The Econometrics of Stock Market Returns. An introduction

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1 The Standard Static Asset Allocation problem

Consider the case of an investor who adopts a buy and hold portfolio strategy for a single period of any fixed length (the length is not a decision variable in the asset allocation) from time t to time T. Let us indicate with r the random vector of linear total returns from time t to time T from a given set of (k) risky assets for the time period, $r \sim (\mu, \Sigma)$.

The investor has also available at time t a security whose price at T is known at t (typically a non defaultable bond) called 'risk free security'. Let r^f be the (linear) non random return from this investment over the period.

The investor's strategy is to invest in the risk free security and in the stocks at time t and then liquidate the investment at time T. The relative amounts of each investment in stocks are in the column vector w, while $(1 - w'e)$ is the relative amount invested in the risk free security (e is a column vector of ones). Given a degree of risk aversion λ a standard description of this allocation problem is the following:

$$\max_w (1 - w'e) r^f + w'\mu - 0.5\lambda * w'\Sigma w$$
$$r \sim (\mu, \Sigma)$$

In this setup the distribution of returns is fully described by the first two moments and the miniminization of the variance of the portfolio implies that by ruling out big losses also big gains are ruled out. The solution of this problems determines portfolio's weight in terms of the preferences of the investor, and the (known) mean and the variance covariance matrix of the joint distribution of returns.

$$\hat{w} = \frac{1}{\lambda}\Sigma^{-1}\left(\mu - er^f\right)$$

In order to make the approach operational knowledge of λ needs to be paired with estimates of Σ and μ.

Consider now the special case in which $\hat{w}'e = 1$, that is no investment in the risk free security. In other words we are selecting the so called 'tangency' portfolio. In this case we have:

$$\hat{w} = \frac{\Sigma^{-1}\left(\mu - er^f\right)}{e'\Sigma^{-1}\left(\mu - er^f\right)}$$

In order to make the approach operational knowledge of λ needs to be paired with some estimates for Σ and μ.

2 Asset Allocation with the Simplest Empirical Model

The simplest approach to the solution of the problem of finding numerical counterparts for Σ and μ is the use of historical moments. No econometrics is needed to this end. This approach is justified by the view prevalent in the 70s' on the behaviour of asset prices and financial returns that Cochrane (1999) summarized as follows:

- CAPM is a good measure of risk and thus a good explanation of why some stocks earn higher average returns than others
- Excess Returns are close to unpredictable: any predictability is a statistical artifact or cannot be exploited after transaction costs
- Volatility is constant

What ia the econometric specification that sustain the view from the 70s?

Consider the following simultaneous equation linear regression model for a sample of size T of observations on a vector of G returns

$$\mathbf{y}^{+} = \mathbf{x}^{+}\boldsymbol{\delta}^{+} + \mathbf{u}^{+}, \tag{1}$$

where $\mathbf{y}^+$ is a $(GT \times 1)$ vector, $\mathbf{x}^+$ is a $\left(GT \times \sum_{i=1}^{G} K_i\right)$ matrix, $\boldsymbol{\delta}^+$ is a $\left(\sum_{i=1}^{G} K_i \times 1\right)$, $\mathbf{u}^+$ is a $(GT \times 1)$ vector of residuals:

$$\mathbf{y}^+ = \begin{pmatrix} \mathbf{y}_1 \\ . \\ . \\ \mathbf{y}_G \end{pmatrix}, \quad \mathbf{x}^+ = \begin{pmatrix} \mathbf{x}_1 & 0 & 0. & 0 \\ 0 & \mathbf{x}_2 & . & . \\ 0 & . & . & . \\ 0 & . & . & \mathbf{x}_G \end{pmatrix},$$

$$\boldsymbol{\delta}^+ = \begin{pmatrix} \boldsymbol{\delta}_1 \\ . \\ . \\ \boldsymbol{\delta}_G \end{pmatrix}, \quad \mathbf{u}^+ = \begin{pmatrix} \mathbf{u}_1 \\ . \\ . \\ \mathbf{u}_G \end{pmatrix}.$$

