Università Commerciale Luigi Bocconi

An Introduction to the Use of Realized Variance and Covariance in Risk Management

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20541 – Advanced Quantitative Methods for Asset Pricing and Structuring

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Plan of the Lecture

- From parametric non parametric methods
- The realized variance estimator
- The effects of microstructure noise and sparse estimation
- Adjusting RV estimates for the presence of jumps
- Forecasting realized variance
- The Heterogeneous Autoregressive Realized Variance model
- The realized covariance estimator

Key Idea: Realized Moments as Nonparametric Estimators

- The methods examined in 20192 to model/forecast conditional second moments are parametric ⇒ they write out complete functional and distributional specifications for the random variables
  - This includes stochastic volatility covered in the first part of this course
- However, most of the standard parametric, latent volatility models fail to satisfactorily capture a number of stylized facts
- Key idea: **if continuously observed prices were available and transaction costs did not exist, realized returns and therefore their variation over time would be measured without error**
  - Their realized variance (RV) could be treated as an observable variable
- In reality, prices may only be observed in discrete time, but Merton (1980) noted that the variance over a fixed interval can be estimated fairly accurately as the sum of squared returns, provided that returns are available at a sufficiently high-sampling frequency
- Given the increase in the availability of high-quality transaction data, the use of high-frequency returns to construct ex post (realized) measures of (daily) volatility has gained popularity

Quadratic Variation

- Suppose that the logarithmic price of an asset follows the continuous time semi-martingale (to rule out arbitrage opportunities) diffusion:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t)$$

  - $\sigma(t)$ is strictly positive and square integrable instantaneous volatility
  - W(t) is a standard Brownian motion
- The continuous compounded return over the time interval from t-*k* to t is:

$$R(t,k) = p(t) - p(t-k) = \int_{t-k}^{t} \mu(\tau)d\tau + \int_{t-k}^{t} \sigma(\tau)dW(\tau)$$

- Last term can be seen as the square root of integrated variance ($IV_{t,k}$) also called quadratic variation ($QV_{t,k}$): $\int_{t-k}^{t} \sigma^2(\tau)d\tau$
- Can high frequency data be used to estimate QV/IV? Yes, with RV!

DEFINITION 10.1 (Realized variance)

RV is the sum of squared returns over a period t (usually 1 day):

$$RV_{t,k} = \sum_{j=1}^{m} R^2_{t-k+j/m}, \quad j = 1, \ldots, m,$$

where $R_{t-k+j/m} = p_{t-k+j/m} - p_{t-k+(j-1)/m}$ is the return in the subinterval $[t-k+(j-1)/m, t-k+j/m]$, where $\left\{t-k+\frac{j}{m}, j=1,\ldots,m\right\}$ is the partition of the interval $[t-k, t]$.

The Realized Variance Estimator

Under fairly general conditions (the log-price following a semi-martingale will suffice), RV is a consistent estimator of QV = IV

- E.g., compute daily RV (where the day is assumed to contain 6 trading hours, that is 360 mins) at a sampling frequency of 5 mins
- We divide the 360 min in 72 periods of length equal to 5 min and measure the return over each of them; next, we sum the 72 squared returns
- Therefore m = 72 and the length of the interval is k (equal to 1 day, that is 360 min) divided by $m \Rightarrow$ 5 mins
- This sampling scheme that implies intervals that are equidistant in calendar time is also known as calendar time sampling

- Under fairly general assumptions, RV is a consistent estimator of QV, such that $RV_{t,k} \rightarrow^{p} QV_{t,k}$
  - The sample path variation is not affected by innovations to the drift component $\Rightarrow$ do not need to subtract the mean of the intraday returns
  - This is because the mean term μ(t)dt is of a lower order in terms of second-order properties than the diffusive innovations, $\sigma(t)$dW(t)
  - The nonparametric estimator is based on uncentered squared returns and this is convenient as specifying conditional mean is always hard

