Caution and Reference Effects[*]

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Abstract

We introduce Cautious Utility, a new model based on the idea that individuals are unsure of trade-offs between goods and apply caution. The model yields an endowment effect, even when gains and losses are treated symmetrically. Moreover, it implies either loss aversion or loss neutrality for risk, but in a way unrelated to the endowment effect, and it captures the certainty effect, providing a novel unified explanation of all three phenomena. Cautious Utility can help organize empirical evidence, including some that directly contradict leading alternatives.

Keywords: Cautious Utility, Endowment Effect, Loss Aversion, Certainty Effect, Non-Expected Utility, Cumulative Prospect Theory.

1 Introduction

A prominent place in behavioral economics is held by the *endowment effect*: the widely documented observation that the maximum price individuals are willing to pay to acquire a good (WTP) is often below the minimum price they are willing to accept to sell the same good if they owned it (WTA) (Kahneman et al., 1991). The far dominant explanation in economics ascribes it to an asymmetry in the treatment of gains and losses: if selling a good is perceived as a loss, and losses are overweighted, then individuals are reluctant to sell, creating the endowment effect.

This paper introduces a new model of the endowment effect built on a different idea: individuals are *unsure of the trade-off* they should apply—for example, they may be unsure whether a mug is worth \$3 or \$4—and, facing this uncertainty, apply a criterion of *caution*. We call it *Cautious Utility*.

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We show the following features. First, Cautious Utility leads to the endowment effect *even when utilities are symmetric for gains and losses*; its extent depends on the degree of uncertainty about trade-offs. Second, the model yields loss aversion for risky lotteries (subjects reject bets that return identical gains and losses with equal probability, Kahneman and Tversky 1979), but this is not necessarily related to the endowment effect, in the sense that individuals may exhibit the endowment effect and be loss neutral, or vice-versa. Third, Cautious Utility captures the certainty effect and other forms of Non-Expected Utility, therefore providing a novel, *unified explanation* to Non-Expected Utility and reference effects. In fact, relying on our previous work (Cerreia-Vioglio et al., 2015), we show that Cautious Utility can be *derived* from a behavioral property that imposes a form of certainty effect over bundles, therefore showing a high-level connection between the endowment effect and the certainty effect. Finally, Cautious Utility can help organize existing evidence, including where the endowment effect should be prevalent, as well as observations directly at odds with leading alternatives.

The endowment effect is widely and robustly documented in the lab and in the field (Horowitz and McConnell, 2002; DellaVigna, 2009; Anagol et al., 2018; O'Donoghue and Sprenger, 2018; Chapman et al., 2023a). It has acquired a prominent role in behavioral economics both for its conceptual importance, as it contradicts the standard assumption in economics that a unique value regulates trade and purchasing decisions, and for its practical implications, as it leads to regions of no trade that reduce the efficiency of markets. Understanding its origin and how to model it correctly is key to any attempt to study its consequences, predict where it should be most prevalent, and design policies to reduce it.

Cautious Utility. In our model, like in standard models of reference dependence, individuals consider changes with respect to a given reference point; specifically, they evaluate lotteries over these changes for bundles in $\mathbb{R}^k$, where the first dimension is money. Contrary to most models, in Cautious Utility individuals have not one, but a *set* of utilities $\mathcal{W}$, and use the most pessimistic one to evaluate each option. Specifically, if v is a utility function and p a lottery, call $c(p, v)$ the monetary certainty equivalent of that lottery using v—the amount of money indifferent to p for utility v. Cautious Utility assigns to p the value

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v).$$

The key ideas are *i*) individuals may be unsure of how to evaluate bundles—they may entertain multiple utility functions as plausible; and *ii*) facing this multiplicity, they choose with *caution*—using the utility that returns the lowest monetary certainty equivalent.

To illustrate, consider individuals who evaluate bundles of money (dimension 1) and mugs (dimension 2), as in the famous experiment of Kahneman et al. (1990). These individuals contemplate two utilities: $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. It is as if they are unsure whether a mug is worth \$1 (under v_1) or \$2 (under v_2). The multiplicity of utilities captures the uncertainty about trade-offs. Cautious Utility stipulates that, in the face of this uncertainty, the individuals use the utility that returns the lowest value (in terms of monetary certainty equivalents).

We show that Cautious Utility implies the endowment effect. For an intuition, consider our example, and note that the worst-case scenario when buying a mug is when it is least valuable: thus, the WTP is calculated using v_1 and equals \$1. The worst case when selling the mug is instead when it has the highest value: the WTA is calculated using v_2 and equals \$2. Thus, WTA>WTP. Despite its simplicity, this example captures a broader result: we show that, whenever there is some uncertainty on the trade-off between a good and money, the WTA is strictly above the WTP, generating the endowment effect.

Our approach differs from the standard explanation based on overweighting losses: note how in the example above, all utilities are *symmetric* for gains and losses. The key drivers are instead the uncertainty about trade-offs and caution. This is a different and independent channel from any asymmetry in the utilities. To make explicit the role of caution, in parts of the paper we focus on *symmetric sets* of utilities—either all utilities are symmetric for gains and losses, or, if one is not, the set also includes the specular function; we call this case *Symmetric Cautious Utility*. We also show that 'incautious' individuals, those who use the sup instead of the inf, exhibit the opposite of the endowment effect (as well as of the other effects discussed below, loss aversion for risk and the certainty effect).

Aside from the general model, we present two special cases—with linear and power utilities—that involve few parameters, making applications and estimations much easier while capturing our behaviors of interest. We also show how our approach extends to exchange asymmetries and stochastic reference points and can easily generate the endowment effect for lottery tickets.

Loss Aversion for Risk. Kahneman and Tversky (1979) note how individuals often reject bets that return identical gains and losses with equal probability; call this *loss aversion for* *risk*. Cautious Utility generates (weak) loss aversion for risk, even with symmetric sets of utilities, ruling out the opposite, even locally.[1]

Importantly, under Cautious Utility loss aversion for risk is unrelated to the endowment effect: we may have the endowment effect even with loss neutrality for risk, or loss aversion for risk and no endowment effect; in our example above, individuals are loss neutral for risk (because each utility is risk neutral), yet WTA>WTP.

Non-Expected Utility: A Unified Explanation. Cautious Utility generalizes Cautious Expected Utility of Cerreia-Vioglio et al. (2015) and, like that model, captures the certainty effect and Allais' paradoxes. Intuitively, degenerate lotteries that return a given amount of money have the same certainty equivalent with any utility, making caution irrelevant. But caution does matter for general lotteries, lowering their value and generating an advantage for sure amounts. Indeed, the same forces that generate the endowment effect—uncertainty about the utility and caution—also give the certainty effect and loss aversion for risk, providing a unified explanation.

Empirical Evidence. In Section 4, we show that several documented empirical patterns are compatible with Cautious Utility and not with other models, and vice-versa. Most important is the nature of these patterns: Do they test core aspects of the model? Do they represent important behavioral regularities that we want to capture? Cautious Utility is distinct from leading alternatives also in these more critical dimensions.

First, recent evidence contradicts the core idea that the endowment effect is due to loss aversion. If the endowment effect derives from the overweighting of losses, then it should be highly correlated with loss aversion for risk—as this is the most direct manifestation of overweighting of losses. However, Chapman et al. (2023a) robustly show that the endowment effect and loss aversion for risk are *not* correlated, with a sizable fraction of subjects exhibiting the endowment effect while being loss neutral for risk (for the same good). This is in direct contradiction to the core idea of loss-aversion-based explanations. Moreover, several papers show that the endowment effect holds robustly in several contexts, while the evidence of loss aversion for risk is much less robust. Cautious Utility, instead, decouples the endowment effect and loss aversion for risk, allowing for any correlation and loss neutrality for risk despite an endowment effect.

[1]We add 'for risk' to the standard term 'loss aversion' to avoid confusion, as the same term is also used to denote the asymmetry parameter of Cumulative Prospect Theory (the coefficient λ; Section 1.1).

Second, Cautious Utility can help organize the evidence of where the endowment effect is strongest and how it reacts to external events. If the endowment effect depends on the uncertainty about trade-offs, as in Cautious Utility, then it should *i*) vary with how familiar the goods are, with higher endowment effect when trade-offs are more uncertain; and *ii*) vary with information about the value of the objects, as it affects the trade-offs that subjects consider, with information pointing to the middle of the range shrinking the effect. Robust evidence gives strong empirical support to both predictions: meta-studies of the decades of research on the endowment effect document how the strength and frequency of the endowment effect vary substantially across goods, decreasing with familiarity; and many papers show how it is also heavily affected by information, diminishing to the point of disappearing in some cases. Neither pattern is predicted by loss-aversion-based explanations—one would need to assume that the pain of losing an object is high for unfamiliar goods, and decreases with information.

Foundation: Endowment Effect from the Certainty Effect. Dillenberger (2010) introduced the property of Negative Certainty Independence to capture the certainty effect over money. We show that Cautious Utility is characterized by an extension to a form of *certainty effect over bundles*, together with basic postulates (e.g., monotonicity). Paired with our result that Symmetric Cautious Utility returns the endowment effect and loss aversion for risk, our final result shows that, given symmetry and basic postulates, a form of certainty effect for bundles *formally implies* the endowment effect and loss aversion for risk. To our knowledge, such a formal relationship between Non-Expected Utility and reference effects is novel.

1.1 Related Theoretical Literature

(Cumulative) Prospect Theory. The most popular model to study our behaviors of interest is Cumulative Prospect Theory (Tversky and Kahneman, 1992), henceforth CPT, which extends the original Prospect Theory (Kahneman and Tversky, 1979). Violations of Expected Utility are captured by probability weighting. Reference dependence is captured separately, by positing that individuals evaluate changes relative to a reference point and that 'losses loom larger than gains.' The latter is formalized by assuming that the utility is not symmetric for gains and losses and losses weigh more—a common approach is to take $\lambda > 1$ and $v(-x) = -\lambda v(x)$ for $x > 0$. This asymmetry reduces the value of even bets around zero and generates a gap between WTA and WTP.

Cautious Utility is different. Probabilities are taken at face value, not weighted, and instead of a single asymmetric utility, we have many utilities—possibly all symmetric. All three biases come from the same source, uncertainty about the utility and caution. The two models are not only conceptually different but also behaviorally distinct: we show that the only preferences compatible with both our model and CPT are standard Expected Utility with no reference effects. In Section 4, we discuss several implications of this difference, including documented behaviors that are compatible with one model but not the other.

Cautious Expected Utility. Our approach builds on Cerreia-Vioglio et al. (2015), which studies preferences over monetary lotteries on a bounded interval that admit the following *Cautious Expected Utility* representation: there exists a set $\mathcal{W}$ of strictly increasing and continuous functions over money such that the value of a lottery p is given by $\inf_{v\in\mathcal{W}} c\,(p,v)$. We extend this model and its characterization to bundles of goods (explicitly discussing gains/losses and symmetry) and to unbounded spaces. We show that, in this extension, the same forces that generate the certainty effect over money also generate the endowment effect and loss aversion for risk, providing a new model for these phenomena. Cerreia-Vioglio (2009) characterizes preferences that satisfy convexity and shows that they can be represented with a set of utilities and pessimism, connecting convexity with a preference for hedging in the face of uncertainty about the value of outcomes, future tastes, or the degree of risk aversion. Our model is a special case, as our preferences are convex.

Incomplete Preferences, Preference Imprecision, and Perception. An alternative approach to studying reference effects is via incomplete preferences (Bewley, 1986; Masatlioglu and Ok, 2005, 2014; Ortoleva, 2010; Ok et al., 2015). (These papers are typically silent on loss aversion for risk or the certainty effect as they do not study risk preferences.) Agents have an incomplete preference relation and deviate from their reference point (or status quo) only if an alternative is better according to that relation, generating status quo bias and the endowment effect. As incomplete preferences can be represented using multiple utilities, here, too, the endowment effect is related to the inability to compare bundles.

Cerreia-Vioglio et al. (2015) show that Cautious Expected Utility can be derived as a completion of an incomplete relation, and the same is true here. Indeed, the literature on incomplete preferences was an inspiration for our work. However, there are three critical differences. First, our preferences are complete: our agent uses caution as a criterion to complete them, and this criterion drives our results. Second, risk plays a central role in our paper: we *derive* reference effects from a form of certainty effect and connect the different phenomena; there is no similar link in the models above. Third, the models above only specify behavior when the status quo, or the reference point, is available in the choice set or when there is no status quo; here, instead, the behavior is specified independently of the availability of the status quo.

Sagi (2006) introduces a notion of acyclicity for complete preferences relations indexed by a reference point, similar to status quo bias and no regret. He shows that this necessitates a form of local linearity and that, when extended under risk, it is violated by a specific extension of CPT to stochastic reference points. The conclusion of the paper notes that the property would be satisfied by a model similar to ours, except that it does not include certainty equivalents and distorts probabilities using probability weighting.[2]

The difficulty in making comparisons also relates to the literature on preference imprecision (Dubourg et al., 1994, 1997; Butler and Loomes, 2007, 2011; Cubitt et al., 2015), imprecise perception and rational inattention (Gabaix and Laibson, 2017; Woodford, 2020), or cognitive uncertainty (Enke and Graeber, 2023). However, almost none of these papers studies the endowment effect,[3] and none includes the central contribution of our paper, caution as a rule of choice; we show that it is precisely caution that yields the endowment effect and loss aversion (while "incaution" yields the opposite).

Other Explanations. Other accounts of the endowment effect are based on memory (Johnson et al., 2007), while versions of saliency can generate all our behaviors of interest (Bordalo et al., 2012). Weaver and Frederick (2012) propose that an endowment effect emerges for individuals who consider both a value v and a reference price p, and *i*) do not want to pay more than the smallest of the two, and *ii*) are not willing to accept less than the largest. They are unwilling to pay more/accept less than their value, but they think that paying more/accepting less than the reference price would be a bad deal. This approach is reminiscent of ours because individuals use the maximum of two values for the WTA and the minimum for WTP. Our approach, however, is built on uncertainty about values, absent in Weaver and Frederick (2012) (the value and the reference price are assumed to be known),

[2]A working paper version, gently shared by the author, extends on this and shows that a version without probability weighting (and no certainty equivalents) can be axiomatized adding a weakening of independence. This model is closely related to Maccheroni (2002) and exhibits a form of status quo bias and a form of loss aversion but not the certainty effect. The paper does not include a formal analysis of the endowment effect.

[3]One exception is the experiment in Dubourg et al. (1994), which finds that preference imprecision has (some) relation to the endowment effect and that subjects tend to select above the middle of the range for WTA and below the middle for WTP; both findings are in line with our results.

as well as caution, with no reference to market mechanisms.

We conclude by noting that, while some aspects of our model are reminiscent of existing approaches, to our knowledge, new to the literature is our result on the connection between the certainty effect for bundles and the endowment effect.

2 Cautious Utility

Before we introduce our model, a brief methodological discussion may be useful, for there are two approaches we can take.

One approach, common in decision theory, is to start from a general functional form built on the least restrictive axiomatic foundation and show that, despite its generality, this model delivers the desired reference effects. For this analysis, one would like the most general model, derived from the weakest axioms, because it gives the strongest results on its implications.

General models, however, are often hard to use in applications. This is why many papers in behavioral economics take a different approach: they propose much more restrictive functional forms with few parameters that are easy to apply and estimate. (This literature is rarely concerned with axiomatic foundations.)

In this paper, we try to achieve both goals—the generality of decision theory and the applicability of behavioral economics. We begin by defining the general Cautious Utility model, which, as we show in Section 5, is derived from a weak set of axioms; in Section 3, we show that this general form delivers the desired reference effects, proving the strongest link between weak axioms and reference effects. At the same time, later in this section, we also define two special cases of our model, Linear Cautious Utility and Power Cautious Utility, which are easy to use in applications and involve very few parameters. Crucially, we will show that these special cases can capture our main behaviors of interest.

2.1 The general model

Given $k \in \mathbb{N}$, consider the space of k-dimensional bundles (money, mugs, pens, ...). For ease of reference, the first dimension denotes money and will be used as the unit of account with which we measure the value of all alternatives.[4] Because we are interested in refer-

[4]Unlike a numeraire, the choice of money is not entirely without loss. We adopt money because this is the dimension on which we will impose our key axiom of certainty bias (M-NCI; see Section 5). We could pick any dimension for which the equivalent postulate holds, which is also any dimension for which there is no

ence effects, we need to incorporate reference points. We take the standard approach in reference-dependent preferences, the same used in Prospect Theory, and define Cautious Utility on *relative changes* with respect to a given reference point: if y is the final allocation and r the reference bundle, then each bundle is viewed as $x = y - r$. For example, if the reference point is the current endowment, a bundle that returns an extra $3 and takes away 2 mugs is evaluated as $(3, -2)$. In this aspect, Cautious Utility coincides with Prospect Theory, allowing for direct comparisons with other models and giving complete flexibility on the reference point—it could be the endowment, the allocation of others, the expectation, *etc*. For now, we assume that the reference point is deterministic; we extend to stochastic reference points in Section 6.

Formally, we consider allocations in $\mathbb{R}^k$ and let Δ be the set of all lotteries, that is, (Borel) probability measures over $\mathbb{R}^k$ with compact support. We study a preference relation $\succcurlyeq$ over Δ. Denote by ae_i the bundle whose i-th coordinate takes value $a \in \mathbb{R}$ while all the others are 0. With a small abuse of notation, denote by 0 both the number and the vector whose components are all zero. Given $x \in \mathbb{R}^k$, we interchangeably use x and δ_x to denote the degenerate lottery that pays x with certainty. If $p \in \Delta$ and $v : \mathbb{R}^k \to \mathbb{R}$ is strictly increasing and continuous, then $\mathbb{E}_p(v)$ denotes the expected utility using v, i.e., $\int v \mathrm{d}p$, while $c(p, v) \in \mathbb{R}$ indicates its monetary certainty equivalent (if it exists), i.e., the unique monetary value such that $\mathbb{E}_p(v) = v(c(p, v)e_1)$.

The following is the most general version of our model.

Definition 1. A preference relation $\succcurlyeq$ admits a Cautious Utility representation if there exists a set $\mathcal{W}$ of strictly increasing and continuous utility functions $v : \mathbb{R}^k \to \mathbb{R}$ with $v(0) = 0$, such that (i) for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ satisfying $v(y + me_1) \geq v(x) \geq v(y - me_1)$ for all $v \in \mathcal{W}$; and (ii) the function $V : \Delta \to \mathbb{R}$, defined as

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \tag{1}$$

is a continuous utility representation of $\succcurlyeq$.[5]

Cautious Utility builds upon two key tenets. First, agents have not one but a *set* of utilities: they may be unsure of which utility to use. For example, agents may be unsure of the

uncertainty about the trade-off with money (e.g., using dollars vs. euros as the unit of account should not change anything).

