Università Commerciale
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# Lecture 5 – Introduction to Mean-Variance Analysis: the Opportunity Set and the Efficient Frontier

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Prep Course in Quant Methods for Finance

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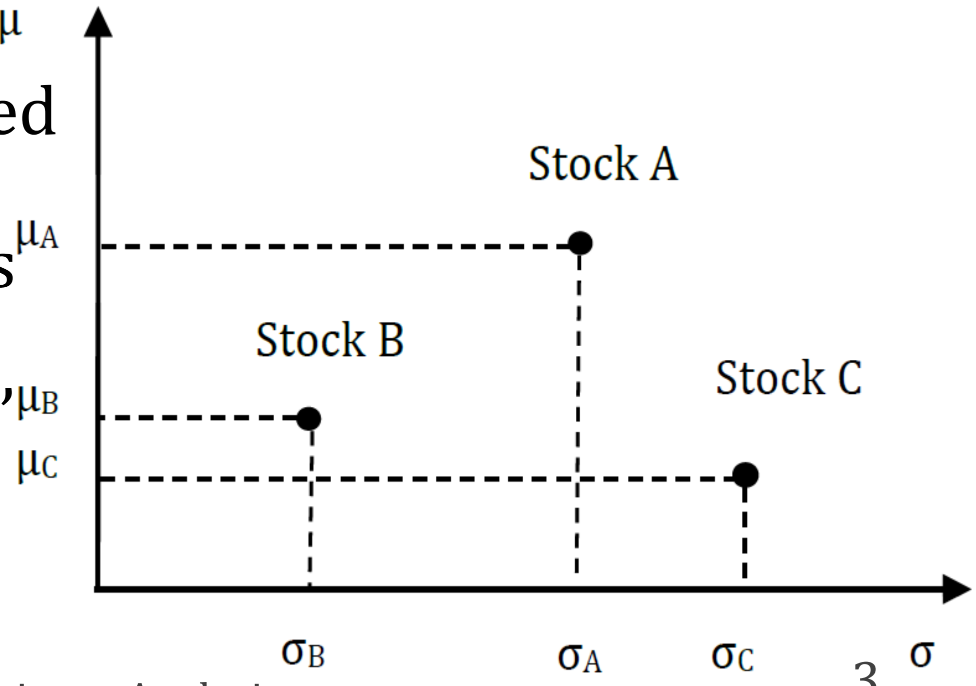
# Outline and objectives

- Mean-variance and efficient frontiers: logical meaning
  - Guidolin-Pedio, chapter 3, sec. 1
- The case of no borrowing and lending and two risky assets
  - Guidolin-Pedio, chapter 3, sec. 1.1
- Generalizations to the case of *N* risky assets
  - Guidolin-Pedio, chapter 3, sec. 1.2
- Two-fund separation result
  - Guidolin-Pedio, chapter 3, sec. 1.2
- Extension to unlimited borrowing and lending
  - Guidolin-Pedio, chapter 3, sec. 2
- Limited borrowing and lending
  - Guidolin-Pedio, chapter 3, sec. 2
- Short-sale constraints
  - Guidolin-Pedio, chapter 3, sec. 3

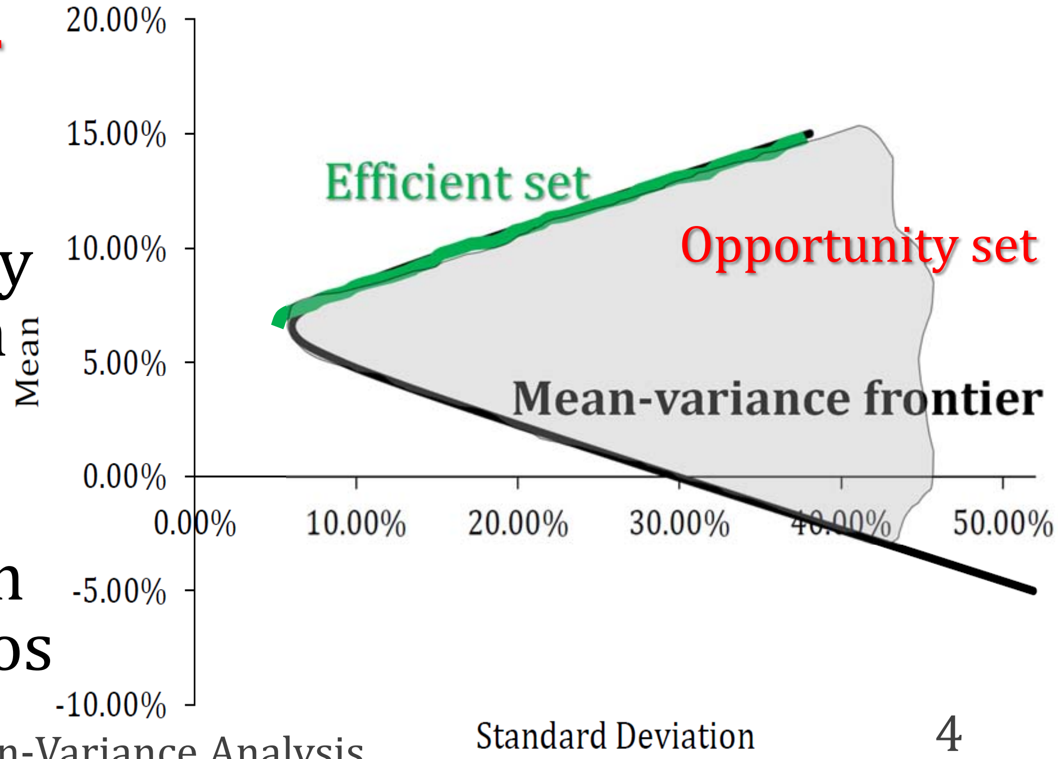
# Mean Variance Framework: Key Concepts

- We review the development of the celebrated mean-variance framework introduced by Markowitz in the 1950s
- Initially at least, risky assets only, no borrowing or lending
- Assume that for some reasons, the joint distribution of asset returns is completely characterized by their means, variances, and covariances
- We represent each asset (and portfolio of assets) in a two-dimensional diagram, where expected portfolio return is plotted on the vertical axis and standard deviation is on the horizontal axis
- Not all securities may be selected, e.g., stock C is dominated by the remaining two stocks in terms of MV dominance