Microstructure Noise

- The enemy of the applied use of RV is **microstructure noise**: theoretically, an arbitrary precision in the estimate of the QV can be reached by increasing the frequency of the observations.
- High-frequency data are generally plagued by micro-structure noise that may arise from the bid-ask bounce, asynchronous and infrequent trading, slow response of prices to block trading, etc.
- These microstructure effects may induce spurious autocorrelations in (ultra) high-frequency returns, which can in turn unduly inflate the RV measure over what "pure" measures of prices would justify
- Suppose that $p_{t,j} = p^*_{t,j} + \varepsilon_{t,j}$, where p^* is the latent, true price and ε is the noise $\Rightarrow$ the intraday asset return can be written as:

$$p_{t,j} - p_{t,j-1} = R_{t,j} = R^*_{t,j} + \varepsilon_{t,j} - \varepsilon_{t,j-1} = R^*_{t,j} + v_{t,j}$$

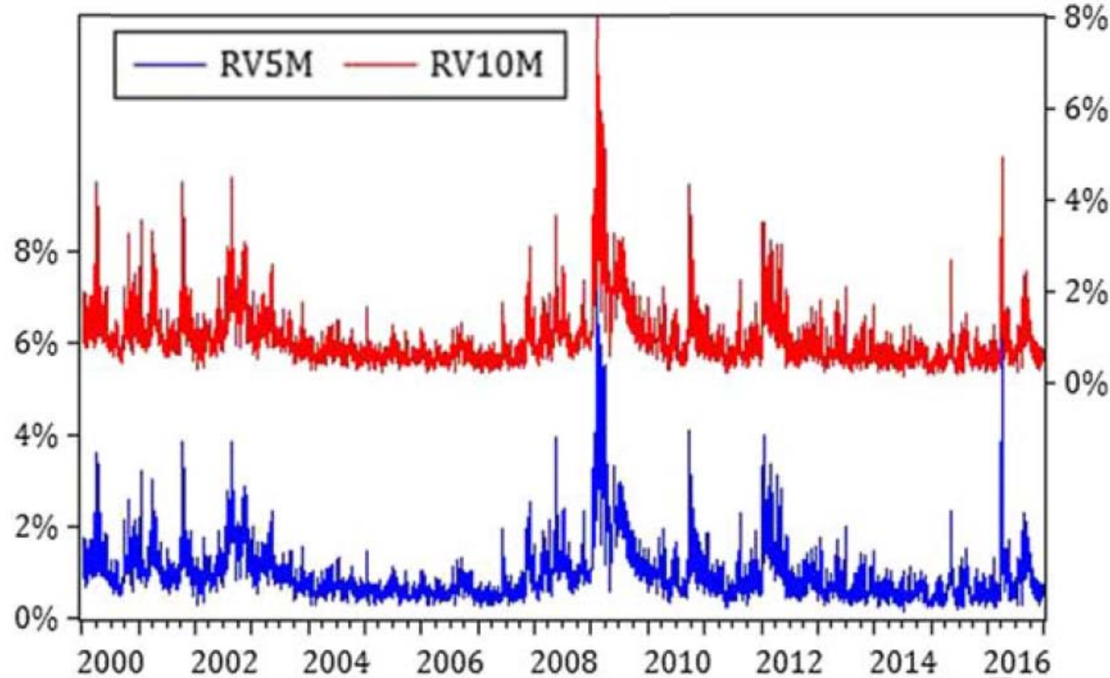
  - Because this is a IMA process, not difficult to say that it is autocorrelated
- Because of MA(1) nature, **this return process is autocorrelated** $\Rightarrow$

$$RV_t = \sum_{j=1}^{m} (R^*_{t,j})^2 + 2\sum_{j=1}^{m} R^*_{t,j} v_{t,j} + \sum_{j=1}^{m} v^2_{t,j} \Rightarrow E\left[RV_t | R^*_t\right] = RV^*_t + \underbrace{2mE\left[\varepsilon^2_{t,j}\right]}_{\text{upward bias}}$$

Microstructure Noise and Sparse Estimation

Under many types of microstructural bias, RV is an upward-biased estimate of QV but using sparse RV provides simple remedy

