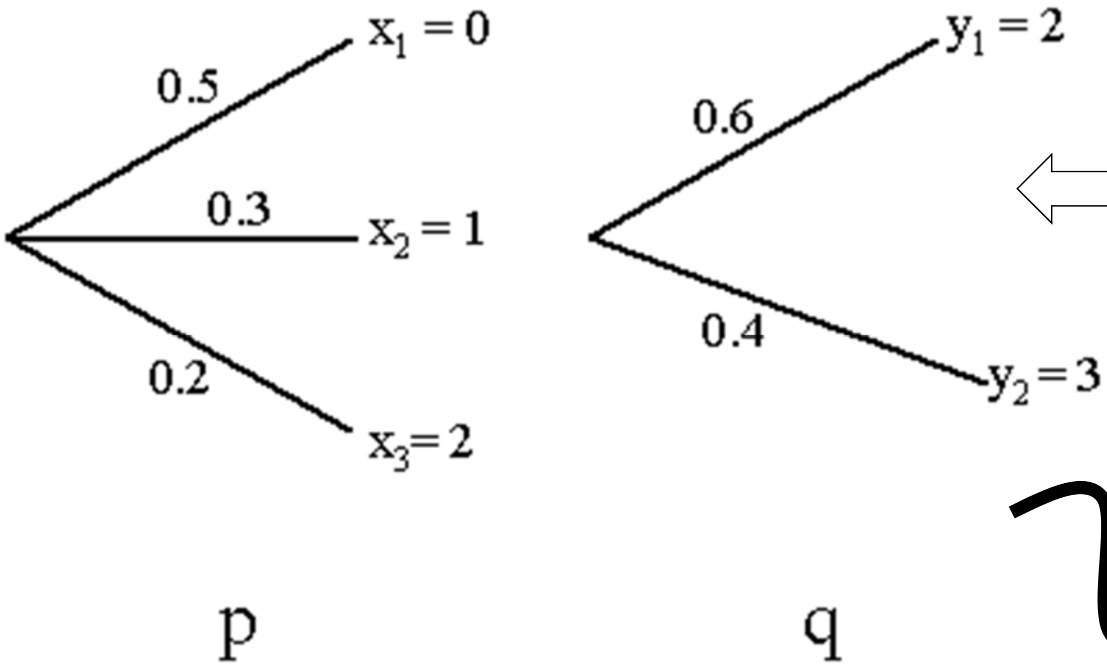
$E[lnR_i]$	$\boxed{2.693}$		2.477		2.635		2.483	

- What are the axioms supporting the EUT?
- These concerns lotteries (x, y; π), which indicates a game that offers payoff x with probability π and payoff y with probability 1 - π

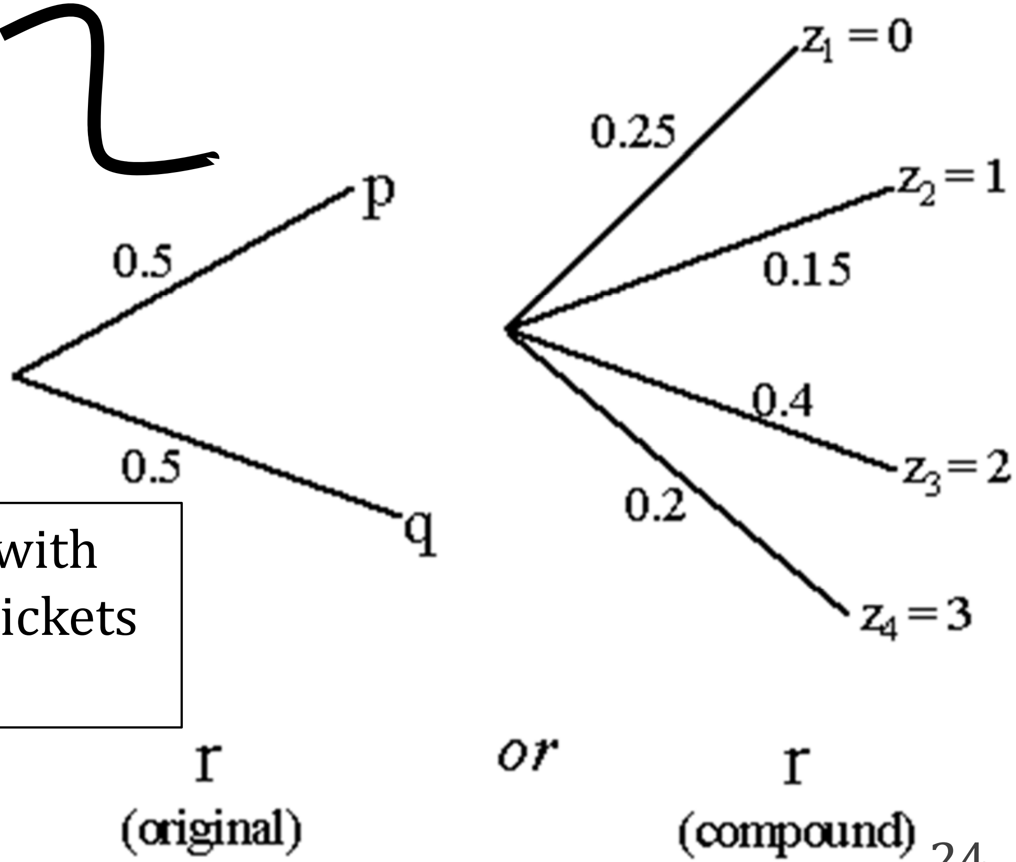
  1 Lottery reduction and consistency: (i) (x, y; 1) = x; (ii) (x, y; π) = (y, x; 1 - π); (iii) (x, z; π) = (x, y; π +(1 - π)q) if z = (x, y; q)
- This axiom means that investors are concerned with the net cumulative probability of each outcome and are able to see through the way the lotteries are set up