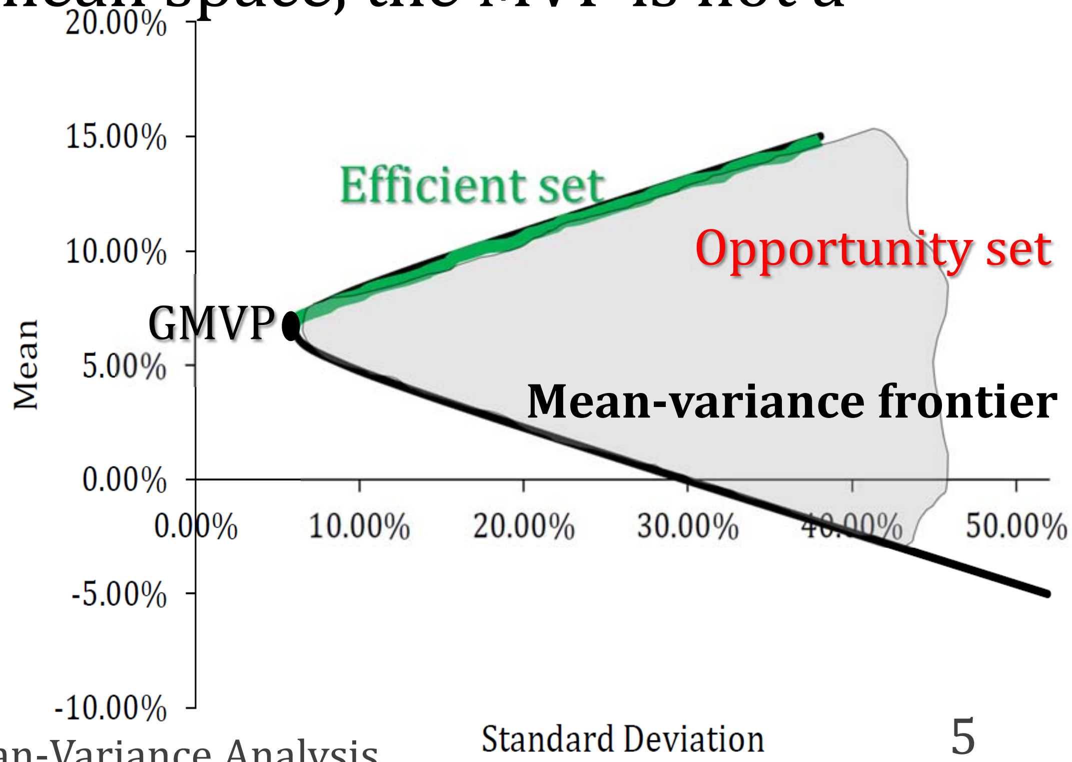
# Mean Variance Framework: Key Concepts

- According to MV criterion a portfolio is efficient if and only if there is no other portfolio that allows the investor to achieve the same expected return with a lower level of risk or a higher level of expected return with the same level of risk
- Three key notions: i) the opportunity set (feasible region), which includes all the portfolios (both efficient and inefficient) that the investor is able to build given securities in the asset menu
- (ii) the mean-variance frontier (aka minimum variance frontier, MVF) subset of the opportunity set containing only the portfolio(s) with minimum variance for any target level of expected returns
- (iii) the efficient frontier, which only includes efficient portfolios



# Mean Variance Framework: Key Concepts

- Because it is possible that a portfolio exists which has a higher return than another portfolio with the same level of risk, only portfolios that have a higher expected return than the global minimum variance portfolio (GMVP) are efficient
- The preferences of the investor(s) for risk are not relevant to the determination of the efficient frontier
- In the classical standard dev.-mean space, the MVF is not a function, but a «right-rotated hyperbola”
- The GMPV is Global Minimum Variance Ptf. and it is of high interest because «separates» the efficient set from the MVF
- The structure GMVP does not depend on expected returns



# The Efficient Frontier with Two Risky Assets

- Assume no borrowing or lending at the risk-free rate
- Re-cap of a few basic algebraic relationships that exploit the fact that with two risky assets, $\omega_B = 1 - \omega_A$
  - See textbook for detailed derivations $\mu_P = \omega_A \mu_A + (1-\omega_A)\mu_B$
  - Portfolio mean & variance: $\sigma_P^2 = \omega_A^2 \sigma_A^2 + (1-\omega_A)^2 \sigma_B^2 + 2\omega_A(1-\omega_A)\sigma_{AB}$
  - Using the definitions of correlation and of standard deviation:

$$\sigma_P = \sqrt{\omega_A^2 \sigma_A^2 + (1-\omega_A)^2 \sigma_B^2 + 2\omega_A(1-\omega_A)\rho_{AB}\sigma_A\sigma_B}$$

  - Solve std. dev equation for $\omega_A$ and plug the result into mean equation ⇒ a system of 2 equations in 2 unknowns
  - The system has in general a unique solution ⇒ the opportunity set is a curve and it coincides with the mean-variance frontier (there is only one possible level of risk for a given level of return)
  - The shape of set depends on the correlation between the 2 securities
- Three possible cases: (i) $\rho_{AB} = +1$; (ii) $\rho_{AB} = -1$; (iii) $\rho_{AB} \in (-1,1)$
- Case (i): $\rho_{AB} = +1$: the expression for $\sigma^2_P$ becomes a perfect square sum and this simplifies the algebra

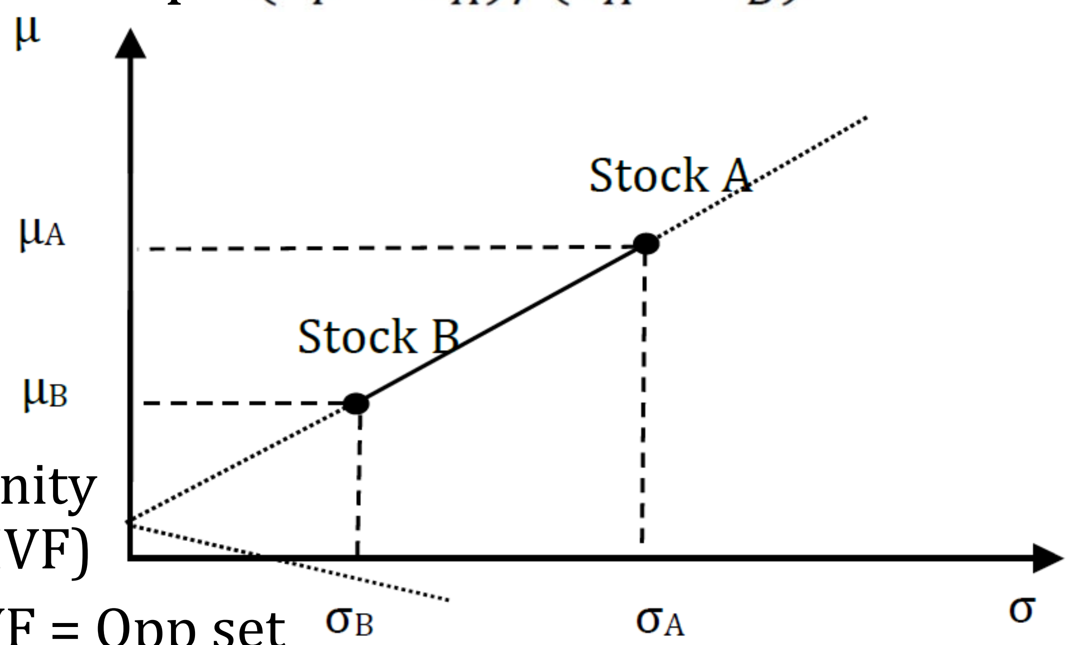
# The Efficient Frontier with Two Risky Assets

- After algebra (see textbook), we have:

$$\mu_P = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\mu_A + \left(1 - \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\,(\mu_A - \mu_B)$$

  the equation of a straight line, with slope $(\sigma_P - \sigma_A)/(\sigma_A - \sigma_B)$

  - In the picture, dashed lines == portfolios require short selling
  - Without short sales, the least risky stock == GMVP
  - With short sales, the GMVP has zero risk
  - In this special case, the opportunity set = mean-variance frontier (MVF)
  - With no short sales, EffSet = MVF = Opp set




- Case (ii): $\rho_{AB}$ = -1: the expression for $\sigma^2_P$ becomes a perfect square difference and this simplifies the algebra (see textbook) to yield:

$$\sigma_P = \omega_A\sigma_A - (1 - \omega_A)\sigma_B \quad \text{or to} \quad \sigma_P = -\omega_A\sigma_A + (1 - \omega_A)\sigma_B$$

- Yet, each of the equations only holds when the RHS is positive

# The Efficient Frontier with Two Risky Assets

- The opportunity set is a straight line, but its slope depends on which of the equations above holds
- If the first equation applies, the opportunity set is equal to:

$$\mu_P = \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\mu_A + \left(1 - \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_P + \sigma_B}{\sigma_A + \sigma_B}(\mu_A - \mu_B)$$

