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SECOND PART, LECTURE 7: SIMULATION-BASED METHODS IN RISK MANAGEMENT

OVERVIEW

1) Historical Simulations (IID bootstrap): pros and cons

2) Weighted Historical Simulations

3) Monte Carlo Simulation Methods

4) Filtered Historical Simulation Methods

HISTORICAL SIMULATION (HS) METHODS

- Consider the definition of portfolio returns

$$R_{PF,t+1} = \sum_{i=1}^{n} w_i R_{i,t+1}$$

and consider the availability of a past sequence of *m* daily hypothetical ptf. returns, calculated using past returns on the underlying assets, but using today's ptf. weights:

$$\{R_{PF,t+1-\tau}\}_{\tau=1}^{m} \equiv \left\{\sum_{i=1}^{n} w_i R_{i,t+1-\tau}\right\}_{\tau=1}^{m}$$

- The HS technique simply assumes that the distribution of tomorrow's portfolio returns, $R_{PF,t+1}$, is well approximated by the empirical distribution of the past m observations, that is, the histogram of $\{R_{PF,t+1-\tau}\}_{\tau=1}^{m}$
  - The value at risk (VaR) with coverage rate, p, is then simply calculated as 100pth percentile of the sequence of past portfolio returns: $VaR_{t+1}^{p} = -Percentile\left\{\{R_{PF,t+1-\tau}\}_{\tau=1}^{m}, 100p\right\}$

Historical Simulation (HS) Methods

- $Percentile\left\{\left\{R_{PF,t+1-\tau}\right\}_{\tau=1}^{m}, 100p\right\}$ is simply computed by sorting the past *m* returns in ascending order and selecting the number such that only 100p% of the observations are smaller than $Percentile\left\{\left\{R_{PF,t+1-\tau}\right\}_{\tau=1}^{m}, 100p\right\}$
  - As the VaR typically falls in between two observations, linear interpolation can be used to calculate the exact number
- Also called IID bootstrap; HS is widely used in practice. Main reasons: (1) ease of implementation, (2) its model-free nature
  - No parameters have to be estimated by maximum likelihood or any other method: therefore, no numerical optimization has to be performed.
  - The model-free nature of HS also has serious drawbacks: (i) how large should *m* be? (ii) originates box-shaped patterns that arise from the abrupt in/ex-clusion of large losses, (iii) scaling issues over time

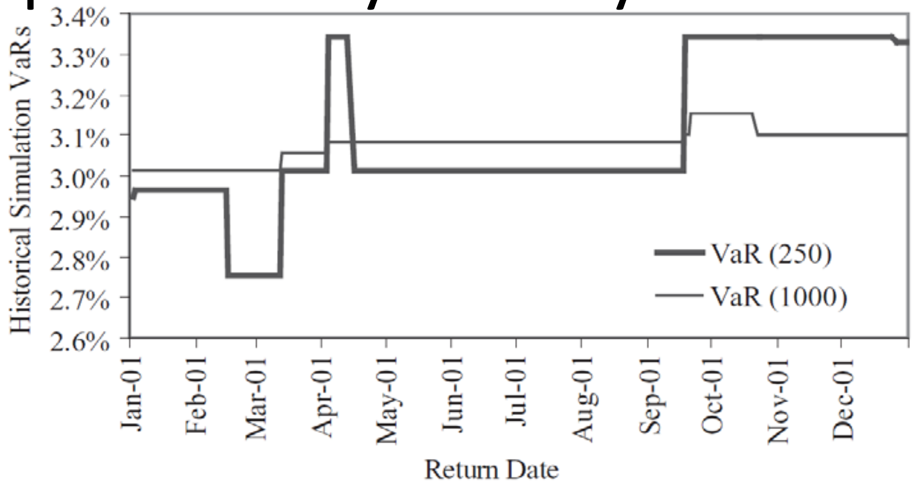
Historical Simulation (HS) Methods

- If m is too large, then most recent observations, which presumably are most relevant for tomorrow's distribution, will carry very little weight, and the VaR will tend to look very smooth over time
- If m is chosen to be too small, then the sample may not include enough large losses to enable the risk manager to calculate, say, a 1% VaR with any precision
- Conversely, the most recent past may be very unusual, so that tomorrow's VaR will be too extreme