The following properties hold for $\mathbf{u}^+$:

$$E\left(\mathbf{u}^+\right) = 0,$$

$$E\left(\mathbf{u}^+\mathbf{u}^{+\prime}\right) = \begin{pmatrix} E\left(\mathbf{u}_1\mathbf{u}_1'\right) & E\left(\mathbf{u}_1\mathbf{u}_2'\right) & .. & E\left(\mathbf{u}_1\mathbf{u}_G'\right) \\ E\left(\mathbf{u}_2\mathbf{u}_1'\right) & E\left(\mathbf{u}_2\mathbf{u}_2'\right) & .. & . \\ . & . & .. & . \\ . & . & .. & . \\ E\left(\mathbf{u}_G\mathbf{u}_1'\right) & . & .. & E\left(\mathbf{u}_G\mathbf{u}_G'\right) \end{pmatrix},$$

where each block of the above matrix is $(T \times T)$.

The simplest model adopted specifies $\mathbf{x}_1 = \mathbf{x}_2 = ... = \mathbf{x}_G = e$ and assumes that all residuals are contemporaneously but not serially correlated, with non-singular variance-covariance matrix Σ, we have:

$$\begin{aligned} E\left(\mathbf{u}^+\mathbf{u}^{+\prime}\right) &= \Sigma \otimes I_T \\ &= \begin{pmatrix} \sigma_{11}I_T & \sigma_{12}I_T & .. & \sigma_{1G}I_T \\ \sigma_{21}I_T & \sigma_{22}I_T & .. & . \\ . & . & .. & . \\ . & . & .. & . \\ \sigma_{G1}I_T & . & .. & \sigma_{GG}I_T \end{pmatrix}. \end{aligned}$$

within specification paramers are validly estimated by applying OLS to each equations. Consider for example the observations on the first return:

$$\mathbf{y}_1 = X\beta + u$$

$$\mathbf{y}_1 = \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_T \end{bmatrix}, X = e = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix}$$

The OLS estimates of the relevant parameters are then

$$\hat{\beta} = \frac{1}{T}\sum_{t=1}^{T} x_t, \ \hat{\sigma}_u^2 = \sum_{t=1}^{T}\frac{1}{T}\left(x_t - \hat{\beta}\right)^2$$

3 The Black-Littermann Approach

The traditional simplest approach to portfolio allocation can lead to dramatic swings in the optimal portfolio weights for small changes in investment views as given by the estimates of μ and Σ.There is a simple reason for ths fact: too much sampling error in the estimation of the expected return and, due to this, an asset allocation which is idiosyncratic to the specific estimation sample.

The Black-Litterman model (see Black and Litterman, 1990, and Black and Litterman, 1991) was developed to provide a systematic resolution to this problem. The basic idea is that of using as a starting point the market allocation and express the investor's 'views' as departures from this allocation. The main contribution of the method it to discipline the asset manager action. A numerical specification of the views and of the confidence in such views is required. The method ensures the most efficient mplementation of the expressed views.

The basic idea of the Black and Litterman model is that expected returns are not estimated. Given the knowledge of capitalization of different markets it is possible to obtain returns by reverse engineering of the optimal portfolio allocation formula. In practice, given the knowledge of the market capitalization and therefore of the market weights w_{mkt} and some estimates of the variance-covariance matrix of returns, we can use the optimal portfolio allocation condition to derive the expected returns consistent with the market capitalization:

$$\mu_{mkt} = \lambda\Sigma w_{mkt} + er^f$$

Assume now that the portoflio allocator hold some views on a subset of size q of the k returns included in the market portfolio:

$$P\mu_r \sim N_q\left(V, \Gamma\right)$$

where μ_r is the vector of k returns, and P is a selection matrix (qxp) that selects the subset of returns on which there are views. The views are expressed as a vector of mean expected returns V and a diagonal variance-covariance matrix Γ, expressing the confidence on the views.

This views have to be baanced against the distribution of returns implied in the market capitalization:

$$\mu_r \sim N_p\left(\mu_{mkt}, \tau\Sigma\right)$$

where τ is a scalar smaller than one (and conventinally set to 1/3) to filter out of the estimated variance-covariance matrix of returns the impact of their random variation (i.e. to take into account the effect of noise in small samples).