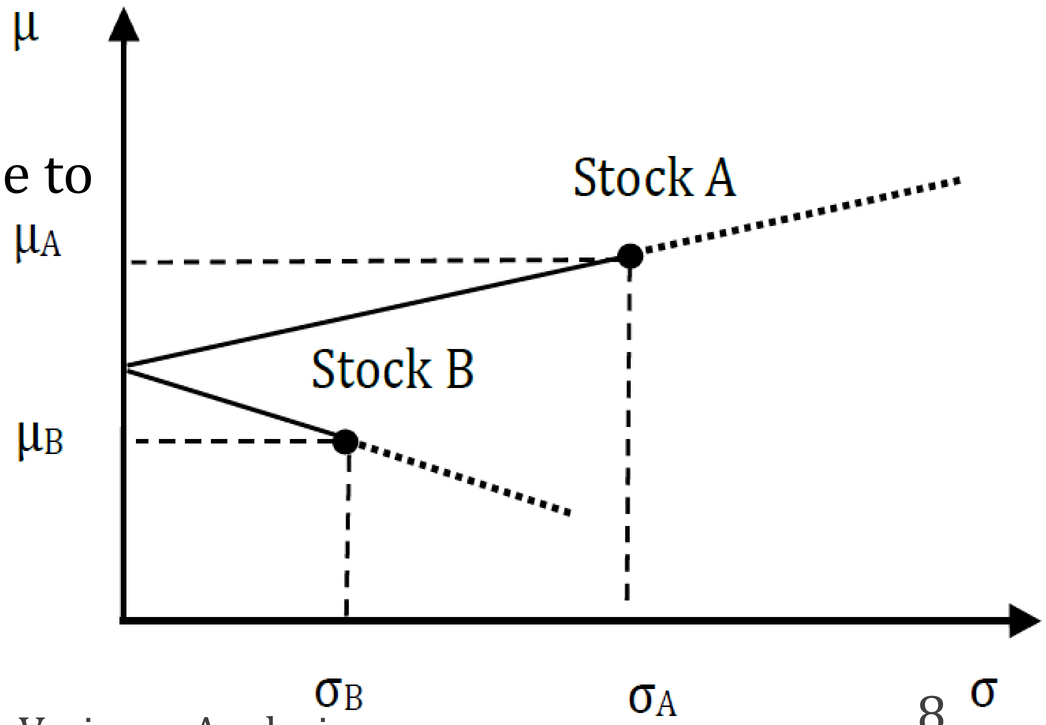
while if the second equation holds, the opportunity set is equal to:

$$\mu_P = \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\mu_A + \left(1 - \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}\right)\mu_B = \mu_B + \frac{\sigma_B - \sigma_P}{\sigma_A + \sigma_B}(\mu_A - \mu_B)$$

  - In the picture, dashed lines == portfolios require short selling
  - Even without short sales, possible to find a combination that has zero variance, i.e., it is risk-free
  - Such a riskless portfolio is GMVP
  - The expression for such a ptf. is:

$$\omega_A^{GMVP}\sigma_A - (1 - \omega_A^{GMVP})\sigma_B = 0 \text{ or}$$

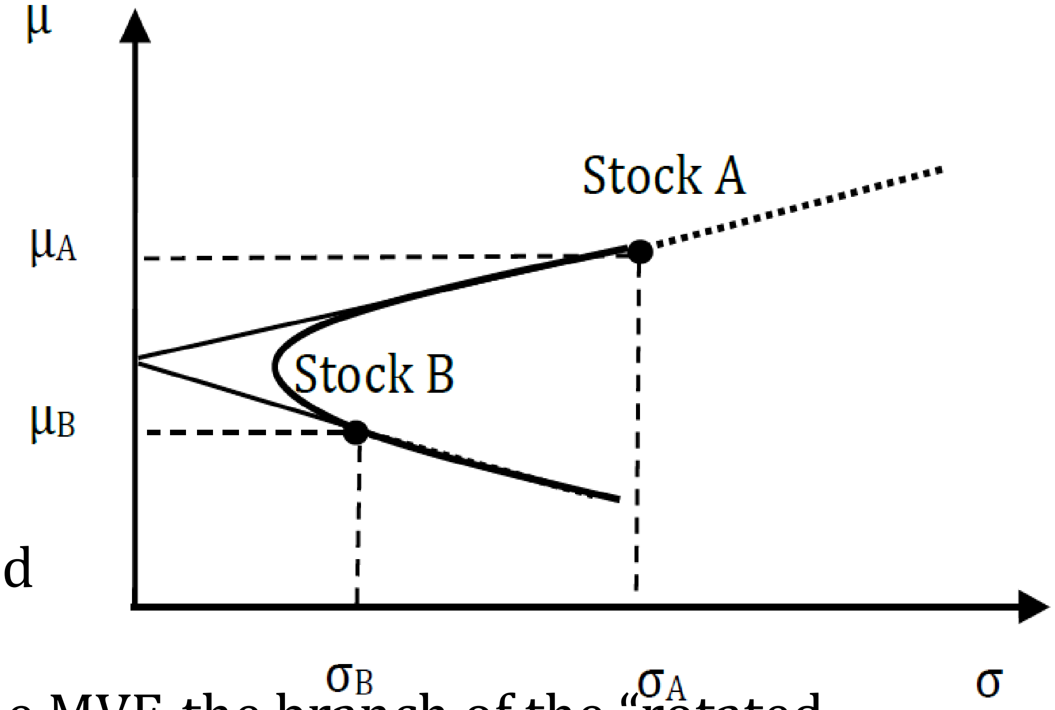
$$-\omega_A^{GMVP}\sigma_A + (1 - \omega_A^{GMVP})\sigma_B = 0$$



# The Efficient Frontier with Two Risky Assets

$$\Longrightarrow \quad \omega_A^{GMVP} = \frac{\sigma_B}{\sigma_A + \sigma_B}$$

- Case (iii): $\rho_{AB} \in (-1,1)$: In this case, although tricks exist to trace it out, the MVF does not have a closed-form expression
  - The MVF is non-linear, a parabola (i.e., a quadratic function) in the variance-mean space
  - Or a a (branch of) hyperbola in standard deviation-mean space
  - In such a space, the MVF is not a function, it is just a «correspondence», a "right-rotated hyperbola"