The Expected Utility Theorem: Supporting Axioms



These are two very simply lotteries, called *p* and *q*

$\sim$



These is the compound lottery of *p* and *q* with probabilities 0.5 and 0.5, think of lottery tickets that make you win other lottery tickets

The Expected Utility Theorem: Supporting Axioms

The axioms supporting the EUT are (i) lottery reduction, (ii) completeness, (iii) transitivity, (iv) continuity, (v) independence of irrelevant alternatives; (vi) certainty equivalence

- This is demanding in terms of computational skills required of investors

② Completeness: The investor is always able to decide whether she prefers z to l, l to z, or both, in which case she is indifferent

③ Transitivity: For any lotteries z, l, and h, if $z \succsim l$ and $l \succsim h$, then $z \succsim h$

④ Continuity: The preference relation is continuous as established earlier

⑤ Independence of irrelevant alternatives: Let (x, y; π) and (x, z; π) be any two lotteries; then, $y \succsim z$ if and only if $(x, y; \pi) \succsim (x, z; \pi)$; this implies that $(x, y; \pi_1) \succsim (x, z; \pi_2)$ if and only if $\pi_1 \geq \pi_2$, i.e., preferences are independent of beliefs, as summarized by state probabilities

- A bundle of goods or monetary amount remains preferred even though it is received under conditions of uncertainty, through a lottery

⑥ Certainty equivalence: Let x, y, z be payoffs for which $x > y > z$, then there exists a monetary amount CE (certainty equivalent) such that $(x, z; \pi) \sim$ CE

- Arbitrary monotone transformations of cardinal utility functions do not preserve ordering over lotteries

The EUT: Linear Affine Transformations

Any linear affine, monotone increasing transformation of a VNM utility function ($V(\cdot) = a + bU(\cdot)$, $b > 0$) represents the same preferences