[5]Assuming $v(0) = 0$ is a convenient normalization; our results hold without it except for point 2 of Proposition 3 where the condition $-v(-ae_i) \neq v(ae_i)$ becomes $v(0) - v(-ae_i) \neq v(ae_i) - v(0)$.

trade-off between goods or how risk averse they should be. Second, agents act with *caution*: to evaluate each alternative, they use the utility with the *lowest* monetary certainty equivalent. Using certainty equivalents guarantees that the comparison across utilities is made after bringing each dimension to the same unit of account (monetary amounts), avoiding comparisons across utilities, for which normalizations matter. Condition (i) in the definition guarantees that monetary certainty equivalents are always well-defined.

To illustrate the role of caution, it is useful to define a model that takes the opposite approach, where the agent uses the utility with the *highest* monetary certainty equivalent. We say that a preference relation admits an *Incautious Utility* representation if it is represented by (1) where sup replaces the inf.

Remark 1. *Despite the use of the most pessimistic utility, under Cautious Utility agents can be risk averse, seeking, or have varying risk attitudes: as in Expected Utility, this depends on the curvature of the utilities in $\mathcal{W}$. Cerreia-Vioglio et al. (2015) show that agents are risk averse when all functions are concave and risk seeking when all are convex. Similarly, if utilities are all concave for gains and convex for losses, the individual is risk averse for gains and risk seeking for losses. Overall, Cautious Utility does not restrict risk attitudes.*

Remark 2. *Using the* inf *in Cautious Utility may at first appear too pessimistic, and one may wish to consider milder formulations. For example, the agent may use some weighted average of the* inf *and the* sup*. A few considerations are in order. First, the set of utilities is subjective: it reflects the agent's preferences and is not the set of all possible utilities. Thus, the* inf *is taken only over a restricted collection. Second, this representation is not necessarily very pessimistic. For example, take a finite set $\mathcal{W}$ of quasi-linear utilities and some $u \in \mathcal{W}$, and an individual who uses the most pessimistic utility in $\mathcal{W}' = \{(1-\gamma)\, u + \gamma v : v \in \mathcal{W}\}$. Here γ can be understood as a 'pessimism weight:' the larger γ, the larger the span of utilities, the lower the evaluation. When it is small, the individual is only 'mildly' pessimistic, yet these preferences admit a Cautious Utility representation with set $\mathcal{W}'$. Finally, in Section 5 we show that Cautious Utility emerges from a natural axiom on the certainty effect; to the extent that one accepts this requirement, the model* necessarily *follows.*

2.2 Two Convenient Special Cases: Linear and Power Cautious Utility

We now discuss two special cases of Cautious Utility with few parameters and convenient functional forms that are simple to apply and estimate. For ease of exposition, we mostly focus on the case of $k = 2$, but the analysis easily generalizes.

Linear Cautious Utility. We begin by considering additive linear utilities. When $k = 2$, we say that a preference relation $\succcurlyeq$ admits a *Linear Cautious Utility* representation if there exist $\bar{a} > \underline{a} > 0$ such that

$$V(p) = \min\{\mathbb{E}_p[x_1 + \bar{a}x_2], \mathbb{E}_p[x_1 + \underline{a}x_2]\}.$$

Here, utilities take the simple additive and linear form. By considering both $\bar{a}$ and $\underline{a}$, this special case captures the uncertainty about trade-offs in the most direct way: it is as if the individual were unsure if a unit of x_2 is worth \$$\bar{a}$ or \$$\underline{a}$. Note that this version has *only two utilities* and *only two parameters*. When $k = 2$, this is without loss of generality within additive linear representations: since only the highest and lowest tradeoffs matter, given any set of additive linear utilities, we can always focus only on the two "extreme" utilities. (Also without loss of generality is the fact that monetary amounts are unweighted, which means that monetary certainty equivalents coincide with expected values.) The following example, which we already alluded to in the introduction, will be useful to illustrate how this simple form can generate the endowment effect.

Example 1. *Consider $k = 2$, money and mugs. Suppose $\mathcal{W} = \{v_1, v_2\}$ where $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. The agent considers two possible trade-offs between money and mugs: one mug is equivalent to \$1 according to v_1, and to \$2 according to v_2. Because of caution, the value of one mug is $V(0, 1) = \min\{1, 2\} = 1$.*

To extend this model to $k > 2$ goods, consider a finite set $A \subset \mathbb{R}^k_{++}$, with $a_1 = 1$ for all $a \in A$, such that $\succcurlyeq$ is represented by

$$V(p) = \min_{a \in A} \mathbb{E}_p[x_1 + \sum_{i=2,\ldots,k} a_i x_i].$$

Power Cautious Utility. The previous form assumes linearity and, thus, risk neutrality. We now give a form that allows for non-neutral risk attitudes, focusing on CRRA utilities and allowing for different curvatures for gains and losses. For any $\alpha, \beta \in \mathbb{R}_{++}$, consider functions $f_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ of the form

$$f_{\alpha,\beta}(t) = \begin{cases} t^\alpha & \text{if } t \geq 0 \\ -(-t)^\beta & \text{if } t < 0. \end{cases}$$

With $k = 2$, we say that $\succcurlyeq$ admits a *Power Cautious Utility* representation if there exist $a > 0$ and $\bar{\alpha} > \underline{\alpha} > 0$ such that $\succcurlyeq$ is represented by a Cautious Utility representation with

$$\mathcal{W} = \{f_{\alpha,\beta}(x_1) + af_{\alpha,\beta}(x_2) : \alpha, \beta \in \{\underline{\alpha}, \bar{\alpha}\}\}.$$

In Power Cautious Utility, individuals consider different curvatures—as if they were uncertain about how risk-averse they should be. Notably, Power Cautious Utility has the *same* number of parameters as an Expected Utility model that includes one parameter for the weight of the good, one for risk aversion for gains, and one for losses; the extra flexibility here comes from the mix and match combinations of risk attitude parameters, which gives multiple utilities. Perhaps surprisingly, as we will see, uncertainty about utility curvature alone is sufficient to generate the endowment effect (even for riskless objects) and loss aversion for risk. The following example will be useful to illustrate this point later.

Example 2. *Consider again $k = 2$ and assume that the set of utilities is $\mathcal{W} = \{f_{\alpha,\beta}(x_1) + f_{\alpha,\beta}(x_2) : \alpha, \beta \in \{.25, .5\}\}$.*

The Linear and Power Cautious Utility models focus on different aspects—uncertainty about trade-offs between the goods and uncertainty about risk attitudes. They can be easily combined in a general model, which includes only one parameter beyond the baseline, with the following set of utilities: for some $\bar{a}, \underline{a}, \bar{\alpha}, \underline{\alpha} \in \mathbb{R}_{++}$,

$$\mathcal{W} = \{f_{\alpha,\beta}(x_1) + af_{\alpha,\beta}(x_2) : a \in \{\underline{a}, \bar{a}\} \text{ and } \alpha, \beta \in \{\underline{\alpha}, \bar{\alpha}\}\}.$$

3 Cautious Utility and Reference Effects

3.1 The Endowment Effect

We begin by introducing Willingness to Pay (WTP) and Willingness to Accept (WTA).

WTA and WTP. $\text{WTP}_i(m)$ is the maximum amount of money that the agent is willing to pay to purchase m units of good $i \in \{2, \dots, k\}$. Thus, it satisfies

$$0 \sim me_i - \text{WTP}_i(m)e_1.$$

In words, the individual is indifferent between not buying (getting 0) and acquiring m units of good i while foregoing $\text{WTP}_i(m)$ units of money. Similarly, $\text{WTA}_i(m)$ is the minimum amount of money that the agent is willing to accept to sell m units of the good, satisfying

$$0 \sim -me_i + \mathrm{WTA}_i\,(m)\,e_1.$$[6]

We say that a preference $\succcurlyeq$ *exhibits the endowment effect* for good i if $\mathrm{WTA}_i(m) \geq \mathrm{WTP}_i(m)$ for all $m \in \mathbb{R}_+$. It exhibits the endowment effect if this is the case for all $i \in \{2, \ldots, k\}$. It exhibits the *opposite of the endowment effect* when the inequality is reversed.

WTA and WTP under Cautious Utility. Our first result illustrates how Cautious Utility generates WTA and WTP differently. To state this formally, we first define the WTA and WTP induced by a utility function over bundles. For any strictly increasing and continuous utility v, let WTA_i^v denote the WTA of an Expected Utility agent using utility v, that is, the amount such that $v(-me_i + \mathrm{WTA}_i^v(m)e_1) = v(0)$. Define WTP_i^v analogously.

Proposition 1. *If $\succcurlyeq$ admits a Cautious Utility representation with set $\mathcal{W}$, then for each $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$*

$$\mathrm{WTA}_i\,(m) = \sup_{v\in\mathcal{W}} \mathrm{WTA}_i^v\,(m) \quad \text{and} \quad \mathrm{WTP}_i\,(m) = \inf_{v\in\mathcal{W}} \mathrm{WTP}_i^v\,(m)\,.$$

In words, WTA in Cautious Utility is the highest WTA obtained by the utilities in $\mathcal{W}$, while the WTP is the lowest of the WTPs. Caution leads individuals to focus on opposite ends of the range of values, pushing toward the endowment effect. This result follows a simple intuition, illustrated by reconsidering our first example.

Example 1 (cont.). *Consider again Example 1, where $k = 2$ and $\mathcal{W}$ consists of $v_1(x_1, x_2) = x_1 + x_2$ and $v_2(x_1, x_2) = x_1 + 2x_2$. Recall that the WTP is the amount \$$z \geq 0$ that satisfies $(0, 0) \sim (-z, m)$ for $m \geq 0$. Then,*

$$V(0,0) = \min\{0,0\} = 0 \qquad V(-z,m) = \min\{-z+m, -z+2m\} = -z+m$$
$$\Rightarrow \quad 0 = -\mathrm{WTP}_2(m) + m \quad \Rightarrow \quad \mathrm{WTP}_2(m) = m.$$

Similarly, WTA is the amount \$$r \geq 0$ such that $(0, 0) \sim (r, -m)$. Then

$$V(r,-m) = \min\{r-m, r-2m\} = r-2m \quad \Rightarrow \quad 0 = \mathrm{WTA}_2(m) - 2m \quad \Rightarrow \mathrm{WTA}_2(m) = 2m.$$

[6]Formally, for each $i \in \{2, \ldots, k\}$, $\mathrm{WTP}_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\mathrm{WTA}_i : \mathbb{R}_+ \to \mathbb{R}_+$ are defined by $\mathrm{WTP}_i\,(m) = \max\left\{l \in \mathbb{R}_+ : \delta_{me_i - le_1} \succcurlyeq \delta_0\right\}$ and $\mathrm{WTA}_i(m) = \min\left\{l \in \mathbb{R}_+ : \delta_{-me_i + le_1} \succcurlyeq \delta_0\right\}$. In our model, they are always well-defined and satisfy the simpler conditions above.

Therefore, $\mathrm{WTA}_2(m) = 2m > m = \mathrm{WTP}_2(m)$ *for all* $m > 0$*, the endowment effect. Proposition 1 shows how these can be derived also from the WTA and WTP for each utility, which may be computationally simpler. We have:*

$$v_1(-\mathrm{WTP}_2^{v_1}(m), m) = v_1(0,0) \Rightarrow -\mathrm{WTP}_2^{v_1}(m) + m = 0 \Rightarrow \mathrm{WTP}_2^{v_1}(m) = m$$
$$v_2(-\mathrm{WTP}_2^{v_2}(m), m) = v_2(0,0) \Rightarrow -\mathrm{WTP}_2^{v_2}(m) + 2m = 0 \Rightarrow \mathrm{WTP}_2^{v_2}(m) = 2m.$$

Because utilities are symmetric, the WTA is the same as the WTP for each utility. And indeed, WTP_2 *is the smallest of the two, while* WTA_2 *the largest.*

The two utilities in the example are simple linear functions, yet we have an endowment effect. The crucial feature is that they entail a different trade-off, or 'exchange rate,' between money and mugs: a mug is worth either \$1 or \$2. When buying, a cautious agent is pessimistic about the value of mugs, and the WTP is the lowest one, at \$1. When selling, the opposite happens, and the WTA is the highest, at \$2. This creates the endowment effect. (This also generates an 'endowment-effect in mugs-terms.' The minimum number of mugs the agent is willing to accept to give up \$1 is 1, but to obtain \$1, the individual is willing to 'pay' only .5 mugs.)

This example shows that uncertainty about trade-offs and caution push WTA and WTP apart and can yield the endowment effect. However, at this level of generality, one cannot guarantee an endowment effect: if, for example, the set consists of only one utility v for which, given $m > 0$, $\mathrm{WTA}_2^v(m) < \mathrm{WTP}_2^v(m)$ because it *underweights* losses (like $v(x,y) = f(x) + f(y)$ with $f(x) = x$ if $x > 0$ and $f(x) = \frac{1}{2}x$ if $x \leq 0$), then we have the opposite of the endowment effect. The underweighting of losses counters the forces of caution (see Corollary 3 and the discussion thereafter). To highlight the role of caution, we can focus on "symmetric" sets of utilities, which we introduce next.

3.2 Symmetry and Strict Behavior

Recall that a function $v : \mathbb{R}^k \to \mathbb{R}$ is *odd* if $v(x) = -v(-x)$ for all $x \in \mathbb{R}^k$, that is, when there is no asymmetry in the treatment of positive and negative values, and the function is symmetric with respect to the origin. In line with this, we say that a set of functions $\mathcal{W}$ is *odd* if for each $v \in \mathcal{W}$ there exists $v' \in \mathcal{W}$ such that $v(x) = -v'(-x)$ for all $x \in \mathbb{R}^k$. If all utilities are odd, so is the set; $\mathcal{W}$ in Example 1 consists of two odd functions and is thus odd. But a set can also be odd when there are non-odd functions, provided that the specular functions are also included; this is the case of Example 2 above.

We say that $\succcurlyeq$ admits a *Symmetric Cautious Utility representation* if it admits a Cautious Utility representation with an odd set $\mathcal{W}$. *Symmetric Incautious Utility* is defined similarly.

We assume symmetry for some of the results below. This is not because symmetry is necessarily appealing—in many cases, one may want to use utilities and sets that are not odd. However, we want to show that Cautious Utility can generate reference effects *even* with symmetry. As we discuss below (Corollary 3 and the discussion thereafter), adding typical asymmetries will only strengthen our results. Note how any Linear Cautious Utility representation is necessarily Symmetric: each utility it involves is odd, and so is the set. Power Cautious Utility is also always symmetric.

The Endowment Effect with Symmetry. Symmetry guarantees that we do not have any unevenness in the treatment of gains and losses. In turn, this means that the 'span' of WTAs and WTPs is the same. That is, if $\mathcal{W}$ is odd, then $\{\text{WTA}_i^v(m) : v \in \mathcal{W}\} = \{\text{WTP}_i^v(m) : v \in \mathcal{W}\}$ for all $m \in \mathbb{R}_+$. To see this, note that if $v, v' \in \mathcal{W}$ are such that $v(x) = -v'(-x)$ for all x, then $\text{WTA}_i^v = \text{WTP}_i^{v'}$. Combining this observation with Proposition 1 implies the following result.

Proposition 2. *The following statements are true:*

1. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation, then it exhibits the endowment effect. If $\succcurlyeq$ admits a Symmetric Incautious Utility representation, then it exhibits the opposite of the endowment effect.*

2. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation, then for each $i \in \{2, \ldots, k\}$ and $m > 0$, the following statements are equivalent:*

   (i) $\text{WTA}_i(m) > \text{WTP}_i(m)$*;*

   (ii) There exist $v, v' \in \mathcal{W}$ such that $\text{WTA}_i^v(m) \neq \text{WTA}_i^{v'}(m)$*;*

   (iii) There exist $v, v' \in \mathcal{W}$ such that $\text{WTP}_i^v(m) \neq \text{WTP}_i^{v'}(m)$.

Part (1) of Proposition 2 shows that even with symmetry in the treatment of gains and losses, Cautious Utility generates the endowment effect. Caution is crucial: if we consider the Incautious model, we obtain the opposite behavior.

Part (2) characterizes when we can expect a strict endowment effect. It is enough that two utilities in $\mathcal{W}$ differ either in their WTA or in their WTP to create a strict wedge between the WTA and WTP of the agent.

This result can also be expressed using Marginal Rate of Substitutions (MRS). Recall that, given a differentiable utility $v : \mathbb{R}^k \to \mathbb{R}$, the MRS of good i with respect to money is $\text{MRS}_i^v(x) = \frac{v_i(x)}{v_1(x)}$, where v_j is the partial derivative of v with respect to x_j. If two utilities have different MRSs between goods and money, we have a strict endowment effect.

Corollary 1. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation. Given $i \in \{2, ..., k\}$, if each $v \in \mathcal{W}$ is continuously differentiable and there exist $v, v' \in \mathcal{W}$ such that $\text{MRS}_i^v(x) \neq \text{MRS}_i^{v'}(x)$ for all $x \in \mathbb{R}^k$ with $x_1, x_i \neq 0$, then $\text{WTA}_i(m) > \text{WTP}_i(m)$ for all $m \in \mathbb{R}_{++}$.*

Comparisons Across Goods. Cautious Utility allows the endowment effect to vary across goods and also provides simple comparative statics. As is standard, we use the ratio between WTA and WTP to define the strength of the endowment effect. As opposed to other models, this strength can vary with the good or the quantity of each good.

Example 3. *Consider $v_1(x_1, x_2, x_3) = x_1 + x_2 + \alpha x_3$ and $v_2(x_1, x_2, x_3) = x_1 + x_2 + \beta x_3$, with $\alpha > \beta > 0$. If $\mathcal{W} = \{v_1, v_2\}$, there is an endowment effect for good 3 but not for good 2: $\frac{\text{WTA}_3(m)}{\text{WTP}_3(m)} = \frac{\alpha}{\beta} > 1 = \frac{\text{WTA}_2(m)}{\text{WTP}_2(m)}$ for all $m > 0$.*

Example 4. *Consider $v_1(x_1, x_2) = x_1 + \alpha x_2$ (for $\alpha > 0$) and $v_2(x_1, x_2) = x_1 + x_2^3$. If $\mathcal{W} = \{v_1, v_2\}$, the endowment effect varies with the quantity: for $m \neq m'$ with $mm' \neq \alpha$, we have $\frac{\text{WTA}_2(m)}{\text{WTP}_2(m)} \neq \frac{\text{WTA}_2(m')}{\text{WTP}_2(m')}$.*

In general, the strength of the endowment effect depends on the range of possible trade-offs that the agent considers for each good: the endowment effect is more substantial when the range is larger. We will revisit this result when we discuss the empirical evidence in Section 4. For any set A, denote by $\text{co}(A)$ its convex hull.