– Typically m is chosen in practice to be between 250 and 1,000 days corresponding to approximately 1 to 4 years

– Notice also how the dynamic patterns of the HS VaRs are crucially dependent on *m*

– The lack of properly specified dynamics in the HS methodology causes it to ignore well-



Historical Simulation (HS) Methods

established stylized facts on return dependence—most importantly variance clustering
    - This is responsible for the curious box shapes
- – As the HS method does not rely on a well-specified dynamic model, we have no theoretically correct way of extrapolating from the 1-day distribution to get the T-day distribution other than finding more past data
    - While it may be tempting to multiply the 1-day VaR from HS by $(T)^{1/2}$ to obtain a T-day VaR, doing so is only valid under normality, which the HS approach is tailored to avoid
    - If the data were truly normally distributed, then HS would not be an attractive method at all
- – Under HS, if *m* is too small, then we do not have enough observations in the left tail to calculate a precise VaR measure; if m is too large, then the VaR will not be sufficiently responsive to most recent returns

Weighted Historical Simulations (WHS)

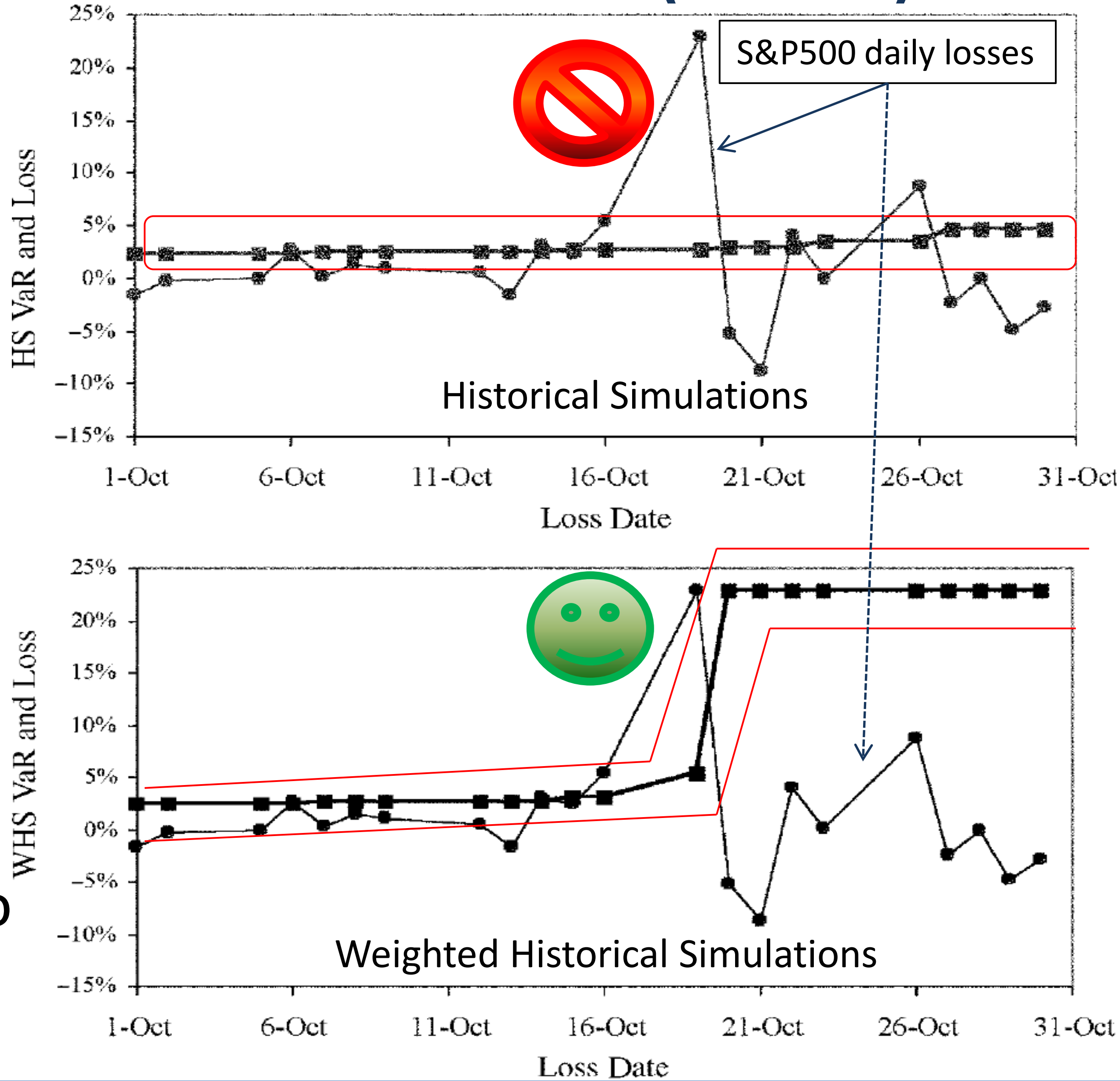
- Weighted historical simulations are designed to relieve the tension in the choice of *m* by assigning relatively more weight to most recent observations and relatively less weight to returns further in the past
  - Typical weighting scheme ξ_τ is declining exponentially:

$$\xi_\tau = \left\{ \eta^{\tau-1}(1-\eta)/\left(1-\eta^m\right) \right\}_{\tau=1}^{m}$$

  - η^τ goes to zero as $\tau \rightarrow \infty$ and the weights τ = 1, 2, …, *m* sum to 1
  - Typically, η is assumed to be a number between 0.95 and 0.99
  - Once η is chosen, the WHS technique still does not require estimation: **100p% VaR is calculated by accumulating the weights of the ascending returns until 100p% is reached**
  - The weighting function builds in some "conditionality" in the technique
  - However, inconsistencies may arise when WHS is computed from the left tail but is applied to SHORT positions

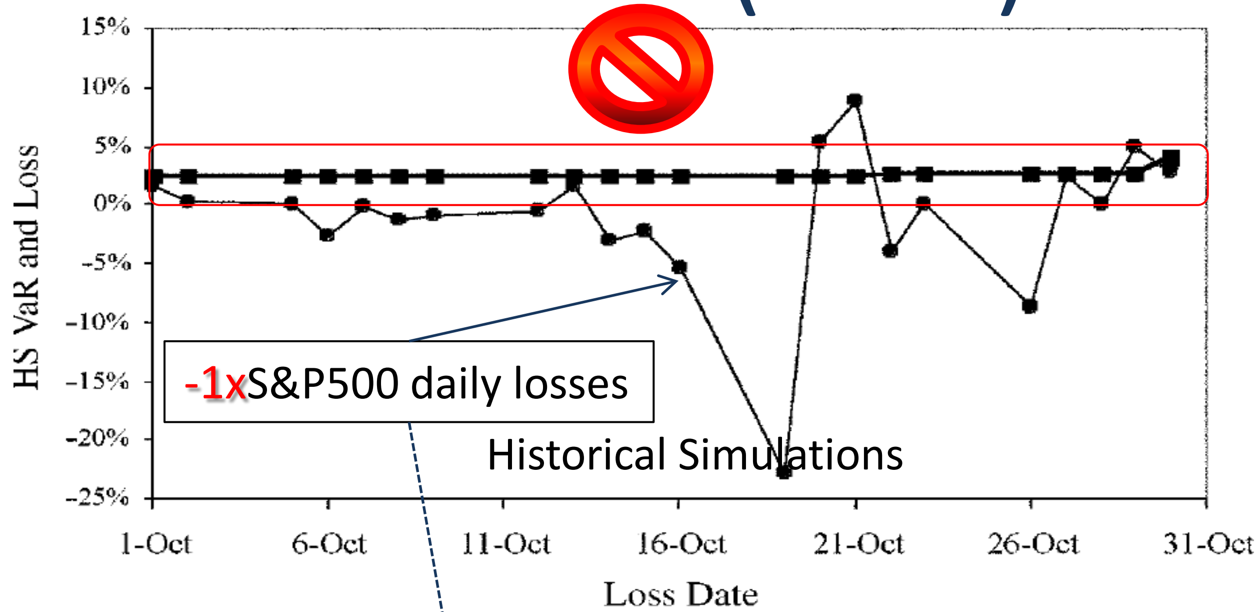
WEIGHTED HISTORICAL SIMULATIONS (WHS)