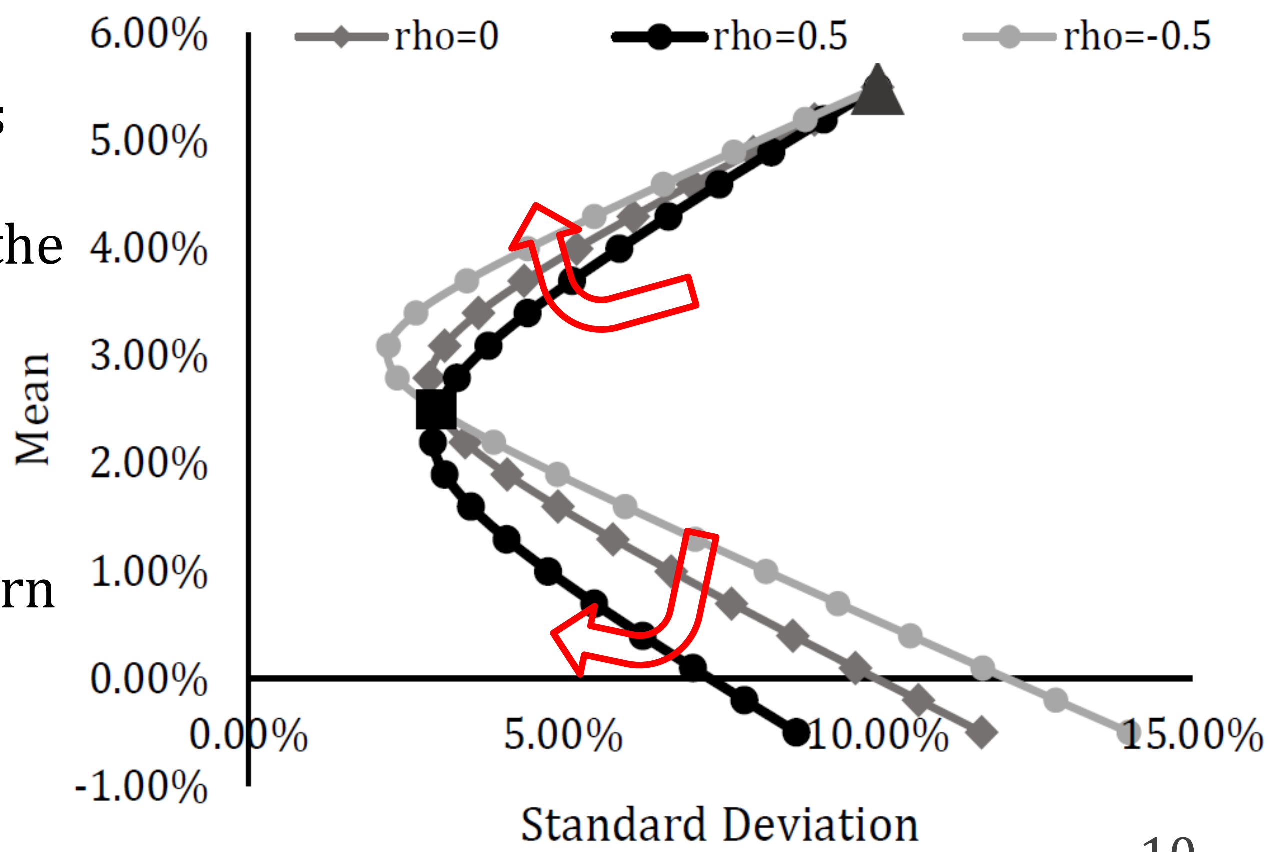



  - The efficient set == a portion of the MVF, the branch of the "rotated hyperbola" that lies above (and includes) the GMVP
  - To distinguish the efficient set from the MVF we have to find the GMVP:

$$\frac{\partial \sigma_P^2}{\partial \omega_A} = 2\omega_A \sigma_A^2 - 2(1-\omega_A)\sigma_B^2 + 2(1-2\omega_A)\rho_{AB}\sigma_A\sigma_B \Longrightarrow \omega_A^{GMVP} = \frac{\sigma_B^2 - \rho_{A,B}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{A,B}\sigma_A\sigma_B}$$

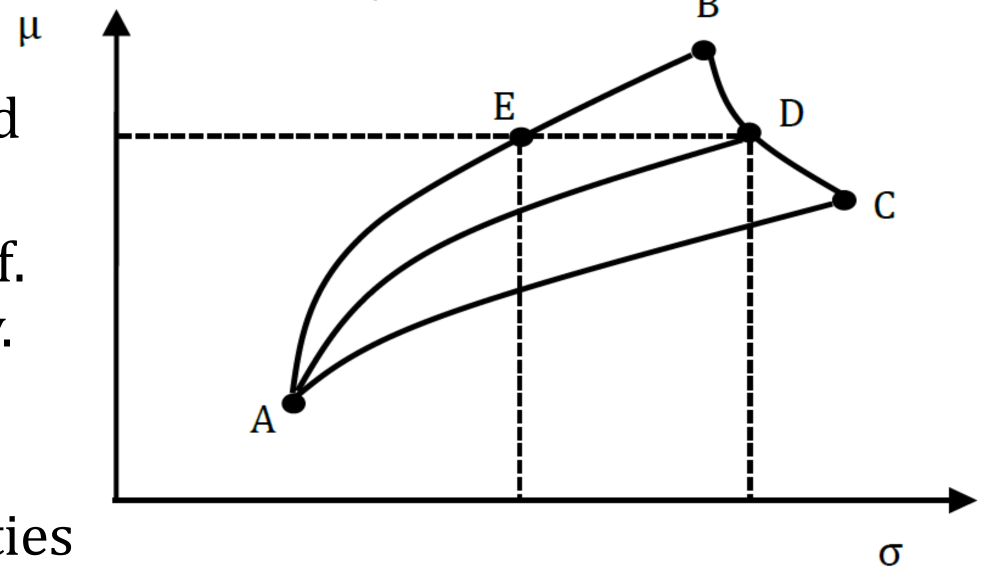
# One Numerical Example

- For instance, consider stock A with $\mu_A = 5.5\%$ and $\sigma_A = 10\%$, and stock B with $\mu_B = 2.5\%$ and $\sigma_B = 3\%$
- Draw MVF in Excel for $\rho_{A,B} = 0$, $\rho_{A,B} = 0.5$, and $\rho_{A,B} = -0.5$
- See textbook for calculations and details and book's website for exercises in Excel related to this case
  - When the $\rho_{A,B} < 0$, it is possible to form ptfs. that have a lower risk than each of the 2 assets
  - Clearly, as $\rho_{A,B}$ declines risk characterizing the GMVP moves towards the left, inward
  - The entire MVF rotates upward, less risk may be borne for identical expected portfolio return
  - Note that the GMVP often needs to include short positions



# The Case of N Risky Assets

- Usually investors choose among a large number of risky securities
  - E.g., allocation among the 500 stocks in the S&P 500
- Extend our framework to the general case, with *N* risky assets
- The MVF no longer coincides with the opportunity set, which now becomes **a region** and not a line
  - Ptf. D, a combination of assets B and C, is not MV efficient
  - It gives the same mean return as ptf. E but implies a higher standard dev. and a risk-averse investor would never hold portfolio D
  - To exclude all the inefficient securities and ptfs., as first step the investor needs to trace out the MVF, i.e., select ptfs. with minimum variance (std. dev.) for each level of μ
  - Only interested in the upper bound of the feasible region



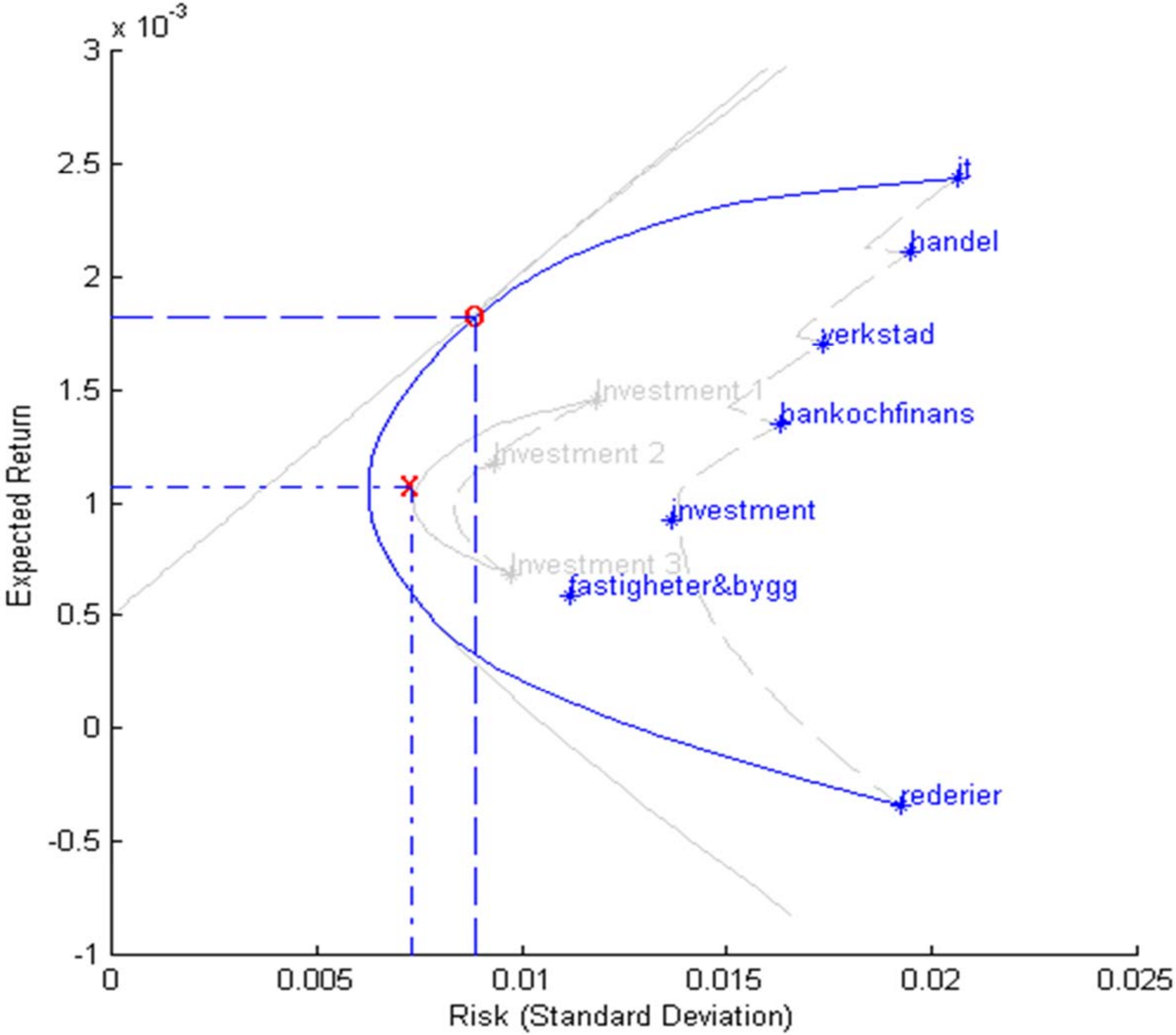

- We solve the following **quadratic programming problem**:

$$\min_{\{\omega\}} \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \quad Subject\ to \quad \boldsymbol{\iota}'\boldsymbol{\omega} = 1, \quad \boldsymbol{\mu}'\boldsymbol{\omega} = \bar{\mu}$$

Nx1 vector of 1s

Target mean

# The Case of N Risky Assets



# The Case of N Risky Assets

- For the time being, no short-sale restrictions have been imposed
  - To solve the program, assume that no pair or general combination of asset returns are linearly dependent
  - $\Rightarrow \Sigma$ is nonsingular and invertible; in fact, $\Sigma$ is (semi-)positive definite
- Under these conditions, it is a constrained minimization problem that can be solved through the use of Lagrangian multiplier method
- See the textbook for algebra and details
- If one defines $A \equiv \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}$, $B \equiv \boldsymbol{\mu}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $C \equiv (\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})$, and $D = BC - A^2$ then the unique solution to the problem, $\boldsymbol{\omega}^*$, is:

$$\boxed{\boldsymbol{\omega}^* = \mathbf{g} + \boldsymbol{h}\,\bar{\mu}} \qquad \mathbf{g} = \frac{1}{D}[B(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})],\ \mathbf{h} = \frac{1}{D}[C(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\iota})]$$

i.e., any combination of MVF ptf. weights gives another MVF ptf.

- Consider two MVF ptfs. $P_1$ and $P_2$ with mean $\mu_{P1}$ and $\mu_{P2}$, and assume that $P_3$ is a generic portfolio on the MVF: always possible to find a quantity *x* such that $\mu_{P_3} = x\mu_{P_1} + (1-x)\mu_{P_2}$
- Other MVF ptf: $\boldsymbol{\omega}_{P_3} = x\omega_{P_1} + (1-x)\omega_{P_2} = x(\mathbf{g} + \mathbf{h}\mu_{P_1}) + (1-x)(\mathbf{g} + \mathbf{h}\mu_{P_2})$

$$= \mathbf{g} + \mathbf{h}(x\mu_{P_1} + (1-x)\mu_{P_2}) = \mathbf{g} + \mathbf{h}\mu_{P_3},$$

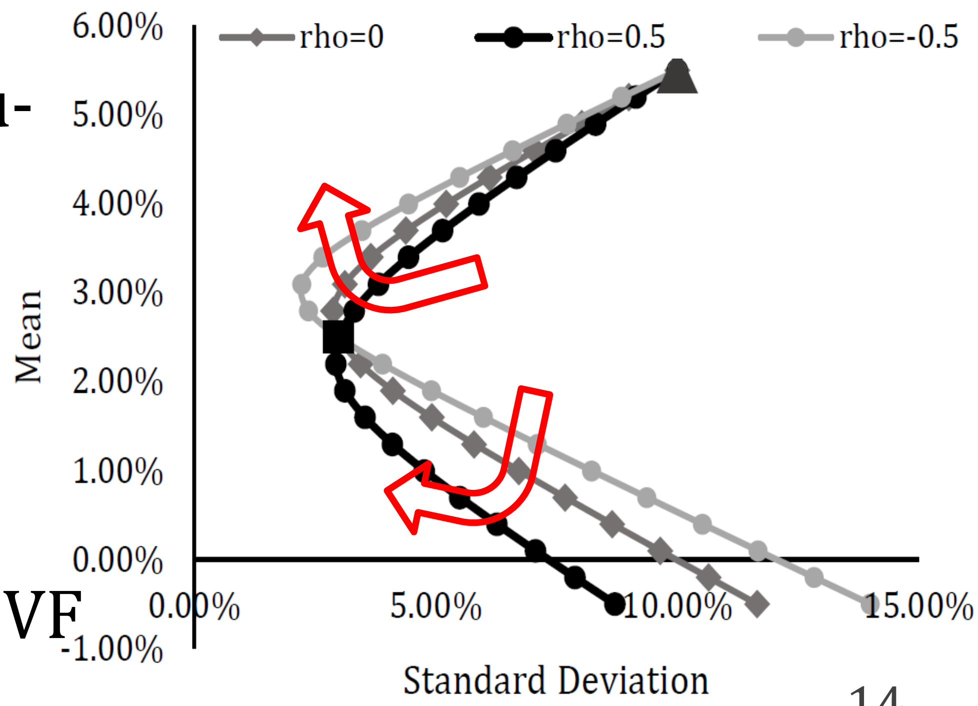
# Recap: The Efficient Frontier, in General

- The GMVP and the entire MVF depend strongly on the correlation structure of security returns: the lower are the correlations (on average), the more the efficient set moves up and to the left, improving the risk-expected return trade-off
- The position and shape of the MVF reflects the diversification opportunities that a given asset menu offers
- Even though, MVF ptfs are solutions of a complex quadratic programming problem, in the absence of constraints, their structure is relatively simple:

$$\boldsymbol{\omega}^* = \mathbf{g} + \boldsymbol{h}\,\bar{\mu}$$

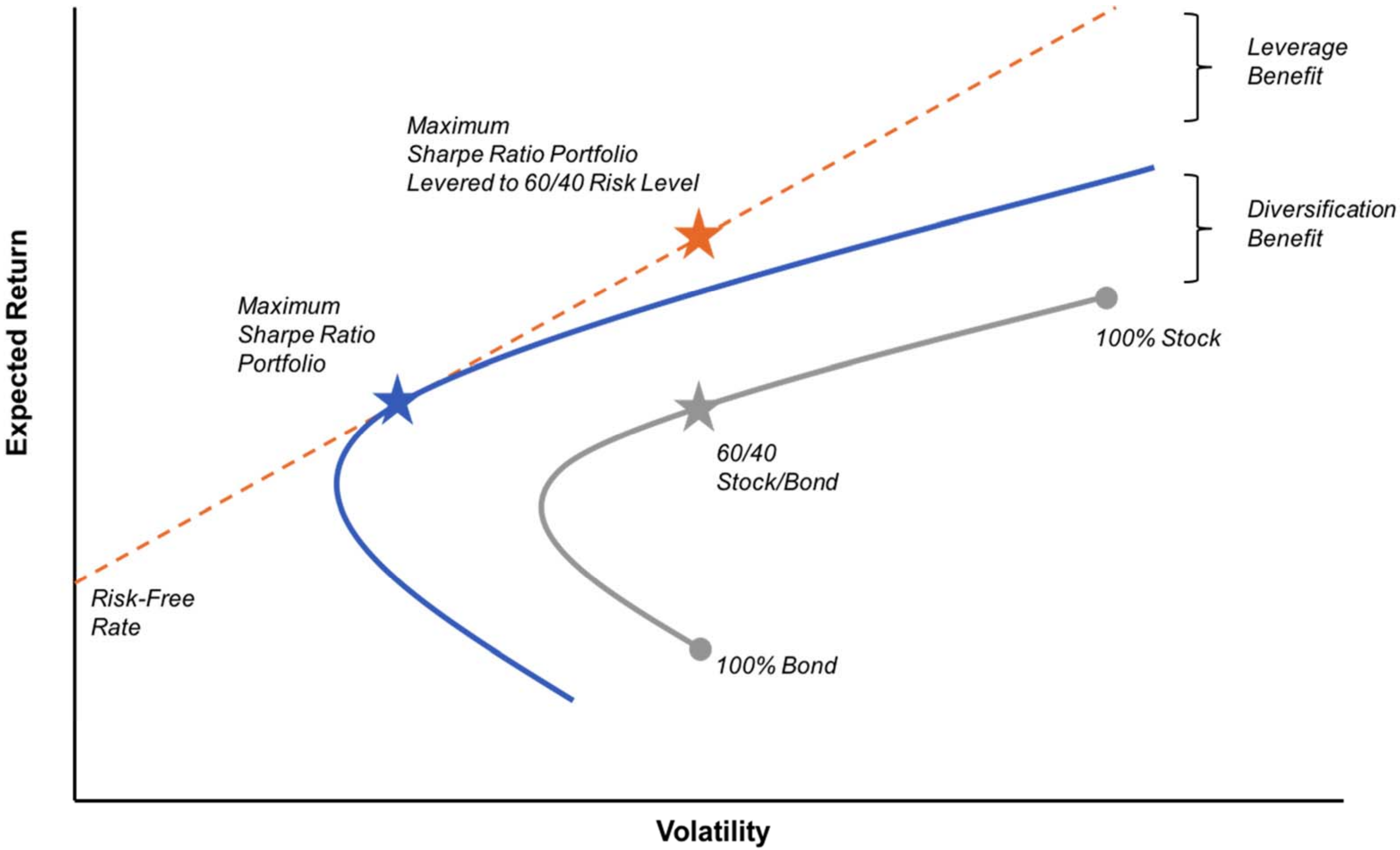
$$\mathbf{g} = \frac{1}{D}\left[B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\right]$$

$$\mathbf{h} = \frac{1}{D}\left[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})\right]$$




- Combinations of MVF ptfs. are MVF

# Recap: The Efficient Frontier, in General



# Two-Fund Separation

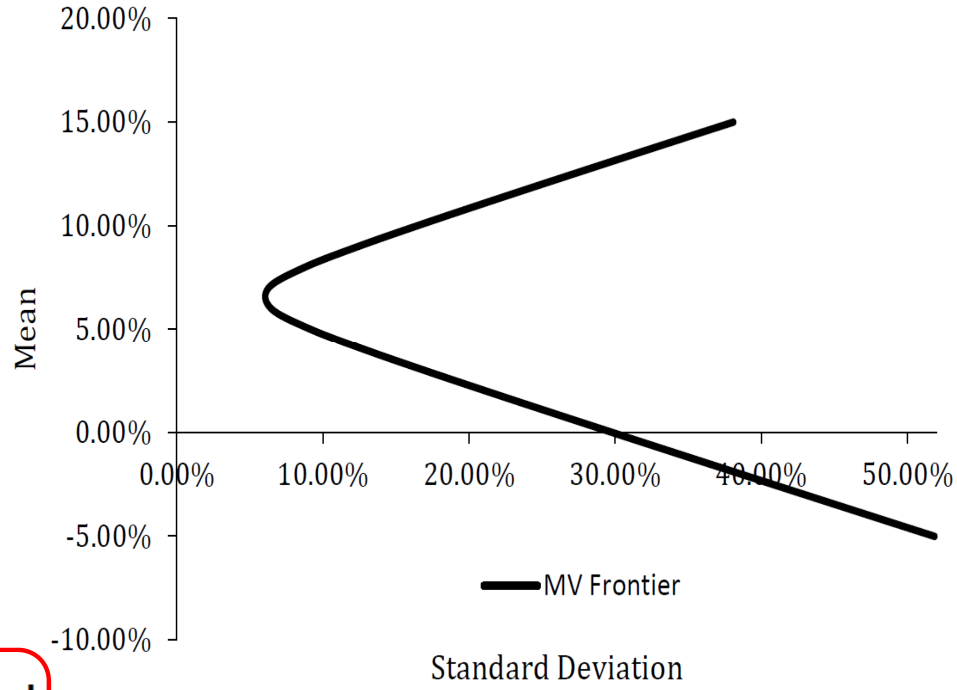
It is sufficient to know two points (portfolios) on the mean-variance frontier to generate all the others

- All MV-optimizers are satisfied by holding a combination of two mutual funds (provided they are MV efficient), regardless of preferences
- Their heterogeneous preferences will only impact the way in which they combine the two funds that they choose to hold
- In equilibrium, if all investors are rational MV optimizers, the market portfolio, being a convex combination of the optimal portfolios of all the investors, has to be an efficient set portfolio
- As for the shape of MVF when *N* assets are available, this is a rotated hyperbola as in case of 2 assets: $$\sigma_P^2 = \frac{1}{D}\left[C(\mu_P)^2 - 2A\mu_P + B\right]$$
  - Equation of a parabola with vertex in $((1/C)^{1/2}, A/C)$, which also represents the global minimum variance portfolio
  - The textbook shows that GMV weights are: $\boldsymbol{\omega}_{GMVP} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{C} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}$,

# One Strategic Asset Allocation Example

- Consider three assets – U.S. Treasury, corporate bonds, and equity – characterized by the mean vector and the variance-covariance matrix:

$$\mu = \begin{bmatrix} 6.00 \\ 7.50 \\ 9.00 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 0.005 & 0.004 & -0.002 \\ 0.004 & 0.008 & 0.003 \\ -0.002 & 0.003 & 0.025 \end{bmatrix}$$