The Black-Litterman approach is aimed at generating a value of for the expected returns μ_{BL} by combining optimally the distribution of returns implied in the market capitalization and the views of the portfolio allocator. This is obtained by solving an optimization problem:

$$\mu_{BL} = \arg\min_{\mu} \left(\mu - \mu_{mkt}\right)' \left(\tau\Sigma\right)^{-1} \left(\mu - \mu_{mkt}\right) + \left(P\mu - V\right)' \Gamma^{-1} \left(P\mu - V\right)$$

This is a weigthed least squares problems, where weigths depend on variance-covaraince matrix. When the diagonal elements of Γ go to zero, that is, when there is infinite confidence in the views, the problem becomes a costrained least squares problem where the relevant constraint is $P\mu_{BL} = V$. On the other hand, when Γ has diagonal elements diverging to infinity (no confidence in the views), the solution to the problem is simply $\mu_{BL} = \mu_{mkt}$.

The first order conditions for the solution of the problem can be written as follows:

$$2\left(\tau\Sigma\right)^{-1}\left(\mu_{BL} - \mu_{mkt}\right) + 2P'\Gamma^{-1}\left(P\mu_{BL} - V\right) = 0$$

from which we can derive:

$$\mu_{BL} = \left(\left(\tau\Sigma\right)^{-1} + P'\Gamma^{-1}P\right)^{-1}\left(\left(\tau\Sigma\right)^{-1}\mu_{mkt} + P'\Gamma^{-1}V\right)$$

which makes clear that μ_{BL} is obtained by combining optimally market views and investor's views.

Note that μ_{BL} could be equivalently written as:

$$\begin{aligned} \mu_{BL} &= \mu_{mkt} + K\left(V - P\mu_{mkt}\right) \\ K &= \left(\tau\Sigma\right)P'\left(P\tau\Sigma P' + \Gamma\right)^{-1} \end{aligned}$$

Given μ_{BL} optimal portfolio weights are obtained by the usual formula:

$$w_{BL} = \frac{\Sigma^{-1}\left(\mu_{BL} - er^{f}\right)}{e'\Sigma^{-1}\left(\mu_{BL} - er^{f}\right)}$$

4 Going to the data: Asset Allocation in Practice

1. Import data in Matlab from the file *Stockint2011.xls.* You should have in your workfile the time series of prices and dividends for the US, UK and German stock markets, of exchange rates, of the yield on German 10-years government bond and of the yield of the German 3-months government bond. Data have monthly frequency and are collected for the sample 05/1977:09/2010.

2. Assume you are a German investor. Compute from your perspective (*use exchange rates!*) monthly total returns (*i. e.* including dividens) in excess of the risk free rate for the the US, UK and German stock markets and for the German 10-years bond. Use the annualized yield of the 3-months government bond as the risk free rate. **Work with log returns, even if in the next sections this choice will lead to some inaccuracy**.

3. Compute and plot cumulative excess returns of all risky assets over the time period 01/1978:12/2003.

4. Assume you want to invest your wealth in the risky assets for the period 01/2004:12/2007. Solve the asset allocation problem using the historical sample 01/1978:12/2003. In order to compute weights, use the solution to the Markowitz mean-variance optimization problem. Base this exercise on unconditional moments.

5. Please comment on the weights delivered by your optimization exercise. Plot the performance of your portfolio against the one of alternative portfolios based on a buy and hold strategy for each of the risky assets over the investment period.

6. Re-estimate the weights by calculating the unconditional moments over the sample 01/2004:12/2007; are the weights equal to the ones computed in question 4?

7. Assume that you have a view, different from the uncondtional moments, on the expected excess returns of stock market indexes over the investment period 01/2004:12/2007. How would you now approach the asset allocation problem?

SOLUTION

This solution is a commented version of the MATLAB code ASSETALLOC.m posted on the course website.