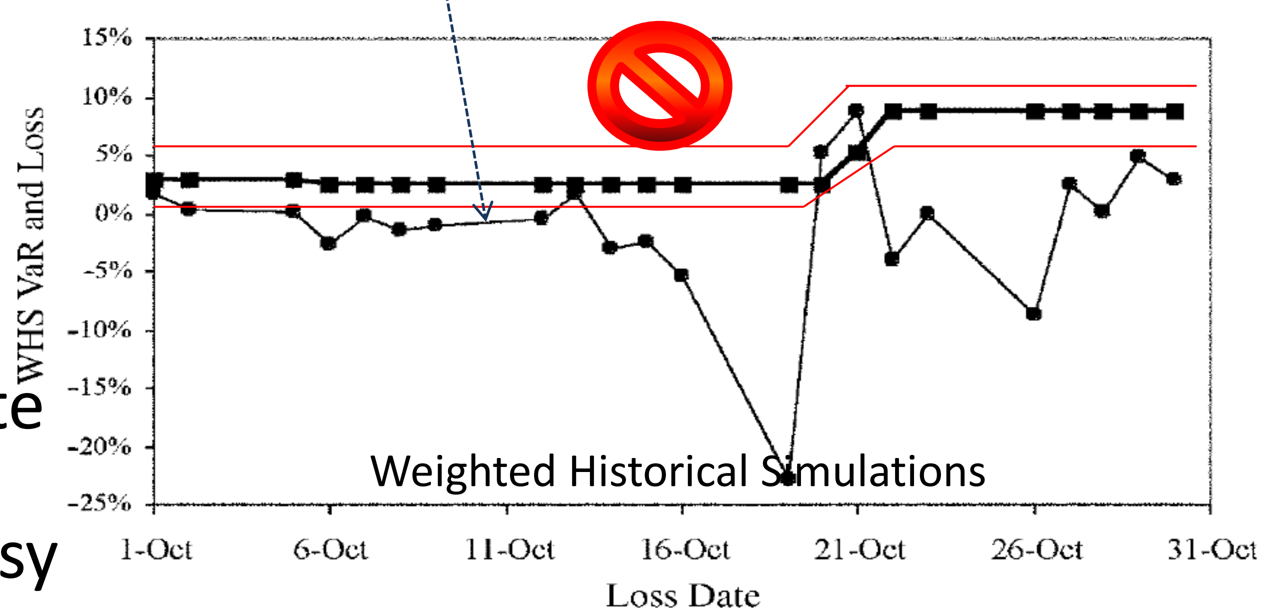
- These are m=250 VaRs computed around the crash of October 1987
- In the case of WHS, $\eta = 0.99$ as often advised
- The VaRs concern a LONG 1$ position in the S&P 500 index
- HS is obviously in-correct and unhelpful
- WHS is instead suffi-ciently "conditional" to help by reacting to the crash…