Corollary 2. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation with finite set $\mathcal{W}$. For each $i, j \in \{2, \ldots, k\}$ and $m, m' \in \mathbb{R}_{++}$,*

$$\text{co}\left(\left\{\text{WTA}_i^v(m) : v \in \mathcal{W}\right\}\right) \supset \text{co}\left(\left\{\text{WTA}_j^v(m') : v \in \mathcal{W}\right\}\right) \implies \frac{\text{WTA}_i(m)}{\text{WTP}_i(m)} > \frac{\text{WTA}_j(m')}{\text{WTP}_j(m')}.$$

We conclude this discussion with a result on the role of symmetry. The following is true even without assuming it.

Corollary 3. *The following statements are true:*

1. *If $\succcurlyeq$ admits a Cautious Utility representation $\mathcal{W}$ and there exists $v \in \mathcal{W}$ and $i \in \{2, \ldots, k\}$ such that* $\mathrm{WTA}_i^v(m) \geq \mathrm{WTP}_i^v(m)$ *for all $m \in \mathbb{R}_+$, then $\succcurlyeq$ exhibits the endowment effect for good i.*

2. *There exists $\succcurlyeq$ which admits a Cautious Utility representation $\mathcal{W}$ such that, for some $i \in \{2, \ldots, k\}$, we have* $\mathrm{WTA}_i^v(m) \leq \mathrm{WTP}_i^v(m)$ *for all $m \in \mathbb{R}_+$ and $v \in \mathcal{W}$, and yet $\succcurlyeq$ exhibits the endowment effect for good i.*

The corollary above should clarify that we consider *Symmetric* Cautious Utility merely to highlight the role of caution even under symmetry. Without symmetry, Cautious Utility gives the endowment effect if *at least one* utility overweights losses (for example, it is concave); this follows immediately from Proposition 1 and is point 1 of Corollary 3. In general, asymmetries of this kind simply add to the other forces highlighted above. It is only if *all* utilities underweight losses that Cautious Utility may not exhibit the endowment effect, depending on the relative strength of the underweight of losses and multiplicity of utilities (point 2 of Corollary 3).[7]

3.3 Loss Aversion for Risk

Following Kahneman and Tversky (1979), we use loss aversion for risk to indicate the rejection of even bets around zero (see also Markowitz, 1952). Formally, a preference $\succcurlyeq$ is *loss averse for risk* on dimension $i \in \{1, \ldots, k\}$, if for each $a \in \mathbb{R}_{++}$

$$\delta_0 \succcurlyeq \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}.$$

It is loss averse for risk if this is the case for all $i \in \{1, \ldots, k\}$. Gain seeking and loss neutrality for risk are defined analogously, with $\succcurlyeq$ replaced by $\preccurlyeq$ and $\sim$, respectively. Finally, $\succcurlyeq$ is strictly loss averse (resp. gain seeking) for risk on dimension i if this also holds strictly for some $a \in \mathbb{R}_{++}$.

Proposition 3. *The following statements are true:*

[7]For example, take $\mathcal{W} = \{v, v'\}$ where $v(x_1, x_2) = f(x_1) + f(x_2)$ and $v'(x_1, x_2) = f(x_1) + \gamma f(x_2)$, with $f(a) = a$ for $a > 0$ and $f(a) = \lambda a$ for $a \leq 0$, with $\lambda > 0$ and $\gamma > 1$. Both utilities underweight losses if $\lambda < 1$, while the bigger γ is, the bigger the uncertainty about the trade-off. Note that for $m \geq 0$, $\mathrm{WTA}_2(m) = m\gamma\lambda$ and $\mathrm{WTP}_2(m) = \frac{m}{\lambda}$, which means $\mathrm{WTA}_2(m) \geq \mathrm{WTP}_2(m)$ if and only if $\lambda \geq \frac{1}{\sqrt{\gamma}}$. Therefore, even when $\lambda < 1$ and all the utilities underweight losses and would give the opposite of the endowment effect, the model may still return the endowment effect if there is enough uncertainty about trade-offs (γ high enough).

1. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation, then it is loss averse for risk. If $\succcurlyeq$ admits a Symmetric Incautious Utility representation, then it is gain seeking for risk.*

2. *Let $\succcurlyeq$ admit a Symmetric Cautious Utility representation. Given $a \in \mathbb{R}_{++}$ and $i \in \{1, \ldots, k\}$, $\succcurlyeq$ is strictly loss averse for risk on dimension i at a, that is, $\delta_0 \succ \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$, if and only if $-v(-ae_i) \neq v(ae_i)$ for some $v \in \mathcal{W}$.*

Part (1) of Proposition 3 shows that Cautious Utility entails (weak) loss aversion for risk, even under symmetry. Once again, this is due to uncertainty about the utility and caution. But while the endowment effect is due to uncertainty about trade-offs between money and goods, loss aversion for risk is due to uncertainty about how to aggregate gains and losses. This difference is easily illustrated by our Example 1. In that case, we have $\mathbb{E}_{\frac{1}{2}\delta_{ae_i}+\frac{1}{2}\delta_{-ae_i}}(v) = v(0)$ for both utilities, thus $V(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}) = \min\{0, 0\} = 0 = V(\delta_0)$ for $i = 1, 2$ and for all $a \in \mathbb{R}_{++}$, giving us loss neutrality for risk. Yet, we have seen that in this case we have the endowment effect. This shows that in Cautious Utility, the endowment effect may emerge even without loss aversion for risk.

Part (2) shows that to obtain strict loss aversion for risk, we need at least one utility to be not odd. This is illustrated by reconsidering our second example.

Example 2 (cont.). *Consider again Example 2. As opposed to Example 1, money and mugs are treated identically, but the individual considers different curvatures. With lottery $\frac{1}{2}\delta_{ae_1} + \frac{1}{2}\delta_{-ae_1}$, $a > 0$, $a \neq 1$, at least one of the utilities must return a negative expected utility, that is,*

$$\min\left\{\frac{1}{2}a^{.25} - \frac{1}{2}a^{.5}, \frac{1}{2}a^{.5} - \frac{1}{2}a^{.25}\right\} < 0,$$

(noting that $\frac{1}{2}a^{.25} - \frac{1}{2}a^{.25} = 0 = \frac{1}{2}a^{.5} - \frac{1}{2}a^{.5}$). But then, since $v(0) = 0$ for all $v \in \mathcal{W}$, also the minimum of the certainty equivalents must be negative, and so must be the value of the lottery, that is,

$$V\left(\frac{1}{2}\delta_{ae_1} + \frac{1}{2}\delta_{-ae_1}\right) < 0 = V(\delta_0).$$

This is strict loss aversion for risk on money, and the same is true for mugs since utilities are the same. For the endowment effect, by Proposition 1, for any $m > 0$

$$\mathrm{WTP}_2(m) = \min\{\sqrt{m}, m^2, m\} \qquad \text{and} \qquad \mathrm{WTA}_2(m) = \max\{\sqrt{m}, m^2, m\}.$$

Hence $\mathrm{WTA}_2(m) > \mathrm{WTP}_2(m)$ for all $m \neq 1$.

In this example, the individual considers several utilities, one for each combination of curvatures. Some of these utilities are not odd, and for those the utility of ae_1 is not minus the utility of $-ae_1$, creating an asymmetry. But the set is odd, and if one utility has an asymmetry in favor of one direction, the other has the opposite. This means that the expected utility of $\frac{1}{2}\delta_{ae_1} + \frac{1}{2}\delta_{-ae_1}$ is negative for at least one utility, and so is the certainty equivalent. Cautious individuals must then assign a negative value to this lottery, giving loss aversion for risk.

We have seen that under Symmetric Cautious Utility, exhibiting a strict endowment effect implies neither strict loss aversion nor loss neutrality on any dimension. The converse is also true: the agent may be strictly loss averse for risk on all dimensions yet exhibit no endowment effect. For example, consider a small variation of Example 2, with $\mathcal{W} = \{f_{\alpha,\beta}(x_1+x_2) : \alpha, \beta \in \{.25, .5\}\}$. Here, we have strict loss aversion for risk on each dimension but no endowment effect. In general, in Cautious Utility, the endowment effect and loss aversion for risk are not necessarily related. We will revisit this observation when discussing the empirical evidence in Section 4.

3.4 The Certainty Effect

Following Kahneman and Tversky (1979), we say that $\succcurlyeq$ exhibits the certainty effect if for all $x, y \in \mathbb{R}$ and $\alpha, \beta \in (0,1)$, if $\alpha\delta_{ye_1} + (1-\alpha)\delta_0 \sim \delta_{xe_1}$, then $\alpha\beta\delta_{ye_1} + (1-\alpha\beta)\delta_0 \succcurlyeq \beta\delta_{xe_1} + (1-\beta)\delta_0$. When the latter holds strictly, this corresponds to the Allais' paradoxes—the Common Ratio or Common Consequence effects.

Cautious Utility exhibits the certainty effect while ruling out the opposite violation of Independence (the case where $\succcurlyeq$ above is reversed and holds strictly at least once). This follows directly from the functional form. Intuitively, while the agent acts with caution when evaluating general lotteries, caution does not play any role when evaluating monetary amounts—the monetary certainty equivalent of a degenerate lottery that yields \$$m$ is m for any utility. The implication is deeper: Section 5 shows that Cautious Utility can be *derived* from positing a form of certainty effect on risk preferences. (We postpone formal statements about these implications to that discussion.)

Conceptually, this feature sets Cautious Utility apart. In most other models, loss aversion for risk and the endowment effect are linked by one parameter, while Non-Expected Utility is conceptually separate. For example, under CPT, the first effects are ascribed to the over-weighting of losses, while the certainty effect is due to probability weighting. In Cautious

Utility, instead, loss aversion for risk, the endowment effect, and violations of Expected Utility are conceptually related and stem from *the same source*—uncertainty about the utility and caution. As such, Cautious Utility offers a unified explanation of three phenomena at the core of behavioral economics. This, however, does not mean that the phenomena must manifest themselves together.

Observation 1. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation, then:*

(i) The agent may exhibit the certainty effect for monetary lotteries yet be loss neutral for risk or exhibit no endowment effect.

(ii) The agent may follow Expected Utility for monetary lotteries yet exhibit the endowment effect. The agent may follow Expected Utility for monetary lotteries with only gains or only losses yet exhibit loss aversion for risk.

We can have violations of Expected Utility without loss aversion for risk or the endowment effect, and the endowment effect independently of the certainty effect on monetary lotteries. This is intuitive: we have seen how the endowment effect can emerge from uncertainty about the trade-off between different goods, while violations of Expected Utility for monetary lotteries are due to uncertainty about how to evaluate monetary amounts.[8]

3.5 Cautious Utility and Prospect Theory are Fully Distinct

We now show that Cautious Utility is not only conceptually different from CPT, but also fully behaviorally distinct, in the sense that the only preferences compatible with both models are those featuring none of the phenomena we are interested in.

To define CPT, we begin with the case in which only monetary lotteries are involved ($k = 1$). Consider a strictly increasing and continuous utility function $v : \mathbb{R} \to \mathbb{R}$ such that $v(0) = 0$, and two probability distortion functions $w^+, w^- : [0,1] \to [0,1]$, that are strictly increasing, continuous, and take value 0 at 0 and 1 at 1. For each lottery p over $\mathbb{R}$ with compact support, denote by F_p its corresponding CDF. Define

$$\mathrm{CPT}_{v,w^+,w^-}(p) = \int_{[0,\infty)} v(x)\, dw^+\left(F_p(x)\right) + \int_{(-\infty,0]} v(x)\, dw^-\left(F_p(x)\right).$$

[8]For (i), take v and v' such that $v(x_1, x_2) = f(x_1 + x_2)$ and $v'(x_1, x_2) = g(x_1 + x_2)$ for some normalized, strictly increasing, continuous, divergent, and odd f and g which are not ranked in terms of risk aversion. For the first part of (ii), use the set $\mathcal{W}$ of Example 1; for the second part, use the same example as in the end of Section 3.3.

This is similar to Expected Utility with utility v, except that probabilities are distorted (in their cumulative distribution). A widely used special case assumes $v(-x) = -\lambda v(x)$ for $x > 0$, where λ denotes the coefficient of loss aversion and regulates the asymmetry in the treatment of gains and losses, with $\lambda > 1$ capturing loss aversion.

We consider two ways to extend CPT to bundles: probability distortions and reference-dependence can be applied to each dimension separately, or the agent can first compute the utility of each bundle, and then compare it to a 'global' reference point with utility zero.

The first approach, widespread in the applied literature and studied by Bleichrodt et al. (2009), considers for each $i \in \{1, \ldots, k\}$ a strictly increasing and continuous utility $u_i : \mathbb{R} \to \mathbb{R}$ with $u_i(0) = 0$. For each lottery p over $\mathbb{R}^k$, let p_i be the marginal distribution over dimension i. Preferences admit an *Additive CPT representation* if they are represented by

$$V(p) = \sum_{i=1}^{k} \mathrm{CPT}_{u_i, w^+, w^-}(p_i) .$$

The second approach was proposed by Tversky and Kahneman (1981, p. 456) and formally derived by Wakker and Tversky (1993). The agent has a strictly increasing and continuous utility over bundles $u : \mathbb{R}^k \to \mathbb{R}$, with $u(0) = 0$. For each lottery p, denote by p_u the distribution it induces over *utility* levels.[9] Preferences admit a *u-CPT representation* if

$$V(p) = \mathrm{CPT}_{v, w^+, w^-}(p_u) .$$

Before stating our result, we need an extra property. A finite Cautious Utility representation is *essential* if for each $\tilde{v} \in \mathcal{W}$ there exists $p \in \Delta$ such that

$$\min_{v \in \mathcal{W}} c(p, v) < \min_{v \in \mathcal{W} \setminus \{\tilde{v}\}} c(p, v) .$$

This guarantees that no utility is redundant and that the set includes only the genuinely relevant elements. In all our examples above, it can be shown that the set is essential.

Proposition 4. *If $\succcurlyeq$ admits a Symmetric Cautious Utility representation as well as either an Additive CPT or a u-CPT representation, then $\succcurlyeq$ admits an Expected Utility representation. Moreover, if the representation is also finite and essential, then $\succcurlyeq$ is loss neutral for risk and exhibits no endowment effect.*

[9] That is, for all Borel subsets B of $\mathbb{R}$ and for all $p \in \Delta$, $p_u(B) = p(\{x \in \mathbb{R}^k : u(x) \in B\})$.

Proposition 4 extends a similar result in Cerreia-Vioglio et al. (2015) that shows how Cautious Expected Utility and Rank Dependent Expected Utility are fully distinct.[10]

4 Cautious Utility and Empirical Evidence

We now relate Cautious Utility to empirical evidence. We focus on contrasting our model with loss-aversion-based explanations as the most prominent alternative, even though some of the patterns we discuss are compatible with other models (e.g., Bordalo et al. 2012). Our aim is not to run a competition between models but to demonstrate the merit of considering uncertainty about trade-offs and caution as a potential source of reference effects. We argue how this approach may help us capture empirical regularities that are not just smartly designed tests but either contradict the core aspects of alternative models or constitute patterns of substantive importance. As the evidence on our phenomena of interest is immense, we focus on differentiating aspects and refer to DellaVigna (2009) and O'Donoghue and Sprenger (2018) for recent surveys.

The Endowment Effect and Loss Aversion for Risk. If the endowment effect is due to overweighting of losses, then it must be correlated with loss aversion for risk. This is because the latter is the most direct manifestation, and a direct test, of any asymmetric weighting of gains and losses, as originally noted by Kahneman and Tversky (1979) when loss aversion was introduced. (See Chapman et al. 2023a for an analysis of why this is the case also when goods are lottery tickets, under classical Prospect Theory or variants like Köszegi and Rabin 2006, 2007 or third generation prospect theory, Schmidt et al. 2008.) For the same reason, we should not observe an endowment effect without loss aversion for risk.

These predictions do not find empirical support. While Gächter et al. (2022) and Dean and Ortoleva (2019) find a (mild) positive correlation on specific samples, Chapman et al. (2023a) test it in four large representative samples, considering different measures of loss aversion for risk and of the endowment effect for the same good and adopting several techniques to reduce measurement error. They do not find a positive relationship between the endowment effect and loss aversion for risk in any of their tests. This directly contradicts loss-aversion-based explanations.

[10]To see why essentiality matters, suppose $k = 1$ and $\mathcal{W} = \{v, v'\}$, where v is strictly increasing, concave, and such that $v(0) = 0$ and v' is such that $v(x) = -v'(-x)$ for all $x \in \mathbb{R}$. $\mathcal{W}$ is odd, but since v' is convex, it is never used. Preferences are thus Expected Utility with utility v, which, by concavity, is loss averse for risk.

Moreover, while the empirical evidence of the endowment effect is very robust, the same cannot be said of loss aversion for risk. Even though many papers document it (Camerer, 1995; Starmer, 2000), several studies show it is fragile (Ert and Erev, 2008, 2013), while others find only a minority of loss averse individuals, especially in representative samples: see Chapman et al. (2023b) and many references therein. Moreover, Oprea (2022) shows how an important fraction of what we call loss aversion for risk may be due to complexity rather than genuine features of the utility.

Instead, we have seen that Cautious Utility does not entail a relationship between loss aversion for risk and the endowment effect; each may exist without the other, or they may coexist and be unrelated.[11]

The Endowment Effect across Goods. Under Cautious Utility, the endowment effect is due to uncertainty about trade-offs. If this is the case, we would expect the extent of the disparity between WTA and WTP to vary with the types of objects: larger when there is more uncertainty about values, as in goods or services rarely traded, like life-insurance contracts; smaller for goods whose value is less in doubt, like frequently traded goods. Corollary 2 and the corresponding examples in Section 3.2 show how this is a prediction of our model.

This prediction finds strong empirical support. Decades of research studied the endowment effect for several types of goods and shows that the effect is strongest with less common goods, is reduced for ordinary market goods that individuals regularly trade, and disappears for objects of known value, like monetary tokens.[12]

By contrast, loss-aversion-based explanations are compatible with different intensities of the endowment effect across goods but do not predict such a pattern. To explain it in that model, one needs to assume that the disutility of losing is stronger for unfamiliar goods and decreases for familiar ones. As such, Cautious Utility can help organize critical evidence of the Endowment Effect: where it is more common and where it is more intense.

[11] It is also easy to construct examples of Cautious Utility in which loss aversion for risk over money is independent of the endowment effect for monetary lottery tickets (in the sense that we can identify a parametric family and a distribution of parameters in the population such that the endowment effect is distributed independently of the distribution of loss aversion); see Chapman et al. (2023a, Appendix D).