- In practice, in the presence of a noise of empirically reasonable size, RV is a biased estimator of QV
- One way of dealing with microstructure noise is to construct a grid of intraday returns sampled at a frequency lower than 1-min (the classical "all RV"), as biases cancel out as the frequency declines
- This simple procedure is known as sparse sampling
- For the S&P 500, the (square root of) RV10 (right axis) is slightly noisier than its 5-min counterparty (left axis)
- This is because by choosing a lower sampling frequency, we discard information, thus reducing the efficiency of the estimator



Bias-Adjusted Estimators

The average RV estimator is the arithmetic mean of s alternative, standard RV estimators and better trades-off efficiency and bias

- While sparse sampling helps to reduce microstructure biases, the cost is to lose (potentially important) information
- An intuitive idea is that instead of using sparse estimators with (lower) frequency *s*,

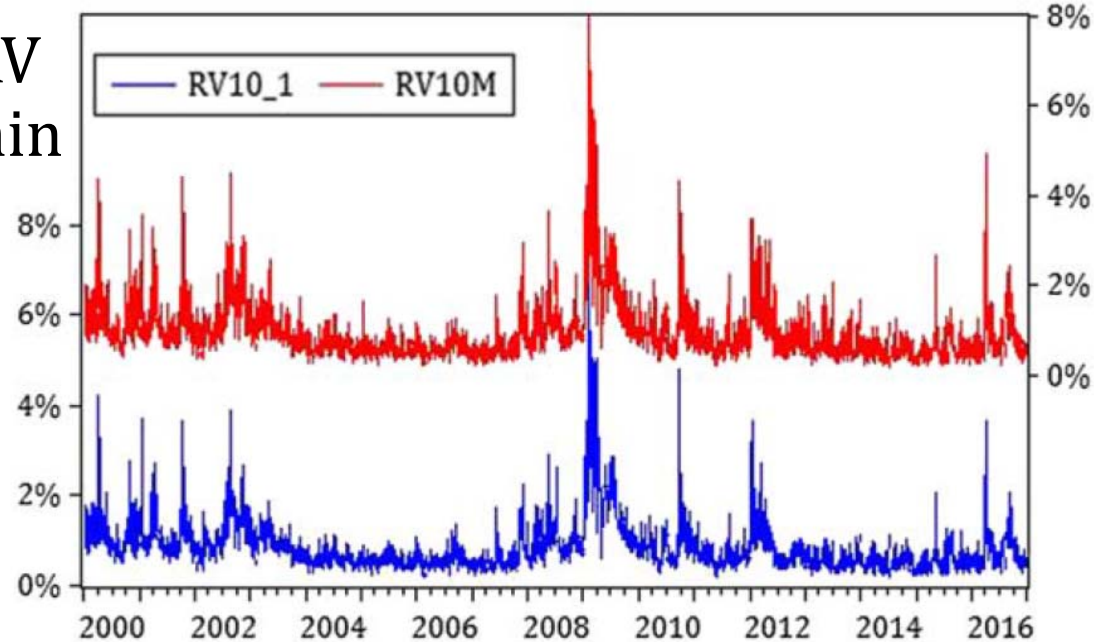
$$RV_t^s = \sum_{j=1}^{m/s} R^2_{t-1+js/m}$$

we average *s* overlapping estimators with *m* frequency each starting at different times: $RV_t^{Avg} = \frac{1}{s}\sum_{i=1}^{s} RV_t^{s,i}.$

  - E.g., to compute the first estimator, start, say, at 8 a.m. and use a 10-min grid to sample the next price at 8:10 a.m. and to compute the first return
  - Next, instead of starting at 8:00 a.m. to compute the second estimator, we shall start at 8:01 a.m., and use again the 10-min grid, by sampling the price to compute the first return at 8:11 a.m.
  - Repeat till we compute the 10th sparse RV estimator starting at 8:09 a.m.
  - Average the resulting *s* sparse RV estimators