Weighted Historical Simulations (WHS)

- Again m=250 and η = 0.99 in WHS
- The VaRs concern a SHORT 1$ position in the S&P 500 index
- HS is obviously incorrect and unhelpful
- WHS also does not appear to sufficiently react…
- Solution is simple and yet mind-bogging: for short positions compute VaR on GAINS and not index losses… Just messy





An Example

Data		Long	Short	Long
		HS with m = 5		WHS with m = 5, η=0.9

Date	Return	Date	20% VaR/short position	20% VaR/long position	Date	20% VaR
February 2008	+1.38%	February 2008	–	–	February 2008	–
March 2008	-8.49%	March 2008	–	–	March 2008	–
April 2008	+0.37%	April 2008	–	–	April 2008	–
May 2008	+4.58%	May 2008	–	–	May 2008	–
June 2008	-3.21%	June 2008	4.58%	8.49%	June 2008	3.21%
July 2008	-1.03%	July 2008	4.58%	8.49%	July 2008	3.21%
August 2008	-4.59%	August 2008	4.58%	4.59%	August 2008	4.59%
September 2008	-28.10%	September 2008	4.58%	28.10%	September 2008	28.10%
October 2008	-7.89%	October 2008	-1.03%	28.10%	October 2008	28.10%
November 2008	-3.07%	November 2008	-1.03%	28.10%	November 2008	7.89%
December 2008	-6.88%	December 2008	-3.07%	28.10%	December 2008	7.89%
January 2009	-8.63%	January 2009	-3.07%	28.10%	January 2009	8.63%
February 2009	-0.63%	February 2009	-0.63%	8.63%	February 2009	8.63%

Monte Carlo Simulation Methods (MCS)

- Simple idea, almost the opposite of HS and WHS: if you are ready to write down and estimate a complete parametric model of the dynamic (conditional) stochastic process for asset returns, then
  - Computing VaR proceeds in the same way as under HS, but
  - The *m* historical ptf. returns are replaced by MC simulated returns from the fully parameterized model
  - Let's work through a complete example
  - Consider a Gaussian GARCH(1,1):

$$R_{t+1} = \sigma_{t+1} z_{t+1}, \quad \text{with } z_{t+1} \sim N(0, 1)$$

    Fully parametric because we have specified a Gaussian density
  - Using random number generators generate a set of artificial random numbers $\check{z}_{i,1}$ i = 1, 2, . . . , MC, drawn from the standard normal distribution, N(0, 1)
    - MC denotes the number of draws, e.g., 10,000

Monte Carlo Simulation Methods (MCS)

- From these random numbers we can calculate a set of hypothetical returns for tomorrow $\check{R}_{i,t+1} = \sigma_{t+1}\check{z}_{i,1}$
- Given these hypothetical returns, we can update the variance to get a set of hypothetical variances for the day after tomorrow, t+2, as follows: $\check{\sigma}^2_{i,t+2} = \omega + \alpha\check{R}^2_{i,t+1} + \beta\sigma^2_{t+1}$
- Given a new set of artificial random numbers drawn from the N(0, 1) distribution, $\check{z}_{i,2}$ i = 1, 2, . . . , MC, drawn from N(0, 1), simulate the return on day t + 2 as $\check{R}_{i,t+2} = \check{\sigma}_{i,t+2}\check{z}_{i,2}$ and variance is now updated by $\check{\sigma}^2_{i,t+3} = \omega + \alpha\check{R}^2_{i,t+2} + \beta\check{\sigma}^2_{i,t+2}$
- From these hypothetical returns, we can calculate the hypothetical K-day return:

$$
\begin{array}{lcccc}
 & \check{z}_{1,1} \to \check{R}_{1,t+1} \to \check{\sigma}^2_{1,t+2} & \check{z}_{1,2} \to \check{R}_{1,t+2} \to \check{\sigma}^2_{1,t+3} & \cdots & \check{z}_{1,K} \to \check{R}_{1,t+K} \\
\nearrow & \check{z}_{2,1} \to \check{R}_{2,t+1} \to \check{\sigma}^2_{2,t+2} & \check{z}_{2,2} \to \check{R}_{2,t+2} \to \check{\sigma}^2_{2,t+3} & \cdots & \check{z}_{2,K} \to \check{R}_{2,t+K} \\
\sigma^2_{t+1} \longrightarrow & \cdots & \cdots & \cdots & \cdots \\
\searrow & \cdots & \cdots & \cdots & \cdots \\
 & \check{z}_{MC,1} \to \check{R}_{MC,t+1} \to \check{\sigma}^2_{MC,t+2} & \check{z}_{MC,2} \to \check{R}_{MC,t+2} \to \check{\sigma}^2_{MC,t+3} & \cdots & \check{z}_{MC,K} \to \check{R}_{MC,t+K}
\end{array}
$$

MC sequences of simulated returns between t + 1 and t + K

MONTE CARLO SIMULATION METHODS (MCS)

$$\check{R}_{i,t+1:t+K} = \sum_{k=1}^{K} \check{R}_{i,t+k}, \quad \text{for } i = 1, 2, \ldots, MC$$

- Collect the MC simulated K-day returns in a set $\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}$
- Calculate K-period VaR by calculating the 100p percentile:

$$VaR^{p}_{t+1:t+K} = -Percentile\left\{\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}, 100p\right\}$$

- The key advantage of MCS is its flexibility: you can use it for any assumed distribution of standardized returns—normality is not required (here it is just an example)
  - Key problem: it is slow, especially when MC is set to a large number (e.g., 20,000 or 50,000, as it is advisable)
- Logical incoherence in our approach so far: on the one hand, HS and WHS are model-free but cannot enforce "conditionality" because they lack a time series model
- On the other hand, MCS is tightly parametric

Filtered Historical Simulation (FHS)

  - Taking a model-based approach (MCS) is good if the model is a fairly accurate description of reality; taking a model-free approach (HS) is sensible if observed data may capture features of returns that are not captured by any parametric model
- Anything in between the two?
- Yes, something called Filtered Historical Simulation
- Basic idea is simple: draw not from some assumed parametric distribution but from the past history of (standardized) residuals obtained from a parametric model
- Also called parametric bootstrap
  - Let's illustrate with one example: you have estimated a GARCH(1,1) of ptf. variance; although you are comfortable with the variance model, you are not comfortable making a specific distributional assumption about standardized returns
  - Instead, you would like the past returns data to inform you about the distribution directly without further assumptions

FILTERED HISTORICAL SIMULATION (FHS)

– Given a sequence of past returns, $\{R_{t+1-\tau}\}^{m}_{\tau=1}$, we can estimate the GARCH and calculate past standardized returns from observed returns and from the estimated standard deviations as:

$$\hat{z}_{t+1-\tau} = R_{t+1-\tau}/\sigma_{t+1-\tau}, \quad \text{for } \tau = 1, 2, \ldots, m$$

– At the end of day t we obtain R_t and we can calculate σ^2_{t+1}; instead of drawing random number from some specific distribution, we can **draw with replacement** from our own database of past standardized residuals, $\left\{\hat{z}_{t+1-\tau}\right\}^{m}_{\tau=1}$