[12] In their widely-cited metastudy of the empirical literature, Horowitz and McConnell (2002, p. 427) describe the heterogeneity of ratios between WTAs and WTPs across forty-five studies and note: "With regard to patterns in the observed ratios, we find that, on average, the less the good is like an 'ordinary market good,' the higher the ratio. The ratio is highest for public and non-market goods, next highest for ordinary private goods, and lowest for experiments involving forms of money. A generalization of this pattern holds even when we account for differences in survey design: ordinary goods have lower ratios than non-ordinary ones. This pattern is the major result we discover."

The Endowment Effect and Information. If the endowment effect is due to uncertainty about trade-offs, it should also be affected by information about values, like market value. For example, individuals who consider a range of values and are told that the market value lies in the middle of the range may well incorporate this knowledge and shrink the range of values—naturally, the information reduces the uncertainty about trade-offs.

This finds substantial support. Weaver and Frederick (2012) show that the endowment effect is severely reduced when subjects are given information on market values pointing to an intermediate price between typical WTAs and WTPs; it is instead higher when the information suggests either a high price, above typical WTAs, or a very low one, below typical WTPs.[13] Shogren et al. (1994) and List (2004a) find that the endowment effect is reduced by showing continuous trading in a public auction or by providing trading experience; List (2003, 2004b) shows how experienced traders exhibit much less endowment effect for goods they frequently trade.

Loss-aversion-based explanations are, in principle, compatible with these patterns but require that the 'pain of losing' varies substantially and non-monotonically with information—it disappears when subjects observe trading or are informed of intermediate prices, it increases if told very high or very low prices. This seems less plausible.

Violations of Expected Utility. We briefly review how Cautious Utility relates to the evidence of Non-Expected Utility and refer to Cerreia-Vioglio et al. (2015) for in-depth discussion. Cautious Utility is compatible with Allais-type behavior when one option is risk-free, with less frequent violations when no option is risk-free, and with *mixed fanning*—indifference curves becoming flatter towards better prizes (Camerer, 1995). It also allows for the strength of the certainty effect to vary with stake sizes (Conlisk, 1989; Camerer, 1989; Burke et al., 1996; Fan, 2002; Huck and Müller, 2012), which is incompatible with CPT. Kahneman and Tversky (1979) document the opposite of the certainty effect for losses, which is incompatible with Cautious Utility, but this has received much less attention.[14]

[13]While formalizing a complete dynamic model is outside the scope of this paper, let us illustrate how this patterns can be easily generated. Consider an individual with utilities $x_1 + \alpha x_2$ and $x_1 + \beta x_2$, with $0 < \alpha < \beta$, giving us $\text{WTA}_2(1) = \beta$ and $\text{WTP}_2(1) = \alpha$. Suppose they observe some people evaluate mugs as equal $\$i > 0$, but they are unsure how much to trust them and consider a range of trust parameters $[\lambda_1, \lambda_2]$ with $0 < \lambda_1 \leq \lambda_2 < 1$. They then "update" their set of utilities, which becomes $\{x_1 + (\lambda i + (1-\lambda)a)x_2 : a \in \{\alpha, \beta\}$ and $\lambda \in [\lambda_1, \lambda_2]\}$. If $i \in [\alpha, \beta]$, after information $\text{WTA}_2(1) = \lambda_1 i + (1-\lambda_1)\beta$ and $\text{WTP}_2(1) = \lambda_1 i + (1-\lambda_1)\alpha$, so the endowment effect *shrinks*. If $i \notin [\alpha, \beta]$, the endowment effect can grow if i is sufficiently distant from α or β and there is a large enough range of λs.

[14]Ruggeri et al. (2020) conducts a large-scale replication of the experiments in Kahneman and Tversky (1979) and finds that, while most effects replicate, this is not the case for the evidence of the opposite of the

While Cautious Utility allows for risk aversion for gains and risk seeking for losses, it is not compatible with the '4-fold' pattern, which includes risk seeking for gains of small probability and risk aversion for losses of small probability, instead easily captured by CPT with S-shaped probability weighting.[15] Finally, as we will see in Section 6, Cautious Utility is compatible with the documented preferences for randomization (Agranov and Ortoleva, 2017, 2022, forthcoming), but not with a strict preference for randomization with degenerate monetary amounts, which are also documented.

5 Axiomatic Foundation

We now provide the behavioral foundation of Cautious Utility. Endow $\mathbb{R}^k$ with the usual Euclidean topology and Δ with a version of the weak topology.[16] Consider a binary relation $\succcurlyeq$ on Δ, on which we impose the following axioms.

Axiom 1 (Weak Order). *The relation $\succcurlyeq$ is complete and transitive.*

Axiom 2 (Continuity). *For each $q \in \Delta$ the sets $\{p \in \Delta : p \succcurlyeq q\}$ and $\{p \in \Delta : q \succcurlyeq p\}$ are closed.*

Axiom 3 (Monotonicity). *For each $x, y \in \mathbb{R}^k$*

$$x > y \implies \lambda\delta_x + (1-\lambda)\, r \succcurlyeq \lambda\delta_y + (1-\lambda)\, r \qquad \forall\lambda \in (0,1]\,, \forall r \in \Delta$$

and $\lambda\delta_x + (1-\lambda)\, r \succ \lambda\delta_y + (1-\lambda)\, r$ for some $\lambda \in (0,1]$ and for some $r \in \Delta$.[17]

Axiom 4 (Monetary equivalent). *For each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$\lambda\delta_{y+me_1} + (1-\lambda)\, r \succcurlyeq \lambda\delta_x + (1-\lambda)\, r \succcurlyeq \lambda\delta_{y-me_1} + (1-\lambda)\, r \qquad \forall\lambda \in (0,1]\,, \forall r \in \Delta.$$

The first three postulates are standard. Monetary equivalent stipulates that for any two bundles $x, y \in \mathbb{R}^k$, there is a monetary amount m large enough that receiving that amount on top of y is better than x and losing that amount is worse than x, and this remains true even

certainty effect for losses: the majority of subjects exhibit a behavior compatible with Expected Utility in this range (a pattern compatible with Cautious Utility and with CPT with no probability weighting for losses).

[15] Recent papers argue that risk seeking for gains with small probabilities may be due to misunderstandings or complexity, not to a feature of risk preferences (Hertwig et al., 2004; Abdellaoui et al., 2011; Oprea, 2022).

[16] A generalized sequence $\{p_\alpha\}_{\alpha\in A}$ in Δ converges to p if and only if $\mathbb{E}_{p_\alpha}(v) \to \mathbb{E}_p(v)$ for all $v \in C\left(\mathbb{R}^k\right)$.

[17] $x > y$ means that $x_i \geq y_i$ for all i, where at least one of the inequalities is strict.

if we mix with some other lottery r. This guarantees that monetary certainty equivalents, WTAs, and WTPs are well-defined.

The next axiom is our key assumption. It extends the Negative Certainty Independence (NCI) axiom of Dillenberger (2010) and Cerreia-Vioglio et al. (2015) to multi-dimensional bundles and generalizes the definition of certainty effect of Kahneman and Tversky (1979).

Axiom 5 (Multi-Dimensional Negative Certainty Independence (M-NCI)). *For each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$p \succcurlyeq \delta_{me_1} \implies \lambda p + (1-\lambda) r \succcurlyeq \lambda \delta_{me_1} + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta.$$

Like the original NCI, this property states that if a sure amount of money m is not preferred to a lottery p, then this ranking does not change if we mix both with another lottery. M-NCI is a weakening of standard Independence that captures the certainty effect. Intuitively, mixing m with a lottery eliminates its certainty appeal. Therefore, if m is worse than p when certain, it will remain so after the mixture.

For ease of comparison, it is also helpful to consider the inverse postulate that rules out the certainty effect while allowing for the opposite.

Axiom 6 (Multi-Dimensional Positive Certainty Independence (M-PCI)). *For each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$\delta_{me_1} \succcurlyeq p \implies \lambda \delta_{me_1} + (1-\lambda) r \succcurlyeq \lambda p + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta.$$

Our characterization theorem below focuses on *canonical* representations. To define them, we first introduce the following subrelation $\succcurlyeq'$:

$$p \succcurlyeq' q \overset{\text{def}}{\iff} \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta.$$

Intuitively, $\succcurlyeq'$ captures the rankings of which the agent is sure: $p \succcurlyeq' q$ when not only $p \succcurlyeq q$, but also any mixture featuring p is better than the corresponding mixture with q. It is easy to verify that $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies the Independence axiom of Expected Utility, and that it is incomplete (yet still transitive) whenever preferences are not Expected Utility. We say that a Cautious Utility representation $\mathcal{W}$ (see Definition 1) is *canonical* if it also represents $\succcurlyeq'$, in the sense that

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}.$$

Theorem 1. *A binary relation $\succcurlyeq$ on Δ satisfies Axioms 1-5 if and only if it admits a canonical Cautious Utility representation. Moreover, a canonical representation is unique up to the closed convex cone hull.*[18]

This theorem shows that Cautious Utility can be derived from an axiom that postulates the certainty effect, M-NCI, together with basic properties. It is routine to show that Incautious Utility is characterized by the same axioms with M-NCI replaced by M-PCI. This result extends the main representation theorem of Cerreia-Vioglio et al. (2015) to a setup of lotteries over multi-commodity bundles and to an unbounded domain, necessary to define monetary certainty equivalents.

In the main text, we discussed Cautious Utility representations which are not necessarily canonical. When $\mathcal{W}$ is finite, as in most applications, preferences represented in this way satisfy all the above axioms (Axioms 1-5; see Remark 3 in the Appendix). Without additional structure, Monotonicity is guaranteed only in a weaker form, that is,

$$x \geq y \implies \lambda\delta_x + (1-\lambda)\, r \succcurlyeq \lambda\delta_y + (1-\lambda)\, r \qquad \forall \lambda \in (0,1]\,, \forall r \in \Delta.$$

Foundation of Symmetry. We now give a simple foundation to our symmetry assumption. For each $p \in \Delta$, denote by $\sigma(p) \in \Delta$ the lottery that, compared to p, swaps gains with losses, that is, $\sigma(p)\,(B) = p(-B)$ for all Borel subsets B of $\mathbb{R}^k$.

A natural form of symmetry posits that if p is better than q, then $\sigma(q)$ is better than $\sigma(p)$ (the two must be swapped as we are inverting gains and losses). But this would be too strong, as it rules out strict loss aversion for risk: since $\delta_0 = \sigma(\delta_0)$ and $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} = \sigma\left(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}\right)$, we would get $\delta_0 \sim \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. A weaker version posits that if not only $p \succcurlyeq q$, but also each mixture of p is better than the corresponding mixture of q, that is $p \succcurlyeq' q$, then we obtain $\sigma(q) \succcurlyeq \sigma(p)$. This is exactly the form of symmetry corresponding to the Symmetric Cautious (or Incautious) Utility model.[19]

Axiom 7 (Weak Symmetry)**.** *For each $p, q \in \Delta$, $p \succcurlyeq' q \implies \sigma(q) \succcurlyeq \sigma(p)$.*

Proposition 5. *A binary relation $\succcurlyeq$ on Δ satisfies Axioms 1-5 and 7 if and only if it admits a canonical Symmetric Cautious Utility representation.*

[18]More formally, if $\mathcal{W}_1, \mathcal{W}_2 \subseteq C\left(\mathbb{R}^k\right)$ are two canonical representations, then $\overline{\text{cone}}\,(\mathcal{W}_1) = \overline{\text{cone}}\,(\mathcal{W}_2)$ where cone $(\mathcal{W}_i)$ denotes the smallest convex cone containing $\mathcal{W}_i$ and $\overline{\text{cone}}$ denotes its closure in $C\left(\mathbb{R}^k\right)$ with respect to the topology of uniform convergence over compacta.

[19]The further restriction that all utilities in a canonical Symmetric Cautious Utility representation are odd is characterized by adding a global form of loss neutrality, that is, $\frac{1}{2}\delta_x + \frac{1}{2}\delta_{-x} \sim \delta_0$ for all $x \in \mathbb{R}^k$. (The proof follows by the same arguments of point 2 of Proposition 3.)

Certainty Effect and Reference Effects. We conclude our discussion on the foundations by highlighting a key implication of our results.

Corollary 4. *If $\succcurlyeq$ satisfies Axioms 1-4 and 7, then:*

1. *If $\succcurlyeq$ satisfies M-NCI, then it exhibits the endowment effect and it is loss averse for risk;*
2. *If $\succcurlyeq$ satisfies M-PCI, then it exhibits the opposite of the endowment effect and it is gain seeking for risk.*

This corollary follows immediately from Propositions 2, 3, and 5. It states that under Weak Symmetry and basic axioms, ruling out the opposite of the certainty effect over bundles, as encoded by M-NCI, *formally implies* loss aversion for risk and the endowment effect. The opposite postulate—M-PCI, which allows for the opposite of the certainty effect over bundles—gives the opposite of the endowment effect and loss aversion. This result shows a formal connection between violations of Expected Utility and reference effects, which, to our knowledge, is new.

6 Additional Properties and General Discussion

Stochastic Reference Points. What if the reference point is stochastic? For example, it may be the current portfolio of financial assets or a distribution of payoffs the individual expects to receive. We defined changes relative to a fixed reference point by 'subtracting' it, and we can do the same when the reference point is a lottery. We proceed in steps. Given a reference lottery r that pays x_i with probability $r(x_i)$, the (degenerate) final allocation y is evaluated as the lottery that pays $y - x_i$ with probability $r(x_i)$: for example, if $k = 1$ and the reference point is $r = \frac{1}{2}\$10 + \frac{1}{2}\$0$, the final allocation \$7 is evaluated by Cautious Utility as the lottery $\frac{1}{2}(-\$3) + \frac{1}{2}\$7$. Intuitively, it is as if the agent were 'issuing' the reference lottery and paying its prizes in every contingency. Denote the subtraction of a lottery $r \in \Delta$ from $y \in \mathbb{R}^k$ as $y - r$; $r - y$ is defined similarly.

To extend to stochastic final allocations, we need to consider the correlation with the reference lottery. Consider a final allocation q and a reference lottery r, suppose both are simple lotteries, and denote by $P_{q,r}(x, y)$ the joint probability that q returns x and r returns y. Then, define $q - r \in \Delta$ simply as $\sum_{x,y} P_{q,r}(x, y)\delta_{x-y}$.[20] For example, suppose the final

[20]In general, define the map $T : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ by $T(x, y) = x - y$ for all $x, y \in \mathbb{R}^k$. Given $q, r \in \Delta$ with joint probability $P_{q,r}$, denote $q - r \in \Delta$ by $(q - r)(B) = P_{q,r}(T^{-1}(B))$ for all Borel sets B of $\mathbb{R}^k$.

allocation is $\frac{1}{2}\delta_x + \frac{1}{2}\delta_y$ and the reference lottery is $\frac{1}{2}\delta_z + \frac{1}{2}\delta_w$. If the two lotteries are independent, the stochastic final allocation will be evaluated as $\frac{1}{4}\delta_{x-z} + \frac{1}{4}\delta_{x-w} + \frac{1}{4}\delta_{y-z} + \frac{1}{4}\delta_{y-w}$; and if the two lotteries are perfectly correlated, so that q returns x if and only if r returns z, it will be evaluated as $\frac{1}{2}\delta_{x-z} + \frac{1}{2}\delta_{y-w}$. Importantly, the value of the final allocation p when the reference point is p itself is 0: this is relevant, for example, for calculating the WTA of lottery tickets, as it implies that keeping the lottery corresponds to 0.[21]

Endowment Effect for Lottery Tickets. Our results extend to the endowment effect for lotteries, widely documented empirically. Similarly to the deterministic case, we define WTA and WTP for a lottery p as

$$\mathrm{WTP}(p) = \max\{l \in \mathbb{R} : p - le_1 \succcurlyeq 0\} \text{ and } \mathrm{WTA}(p) = \min\{l \in \mathbb{R} : le_1 - p \succcurlyeq 0\}.$$

Like above, it can be shown that WTP (p) and WTA (p) are well-defined and that

$$p - \mathrm{WTP}(p)\, e_1 \sim 0 \qquad \text{and} \qquad \mathrm{WTA}(p)\, e_1 - p \sim 0.$$

Given a strictly increasing and continuous utility v, let $\mathrm{WTA}^v(p)$ denote the WTA for p of an Expected Utility maximizer with utility v; define $\mathrm{WTP}^v(p)$ analogously. It is routine to check that they are well-defined under Cautious or Incautious Utility. Our results on WTA and WTP readily extend to this case of lotteries. (The proof follows from arguments identical to those used for Propositions 1 and 2, and is therefore omitted.)

Proposition 6. *If $\succcurlyeq$ admits a Cautious Utility representation $\mathcal{W}$ and $p \in \Delta$, then*

1. $\mathrm{WTA}(p) = \sup_{v\in\mathcal{W}} \mathrm{WTA}^v(p)$ *and* $\mathrm{WTP}(p) = \inf_{v\in\mathcal{W}} \mathrm{WTP}^v(p)$*;*
2. *If $\mathcal{W}$ is odd, then* $\mathrm{WTA}(p) \geq \mathrm{WTP}(p)$.

Choice. Since Kahneman et al. (1990), some experiments measure not only WTA and WTP but also "Choice": the amount of money that makes the agent indifferent with receiving one unit of the object, that is, \$$z$ such that $(z, 0) \sim (0, 1)$. In the data, Choice typically falls between WTA and WTP, though often very close to WTP. This is easy to obtain in Cautious

[21] In accounting for the correlation between the reference lottery and the final allocation, our approach departs from the formulation of Köszegi and Rabin (2006, 2007) and adopts an approach closer to Schmidt et al. (2008). This is evident when the final allocation is the reference lottery itself, evaluated as 0 in our model, while in Köszegi and Rabin (2006, 2007) it is treated as a non-degenerate lottery.

Utility. With the utilities in Example 1, Choice coincides with WTP; with those in Example 2, it is strictly between the WTA and WTP (except for $m = 1$).