Bias-Adjusted Estimators

- The (square root of the) avg. RV estimator, computed at a 10-min frequency but subsampling at a 1-min one is smoother
- Instead of using a lower sampling frequency to remove the autocorrelations, we may try to model them



- For instance, if we were persuaded that return generating process was the one described above, we could correct the RV estimator as:

$$RV_t^{AR(1)} = \sum_{j=1}^{m} (R_{t-1+j/m})^2 + \sum_{j=2}^{m} RV_{t-1+j/m} RV_{t-1+(j-1)/m} + \sum_{j=1}^{m-1} RV_{t-1+j/m} RV_{t-1+(j+1)/m}$$

- Generalizing of this approach, realized kernel estimators of QV
- Kernels are just flexible (weighting) functions to deal with the issue of microstructure noise in high-frequency data, e.g.:

$$RV_t^{HL} = RV_t + 2\sum_{h=1}^{H} K\left(\frac{m}{m-h}\right)\hat{\gamma}_h \qquad \hat{\gamma}_h = m(m-h)^{-1}\textstyle\sum_{j=1}^{m-h} R_{t,j}R_{t,j+h}$$

Bias-Adjusted Estimators

- o Despite being unbiased, this estimator is not consistent, i.e., rather oddly, it has the right expectation for finite *m*, but it diverges away from true, unobserved QV as *m* grows, i.e., when we truly move to continuous time

- Recently, Barndorff-Nielsen, Hansen, Lunde and Shephard (2008) have proposed a class of unbiased and consistent estimators, the flat-top kernel-based estimators:

DEFINITION 10.3 (Flat-top kernel-based estimator)

The flat-top, kernel-based estimator is computed as:

- Bartlett's kernel, $K(x) = 1 - x$
- Second-order kernel, $K(x) = 1 - 2x - x^2$.
- Epanechnikov's kernel, $K(x) = 1 - x^2$.

$$RV_t^{BHLS} = RV_t + \sum_{h=1}^{H} K\left(\frac{h-1}{H}\right)(\hat{\gamma}_h + \hat{\gamma}_{-h}),$$

where the kernel $K(x)$ for $x \in [0,1]$ is a nonstochastic weighting function such that $K(0) = 1$ and $K(1) = 0$, $h = -q, \ldots, q$, RV_t is RV, and $\hat{\gamma}_h$ is again the hth realized autocovariance, computed as earlier.

- It is common knowledge that prices tend to show sudden, discrete movements, when unexpected news hit the market, i.e., jumps:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \xi(t)dq_t$$

- q is a Poisson process uncorrelated with the Brownian motion dW and governed by the jump intensity λ_t, that is, Prob(dq_t = 1) = $\lambda_t$dt

Jumps Processes and Bi-Power Variation

- $\xi(t)$ represents the magnitude of the jump at *t*, should a jump occur
- Under this SDE, the QV of returns over the interval [*t* - *k*, *t*] is given by the sum of the diffusive IV and the cumulative squared jumps:

$$QV_{t,k} = \int_{t-k}^{t} \sigma^2(\tau)d\tau + \sum_{t-k \le \tau \le t} J^2(\tau)$$

Jump counter

- Although the RV estimator remains a consistent measure of the total QV also in presence of jumps, the diffusive and jump volatility components generally display different persistence properties
- As a consequence, Barndorff-Nielsen and Sheppard (2004) suggest to separately estimate the IV_t and the jump components using the *h*-skip bipower variation measure (henceforth BV) to estimate IV:

$$BV(t,k,h,m) = \frac{\pi}{2}\sum_{j=h+1}^{mk}\left|R\left(t-k+\frac{jk}{m},\frac{1}{m}\right)\right|\left|R\left(t-k+\frac{(j-h)k}{m},\frac{1}{m}\right)\right|$$