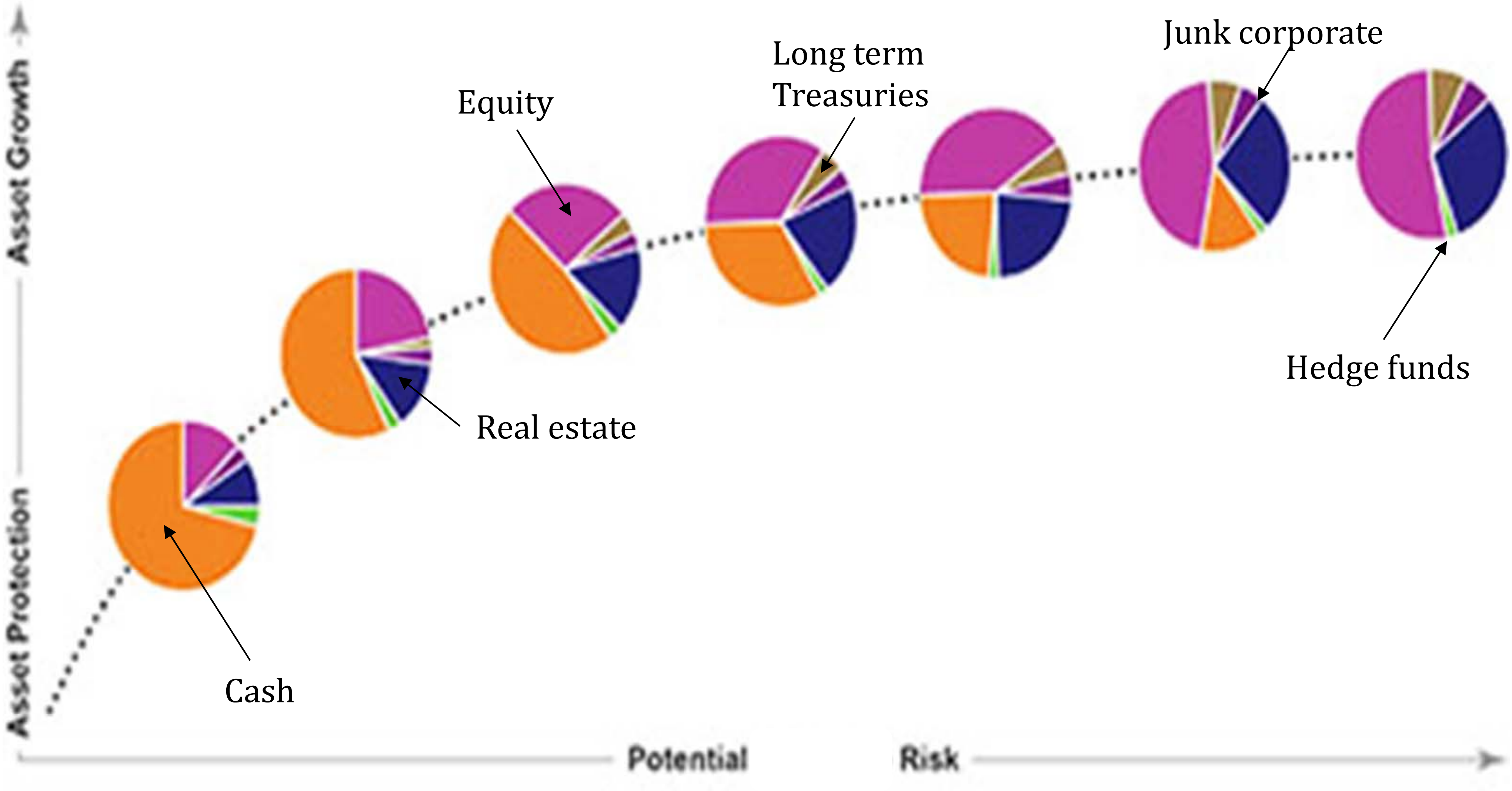

- The textbook guides you to perform calculations of A, B, C, D using Excel:

$$\mathbf{g} = \frac{1}{D}[B(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})]$$

$$= \frac{1}{14.21}\left\{1.26 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 6{,}00 \\ 7{,}50 \\ 9{,}00 \end{bmatrix}\right\} = \begin{bmatrix} 4.63 \\ -3.27 \\ -0.37 \end{bmatrix}$$

$$\mathbf{h} = \frac{1}{D}[C(\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - A(\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})] = \frac{1}{14.21}\left\{282.10 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \begin{bmatrix} 6.00 \\ 7.50 \\ 9.00 \end{bmatrix} - 18.5 \cdot \begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \begin{bmatrix} -57.78 \\ 48.89 \\ 8.89 \end{bmatrix}$$

# Unlimited, Riskless Borrowing and Lending




- Always recall that as you move on the MVF frontier, ptf. structure is affected, often in ways that are hard to guess (see the evolution of the green slice)

# Unlimited, Riskless Borrowing and Lending

- So far, we have ignored the existence of a **risk-free asset** == a security with return $R^f$ known with certainty and zero variance and zero covariance with all risky assets
  - Buying such a riskless asset == lending at a risk-free rate to issuer
  - Assume investor is able to leverage at riskless rate
  - There is no limit to the amount that the investor can borrow or lend at the riskless rate (we shall remove this assumption later)
- Fictional experiment in which the possibility to borrow and lend at $R^f$ is offered to investor who already allocated among N risky assets
- X is the fraction of wealth in an efficient frontier, risky portfolio (A) characterized $\mu_A$ and $\sigma_A$, respectively; a share 1 - X is invested in the riskless asset, to obtain mean and standard deviation:

$$\mu_P = X\mu_A + (1 - X)R^f = R^f + X(\mu_A - R^f)$$

$$\sigma_P = \sqrt{\sigma_P{}^2} = \sqrt{X^2\sigma_A{}^2} = X\sigma_A$$

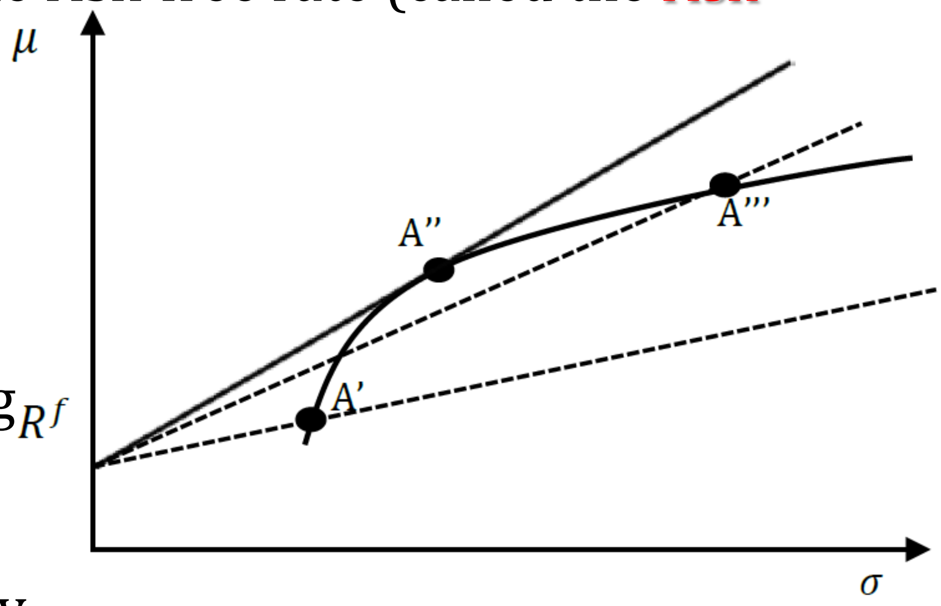
- Solving from X in the first equation and plugging into the second:

$$\mu_P = R_F + \frac{\sigma_P}{\sigma_A}(\mu_A - R^f) = R^f + \frac{(\mu_A - R^f)}{\sigma_A}\sigma_P$$

# Unlimited, Riskless Borrowing and Lending

The capital transformation line measures at what rate unit risk (st. dev.) can be transformed into average excess return (risk premium)

- The equation of a straight line with intercept $R^f$ and slope $(\mu_A - R^f)/\sigma_A$
- This line is sometimes referred to as **capital transformation line**
- The term $(\mu_A - R^f)/\sigma_A$ is called **Sharpe ratio** (SR), the total reward for taking a certain amount of risk, represented by the st. dev.
  - SR is the mean return in excess of the risk-free rate (called the **risk premium**) per unit of volatility
  - The plot shows 3 transformation lines for 3 choices of the risky benchmark A (A', A'', and A''') on the efficient frontier
  - Points to the left of A involve lending at the risk-free rate while the ones to the right involve borrowing
  - As investors prefer more to less, they will welcome a "rotation" of the straight line passing through $R^f$ as far as possible in a counterclockwise direction, **until tangency**



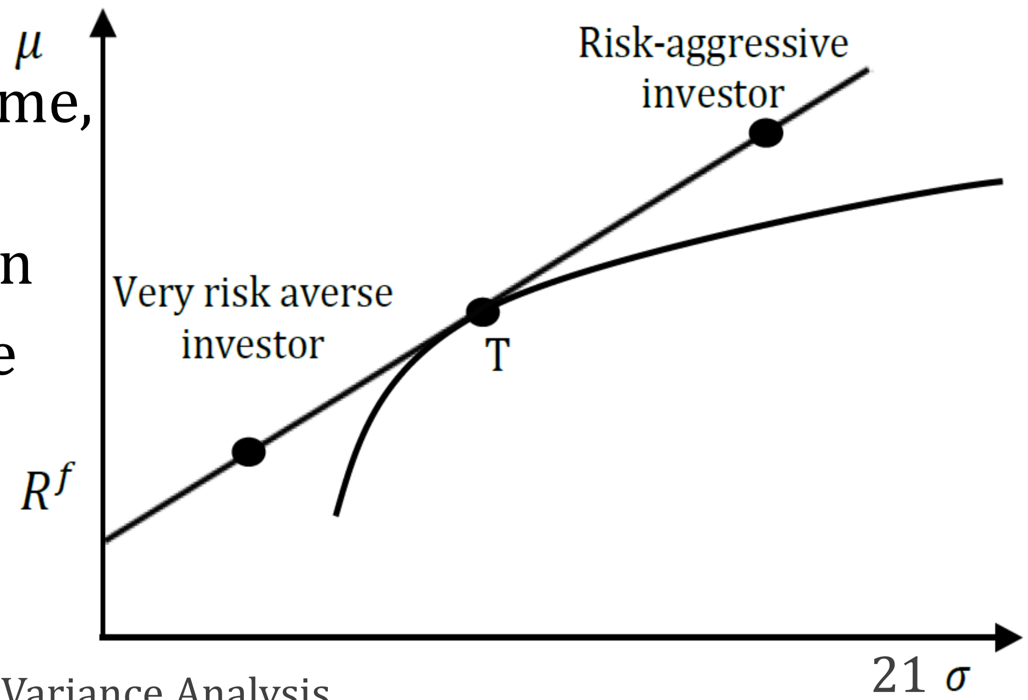
# The Tangency Portfolio and the Capital Market Line