Exchange Asymmetries and Status Quo Bias. It is widely documented that individuals are status quo biased and often reject exchanges favoring to keep their current endowment (status quo) even when no money is involved (e.g., Knetsch, 1989). For example, an individual may be given a mug and asked to exchange it for a chocolate bar, or given a chocolate bar and asked to exchange it for a mug, and reject both. Under Cautious Utility, this happens whenever the individual considers a utility for which the mug is better, and another for which the chocolate bar is better. As a simple example, if mugs and chocolate bars are dimensions 2 and 3, extend Example 1 and suppose $\mathcal{W} = \{v_1, v_2\}$ with $v_1(x_1, x_2, x_3) = x_1 + 2x_2 + x_3$ and $v_2(x_1, x_2, x_3) = x_1 + x_2 + 2x_3$. Then, $V(0, -1, 1) = \min\{-2 + 1, -1 + 2\} = -1 < 0 = V(0, 0, 0)$ and $V(0, 1, -1) = \min\{2 - 1, -2 + 1\} = -1 < 0 = V(0, 0, 0)$.[22]

Randomization. Preferences under Cautious Utility are convex in probabilities, allowing for strict preference for randomization while ruling out the opposite (see also Cerreia-Vioglio et al. 2019). To illustrate, consider the same example above of an individual unsure about the trade-offs between mugs and chocolate bars. The individual is indifferent between one mug and one chocolate bar, since $V(0, 1, 0) = \min\{2, 1\} = 1$ and $V(0, 0, 1) = \min\{1, 2\} = 1$, but strictly prefers a 50/50 lottery p between the two, since $V(p) = \min\{0.5 \cdot 2 + 0.5 \cdot 1, 0.5 \cdot 1 + 0.5 \cdot 2\} = 1.5$. Unsure which is best, our individual prefers to 'hedge.'

Relation to Loss Aversion in Riskless Choice. Tversky and Kahneman (1991) introduce a behavioral definition of loss aversion for preferences over bundles without risk. Assuming only two dimensions ($k = 2$), an individual exhibits *loss aversion for bundles* if, for all $x, y, r, s \in \mathbb{R}^k$ such that $x_1 \geq r_1 > s_1 = y_1$, $y_2 > x_2$, $r_2 = s_2$, if $x - s \succcurlyeq y - s$ then $x - r \succ y - r$; and the same holds if the subscripts 1 and 2 are interchanged. Intuitively, x is best in dimension 1, y is best in dimension 2, while r is better than s in dimension 1 and they coincide in dimension 2. Then, if x is at least as good as y when s is the reference point ($x - s \succcurlyeq y - s$), it must be strictly better when r is the reference point ($x - r \succ y - r$).

Cautious Utility allows for loss aversion for bundles but does not require it to hold (in its original strict form) everywhere. Consider our Example 1 and $x = (5, -\frac{1}{2})$, $y = (3, 1)$,

[22]This example also shows that, even when each utility function v is linear and we restrict ourselves to the positive orthant (so no reference effects are considered), Cautious Utility can differ from the standard model (with perfect substitutes). Both utilities are relevant for computing V.

$r = (5, -1)$, and $s = (3, -1)$: we have $x - s \succ y - s$ and $x - r \succ y - r$, in line with the definition. However, if $x' = (1, 4)$, $y' = (0, 5)$, $r' = (1, 0)$, and $s' = (0, 0)$, we have $x' - s' \sim y' - s'$ but also $x' - r' \sim y' - r'$, giving us loss neutrality in this instance. (This is not exclusive to Cautious Utility: the same holds with several linear forms of Prospect Theory.)

7 Conclusion

We introduce a new way of modeling the endowment effect: uncertainty about trade-offs and caution. This approach also captures loss aversion for risk and the certainty effect, providing a unified way to study three phenomena that are at the core of behavioral economics.

Conceptually, the caution criterion can be viewed as a heuristic adopted when agents are unsure of what to do. As such, caution can be understood as a form of 'uncertainty aversion' even to choices with no objective risk—like choosing the price to pay for a given object—where individuals may feel subjective uncertainty. Applied to resolve uncertainty about trade-offs, how to aggregate gains and losses, or risk aversion over money, caution yields the endowment effect, loss aversion for risk, and the certainty effect, respectively.

Our approach is both conceptually and behaviorally different from leading alternatives, and this difference is not only theoretical but also has practical consequences. The empirical evidence points to behaviors compatible with one model and not the other; but this, by itself, is of limited interest, since most models are imperfect and smartly designed tests can document violations. More importantly, recent tests find evidence against the core aspect of the loss-aversion-based explanations—as the endowment effect appears unrelated to loss aversion for risk, the most direct test of asymmetry of gains and losses. As this evidence negates the fundamental premise of these models, it further points to the need for alternatives, of which Cautious Utility is one compatible with the evidence.

Moreover, decades of research have identified important empirical regularities of the endowment effect: a stronger disparity for rarely traded goods, a lighter one when objects are routinely traded; and a strong effect of information about values. A satisfactory model should be able to capture and rationalize such regularities. While leading alternatives are typically silent about these patterns, the approach underlying Cautious Utility—uncertainty about trade-offs—can instead help organize this evidence.

Cautious Utility is novel and has not yet been subject to equally rigorous testing. Moreover, several avenues are left to be explored, such as a fully dynamic model of how trade-offs are updated with information. Our goal here has been to present a model that can be simple and parsimonious and that, for its unified explanation and empirical fit, may capture some aspect of three phenomena at the core of behavioral economics.

Appendix: Proof of the Main Results

In this appendix, we prove all results except Proposition 4 and a few ancillary facts needed for Theorem 1. All missing results appear in the Online Appendix. We begin with a remark on the necessity of the axioms even for noncanonical representations.

Remark 3. *Consider a Cautious Utility representation $\mathcal{W}$ for $\succcurlyeq$ (not necessarily canonical). By definition, we have that $\mathcal{W}$ is a set of strictly increasing and continuous utility functions $v : \mathbb{R}^k \to \mathbb{R}$ such that for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$v(y + me_1) \geq v(x) \geq v(y - me_1) \quad \forall v \in \mathcal{W}, \tag{2}$$

$v(0) = 0$ for all $v \in \mathcal{W}$, and $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \quad \forall p \in \Delta,$$

is a continuous utility function for $\succcurlyeq$. It is then immediate to observe that $\succcurlyeq$ satisfies Weak Order and Continuity. As for the other axioms, define the binary relation

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}.$$

Clearly, $\succcurlyeq^*$ is a preorder that satisfies Independence. Facts 1–4 below follow from immediate computations and the definition of $\succcurlyeq^*$. Fact 5 follows by the second part of Proposition 9 in the Online Appendix and the discussion thereafter, provided $\mathcal{W}$ is odd:

1. *For each $p, q \in \Delta$*

$$p \succcurlyeq^* q \implies p \succcurlyeq q.$$

2. *For each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$p \succcurlyeq \delta_{me_1} \implies p \succcurlyeq^* \delta_{me_1}.$$

3. *For each $x, y \in \mathbb{R}^k$*

$$x > y \implies \delta_x \succ^* \delta_y.$$

4. *For each* $x, y \in \mathbb{R}^k$ *there exists* $m \in \mathbb{R}_+$

$$\delta_{y+me_1} \succcurlyeq^* \delta_x \succcurlyeq^* \delta_{y-me_1} \text{ and } \delta_{y+me_1} \succcurlyeq \delta_x \succcurlyeq \delta_{y-me_1}.$$

5. *For each* $p, q \in \Delta$

$$p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq^* \sigma(p).$$

By point 1 and since $\succcurlyeq'$ *is the largest subrelation of* $\succcurlyeq$ *that satisfies Independence, we have that* $\succcurlyeq^*$ *is a subrelation of* $\succcurlyeq'$*, that is,* $p \succcurlyeq^* q \implies p \succcurlyeq' q$*. In light of this and given the definition of* $\succcurlyeq'$*, point 2 (resp. point 4) implies that* $\succcurlyeq$ *satisfies M-NCI (resp. Monetary equivalent). Point 3 implies a weaker form of the Monotonicity axiom with strict inequalities replaced by weak ones.*[23] *If the set* $\mathcal{W}$ *is also finite (as in all our examples), then Monotonicity holds as stated: with strict inequalities. Finally, points 1 and 5 imply that* $\succcurlyeq$ *satisfies a weaker form of symmetry, that is* $p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq \sigma(p)$*, whenever* $\mathcal{W}$ *is odd. This form of symmetry is sufficient to obtain our results on the endowment effect and loss aversion for risk.*

Proof of Proposition 1. Consider a Cautious Utility representation $\mathcal{W}$ for $\succcurlyeq$ (not necessarily canonical). For each $i \in \{2, ..., k\}$ recall that $\text{WTA}_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $\text{WTP}_i : \mathbb{R}_+ \to \mathbb{R}_+$ are the functions defined by

$$\text{WTA}_i(m) = \min\left\{l \in \mathbb{R}_+ : \delta_{-me_i+le_1} \succcurlyeq \delta_0\right\} \qquad \forall m \in \mathbb{R}_+ \tag{3}$$

and

$$\text{WTP}_i(m) = \max\left\{l \in \mathbb{R}_+ : \delta_{me_i-le_1} \succcurlyeq \delta_0\right\} \qquad \forall m \in \mathbb{R}_+. \tag{4}$$

By points 1, 3, and 4 of Remark 3 and since $\succcurlyeq$ is represented by a continuous utility, these functions are well-defined and $\delta_{-me_i+\text{WTA}_i(m)e_1} \sim \delta_0$ as well as $\delta_{me_i-\text{WTP}_i(m)e_1} \sim \delta_0$ for all $m \in \mathbb{R}_+$ and for all $i \in \{2, ..., k\}$. Given $v \in \mathcal{W}$, recall that we define WTA_i^v and WTP_i^v according to definitions (3) and (4) for the corresponding Expected Utility preference with Bernoulli utility v. By (2) and since each $v \in \mathcal{W}$ is strictly increasing and continuous, it is immediate to see that $\text{WTA}_i^v(m)$ and $\text{WTP}_i^v(m)$ are the unique solutions of the equations $v(-me_i + le_1) = 0$ and $v(me_i - le_1) = 0$. By (2) and since v is strictly increasing and continuous, this implies that both WTA_i^v and WTP_i^v are continuous functions. Fix $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$. By point 2 of Remark 3 and the definition of $\succcurlyeq^*$, and since each $\hat{v}$ in

[23] That is, given $x, y \in \mathbb{R}^k$, $x \geq y$ implies $\lambda\delta_x + (1-\lambda)r \succcurlyeq \lambda\delta_y + (1-\lambda)r$ for all $\lambda \in (0, 1]$ and for all $r \in \Delta$.

$\mathcal{W}$ satisfies $\hat{v}(0) = 0$, we have that $\delta_{-me_i + \text{WTA}_i(m)e_1} \succcurlyeq^* \delta_0$ and $\delta_{me_i - \text{WTP}_i(m)e_1} \succcurlyeq^* \delta_0$, that is, $\hat{v}(-me_i + \text{WTA}_i(m)e_1) \geq 0$ and $\hat{v}(me_i - \text{WTP}_i(m)e_1) \geq 0$ for all $\hat{v} \in \mathcal{W}$. By the definitions of WTA_i^v and WTP_i^v and since each $\hat{v}$ in $\mathcal{W}$ is strictly increasing, this implies that $\text{WTA}_i(m) \geq \text{WTA}_i^{\hat{v}}(m)$ and $\text{WTP}_i^{\hat{v}}(m) \geq \text{WTP}_i(m)$ for all $\hat{v} \in \mathcal{W}$, yielding that

$$\text{WTA}_i(m) \geq \sup_{\hat{v} \in \mathcal{W}} \text{WTA}_i^{\hat{v}}(m) \text{ and } \inf_{\hat{v} \in \mathcal{W}} \text{WTP}_i^{\hat{v}}(m) \geq \text{WTP}_i(m). \tag{5}$$

Vice-versa, by the definitions of WTA_i^v and WTP_i^v and since each v in $\mathcal{W}$ is strictly increasing, we have that $v\left(-me_i + \sup_{\hat{v} \in \mathcal{W}} \text{WTA}_i^{\hat{v}}(m)e_1\right) \geq 0$ and $v\left(me_i - \inf_{\hat{v} \in \mathcal{W}} \text{WTP}_i^{\hat{v}}(m)e_1\right) \geq 0$ for all $v \in \mathcal{W}$.

By the definition of $\succcurlyeq^*$ and point 1 of Remark 3, we obtain that $\delta_{-me_i + \sup_{\hat{v} \in \mathcal{W}} \text{WTA}_i^{\hat{v}}(m)e_1} \succcurlyeq^* \delta_0$ and $\delta_{me_i - \inf_{\hat{v} \in \mathcal{W}} \text{WTP}_i^{\hat{v}}(m)e_1} \succcurlyeq^* \delta_0$, and, in particular, $\delta_{-me_i + \sup_{\hat{v} \in \mathcal{W}} \text{WTA}_i^{\hat{v}}(m)e_1} \succcurlyeq \delta_0$ and $\delta_{me_i - \inf_{\hat{v} \in \mathcal{W}} \text{WTP}_i^{\hat{v}}(m)e_1} \succcurlyeq \delta_0$. By the definitions of WTA_i and WTP_i, this implies that

$$\text{WTA}_i(m) \leq \sup_{\hat{v} \in \mathcal{W}} \text{WTA}_i^{\hat{v}}(m) \text{ and } \inf_{\hat{v} \in \mathcal{W}} \text{WTP}_i^{\hat{v}}(m) \leq \text{WTP}_i(m).$$

Since m and i were arbitrarily chosen, we can conclude that

$$\text{WTA}_i(m) = \sup_{\hat{v} \in \mathcal{W}} \text{WTA}_i^{\hat{v}}(m) \text{ and } \text{WTP}_i(m) = \inf_{\hat{v} \in \mathcal{W}} \text{WTP}_i^{\hat{v}}(m) \quad \forall m \in \mathbb{R}_+, \forall i \in \{2, ..., k\},$$

proving the statement. ■

Proof of Proposition 2. We begin with a part which is common to both models. Consider $i \in \{2, \ldots, k\}$, $m \in \mathbb{R}_+$, and $v : \mathbb{R}^k \to \mathbb{R}$ strictly increasing, continuous, and such that $v(0) = 0$. Recall that $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$. In particular, $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. Then,

$$v\left(-me_i + \text{WTA}_i^v(m)e_1\right) = 0 \iff \bar{v}\left(me_i - \text{WTA}_i^v(m)e_1\right) = 0 \iff \text{WTA}_i^v(m) = \text{WTP}_i^{\bar{v}}(m). \tag{6}$$

We can now prove points 1 and 2. For point 1, we first prove the statement for the Cautious Utility model and then we move to the Incautious one.

1. **Cautious Utility**. Consider $i \in \{2, \ldots, k\}$ and $m \in \mathbb{R}_+$. Let $v', v'' \in \mathcal{W}$. Without loss of generality, we can assume that $\text{WTA}_i^{v'}(m) \geq \text{WTA}_i^{v''}(m)$. By Proposition 1 and (6), and

since $\bar{v}'' \in \mathcal{W}$, we have that

$$\text{WTA}_i(m) = \sup_{v \in \mathcal{W}} \text{WTA}_i^v(m) \geq \text{WTA}_i^{v'}(m) \geq \text{WTA}_i^{v''}(m) =$$
$$= \text{WTP}_i^{\bar{v}''}(m) \geq \inf_{v \in \mathcal{W}} \text{WTP}_i^v(m) = \text{WTP}_i(m).$$

Since $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$ were arbitrarily chosen, the statement follows.

Incautious Utility. We first discuss how Proposition 1 changes for Incautious Utility. If $\succcurlyeq$ admits an Incautious Utility representation, then for each $m \in \mathbb{R}_+$ and for each $i \in \{2, ..., k\}$

$$\text{WTA}_i(m) \leq \inf_{v \in \mathcal{W}} \text{WTA}_i^v(m) \quad \text{and} \quad \text{WTP}_i(m) \geq \sup_{v \in \mathcal{W}} \text{WTP}_i^v(m).$$

In order to derive these inequalities, we only need to observe that, for Incautious Utility, point 2 of Remark 3 becomes: for each $p \in \Delta$ and for each $m \in \mathbb{R}$, $\delta_{me_1} \succcurlyeq p \implies \delta_{me_1} \succcurlyeq^* p$. The inequalities above then follow by a specular argument, to the one in the proof of Proposition 1, up to (5). We can now prove the statement. Consider $i \in \{2, \ldots, k\}$ and $m \in \mathbb{R}_+$. Let $v', v'' \in \mathcal{W}$. Without loss of generality, we can assume that $\text{WTA}_i^{v'}(m) \geq \text{WTA}_i^{v''}(m)$. By the inequalities above and (6) and since $\bar{v}' \in \mathcal{W}$, we have that

$$\text{WTA}_i(m) \leq \inf_{v \in \mathcal{W}} \text{WTA}_i^v(m) \leq \text{WTA}_i^{v''}(m) \leq \text{WTA}_i^{v'}(m) =$$
$$= \text{WTP}_i^{\bar{v}'}(m) \leq \sup_{v \in \mathcal{W}} \text{WTP}_i^v(m) \leq \text{WTP}_i(m).$$

Since $m \in \mathbb{R}_+$ and $i \in \{2, ..., k\}$ were arbitrarily chosen, the statement follows.

2. Fix $i \in \{2, ..., k\}$ and $m > 0$. Given $v \in \mathcal{W}$, recall again that $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}(x) = -v(-x)$ for all $x \in \mathbb{R}^k$. Since $\mathcal{W}$ is odd, $\bar{v} \in \mathcal{W}$. Moreover, it is immediate to check that $\bar{\bar{v}} = v$ for all $v \in \mathcal{W}$. By (6), we have that for each $v \in \mathcal{W}$

$$\text{WTA}_i^v(m) = \text{WTP}_i^{\bar{v}}(m). \tag{7}$$

Since $\bar{v} \in \mathcal{W}$ and $\bar{\bar{v}} = v$ for all $v \in \mathcal{W}$, we can conclude that for each $v \in \mathcal{W}$

$$\text{WTA}_i^{\bar{v}}(m) = \text{WTP}_i^{\bar{\bar{v}}}(m) = \text{WTP}_i^v(m). \tag{8}$$

(i) implies (ii). By Proposition 1 and since $\text{WTA}_i(m) > \text{WTP}_i(m)$, we have that $\sup_{v \in \mathcal{W}} \text{WTA}_i^v(m) = \text{WTA}_i(m) > \text{WTP}_i(m) = \inf_{v \in \mathcal{W}} \text{WTP}_i^v(m)$. By (8), this implies that there exist $v, v' \in \mathcal{W}$

such that $\mathrm{WTA}_i^{v'}(m) > \mathrm{WTP}_i^{v}(m) = \mathrm{WTA}_i^{\bar{v}}(m)$. Since $\bar{v} \in \mathcal{W}$, this proves the implication.

(ii) implies (iii). By assumption, there exist $v, v' \in \mathcal{W}$ such that $\mathrm{WTA}_i^{v}(m) \neq \mathrm{WTA}_i^{v'}(m)$. By (7), we have that $\mathrm{WTP}_i^{\bar{v}}(m) = \mathrm{WTA}_i^{v}(m) \neq \mathrm{WTA}_i^{v'}(m) = \mathrm{WTP}_i^{\bar{v}'}(m)$. Since $\bar{v}, \bar{v}' \in \mathcal{W}$, this proves the implication.