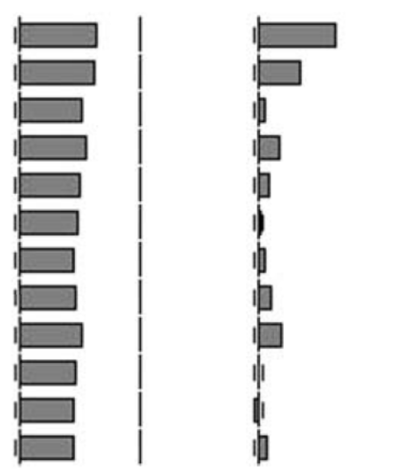
  - When *h* = 1, this is the famous realized bipower variation
- These measures, as *k* and *h* change, are robust to presence of jumps

Forecasting Realized Variance

- If we subtract realized BV from RV, we obtain a consistent estimate of the cumulative squared jump component:

$$RV(t,k,m) - BV(t,k,m) \xrightarrow{m\to\infty} QV(t,k) - IV(t-k) = \sum_{t-k\leq\tau\leq t} J^2(\tau)$$

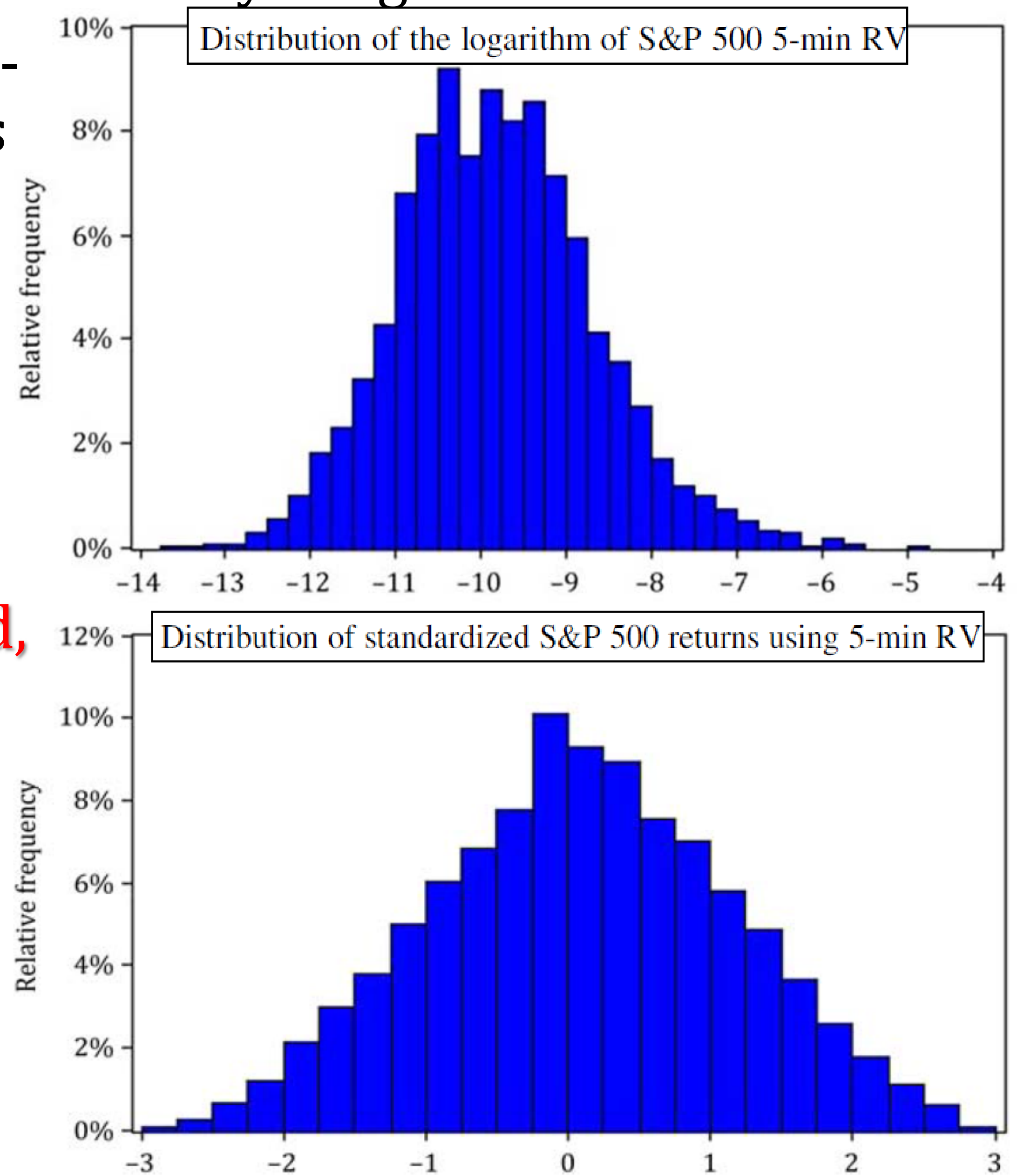
- If the diffusive and the jump components display different persistence properties, separately forecasting each component should lead to better predictions
- How do we forecast RV? Unfortunately, RV is an estimate of the true, unobserved ex post (daily) return variation ⇒ we will be able to compute the RV of day *t* only after the market has closed
- More useful to traders to forecast the *h*-step-ahead RV
- RV is predictable using ARMA models because persistent
- A second stylized fact concerns the distribution of log-RV that is approx. normally distributed