– We build up a distribution of hypothetical future returns and from these we calculate the hypothetical K-day returns as

$$\hat{R}_{i,t+1:t+K} = \sum_{k=1}^{K} \hat{R}_{i,t+k}, \quad \text{for } i = 1, 2, \ldots, FH$$

$$\begin{array}{llllll}
 & \hat{z}_{1,1} \to \hat{R}_{1,t+1} \to \hat{\sigma}^2_{1,t+2} & \hat{z}_{1,2} \to R_{1,t+2} \to \hat{\sigma}^2_{1,t+3} & \cdots & \hat{z}_{1,K} \to \hat{R}_{1,t+K} \\
\nearrow & \hat{z}_{2,1} \to \hat{R}_{2,t+1} \to \hat{\sigma}^2_{2,t+2} & \hat{z}_{2,2} \to \hat{R}_{2,t+2} \to \hat{\sigma}^2_{2,t+3} & \cdots & \hat{z}_{2,K} \to \hat{R}_{2,t+K} \\
\sigma^2_{t+1} \longrightarrow & \cdots & \cdots & \cdots & \cdots \\
\searrow & \cdots & \cdots & \cdots & \cdots \\
 & \hat{z}_{FH,1} \to \hat{R}_{FH,t+1} \to \hat{\sigma}^2_{FH,t+2} & \hat{z}_{FH,2} \to \hat{R}_{FH,t+2} \to \hat{\sigma}^2_{FH,t+3} & \cdots & \hat{z}_{FH,K} \to \hat{R}_{FH,t+K}
\end{array}$$

FH sequences of simulated returns between t + 1 and t + K

Filtered Historical Simulation (FHS)

– Collect the FH hypothetical period returns in a set $\left\{\hat{R}_{i,t+1:t+K}\right\}_{i=1}^{FH}$ then you can calculate the K-period VaR as

$$VaR^p_{t+1:t+K} = -Percentile\left\{\left\{\hat{R}_{i,t+1:t+K}\right\}_{i=1}^{FH}, 100p\right\}$$

– Of course, once you have the FHS distribution of returns, you are not restricted to only computing VaR

– E.g., expected shortfall, $ES^p_{t+1:t+K} = -E_t\left[R_{t+1:t+K} | R_{t+1:t+K} < -VaR^p_{t+1:t+K}\right]$ which in this case will be:

=1 when $\hat{R}_{i,t+1:t+K}$ is below -VaR^p

$$ES^p_{t+1:t+K} = \frac{-1}{p * FH} * \sum_{i=1}^{FH} \hat{R}_{i,t+1:t+K} * \mathbf{1}\left(\hat{R}_{i,t+1:t+K} < -VaR^p_{t+1:t+K}\right)$$

– The ES risk measure can be similarly calculated from MCS

– However it is not obvious how to calculate ES from HS or WHS

– An alternative consists in computing FHS GARCH option prices: e.g., feed the FHS paths in slide 14 into

$$\exp(-r\tilde{T})\frac{1}{MC}\sum_{i=1}^{MC} Max\left\{\check{S}^*_{i,t+\tilde{T}} - X, 0\right\} \quad \check{S}^*_{i,t+\tilde{T}} = S_t \exp\left(\sum_{j=1}^{\tilde{T}} \check{R}^*_{i,t+j}\right)$$

i = 1,2,…,FHS

Reading List/How to Prepare the Exam

- YOU NEED TO GET A FULL GRASP OF CHAPTERS 2 AND 6.4 IN CHRISTOFFERSEN'S BOOK
  - Every sentence/equation must (eventually) make sense to you
  - A set of exercises/questions posted on our class web page will help you: WORK ON IT!
- YOU NEED TO WORK THROUGH THE SAMPLE MATLAB CODE AND PRACTICE PROBLEM POSTED ON THE WEB
  - Two passes: in the first one pay attention to the results; in the second start looking at the Matlab code
- WORK ON THE SHORT SET OF REVIEW QUESTIONS
- Barone-Adesi, G., K., Giannopoulos, and L., Vosper. (1999). "VaR without Correlations for Nonlinear Portfolios," *Journal of Futures Markets*, 19, 583-602.
- Christoffersen, P., F. Diebold, and T. Schuermann. (1998, October). "Horizon Problems and Extreme Events in Financial Risk Management," *Economic Policy Review*, NY FED, 109-118.