(iii) implies (i). By assumption, there exist $v, v' \in \mathcal{W}$ such that $\mathrm{WTP}_i^{v}(m) \neq \mathrm{WTP}_i^{v'}(m)$. Without loss of generality, we can assume that $\mathrm{WTP}_i^{v}(m) > \mathrm{WTP}_i^{v'}(m)$. By Proposition 1 and (8) and since $\bar{v} \in \mathcal{W}$, we have that $\mathrm{WTA}_i(m) = \sup_{v\in\mathcal{W}} \mathrm{WTA}_i^{v}(m) \geq \mathrm{WTA}_i^{\bar{v}}(m) = \mathrm{WTP}_i^{v}(m) > \mathrm{WTP}_i^{v'}(m) \geq \inf_{v\in\mathcal{W}} \mathrm{WTP}_i^{v}(m) = \mathrm{WTP}_i(m)$, proving the implication. ■

Proof of Corollary 1. Fix $i \in \{2, ..., k\}$. Consider $v, v' \in \mathcal{W}$ which are continuously differentiable and such that $\mathrm{MRS}_i^{v}(x) \neq \mathrm{MRS}_i^{v'}(x)$ for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$. By definition of WTA_i^{v} and $\mathrm{WTA}_i^{v'}$ and since v and v' are strictly increasing, $\mathrm{WTA}_i^{v}(m) > 0$ and $\mathrm{WTA}_i^{v'}(m) > 0$ for all $m > 0$. In particular, given $m > 0$, we have that if $x = -me_i + \mathrm{WTA}_i^{v}(m)\, e_1$, then $x_1 \neq 0$ and $x_i \neq 0$. Since MRS_i^{v} and $\mathrm{MRS}_i^{v'}$ are well-defined for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$ and v and v' are strictly increasing, we have that the partial derivative with respect to the first component is strictly positive for both v and v' for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$. By the Implicit Function Theorem and the definition of WTA_i^{v} and since v is strictly increasing, we have that WTA_i^{v} is continuously differentiable on $(0, \infty)$ and the derivative at $m > 0$ is $\mathrm{MRS}_i^{v}\left(-me_i + \mathrm{WTA}_i^{v}(m)\, e_1\right)$. For ease of notation, define $f_v, f_{v'} : (0, \infty) \to \mathbb{R}$ by $f_v(m) = \mathrm{MRS}_i^{v}\left(-me_i + \mathrm{WTA}_i^{v}(m)\, e_1\right)$ and $f_{v'}(m) = \mathrm{MRS}_i^{v'}\left(-me_i + \mathrm{WTA}_i^{v}(m)\, e_1\right)$ for all $m > 0$. Since v and v' are continuously differentiable and $\mathrm{MRS}_i^{v}(x) \neq \mathrm{MRS}_i^{v'}(x)$ for all $x \in \mathbb{R}^k$ with $x_1 \neq 0$ and $x_i \neq 0$, we can conclude that f_v and $f_{v'}$ are continuous on $(0, \infty)$ and such that $f_v(m) \neq f_{v'}(m)$ for all $m > 0$. By the Intermediate Value Theorem, this implies that either $f_v(m) < f_{v'}(m)$ for all $m > 0$ or $f_v(m) > f_{v'}(m)$ for all $m > 0$. Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(m) = v'\left(-me_i + \mathrm{WTA}_i^{v}(m)\, e_1\right)$ for all $m \geq 0$. Since v' and $m \mapsto \mathrm{WTA}_i^{v}(m)$ are continuous and $\mathrm{WTA}_i^{v}(0) = 0$, note that h is continuous and $h(0) = 0$. Since v' is continuously differentiable and so is $\mathrm{WTA}_i^{v}(m)$ on $(0, \infty)$, we have that h is continuously differentiable on $(0,\infty)$ and

$$\begin{aligned} h'(m) &= \frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) f_v(m) - \frac{\partial v'}{\partial x_i}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) \\ &= \frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)\left(f_v(m) - \frac{\frac{\partial v'}{\partial x_i}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)}{\frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)}\right) \\ &= \frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right)\left(f_v(m) - f_{v'}(m)\right) \quad \forall m > 0. \end{aligned}$$

Since $\frac{\partial v'}{\partial x_1}\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) > 0$ for all $m > 0$, we can conclude that either $h'(m) < 0$ or $h'(m) > 0$ for all $m > 0$. In the first (resp. second) case, since h' is continuous on $(0,\infty)$, we have that $h(m) - h(m/2n) = \int_{m/2n}^{m} h'(t)\, dt < 0$ (resp. > 0) for all $m > 0$ and for all $n \in \mathbb{N}$. Since h is continuous, $h(0) = 0$, and the sequence is $\{h(m) - h(m/2n)\}_{n\in\mathbb{N}}$ is decreasing (resp. increasing), we have that $v'\left(-me_i + \mathrm{WTA}_i^v(m)\, e_1\right) = h(m) = \lim_n \left[h(m) - h(m/2n)\right] < 0$ (resp. > 0) for all $m > 0$. In the first (resp. second) case, by definition of $\mathrm{WTA}_i^{v'}(m)$ and since v' is strictly increasing, we have that $\mathrm{WTA}_i^v(m) < \mathrm{WTA}_i^{v'}(m)$ (resp. $>$) for all $m > 0$. By point 2 of Proposition 2 and since $v, v' \in \mathcal{W}$, this implies the statement. ■

Proof of Corollary 2. Consider $l \in \{2, ..., k\}$ and $m'' \in \mathbb{R}_{++}$. By (6) and since $\mathcal{W}$ is odd, we have that $\left\{\mathrm{WTA}_l^v(m'') : v \in \mathcal{W}\right\} = \left\{\mathrm{WTP}_l^{\bar{v}}(m'') : v \in \mathcal{W}\right\} = \left\{\mathrm{WTP}_l^v(m'') : v \in \mathcal{W}\right\}$. By Proposition 1 and since $\mathcal{W}$ is finite, this implies that

$$\mathrm{WTA}_l(m'') = \max_{v\in\mathcal{W}} \mathrm{WTA}_l^v(m'') = \max \mathrm{co}\left(\left\{\mathrm{WTA}_l^v(m'') : v \in \mathcal{W}\right\}\right)$$

and

$$\begin{aligned} \mathrm{WTP}_l(m'') &= \min_{v\in\mathcal{W}} \mathrm{WTP}_l^v(m'') = \min \mathrm{co}\left(\left\{\mathrm{WTP}_l^v(m'') : v \in \mathcal{W}\right\}\right) \\ &= \min \mathrm{co}\left(\left\{\mathrm{WTA}_l^v(m'') : v \in \mathcal{W}\right\}\right). \end{aligned}$$

We can conclude that if $\mathrm{co}\left(\left\{\mathrm{WTA}_i^v(m) : v \in \mathcal{W}\right\}\right) \supset \mathrm{co}\left(\left\{\mathrm{WTA}_j^v(m') : v \in \mathcal{W}\right\}\right)$, then

$$\mathrm{WTA}_i(m) \geq \mathrm{WTA}_j(m') \text{ and } \mathrm{WTP}_i(m) \leq \mathrm{WTP}_j(m') \tag{9}$$

and one of the two inequalities is strict, since the inclusion is proper. Since $\mathcal{W}$ is finite and each $v \in \mathcal{W}$ is strictly increasing, then $\mathrm{WTP}_i(m) = \mathrm{WTP}_i^v(m) > 0$ for some $v \in \mathcal{W}$. By (9) and since $\mathrm{WTP}_i(m) > 0$, we have that $\mathrm{WTP}_j(m') > 0$, proving the statement. ■

Proof of Proposition 3. We first prove point 1, in particular, the statement for the Cautious Utility model, then we move to the Incautious one, and finally we prove point 2.

1. **Cautious Utility**. Consider $i \in \{1, \ldots, k\}$ and $a \in \mathbb{R}_{++}$. By contradiction, assume that $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} \succ \delta_0$. By point 2 of Remark 3, we have that $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} \succcurlyeq^* \delta_0$. By point 5 of Remark 3, $\delta_0 = \sigma(\delta_0) \succcurlyeq^* \sigma\left(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}\right) = \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. By point 1 of Remark 3, we can conclude that $\delta_0 \succcurlyeq \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$, a contradiction.

Incautious Utility. We begin by recalling that for Incautious Utility points 1 as well as 3–5 of Remark 3 hold while point 2 becomes: for each $p \in \Delta$ and for each $m \in \mathbb{R}$

$$\delta_{me_1} \succcurlyeq p \implies \delta_{me_1} \succcurlyeq^* p. \tag{10}$$

Consider $i \in \{1, \ldots, k\}$ and $a \in \mathbb{R}_{++}$. By contradiction, assume that $\delta_0 \succ \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. By (10), $\delta_0 \succcurlyeq^* \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. By point 5 of Remark 3, $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} = \sigma\left(\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}\right) \succcurlyeq^* \sigma(\delta_0) = \delta_0$. By point 1 of Remark 3, $\frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i} \succcurlyeq \delta_0$, a contradiction. ■

2. Set $p = \frac{1}{2}\delta_{ae_i} + \frac{1}{2}\delta_{-ae_i}$. Since $\succcurlyeq$ admits a Symmetric Cautious Utility representation and, in particular, $\mathcal{W}$ is odd, we have that

$$\begin{aligned}
\delta_0 \succ p \iff 0 > \inf_{v \in \mathcal{W}} c(p, v) &\iff \exists v \in \mathcal{W} \quad 0 > c(p, v) \\
\iff \exists v \in \mathcal{W} \quad 0 = v(0) > \frac{1}{2}v(ae_i) + \frac{1}{2}v(-ae_i) \\
\iff \exists v \in \mathcal{W} \quad -v(-ae_i) > v(ae_i), \\
\iff \exists v \in \mathcal{W} \quad -v(-ae_i) \neq v(ae_i)
\end{aligned}$$

proving the statement. ■

Proof of Theorem 1. "Only if." We proceed by steps.

Step 1. There exists a continuous utility function $u : \Delta \to \mathbb{R}$ for $\succcurlyeq$ such that $u\left(\delta_{me_1}\right) = m$ for all $m \in \mathbb{R}$.

Proof of the Step. Let $p \in \Delta$. Since p has compact support, there exists $n \in \mathbb{N}$ such that $[-ne, ne]$ contains the support of p. By Lemma 1 and since $\succcurlyeq$ satisfies Weak Order and Monetary equivalent, we have that there exist $m', m'' \in \mathbb{R}_+$ such that $\delta_{m'e_1} \succcurlyeq' \delta_{ne} \succcurlyeq' \delta_{-m'e_1}$ and $\delta_{m''e_1} \succcurlyeq' \delta_{-ne} \succcurlyeq' \delta_{-m''e_1}$. By Lemma 1 in the Online Appendix and since $\succcurlyeq$ satisfies Weak Order and Monotonicity, if we set $m = \max\{m', m''\}$, we obtain that $\delta_{me_1} \succcurlyeq' \delta_{ne}, \delta_{-ne} \succcurlyeq' \delta_{-me_1}$. By Proposition 10 in the Online Appendix and since $\succcurlyeq$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent and each element of $\mathcal{W}_{\max}(\succcurlyeq')$

is increasing, we have that $\delta_{me_1} \succcurlyeq' \delta_{ne} \succcurlyeq' p \succcurlyeq' \delta_{-ne} \succcurlyeq' \delta_{-me_1}$. Since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, we can conclude that $\delta_{me_1} \succcurlyeq p \succcurlyeq \delta_{-me_1}$. Consider the sets $U = \left\{m \in \mathbb{R} : \delta_{me_1} \succcurlyeq p\right\}$ and $L = \left\{m \in \mathbb{R} : p \succcurlyeq \delta_{me_1}\right\}$. It follows that U and L are nonempty. Since $\succcurlyeq$ satisfies Weak Order, we have that $U \cup L = \mathbb{R}$. By Aliprantis and Border (2006, Lemma 2.52 and Theorem 2.55), the map $x \mapsto \delta_x$ is a (continuous) embedding. Since $\succcurlyeq$ satisfies Continuity, this implies that both U and L are closed. Since $\mathbb{R}$ is connected and $U \cup L = \mathbb{R}$, we can conclude that $U \cap L$ is nonempty and, in particular, $p \sim \delta_{me_1}$ for all $m \in U \cap L$. By Lemma 1 in the Online Appendix and since $\succcurlyeq$ satisfies Weak Order, Monotonicity, and M-NCI, we have that $m \geq m'$ if and only if $\delta_{me_1} \succcurlyeq' \delta_{m'e_1}$ if and only if $\delta_{me_1} \succcurlyeq \delta_{m'e_1}$. This implies that $U \cap L$ is a singleton. We denote by $m_p \in \mathbb{R}$ the unique element such that $p \sim \delta_{m_p e_1}$. Since p was arbitrarily chosen, we define $u : \Delta \to \mathbb{R}$ by $u(p) = m_p$ for all $p \in \Delta$. By construction, we have that $u\left(\delta_{me_1}\right) = m$ for all $m \in \mathbb{R}$. Moreover, since $\succcurlyeq$ satisfies Weak Order, we have that

$$p \succcurlyeq q \iff \delta_{m_p e_1} \succcurlyeq \delta_{m_q e_1} \iff m_p \geq m_q \iff u(p) \geq u(q),$$

proving that u is a utility function for $\succcurlyeq$. Finally, since $\succcurlyeq$ satisfies Continuity, this implies that $\{p \in \Delta : u(p) \geq t\} = \left\{p \in \Delta : u(p) \geq u\left(\delta_{te_1}\right)\right\} = \left\{p \in \Delta : p \succcurlyeq \delta_{te_1}\right\}$ is closed for all $t \in \mathbb{R}$, proving that u is upper semicontinuous. A specular argument yields lower semicontinuity, proving that u is continuous. □

Step 2. $\succcurlyeq'$ is represented by $\mathcal{W}_{\max}(\succcurlyeq')$ which has full image, in particular,

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq') \iff c(p,v) \geq c(q,v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq'). \tag{11}$$

Proof of the Step. By Proposition 10 in the Online Appendix, the first part of (11) follows. Since $\mathcal{W}_{\max}(\succcurlyeq')$ has full image and each element of $\mathcal{W}_{\max}(\succcurlyeq')$ is strictly increasing and continuous, $c(p,v)$ is well-defined for all $p \in \Delta$ and for all $v \in \mathcal{W}_{\max}(\succcurlyeq')$, and also the second part of (11) follows. □

Step 3. For each $p \in \Delta$ we have that $\inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v) \in \mathbb{R}$.

Proof of the Step. Fix $p \in \Delta$. By the same arguments of the first part of Step 1, there exists $m \in \mathbb{R}_+$ such that $\mathbb{E}_{\delta_{me_1}}(v) \geq \mathbb{E}_p(v) \geq \mathbb{E}_{\delta_{-me_1}}$ for all $v \in \mathcal{W}_{\max}(\succcurlyeq')$. It follows that $m \geq \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v) \geq -m$.

Step 4. For each $p \in \Delta$ we have that $u(p) \leq \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v)$.

Proof of the Step. Fix $p \in \Delta$. By Step 3, $\bar{m} = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v)$ is a real number. Pick $m \in \mathbb{R}$ such that $m > \bar{m}$. This implies that there exists $v \in \mathcal{W}_{\max}(\succcurlyeq')$ such that

$c(p,v) < m = c(\delta_{me_1}, v)$. By Step 2, it follows that $p \not\succcurlyeq' \delta_{me_1}$. By Lemma 1 in the Online Appendix and Step 1 and since $\succcurlyeq$ satisfies Weak Order and M-NCI, we have that $\delta_{me_1} \succ p$, yielding that $m = u(\delta_{me_1}) > u(p)$. Since m was arbitrarily chosen to be just strictly greater than $\bar{m}$, we have that $u(p) \leq \bar{m}$, proving the statement. □

Step 5. For each $p \in \Delta$ we have that $u(p) \geq \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v)$.

Proof of the Step. Fix $p \in \Delta$. By Step 3, $\bar{m} = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p,v)$ is a real number. We have that $c(p,v) \geq \bar{m} = c(\delta_{\bar{m}e_1}, v)$ for all $v \in \mathcal{W}_{\max}(\succcurlyeq')$. By Steps 1 and 2 and since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, this implies that $p \succcurlyeq' \delta_{\bar{m}e_1}$ and, in particular, $p \succcurlyeq \delta_{\bar{m}e_1}$, that is, $u(p) \geq u(\delta_{\bar{m}e_1}) = \bar{m}$, proving the statement. □

By imposing $\mathcal{W} = \mathcal{W}_{\max}(\succcurlyeq')$, the implication follows from Steps 1, 2, 4, and 5.

"If." It is routine (cf. Remark 3). ■

As for uniqueness, it follows from the same arguments contained in Evren (2008, Theorem 5), keeping in mind that we further normalized each utility v to be such that $v(0) = 0$.

Proof of Proposition 5. "Only if." By the proof of Theorem 1 and since $\succcurlyeq$ satisfies Axioms 1-5, we have that $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Cautious Utility representation, in particular, $\mathcal{W}_{\max}(\succcurlyeq')$ represents $\succcurlyeq'$ and $\succcurlyeq$. By definition of $\succcurlyeq'$ and since $\succcurlyeq$ satisfies Weak Symmetry, $\succcurlyeq'$ satisfies Independence, and σ is affine and such that $\sigma(\sigma(r)) = r$ for all $r \in \Delta$, this implies that

$$
\begin{aligned}
p \succcurlyeq' q &\implies \lambda p + (1-\lambda)\sigma(r) \succcurlyeq' \lambda q + (1-\lambda)\sigma(r) \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \sigma(\lambda q + (1-\lambda)\sigma(r)) \succcurlyeq \sigma(\lambda p + (1-\lambda)\sigma(r)) \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \lambda\sigma(q) + (1-\lambda)\sigma(\sigma(r)) \succcurlyeq \lambda\sigma(p) + (1-\lambda)\sigma(\sigma(r)) \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \lambda\sigma(q) + (1-\lambda) r \succcurlyeq \lambda\sigma(p) + (1-\lambda) r \quad \forall \lambda \in (0,1], \forall r \in \Delta \\
&\implies \sigma(q) \succcurlyeq' \sigma(p).
\end{aligned}
$$

By Lemma 1 in the Online Appendix and since $\succcurlyeq$ satisfies Weak Order, Monotonicity, and Monetary equivalent, we have that $\succcurlyeq'$ satisfies (12) and (13). By Proposition 9 in the Online Appendix, we can conclude that $\mathcal{W}_{\max}(\succcurlyeq')$ is odd, proving that $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Symmetric Cautious Utility representation.