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Probability
		1	0.665	0.665	1976.54	0.000
		2	0.648	0.368	3850.46	0.000
		3	0.546	0.061	5183.91	0.000
		4	0.567	0.182	6619.98	0.000
		5	0.524	0.083	7848.33	0.000
		6	0.495	0.016	8943.05	0.000
		7	0.472	0.048	9938.75	0.000
		8	0.486	0.104	10993.79	0.000
		9	0.536	0.190	12280.11	0.000
		10	0.490	−0.006	13353.97	0.000
		11	0.465	−0.030	14321.91	0.000
		12	0.467	0.074	15298.69	0.000

Forecasting Realized Variance

- Equivalently, RV is well approximated by a log-normal distribution
- A third fact concerns the distribution of standardized returns (i.e., in this case, returns divided by their realized std. dev.)
- While GARCH or SV models ⇒ standardized returns (z_t) are often not normally distributed (keep display excess kurtosis), **when realized volatility is used, z_t is approximately normal**
- When applied to forecast RV, ARMA(p, q) models tend to be richly parameterized
- Corsi (2009) has proposed an alternative, the heterogeneous autoregressive (HAR) model



The Heterogeneous Autoregressive Model

- The HAR models adds to RV also weekly and monthly moving averages of RV to capture the long-memory dynamics of the process

DEFINITION 10.4 (HAR model)

Given the weekly and the monthly moving averages of RV, that is $RV_{W,t}$ and $RV_{M,t}$, respectively, and daily RV such that $RV_{D,t} \equiv RV_t$, according to the HAR model, the one-step-ahead forecast is:

$$RV_{t+1} = \phi_0 + \phi_D RV_{D,t} + \phi_W RV_{W,t} + \phi_M RV_{M,t} + \varepsilon_{t+1},$$

where

$$RV_{W,t} \equiv RV_{t-4,t} = \frac{[RV_{t-4} + RV_{t-3} + RV_{t-2} + RV_{t-1} + RV_t]}{5},$$

$$RV_{M,t} \equiv \frac{[RV_{t-20} + RV_{t-19} + \cdots + RV_t]}{21},$$

assuming 5 trading days in a week and 21 trading days in a month.