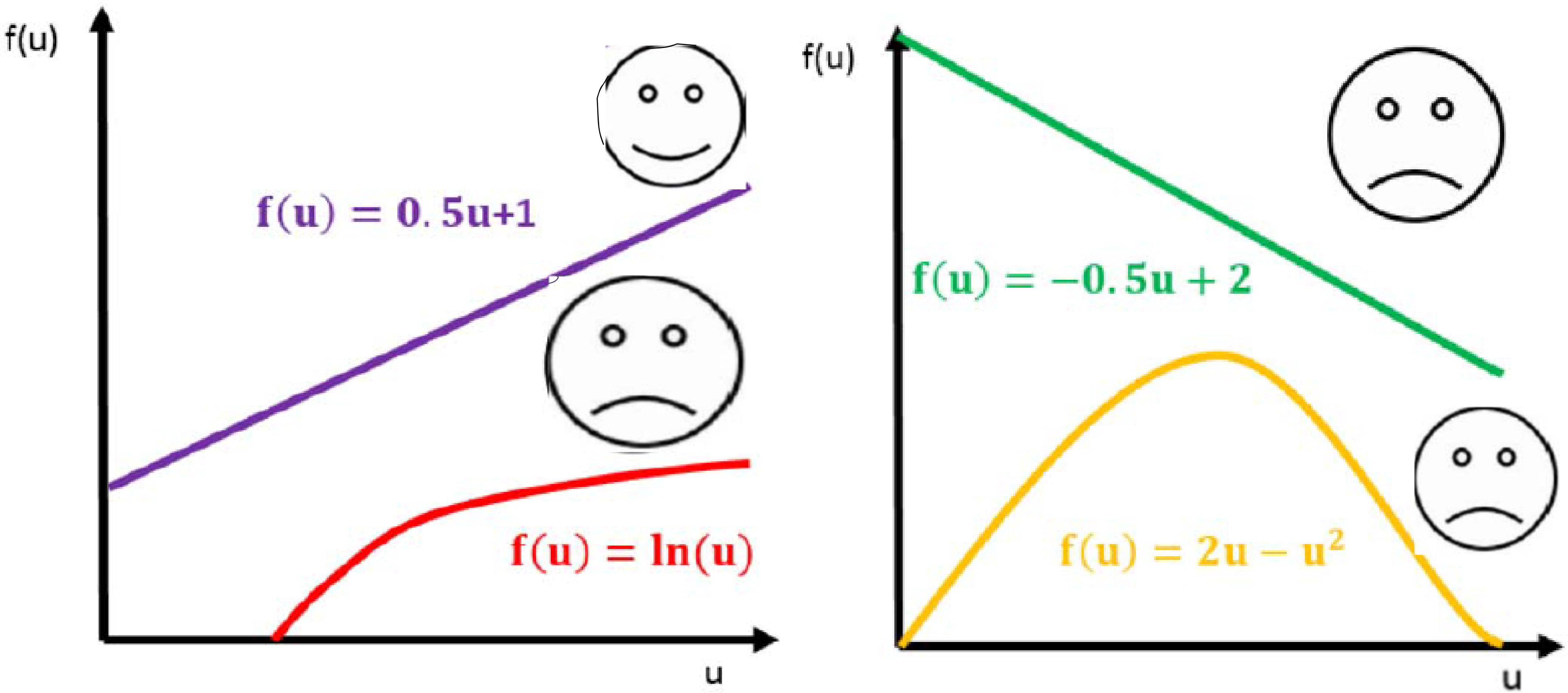
- Arbitrary monotone transformations of cardinal utility functions do not preserve ordering over lotteries
- Are preference defined by the EUT unique up to some kind of transformations as standard $u(\cdot)$ functions were?
- The VNM representation is preserved under linear affine, increasing transformations: if $U(\cdot)$ is a VNM felicity function, then

  $V(\cdot) = a + bU(\cdot) \quad b > 0$ is also a VNM felicity

  - This is because $V((x,y;\pi)) = a + bU((x,y;\pi))$
    $= a + b[\pi U(x) + (1-\pi)U(y)]$
    $= \pi[a + bU(x)] + (1-\pi)[a + bU(y)] = \pi V(x) + (1-\pi)V(y)$
  - E.g., if John's felicity function is $U_{John}(R_i) = \ln(R_i)$ and Mary's is instead $U_{Mary}(R_i) = -2 + 4\ln(R_i)$, Mary and John share the same preferences
  - However, when $U_{Mary}(R_i) = +1000 - \ln(R_i)$ or $U_{Mary}(R_i) = (\ln(R_i))^3$, this will not be the case

The EUT: Linear Affine Transformations

Any linear affine, monotone increasing transformation of a VNM utility function ($V(\cdot) = a + bU(\cdot)$, $b > 0$) represents the same preferences



Completeness of EUT-Induced Rankings

Different VNM felicity functions may induce rather different rankings of lotteries/securities/portfolios, but these will always be complete

- This example shows that the type of felicity function assumed for an investor may matter a lot
- Instead of a log-utility function, assume $U(R_i) = -(R_i)^{-1} = -1/R_i$

State	*Security A*		*Security B*		*Security C*		*Security D*	
	Pay-off	Prob.	Payoff	Prob.	Payoff	Prob.	Payoff	Prob.
i	20	3/15	18	3/15	18	3/15	5	3/15
ii	18	5/15	18	5/15	16	5/15	14	4/15
iii	14	4/15	10	4/15	12	4/15	14	4/15
iv	10	2/15	5	2/15	12	2/15	18	2/15
v	6	1/15	5	1/15	8	1/15	18	1/15
$E[R_i]$	15.47		13.27		14.27		13.00	
$Stdev[R_i]$	4.10		5.33		2.91		4.29	
$-E[1/(R_i)]$	$\boxed{-0.073}$		-0.096		-0.074		-0.094	

- While under a logarithmic utility function, it was security A to be ranked on top of all others, now security A and C are basically on par
- The log and $U(R_i) = -1/R_i$ are related functions but the second implies larger risk aversion