"If." By Theorem 1, Axioms 1-5 follow. Since $\mathcal{W}$ is a canonical Symmetric Cautious Utility representation, $\mathcal{W}$ represents $\succcurlyeq'$. By the second part of Proposition 9 in the Online Appendix and the discussion thereafter, we can conclude that $p \succcurlyeq' q$ if and only if $\sigma(q) \succcurlyeq' \sigma(p)$ for all $p, q \in \Delta$.

Since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, this implies that $p \succcurlyeq' q \implies \sigma(q) \succcurlyeq' \sigma(p) \implies \sigma(q) \succcurlyeq \sigma(p)$ for all $p, q \in \Delta$, proving that $\succcurlyeq$ satisfies Weak Symmetry. ■

The two proofs above provide a foundation for the Cautious Utility model and its symmetric version. In the next remark, we discuss the foundation of the Incautious one.

Remark 4. *Recall that an Incautious Utility representation features the same exact objects of a Cautious one except that the* inf *is replaced by* sup. *It is then important to observe that the Multi-Expected Utility representation of $\succcurlyeq'$ in the Online Appendix and the symmetry property of its representation (Propositions 8–10) have been derived without ever using the M-NCI axiom. The same is true for Steps 1–3 in the proof of Theorem 1 where Step 3 could have been written with* sup *in place of* inf *using the same arguments.*[24] *Thus, substituting M-NCI with M-PCI allows for replacing in Steps 4 and 5 the* inf *with* sup. *Finally, Proposition 5 is a result just in terms of $\succcurlyeq'$ without ever relying on M-NCI.*

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[24]In Step 1, using M-PCI in place of M-NCI seamlessly yields the same result.

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Online Appendix

This appendix includes all the missing proofs and the ancillary facts used in the main body of the paper. We begin with a section on facts instrumental for Theorem 1 and Proposition 5.

Foundation

Recall the definition of $\succcurlyeq'$ in Section 5, that is,

$$p \succcurlyeq' q \overset{\text{def}}{\iff} \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \qquad \forall \lambda \in (0,1], \forall r \in \Delta.$$

The goal of this section is to provide a Multi-Expected Utility representation for $\succcurlyeq'$.

Lemma 1. *Let $\succcurlyeq$ be a binary relation on Δ that satisfies Weak Order. The following statements are true:*

1. *The relation $\succcurlyeq$ satisfies M-NCI if and only if for each $p \in \Delta$ and for each $m \in \mathbb{R}$*

$$p \succcurlyeq \delta_{me_1} \implies p \succcurlyeq' \delta_{me_1}. \qquad \text{(Equivalently } p \not\succcurlyeq' \delta_{me_1} \implies \delta_{me_1} \succ p.\text{)}$$

2. *If $\succcurlyeq$ satisfies Monotonicity, then for each $x, y \in \mathbb{R}^k$*

$$x > y \implies \delta_x \succ' \delta_y. \tag{12}$$

3. *If $\succcurlyeq$ satisfies Monetary equivalent, then for each $x, y \in \mathbb{R}^k$ there exists $m \in \mathbb{R}_+$ such that*

$$\delta_{y+me_1} \succcurlyeq' \delta_x \succcurlyeq' \delta_{y-me_1}. \tag{13}$$

Proof. All three points follow from the definition of $\succcurlyeq'$ and M-NCI, Monotonicity, and Monetary equivalent, respectively. ■

Aumann Utilities and Multi-Expected Utility Representations

In this section, in our formal results, we consider a binary relation $\succcurlyeq^*$ over Δ such that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \tag{14}$$

where $\mathcal{W} \subseteq C\left(\mathbb{R}^k\right)$. Recall that a function $v \in C\left(\mathbb{R}^k\right)$ is an Aumann utility if and only if

$$p \succ^* q \implies \mathbb{E}_p(v) > \mathbb{E}_q(v) \text{ and } p \sim^* q \implies \mathbb{E}_p(v) = \mathbb{E}_q(v).$$

We denote by e the vector whose components are all 1s. We endow $C\left(\mathbb{R}^k\right)$ with the distance $d : C\left(\mathbb{R}^k\right) \times C\left(\mathbb{R}^k\right) \to [0, \infty)$ defined by

$$d(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \min\left\{\max_{x \in [-ne, ne]} |f(x) - g(x)|, 1\right\} \quad \forall f, g \in C\left(\mathbb{R}^k\right).$$

It is routine to show that $\left(C\left(\mathbb{R}^k\right), d\right)$ is separable.[25] Moreover, if $\{f_m\}_{m\in\mathbb{N}} \subseteq C\left(\mathbb{R}^k\right)$ is such that $f_m \xrightarrow{d} f$, then $\{f_m\}_{m\in\mathbb{N}}$ converges uniformly to f on each compact subset of $\mathbb{R}^k$.

Proposition 7. *If $\succcurlyeq^*$ is as in (14) and such that*

$$x > y \implies \delta_x \succ^* \delta_y, \tag{15}$$

then $\succcurlyeq^*$ admits a strictly increasing Aumann utility.

Proof. By (14), observe that $x > y$ implies $v(x) \geq v(y)$ for all $v \in \mathcal{W}$. This implies that each $v \in \mathcal{W}$ is increasing. By Aliprantis and Border (2006, Corollary 3.5), there exists a countable d-dense subset D of $\mathcal{W}$. Clearly, we have that

$$p \succcurlyeq^* q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{16}$$

Vice-versa, consider $p, q \in \Delta$ such that $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in D$. Since p and q have compact support, there exists $\bar{n} \in \mathbb{N}$ such that $[-\bar{n}e, \bar{n}e]$ contains both supports. Consider $v \in \mathcal{W}$. Since D is d-dense in $\mathcal{W}$, there exists a sequence $\{v_l\}_{l\in\mathbb{N}} \subseteq D$ such that $v_l \xrightarrow{d} v$. It follows that v_l converges uniformly on $[-\bar{n}e, \bar{n}e]$. This implies that

$$\begin{aligned}\mathbb{E}_p(v) &= \int_{[-\bar{n}e,\bar{n}e]} v \mathrm{d}p = \lim_l \int_{[-\bar{n}e,\bar{n}e]} v_l \mathrm{d}p = \lim_l \mathbb{E}_p(v_l) \\ &\geq \lim_l \mathbb{E}_q(v_l) = \lim_l \int_{[-\bar{n}e,\bar{n}e]} v_l \mathrm{d}q = \int_{[-\bar{n}e,\bar{n}e]} v \mathrm{d}q = \mathbb{E}_q(v).\end{aligned}$$

[25] A proof is available upon request.

By (14) and (16) and since v was arbitrarily chosen, we can conclude that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in D. \tag{17}$$

Since D is countable, we can list its elements: $D = \{v_m\}_{m\in\mathbb{N}}$. Set $b_l = l + \max\{|v_l(-le)|, |v_l(le)|\}$ for all $l \in \mathbb{N}$ and $a_m = \Pi_{l=1}^m b_l \geq b_m$ for all $m \in \mathbb{N}$. Finally, define $v : \mathbb{R}^k \to \mathbb{R}$ by

$$v(x) = \sum_{m=1}^{\infty} \frac{v_m(x)}{a_m} \qquad \forall x \in \mathbb{R}^k. \tag{18}$$

We first prove that v is a well-defined continuous function. Fix $x \in \mathbb{R}^k$. It follows that there exists $\bar{m} \in \mathbb{N}$ such that $x \in [-me, me]$ for all $m \geq \bar{m}$. Since each v_m is increasing, we have that $|v_m(x)| \leq \max\{|v_m(-me)|, |v_m(me)|\} \leq b_m \leq a_m$ for all $m \geq \bar{m}$. Since $a_m \geq m!$ for all $m \in \mathbb{N}$, it follows that

$$\frac{|v_m(x)|}{a_m} = \frac{|v_m(x)|}{b_m a_{m-1}} \leq \frac{1}{a_{m-1}} \leq \frac{1}{(m-1)!} \qquad \forall m \geq \bar{m}+1.$$

This implies that the right-hand side of (18) converges. Since x was arbitrarily chosen, v is well-defined. Next, consider $n \in \mathbb{N}$. From the same argument above, we have that

$$\frac{|v_m(x)|}{a_m} \leq \frac{1}{(m-1)!} \quad \forall x \in [-ne, ne], \forall m \geq n+1.$$

By Weierstrass' M-test and since $\{v_m/a_m\}_{m\in\mathbb{N}}$ is a sequence of continuous functions, we can conclude that $v = \sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on $[-ne, ne]$, yielding that v is continuous on $[-ne, ne]$. Since n was arbitrarily chosen, it follows that v is continuous.

Finally, assume that $p \succ^* q$ (resp. $p \sim^* q$). By (17), we have that $\mathbb{E}_p(v_m) \geq \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m) = \mathbb{E}_q(v_m)$ for all $m \in \mathbb{N}$). In particular, we have that $\mathbb{E}_p(v_m/a_m) \geq \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$ and $\mathbb{E}_p(v_{\hat{m}}/a_{\hat{m}}) > \mathbb{E}_q(v_{\hat{m}}/a_{\hat{m}})$ for some $\hat{m} \in \mathbb{N}$ (resp. $\mathbb{E}_p(v_m/a_m) = \mathbb{E}_q(v_m/a_m)$ for all $m \in \mathbb{N}$). Since $\sum_{m=1}^{\infty} \frac{v_m}{a_m}$ converges uniformly on compacta and the supports of p and q are compact, we can conclude that

$$\mathbb{E}_p(v) - \mathbb{E}_q(v) = \mathbb{E}_p\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\sum_{m=1}^{\infty} \frac{v_m}{a_m}\right) = \lim_l \sum_{m=1}^{l} \mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \lim_l \sum_{m=1}^{l} \mathbb{E}_q\left(\frac{v_m}{a_m}\right)$$
$$= \lim_l \left[\sum_{m=1}^{l} \left(\mathbb{E}_p\left(\frac{v_m}{a_m}\right) - \mathbb{E}_q\left(\frac{v_m}{a_m}\right)\right)\right].$$

This implies that if $p \succ^* q$ (resp. $p \sim^* q$), then $\mathbb{E}_p(v) > \mathbb{E}_q(v)$ (resp. $\mathbb{E}_p(v) = \mathbb{E}_q(v)$), proving that v is an Aumann utility. In particular, by (15), v is strictly increasing. ■

Consider a binary relation $\succcurlyeq^*$ on Δ. Define $\mathcal{W}_{\max}(\succcurlyeq^*)$ as the set of all strictly increasing functions $v \in C(\mathbb{R}^k)$ such that $v(0) = 0$ and $p \succcurlyeq^* q$ implies $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$. We say that a set $\mathcal{W}$ in $C(\mathbb{R}^k)$ has full image if and only if

$$\forall x, y \in \mathbb{R}^k, \exists m \in \mathbb{R}_+ \text{ s.t. } v(y + me_1) \geq v(x) \geq v(y - me_1) \qquad \forall v \in \mathcal{W}.$$

Proposition 8. *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in (14). If $\succcurlyeq^*$ satisfies (12) and (13), then $\mathcal{W}_{\max}(\succcurlyeq^*)$ is a nonempty convex set with full image that satisfies (14).*

Proof. Consider $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and $\lambda \in (0,1)$. Since both functions are strictly increasing and continuous and such that $v_1(0) = 0 = v_2(0)$, it follows that $\lambda v_1 + (1-\lambda) v_2$ is strictly increasing, continuous, and takes value 0 in 0. Since $v_1, v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$, if $p \succcurlyeq^* q$, then $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$ and $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$. This implies that

$$\mathbb{E}_p(\lambda v_1 + (1-\lambda) v_2) = \lambda \mathbb{E}_p(v_1) + (1-\lambda) \mathbb{E}_p(v_2)$$
$$\geq \lambda \mathbb{E}_q(v_1) + (1-\lambda) \mathbb{E}_q(v_2) = \mathbb{E}_q(\lambda v_1 + (1-\lambda) v_2),$$

proving that $\lambda v_1 + (1-\lambda) v_2 \in \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, $\mathcal{W}_{\max}(\succcurlyeq^*)$ is convex. By Proposition 7, there exists a strictly increasing $\hat{v} \in C(\mathbb{R}^k)$ such that

$$p \succ^* q \Longrightarrow \mathbb{E}_p(\hat{v}) > \mathbb{E}_q(\hat{v}) \text{ and } p \sim^* q \Longrightarrow \mathbb{E}_p(\hat{v}) = \mathbb{E}_q(\hat{v}).$$

Without loss of generality, we can assume that $\hat{v}(0) = 0$ (given $\hat{v}$, set $v = \hat{v} - \hat{v}(0)$) and, in particular, we have that $\hat{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is nonempty. Since $\succcurlyeq^*$ satisfies (13), it follows that $\mathcal{W}_{\max}(\succcurlyeq^*)$ has full image. Since $\succcurlyeq^*$ satisfies (12), v is increasing for all $v \in \mathcal{W}$. This implies that for each $v \in \mathcal{W}$ and for each $n \in \mathbb{N}$ the function

$v_n = \left(1 - \frac{1}{n}\right) v + \frac{1}{n}\hat{v} - \left[\left(1 - \frac{1}{n}\right) v\,(0) + \frac{1}{n}\hat{v}\,(0)\right] \in \mathcal{W}_{\max}\,(\succcurlyeq^*)$. By definition, if $p \succcurlyeq^* q$, then $\mathbb{E}_p\,(v) \geq \mathbb{E}_q\,(v)$ for all $v \in \mathcal{W}_{\max}\,(\succcurlyeq^*)$. Vice-versa, we have that

$$\begin{aligned} &\mathbb{E}_p\,(v) \geq \mathbb{E}_q\,(v) \quad \forall v \in \mathcal{W}_{\max}\,(\succcurlyeq^*) \\ &\qquad \Longrightarrow\ \mathbb{E}_p\left(\left(1 - \frac{1}{n}\right) v + \frac{1}{n}\hat{v}\right) \geq \mathbb{E}_q\left(\left(1 - \frac{1}{n}\right) v + \frac{1}{n}\hat{v}\right) \quad \forall v \in \mathcal{W}, \forall n \in \mathbb{N} \\ &\qquad\qquad \Longrightarrow\ \mathbb{E}_p\,(v) \geq \mathbb{E}_q\,(v) \quad \forall v \in \mathcal{W} \implies p \succcurlyeq^* q, \end{aligned}$$

proving that (14) holds with $\mathcal{W}_{\max}\,(\succcurlyeq^*)$ in place of $\mathcal{W}$. ■

We conclude by discussing Multi-Expected Utility representations which feature odd sets. To do this, we make two simple observations. First, recall the map $\sigma : \Delta \to \Delta$, which swaps gains with losses, defined by

$$\sigma\,(p)\,(B) = p\,(-B) \text{ for all Borel subsets } B \text{ of } \mathbb{R}^k \text{ and for all } p \in \Delta.$$

It is immediate to see that σ is affine and $\sigma\,(\sigma\,(p)) = p$ for all $p \in \Delta$. Second, by the Change of Variable Theorem (see, e.g., Aliprantis and Border 2006, Theorem 13.46), we have that

$$\mathbb{E}_{\sigma(r)}\,(v) = \int_{\mathbb{R}^k} v \mathrm{d}\sigma\,(r) = -\int_{\mathbb{R}^k} \bar{v} \mathrm{d}r = -\mathbb{E}_r\,(\bar{v}) \quad \forall r \in \Delta, \forall v \in C\left(\mathbb{R}^k\right) \tag{19}$$

where $\bar{v} : \mathbb{R}^k \to \mathbb{R}$ is defined by $\bar{v}\,(x) = -v\,(-x)$ for all $x \in \mathbb{R}^k$ and for all $v \in C\left(\mathbb{R}^k\right)$.

Proposition 9. *Let $\succcurlyeq^*$ be a binary relation on Δ represented as in (14) which satisfies (12) and (13). The following statements are equivalent:*

(i) For each $p, q \in \Delta$

$$p \succcurlyeq^* q \iff \sigma\,(q) \succcurlyeq^* \sigma\,(p)\,.$$

(ii) For each $p, q \in \Delta$

$$p \succcurlyeq^* q \implies \sigma\,(q) \succcurlyeq^* \sigma\,(p)\,.$$

(iii) $\mathcal{W}_{\max}\,(\succcurlyeq^*)$ is odd.

Moreover, if $\mathcal{W}$ in (14) is odd, then (i) and (ii) hold.

For the last part of the statement, that is proving that if $\mathcal{W}$ is odd, then (i) and (ii) hold, we can dispense with the assumption that $\succcurlyeq^*$ satisfies (12) and (13). The proof will clarify.

Proof. By Proposition 8, we have that

$$p \succcurlyeq^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq^*).$$

In other words, for the first part of the statement, we can replace $\mathcal{W}$ in (14) with $\mathcal{W}_{\max}(\succcurlyeq^*)$.

(i) implies (ii). It is obvious.

(ii) implies (iii). Fix $v \in \mathcal{W}_{\max}(\succcurlyeq^*)$. By definition of $\bar{v}$ and since each v in $\mathcal{W}_{\max}(\succcurlyeq^*)$ is strictly increasing, continuous, and such that $v(0) = 0$, we have that $\bar{v}$ is strictly increasing, continuous, and such that $\bar{v}(0) = 0$. By assumption and (19), we have that

$$p \succcurlyeq^* q \implies \sigma(q) \succcurlyeq^* \sigma(p) \implies \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \implies -\mathbb{E}_q(\bar{v}) \geq -\mathbb{E}_p(\bar{v}) \implies \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}).$$

By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, we can conclude that $\bar{v} \in \mathcal{W}_{\max}(\succcurlyeq^*)$, proving that $\mathcal{W}_{\max}(\succcurlyeq^*)$ is odd.

(iii) implies (i). By (19) and since $\mathcal{W}$ is odd and represents $\succcurlyeq^*$, we have that

$$\begin{aligned} p \succcurlyeq^* q &\iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \iff \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}) \quad \forall v \in \mathcal{W} \\ &\iff \mathbb{E}_{\sigma(q)}(v) \geq \mathbb{E}_{\sigma(p)}(v) \quad \forall v \in \mathcal{W} \iff \sigma(q) \succcurlyeq^* \sigma(p), \end{aligned}$$

proving the implication (since $\mathcal{W}_{\max}(\succcurlyeq^*)$ represents $\succcurlyeq^*$) and also the second part of the statement. ■

Representing $\succcurlyeq'$

We can finally provide a Multi-Expected Utility representation for $\succcurlyeq'$.