- The HAR model has only four parameters and can be estimated simply by OLS, as it does not contain the MA component
- Because it features long lags of past RV through the monthly and weekly RVs, although very simple it is powerful enough to capture very high and slowly decaying persistence, as typical of fractionally integrated models

The Heterogeneous Autoregressive Model

- Considering the (approximate) log-normality of RV, it is best to estimate HAR models of the log transformation of RV (to obtain more precisely estimated coefficients by OLS), as in (ε_{t+1} IID $N(0, \sigma_\varepsilon^2)$)

$$\ln(RV_{t+1}) = \phi_0 + \phi_D \ln(RV_{D,t}) + \phi_W \ln(RV_{W,t}) + \phi_M \ln(RV_{M,t}) + \varepsilon_{t+1}$$

  - In order to obtain *h*-step-ahead forecast, we "undo" the log transform:

$$RV_{t+1|t} = \exp[\phi_0 + \phi_D \ln(RV_{D,t}) + \phi_W \ln(RV_{W,t}) + \phi_M \ln(RV_{M,t})] \times \exp(\sigma_\varepsilon^2/2) = (RV_{D,t})^{\phi_D}(RV_{W,t})^{\phi_W}(RV_{M,t})^{\phi_M}\exp(\phi_0 + \sigma_\varepsilon^2/2).$$

- Many applications, such as portfolio choice and risk management require measuring and also forecasting correlations
- Multivariate applications of realized second moment estimates are made difficult by delayed reactions of one security to changes in the prices of related assets and by nonsynchronous trading effects
- **The asynchronous nature of intraday prices biase realized cova-riances toward 0, unless an appropriate adjustment is made**
- The downward bias occurs because when trading is infrequent, news that affect a pair of assets will be incorporated at different times simply as a result of asynchronous trading

The Realized Covariance Estimator

- Following the same reasoning that we applied in deriving RV measures, an estimate of the realized daily covariance is:

DEFINITION 10.6 (Realized covariance)

The daily realized covariance between two assets computed from intraday returns recorded at m subintervals of length $1/m$ is simply obtained as the sum of the cross products of the intraday returns, that is:

$$RCov^{m}_{12,t} = \sum_{j=1}^{m} R_{1,t-1+j/m} R_{2,t-1+j/m}. \quad (10.32)$$

Therefore, the realized correlation can be computed as:

$$RCorr^{m}_{12,t} = \frac{RCov^{m}_{12,t}}{\sqrt{RV^{m}_{1,t} RV^{m}_{2,t}}}, \quad (10.33)$$

where $RV^{m}_{1,t}$ and $RV^{m}_{2,t}$ are the daily RVs of the two assets.

- As already discussed in the case of RV, the higher the sampling frequency we use, the more efficient the forecast that we obtain
- However, high-frequency data are plagued by microstructure noise
- Two approaches to tackle the problem of asynchronous trading

① Fix an interval, and use the last quote prior to its beginning or, alternatively, the interpolation of the 1st and last price in the interval

The Realized Covariance Estimator

- At least one quote should be available for both assets in the chosen time interval for this algorithm to be applicable

② Denoting as τ(1) the first point in time when both assets have changed their price at least one time since market opening; as soon as we have identified τ(1) we repeat the exercise and denote by τ(2) the first point in time when both assets have changed their prices again

- We iterate this time labeling algorithm until the end of the day, obtaining τ(*j*) data points with j = 1, …, *M*
- We proceed in this way, until we exhaust the available 1-min price grid for both assets
- At the end of the trading day, the synchronized intraday returns for the two assets are:
$$R_{1,\tau(j)} = \ln(P_{1,\tau(j)}) - \ln(P_{1,\tau(j-1)})$$
$$R_{2,\tau(j)} = \ln(P_{2,\tau(j)}) - \ln(P_{2,\tau(j-1)})$$
- The realized daily covariance between the assets can be computed as:
$$RCov_{12,t}^{\text{Sync}} \equiv \sum_{j=1}^{M} R_{1,\tau(j)} R_{2,\tau(j)}$$

The Realized Covariance Estimator

- To obtain the covariance matrix and the realized correlation, it is necessary to compute the RV estimates for the assets on the same time grid τ(j) that has been employed for the covariance
- This approach can be generalized to any number N > 2 assets but the more are the assets involved, the more will be the burden coming from asynchronicity in trading, as measured by the difference between the number of recorded prices on the high-frequency grid recorded for each of the assets and *M*
- When all the assets are highly illiquid, and *N* becomes very large, it is possible for *M* to decli-ne to a very small number, which entails a considerable loss of efficiency