Under no frictions and homogeneous beliefs, there exists a tangency ptf. that maximizes the slope of the transformation line

- Assuming beliefs are homogeneous, there are no frictions or taxes, and that individuals face the same $R^f$ and identical asset menus, all rational, non-satiated investors hold the same tangency portfolio
- It is combined with a certain share of risk-free lending or borrowing
- While the share of wealth an investor lends or borrows at $R^f$ depends on the investor's preference for risk, the risky portfolio should be the same for all the investors
- The steepest CTL gets a special name, the Capital Market Line (CML)
- Special case of two-fund separation
- To determine the tangency ptf. one needs to solve:

$$\max_{\{\omega\}} \frac{(\boldsymbol{\omega}'\boldsymbol{\mu} - R^f)}{(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega})^{\frac{1}{2}}}$$

$$\textit{subject to} \quad \boldsymbol{\omega}'\boldsymbol{\iota} = 1$$



# The Tangency Portfolio and the Capital Market Line

- The textbook explains how the problem may be written as a simple unconstrained max problem that we can solved by solving the FOCs: $$\max_{\{\omega\}} \frac{\boldsymbol{\omega}'(\boldsymbol{\mu} - R^f \boldsymbol{\iota})}{(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega})^{1/2}}$$

- The resulting vector of optimal ptf. weights is: $\boldsymbol{\omega}_T = \dfrac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - R^f \boldsymbol{\iota})}{A - CR^f}$
- Using the same data as in the strategic asset allocation example on three assets – U.S. Treasury, corporate bonds, and equity – we have:

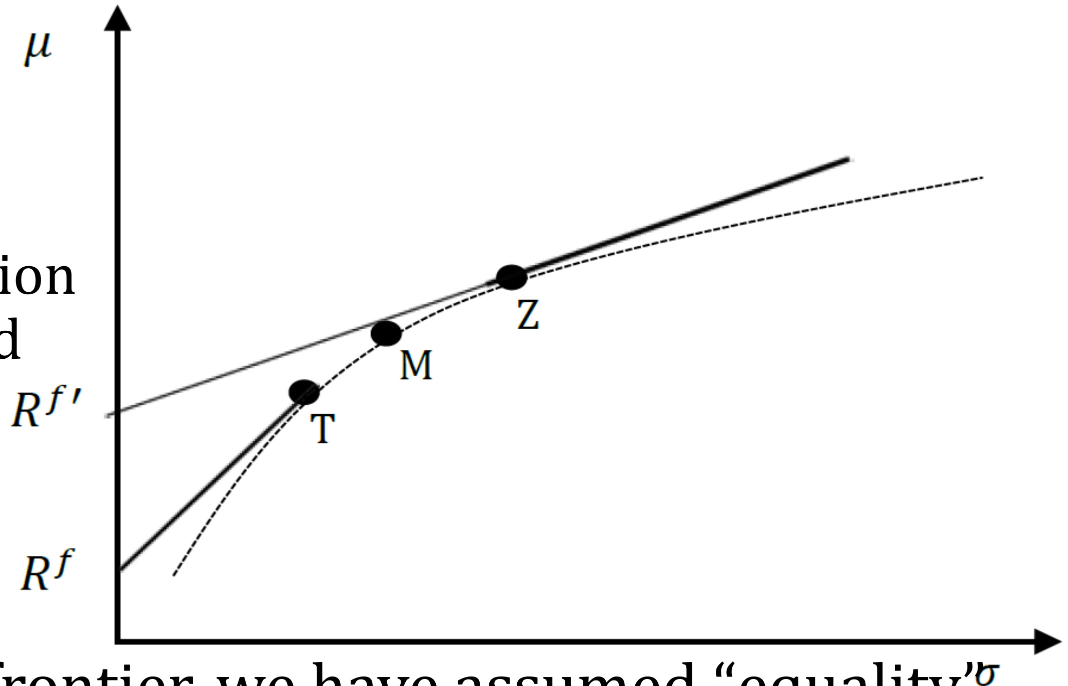
$$\boldsymbol{\omega}_T = \frac{1}{18.5 - 282.11 \cdot 2.5\%}\begin{bmatrix} 402.11 & -223.16 & 58.95 \\ -223.16 & 254.74 & -48.42 \\ 58.95 & -48.42 & 50.53 \end{bmatrix}\begin{bmatrix} 3.50\% \\ 5.00\% \\ 6.50\% \end{bmatrix} = \begin{bmatrix} 0.59 \\ 0.16 \\ 0.25 \end{bmatrix}$$

- Textbook gives indications on how to use Microsoft Excel'sl Solver®
  - The Solver will iteratively change the values of the cells that contain the weights until the value of the Sharpe ratio is maximized
- Up to this point, we have assumed that the investor can borrow money at the same riskless rate at which she can lend
- More reasonable assumption: the investor is able to borrow money, but at a higher rate than the one of the risk free (long) investment

# Unlimited, Riskless Borrowing and Lending

When lending and borrowing is possible at different rates, it is no longer possible to determine a single tangency portfolio

- The figure shows how the CML is modified when borrowing is only possible at a rate $R^{f}{}' > R^{f}$
- There are now two CTLs, both tangent to the efficient frontier
  - All the points falling on the portion of the efficient frontier delimited by T (below) and Z (above) will be efficient even though these do not fall on the straight, CML-type line




- While constructing the efficient frontier, we have assumed "equality" constraints (e.g., portfolio weights summing to one), but no "inequality" constraints (e.g., positive portfolio weights)
- Inequality constraints complicate the solution techniques
- However, unlimited short-selling assumption is often unrealistic (see margin accounts)

# Short-Selling Constraints

- When short-selling is not allowed, portfolio weights should be positive, i.e. the constraint $\boldsymbol{\omega} \geq 0$ (to be interpreted in an element-by-element basis) has to be imposed
  - When $\boldsymbol{\omega}$ has to be positive, the unconstrained maximum may be at a value of that is not feasible
  - Therefore, it is necessary to impose the Kuhn-Tucker conditions
  - The textbook gives a heuristic introduction to what these are
  - Fortunately, Microsoft's Excel Solver® offers the possibility to solve the problem numerically, by-passing these complex analytical details
- Consider again our earlier strategic asset allocation example and let's set $\bar{\mu} = 9\%$
  - In the absence of constraints, the solution is $\boldsymbol{\omega}_T = \begin{bmatrix} -56.66\% \\ 113.33\% \\ 43.33\% \end{bmatrix}$
  - This makes sense because the second asset is characterized by a large Sharpe ratio and hence must be exploited to yield a high mean return by leveraging the first security
  - Selling -57% of the first security is a major hurdle
  - Under nonnegativity constraints we obtain: $\boldsymbol{\omega}_T^{constrain} = \begin{bmatrix} 0\% \\ 0\% \\ 100\% \end{bmatrix}$

# Short-Selling Constraints