Proposition 10. *If $\succcurlyeq$ satisfies Weak Order, Continuity, Monotonicity, and Monetary equivalent, then*

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max}(\succcurlyeq').$$

Moreover, $\mathcal{W}_{\max}(\succcurlyeq')$ is a nonempty convex set with full image.

Proof. By the same techniques of Cerreia-Vioglio (2009, Proposition 22) (see also Cerreia-Vioglio et al. 2017, Lemma 1 and Footnote 10), $\succcurlyeq'$ is a preorder that satisfies Sequential Continuity and Independence.[26] By Evren (2008, Theorem 2), there exists a set $\mathcal{W} \subseteq$

[26]That is, for each two generalized sequences $\{p_\alpha\}_{\alpha \in A}$ and $\{q_\alpha\}_{\alpha \in A}$ in Δ

$$p_\alpha \succcurlyeq' q_\alpha \quad \forall \alpha \in A, p_\alpha \to p, \text{ and } q_\alpha \to q \implies p \succcurlyeq' q.$$

$C\left(\mathbb{R}^k\right)$ such that $p \succcurlyeq' q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$. By Lemma 1 and since $\succcurlyeq$ is a Weak Order which satisfies Monotonicity and Monetary equivalent, we have that $\succcurlyeq'$ satisfies (12) and (13). By Proposition 8 and considering $\succcurlyeq'$ in place of $\succcurlyeq^*$, $\mathcal{W}$ can be chosen to be $\mathcal{W}_{\max}(\succcurlyeq')$, proving the statement. ■

Missing Proofs

In this section, we prove Proposition 4. We begin by showing that if $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical. This fact will be key in proving the aforementioned proposition.

Lemma 2. *If $\succcurlyeq$ admits a finite essential Cautious Utility representation, then it is canonical.*

Proof. Define $\succcurlyeq^*$ to be such that $p \succcurlyeq^* q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all $v \in \mathcal{W}$ where $\mathcal{W}$ is a finite essential Cautious Utility representation of $\succcurlyeq$. Since $\mathcal{W}$ is finite, we have that the smallest convex cone containing $\mathcal{W}$, denoted by $\text{cone}(\mathcal{W})$, is closed with respect to the $\sigma\left(C\left(\mathbb{R}^k\right), \Delta\right)$-topology and so is the set $\text{cone}(\mathcal{W}) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}}$. By definition of $\mathcal{W}_{\max}(\succcurlyeq^*)$, it follows that $\text{cone}(\mathcal{W}) \setminus \{0\} \subseteq \mathcal{W}_{\max}(\succcurlyeq^*)$. By Proposition 8, Remark 3, and (Evren, 2008, Theorem 5) and since $\mathcal{W}$ is a Cautious Utility representation, we have that (where the closure is in the $\sigma\left(C\left(\mathbb{R}^k\right), \Delta\right)$-topology)

$$\begin{aligned}
\text{cone}(\mathcal{W}) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}} &= \text{cl}\left(\text{cone}(\mathcal{W}_{\max}(\succcurlyeq^*)) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}}\right) \\
&\supseteq \text{cl}\left(\mathcal{W}_{\max}(\succcurlyeq^*) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}}\right) \\
&\supseteq \mathcal{W}_{\max}(\succcurlyeq^*) + \left\{\theta 1_{\mathbb{R}^k}\right\}_{\theta\in\mathbb{R}},
\end{aligned}$$

yielding that $\text{cone}(\mathcal{W}) \setminus \{0\} \supseteq \mathcal{W}_{\max}(\succcurlyeq^*)$ and, in particular, $\text{cone}(\mathcal{W}) \setminus \{0\} = \mathcal{W}_{\max}(\succcurlyeq^*)$. Since the functional $v \mapsto c(p, v)$ is quasiconcave over $\text{cone}(\mathcal{W}) \setminus \{0\}$ for all $p \in \Delta$, it is immediate to see that

$$V(p) = \min_{v\in\mathcal{W}} c(p, v) = \min_{v\in\text{cone}(\mathcal{W})\setminus\{0\}} c(p, v) \quad \forall p \in \Delta.$$

By Remark 3 and since $\mathcal{W} = \{v_i\}_{i=1}^n$ is a finite Cautious Utility representation, we have that $\succcurlyeq$ satisfies Axioms 1- 5. By Theorem 1 and its proof, $\mathcal{W}_{\max}(\succcurlyeq')$ is a canonical Cautious

Utility representation for $\succcurlyeq$. In particular, we have that

$$V(p) = \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \text{cone}(\mathcal{W}) \setminus \{0\}} c(p, v) = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(p, v) \quad \forall p \in \Delta.$$

Since $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies the Independence axiom and $p \succcurlyeq^* q$ implies $p \succcurlyeq q$, we have that $\succcurlyeq^*$ is a subrelation of $\succcurlyeq'$ and $\mathcal{W}_{\max}(\succcurlyeq') \subseteq \mathcal{W}_{\max}(\succcurlyeq^*) = \text{cone}(\mathcal{W}) \setminus \{0\}$. By contradiction, assume that $\mathcal{W}_{\max}(\succcurlyeq') \neq \text{cone}(\mathcal{W}) \setminus \{0\}$. Since $\mathcal{W}_{\max}(\succcurlyeq')$ is a convex set closed with respect to strictly positive scalar multiplications, this implies that $\mathcal{W} \not\subseteq \mathcal{W}_{\max}(\succcurlyeq')$. If $\mathcal{W}$ is a singleton, then $\succcurlyeq$ is Expected Utility and, in particular, $\succcurlyeq'$ is complete and coincides with $\succcurlyeq$. This implies that $\mathcal{W} = \{v_1\}$ and $\mathcal{W}_{\max}(\succcurlyeq') = \{\lambda v_1\}_{\lambda > 0} = \text{cone}(\mathcal{W}) \setminus \{0\}$, a contradiction. Assume $\mathcal{W}$ is not a singleton. Consider $\breve{v} \in \mathcal{W} \setminus \mathcal{W}_{\max}(\succcurlyeq')$. Since $\mathcal{W}$ is essential, there exists $\bar{p} \in \Delta$ such that $\min_{v \in \mathcal{W}} c(\bar{p}, v) < \min_{v \in \mathcal{W} \setminus \{\breve{v}\}} c(\bar{p}, v)$. Since $\mathcal{W} = \{v_i\}_{i=1}^n$ and $n \geq 2$, without loss of generality, we can set $\breve{v} = v_n \notin \mathcal{W}_{\max}(\succcurlyeq')$. In particular, we have that

$$\inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = \min_{v \in \mathcal{W}} c(\bar{p}, v) = c(\bar{p}, v_n) < c(\bar{p}, v_i) \quad \forall i \in \{1, \dots, n-1\}. \tag{20}$$

Consider a sequence $\{\hat{v}_m\}_{m \in \mathbb{N}} \subseteq \mathcal{W}_{\max}(\succcurlyeq')$ such that $c(\bar{p}, \hat{v}_m) \downarrow \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. By construction and since $\mathcal{W}_{\max}(\succcurlyeq') \subseteq \text{cone}(\mathcal{W}) \setminus \{0\}$, there exists a collection of scalars $\{\lambda_{m,i}\}_{m \in \mathbb{N}, i \in \{1,\dots,n\}} \subseteq [0, \infty)$ such that $\hat{v}_m = \sum_{i=1}^n \lambda_{m,i} v_i$ for all $m \in \mathbb{N}$. Since $\hat{v}_m$ is strictly increasing, we have that for each $m \in \mathbb{N}$ there exists $i \in \{1, \dots, n\}$ such that $\lambda_{m,i} > 0$. Define $\lambda_{m,\sigma} = \sum_{i=1}^n \lambda_{m,i} > 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and for each $i \in \{1, \dots, n\}$ define also $\bar{\lambda}_{m,i} = \lambda_{m,i} / \lambda_{m,\sigma}$ as well as $\tilde{v}_m = \sum_{i=1}^n \bar{\lambda}_{m,i} v_i = \hat{v}_m / \lambda_{m,\sigma}$. Since $\lambda_{m,\sigma} > 0$ for all $m \in \mathbb{N}$, it is immediate to see that $c(\bar{p}, \tilde{v}_m) = c(\bar{p}, \hat{v}_m)$ for all $m \in \mathbb{N}$ and, in particular, $c(\bar{p}, \tilde{v}_m) \downarrow \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v)$. For each $m \in \mathbb{N}$ denote by $\bar{\lambda}_m$ the $\mathbb{R}^n$ vector whose i-th component is $\bar{\lambda}_{m,i}$. Since $\{\bar{\lambda}_m\}_{m \in \mathbb{R}}$ is a sequence in the $\mathbb{R}^n$ simplex, there exists a subsequence $\{\bar{\lambda}_{m_l}\}_{l \in \mathbb{N}}$ such that $\bar{\lambda}_{m_l,i} \to \bar{\lambda}_i \in [0, 1]$ for all $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n \bar{\lambda}_i = 1$. It is immediate to see that $\tilde{v}_{m_l} = \sum_{i=1}^n \bar{\lambda}_{m_l,i} v_i \overset{\sigma(C(\mathbb{R}^k),\Delta)}{\longrightarrow} \sum_{i=1}^n \bar{\lambda}_i v_i = \tilde{v}$ where $\tilde{v}$ is continuous, strictly increasing, and such that $\tilde{v}(0) = 0$. Moreover, for each $p, q \in \Delta$ we have that $p \succcurlyeq' q$ implies $\mathbb{E}_p(\tilde{v}) \geq \mathbb{E}_q(\tilde{v})$, proving that $\tilde{v} \in \mathcal{W}_{\max}(\succcurlyeq')$. Note that $\bar{\lambda}_n < 1$, otherwise, we would have that $v_n = \tilde{v} \in \mathcal{W}_{\max}(\succcurlyeq')$, a contradiction. By (20) and since $\bar{\lambda}_n < 1$ and the functional $v \mapsto c(p, v)$ is explicitly quasiconcave over $\mathrm{co}(\mathcal{W})$ for all $p \in \Delta$,[27] we have that

$$c(\bar{p}, v_n) < c(\bar{p}, \tilde{v}) = \lim_l c\left(\bar{p}, \tilde{v}_{m_l}\right) = \lim_m c(\bar{p}, \tilde{v}_m) = \inf_{v \in \mathcal{W}_{\max}(\succcurlyeq')} c(\bar{p}, v) = c(\bar{p}, v_n),$$

a contradiction. It follows that $\mathcal{W}_{\max}(\succcurlyeq') = \mathrm{cone}(\mathcal{W}) \setminus \{0\}$ and, in particular, $\mathcal{W}$ represents also $\succcurlyeq'$. This implies that $\mathcal{W}$ is canonical. ■

Proof of Proposition 4. We first prove the first part of the statement assuming $\succcurlyeq$ satisfies u-CPT, and then we will move to the additive case. Since $u(0) = 0$ and u is strictly increasing and continuous, it follows that there exists $\bar{t} > 0$ such that $[-\bar{t}, \bar{t}] \subseteq \mathrm{Im}\, u$. Let $\Delta_0([0, \bar{t}])$ be the set of finitely supported probabilities over $[0, \bar{t}]$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$. By definition, we have that there exist two unique collections $\{t_i\}_{i=1}^n \subseteq [0, \bar{t}]$ and $\{\lambda_i\}_{i=1}^n \subseteq [0, 1]$ such that $\mathrm{supp}\, p = \{t_i\}_{i=1}^n$, $\sum_{i=1}^n \lambda_i = 1$, and $\tilde{p} = \sum_{i=1}^n \lambda_i \delta_{t_i}$. Without loss of generality, we can assume that $t_1 < \ldots < t_n$. We define $\tilde{V} : \Delta_0([0, \bar{t}]) \to \mathbb{R}$ by

$$\tilde{V}(\tilde{p}) = \sum_{j=1}^{n-1} \left(\bar{w}^+ \left(\sum_{i=j}^{n} \lambda_i \right) - \bar{w}^+ \left(\sum_{i=j+1}^{n} \lambda_i \right) \right) v\left(t_j\right) + \bar{w}^+(\lambda_n)\, v(t_n)$$

for all $\tilde{p} \in \Delta_0([0, \bar{t}])$ where $\bar{w}^+ : [0, 1] \to [0, 1]$ is defined by $\bar{w}^+(t) = 1 - w(1 - t)$ for all $t \in [0, 1]$. We next show that for each $\tilde{p} \in \Delta_0([0, \bar{t}])$ and for each $\tilde{t} \in [0, \bar{t}]$, if $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda \tilde{p} + (1 - \lambda)\delta_{\tilde{t}}) = \tilde{V}(\delta_{\tilde{t}})$ for all $\lambda \in (0, 1)$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$ and $\tilde{t} \in [0, \bar{t}]$ such that $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$. Given $\tilde{p} \in \Delta_0([0, \bar{t}])$, since $\{t_i\}_{i=1}^n \subseteq [0, \bar{t}] \subseteq \mathrm{Im}\, u$, there exists $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^k$ such that $u(x_i) = t_i$ for all $i \in \{1, \ldots, n\}$. Consider $p = \sum_{i=1}^n \lambda_i \delta_{x_i}$. It is immediate to see that $\tilde{V}(\tilde{p}) = V(p)$. Since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, there exists $c \in \mathbb{R}$ such that $p \sim \delta_{ce_1}$. This implies that $V(p) = V\left(\delta_{ce_1}\right)$ and, in particular, $u(ce_1) \in [0, \bar{t}]$. Moreover, since u and v are strictly increasing, we have that $u(ce_1) = \tilde{t} \in [0, \bar{t}]$ and $V\left(\delta_{ce_1}\right) = \tilde{V}(\delta_{\tilde{t}})$. By Remark 3 and since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, we have that $\succcurlyeq$ satisfies M-NCI. This yields that $\lambda p + (1 - \lambda)\delta_{ce_1} \sim \delta_{ce_1}$ for all $\lambda \in (0, 1)$. This implies that

$$\tilde{V}(\lambda \tilde{p} + (1 - \lambda)\delta_{\tilde{t}}) = V\left(\lambda p + (1 - \lambda)\delta_{ce_1}\right) = V\left(\delta_{ce_1}\right) = \tilde{V}(\delta_{\tilde{t}}).$$

[27]Formally, see e.g. (Aliprantis and Border, 2006, p. 300), given $p \in \Delta$, for each $h \in \mathbb{N} \setminus \{1\}$, for each $\{v_l\}_{l=1}^h \subseteq \mathrm{co}(\mathcal{W})$, and for each $\{\lambda_l\}_{l=1}^h \subseteq [0, 1]$ such that $\sum_{l=1}^h \lambda_l = 1$ and $\lambda_h < 1$

$$c(p, v_i) > c(p, v_h) \quad \forall i \in \{1, \ldots, h - 1\} \implies c\left(p, \sum_{i=1}^h \lambda_i v_i\right) > c(p, v_h).$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that $\bar{w}^{+}$ is the identity and so is w^{+}. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and w^{+} replaced by w^{-}, yields that w^{-} is the identity. These two facts together allow us to conclude that $p \mapsto V(p) = \mathrm{CPT}_{v,w^{+},w^{-}}(p_u)$ is an Expected Utility functional with utility $v \circ u : \mathbb{R}^k \to \mathbb{R}$.
We next assume that $\succcurlyeq$ admits an Additive CPT representation. As before consider $\bar{t} > 0$. Define $\Delta_0([0, \bar{t}])$ and $\tilde{V}$ as before with v replaced by u_1. For each $\tilde{p} \in \Delta_0([0, \bar{t}])$ define p in Δ to be the product measure $\tilde{p} \otimes \delta_0 ... \otimes \delta_0$. It is immediate to see that $\tilde{V}(\tilde{p}) = V(p)$ for all $\tilde{p} \in \Delta_0([0, \bar{t}])$. As before, we can show that for each $\tilde{p} \in \Delta_0([0, \bar{t}])$ and for each $\tilde{t} \in [0, \bar{t}]$, if $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$, then $\tilde{V}(\lambda\tilde{p} + (1-\lambda)\delta_{\tilde{t}}) = \tilde{V}(\delta_{\tilde{t}})$ for all $\lambda \in (0,1)$. Consider $\tilde{p} \in \Delta_0([0, \bar{t}])$ and $\tilde{t} \in [0, \bar{t}]$ such that $\tilde{V}(\tilde{p}) = \tilde{V}(\delta_{\tilde{t}})$. This implies that $V(p) = V\left(\delta_{\tilde{t}e_1}\right)$, that is, $p \sim \delta_{\tilde{t}e_1}$. By Remark 3 and since $\succcurlyeq$ admits a Symmetric Cautious Utility representation, we have that $\succcurlyeq$ satisfies M-NCI. This yields that $\lambda p + (1-\lambda)\delta_{\tilde{t}e_1} \sim \delta_{\tilde{t}e_1}$ for all $\lambda \in (0,1)$. This implies that

$$\tilde{V}(\lambda\tilde{p} + (1-\lambda)\delta_{\tilde{t}}) = V\left(\lambda p + (1-\lambda)\delta_{\tilde{t}e_1}\right) = V\left(\delta_{\tilde{t}e_1}\right) = \tilde{V}(\delta_{\tilde{t}}).$$

By Bell and Fishburn (2003, Theorem 1) applied to $\tilde{V}$, it follows that $\bar{w}^{+}$ is the identity and so is w^{+}. The same proof, performed with $[-\bar{t}, 0]$ in place of $[0, \bar{t}]$ and w^{+} replaced by w^{-}, yields that w^{-} is the identity. This implies that $\succcurlyeq$ admits an Expected Utility representation with utility $u : \mathbb{R}^k \to \mathbb{R}$ defined by $u(x) = \sum_{i=1}^{k} u_i(x_i)$ for all $x \in \mathbb{R}^k$.

As for the second part of the statement, by Lemma 2 and since $\mathcal{W}$ is a finite essential Cautious Utility representation, we have that $\mathcal{W}$ is a canonical representation, that is, $\mathcal{W} = \{v_i\}_{i=1}^{n}$ represents also $\succcurlyeq'$. Since $\succcurlyeq$ is Expected Utility with utility $v \circ u$ (where in the additive case v is the identity and u is additively separable), we have that $\succcurlyeq'$ coincides with $\succcurlyeq$, yielding that for each $i \in \{1, ..., n\}$ there exists $\lambda_i > 0$ such that $v_i = \lambda_i(v \circ u)$. This implies that $c(p, v_i) = c(p, v \circ u)$ for all $p \in \Delta$ and for all $i \in \{1, ..., n\}$. Since $\mathcal{W}$ is essential, this implies that $\mathcal{W}$ is a singleton. Since $\mathcal{W} = \{v_1\}$ and $\mathcal{W}$ is odd, this implies that v_1 is odd and, in particular, $\succcurlyeq$ is loss neutral for risk and exhibits no endowment effect. ■

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