1. The solution to the first question is based on the folowing lines of the code:

[filename,pathname]=uigetfile('*.xls');
[data,textdata,raw] = xlsread(filename,1);

You load data from the excel spreadsheet into the Matlab workfile. You will now have three objects in the workfile: the matrix *raw* (which shows time series, headers and dates exactly as they appear in the excel file), the matrix *data* (collecting the time series), and the matrix *texdata* (collecting headers and dates). Dates are imported from Matlab as text.

A useful command you might now use is *datenum.* This function allows to transform dates writtem as text in serial numbers.

date=datenum(textdata(3:end,1),'dd/mm/yyyy');

2. To answer this question we have to compute asset returns for each risky asset and then subtract the risk free in order to get excess returns. As a first step we can compute the monthly log risk free rate :

lrf_m = log(1+(data(:,2)/(100*12)));

Bear in mind that yields are always annualized; that's why we are dividing by 12. We are also dividing by 100 for a matter of scaling (in the xls file 1=1%).

We now turn to the German 10-years bond. The time series that we find on the xls file contains the yield of the bond. We therefore have to transform yields into returns.

This can be done easily by computing the duration of the bond and multiplying it by variations in yields. As a first step, we calculate log yields for the bond.

ly_10_m = (log(1+(lag(data(:,1))/(100*12))));

Again, data are rescaled and annualized. We now have to approximate the duration; a way to do it is to use an approximation that can be derived by assuming that coupons are equal to the yield to maturity (just by using of **annuities**).

dur = ((1-(1+(data(:,1)/(100))).^(-10)))./(1-(1+(data(:,1)/(100))).^(-1));

We then calculate returns[1] as:

lret_b_10_m
=(lag(dur).*lag(ly_10_m)-(lag(dur)-1).*ly_10_m);

The excess return is just the difference between the monthly returns of the 10-years bond and the monthly return of the risk free rate.

exlret_b_10 = lret_b_10_m-lrf_m;

For what concerns stocks, log total returns are calculated including both log prices and log dividends (here we just show an example for the German market). Of course, dividend yields need to be annualized.

dy_ger_m = data(:,4)/(100*12);
lret_ger_m = log((p_ger./lag(p_ger))+dy_ger_m);

For the US and UK markets, returns have to be adjusted for the exchange rate; we compute exchange rates as:

r_EUvsDOL = log((data(:,9))./lag(data(:,9)));
r_STRvsDOL = log((data(:,10))./lag(data(:,10)));
r_STRvsEU = r_STRvsDOL - r_EUvsDOL;

In our weird notation, we might not be taking thoroughly into account FX conventions. We therefore specify that: EUvsDOL means 1€ = # .# # # $. For those of you who master FX trading, we can also say that in this example EUR is the base currency.

We the compute returns in EURO terms of the US and UK stock markets:

lret_us_m = lret_us_dol - r_EUvsDOL;
lret_uk_m = lret_uk_dol + r_STRvsEU;

[1] To see how duration and returns have been computed see the box at the end of the chapter

We get excess returns again by just subtracting the log monthly risk free rate.

3. In order to solve this exercise, you just have to compute the cumulative sum (remember we are working with log returns) of excess returns over the chosen sample. As a first step, you have to select the starting and ending date of the sample and count the number of observations in between them. Thus:

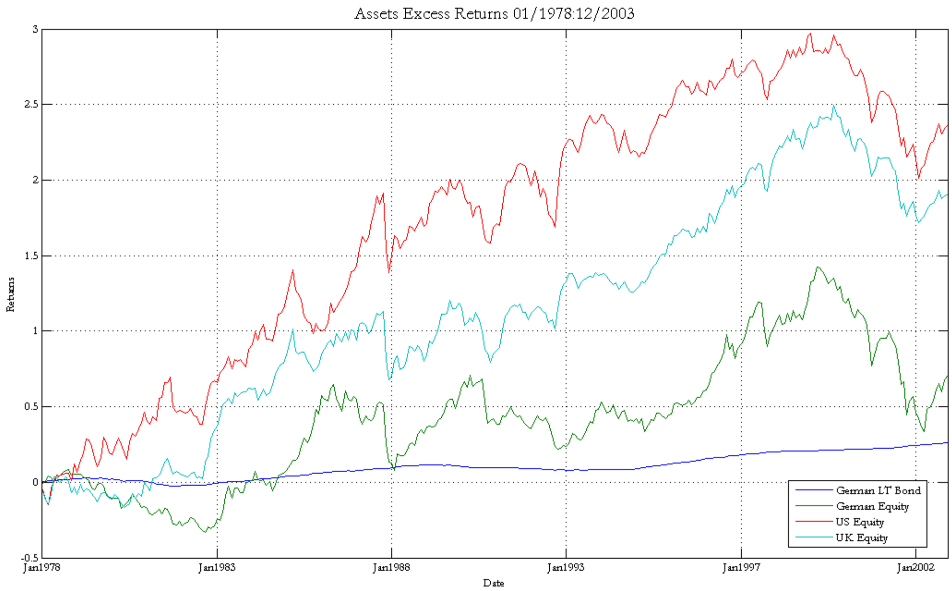
s_start = '01/01/1978';
s_end = '01/12/2003';
date_find=datenum([s_start; s_end],'dd/mm/yyyy');
ss=datefind(date_find(1,1),date);
se=datefind(date_find(2,1),date);

You can now select the relevant observations in the vectors of excess returns, and collect them into a matrix.

R = [exlret_b_10(ss:se) exlret_ger(ss:se) exlret_uk(ss:se) exlret_us(ss:se)];

Cumulated performance is derived as: **Perf = cumsum(R);**

This is the result you should get when plotting the time series:



The code for the plot above is the following:

```
figure(1);
plot(Perf);
title('Assets Excess Returns 01/1978:12/2003','fontname','Garamond','fontsize',
index=1:60:(se-ss+1);
set(gca,'fontname','garamond','fontsize',10);
set(gca,'xtick',index);
set(gca,'xticklabel','Jan1978|Jan1983|Jan1988|Jan1993|Jan1997|Jan2002');
set(gca,'xlim',[1 (se-ss+1)]);
grid;
ylabel('Returns');
xlabel('Date');
h=legend('German LT Bond', 'German Equity','US Equity','UK
Equity',0);
```

A few of brief comments: with the first command you open a new empty figure in Matlab, and set it as figure number 1. In this way, if you open a new figure, you will not overwrite your pervious work. We then plot the matrix **Perf** into the figure. If we are interested in raw results and do not like formatting, our work is done. However, here we discuss some easy commands to improve the appearence of graphs in Matlab. A useful command under this respect is the function **set**, which is used in order to modify figure properties.

In the code, the first thing we do to format the plot is to assign a title and choose its font and size. Then, we assign labels to the x axis. First, we have to state how many ticks we want on the axis and which is the distance of one from the other. That's why we build the linespace **index** and assign it to the command **'xtick'**. After having chosen the number and position of ticks, we can assign labels, as you can see in row 6 of the code. We then define limits for the x axis, we plot a grid on the chart, we assign axis names and plot the legend.

4. The formal solution for the Markowitz asset allocation problem can be found in the handouts of FINANCIAL ECONOMETRICS AND EMPIRICAL FINANCE - MODULE 1, in section 8. In the answer to question 3, we have to find the weights in the Market Portfolio, for the market composed by the four risky assets. The market portfolio coincides with the **tangency porfolio**, whose vector of weights is formally derived in the notes as:

$$w = \frac{\Sigma^{-1}(\mu - r^f)}{1'\Sigma^{-1}(\mu - r^f)}$$

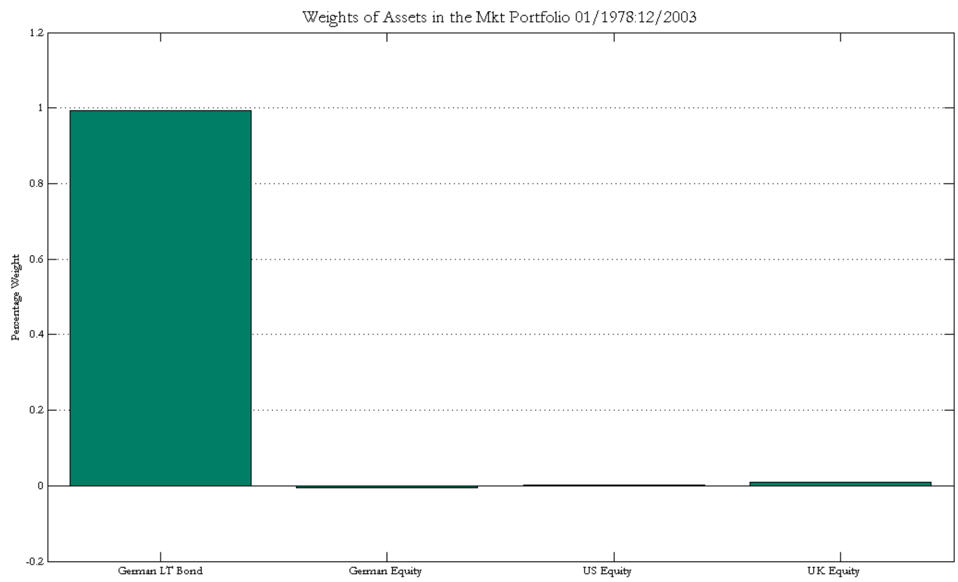
Where Σ is the historical unconditional variance covariance matrix, μ is the vector of unconditional historical means and r^f is the risk free rate; since we are considering a multiperiod investment horizon, we have to multiply both the unconditional mean vector and the unconditional variance-covariance matrix by n, where n is the number of periods (months). This is exactly what we do in the code:

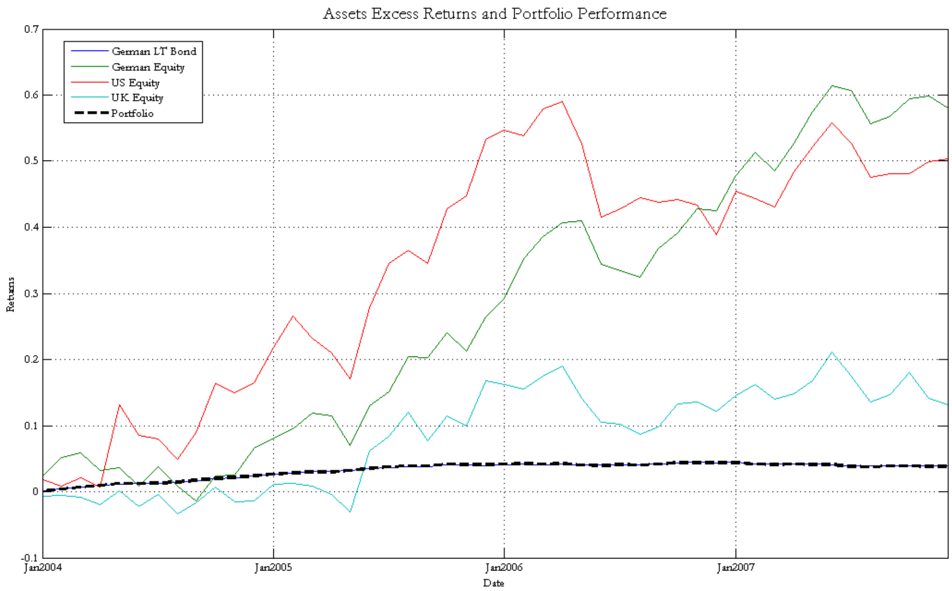
muR = n*mean(R)';
SigmaR = n*cov(R);

Remember that **muR** is a vector of **excess returns**. Then, the vector of weights is:

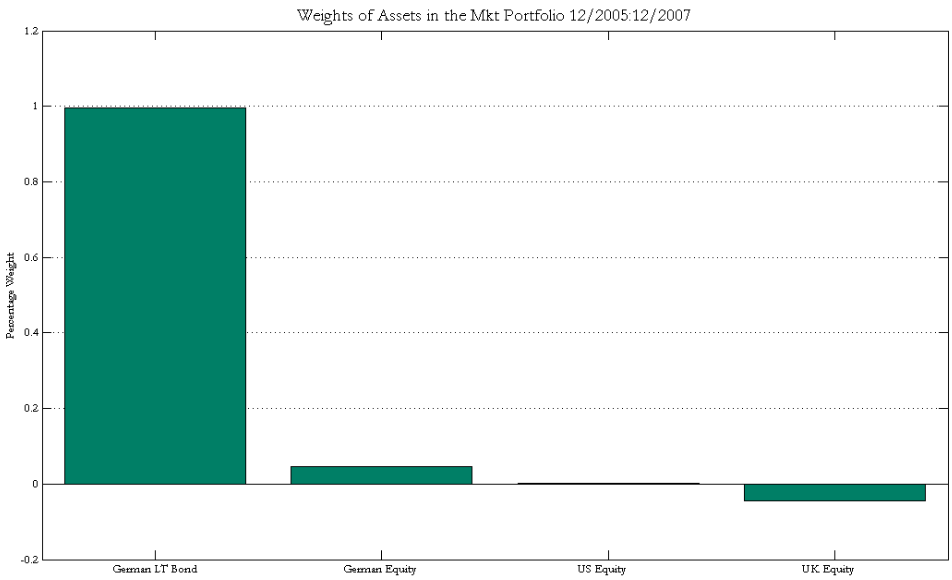
wMP =
((SigmaR^(-1))*muR)./(ones(4,1)'*(SigmaR^(-1)*muR));

5. We now plot the weights and the performance of the portfolio derived using the Markowitz model; as we can see tha portfolio is almost 100% invested in the bond, and over the investment perido dramatically underperforms stocks.





6. We plot weights calculated using data from the sample 01/2004:12/2007; as you can notice their values have slightly changed.



7. You have already met in your studies an asset allocation model able to integrate a view on a certain asset and the Markowitz mean-variance approach. In fact, this solution is discussed in Chapter 12 of the FINANCIAL ECONOMETRICS AND EMPIRICAL FINANCE - MOD-

ULE 1 class notes, and happens to be the Black and Litterman model (hereafter B&L). We now propose an (almost) real life implementation exercise of this framework to your asset allocation problem.

As a first step, we have to reverse engineer market expected returns, starting from the historical matrix Σ (for the sample 01/1978:12/2003), the risk aversion coefficient λ and a vector of observed market weights of the assets in our portfolio. We make the assumption that $\lambda = 2.5$, as it is usually done in B&L implementations. For the traded risky assets, we assume that for German investors the market is equally allocated between the 4 investment opportunities; thus each asset will have an hypothetical observed weight of 25% (note that this assumption doesn't affect you final results very much; try and play around with the numbers). We can then compute the implied expected expected excess returns as:

$$\mu^{IMP} - r^f = \frac{1}{\lambda} w^{IMP} \Sigma$$

In the code:

wMP_i = [0.25 0.25 0.25 0.25]';
lambda = 2.5;
muMP_i = SigmaR*wMP_i./lambda;

As a next step, we have to define the views; here we just say that we expected cumulated excess returns over the whole investment period to be 40% for each stock index. For what concerns the bond, we don't express any view. We therefore specify a selection matrix P:

$$P = \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$$

And a vector of expected values for the views V:

$$V = \begin{bmatrix} 0.4 \\ 0.4 \\ 0.4 \end{bmatrix}$$

To express our confidence in the our insights, we also specify the variance-covariance matrix of the views, and call it Γ. It makes sense to assume that the matrix is diagonal, since it is difficult to make figure out correlation structures between absolute views on different assets (see Chapter 12 for more details). We choose a pretty high value of the standard deviation, which is equal to 0.2 (variance of 0.04) for each one of the equity indexes.

$$\Gamma = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}$$

Finally we have to set a parameter which expresses confidence in the estimate based on historical data of the variance covariance matrix of returns. We set this parameter as $\tau = 0.3$, which is again a value in line with what is usually done in applications of B&L.

Then, we write the solution for the weights:

$$\begin{aligned} s_{BL} &= \lambda(\Sigma^{-1}(\mu^{IMP} - r^f) + \Sigma^{-1}K(V - P(\mu^{IMP} - r^f))) \\ K &= \tau\Sigma P'(\tau P\Sigma P' + \Gamma)^{-1} \end{aligned}$$

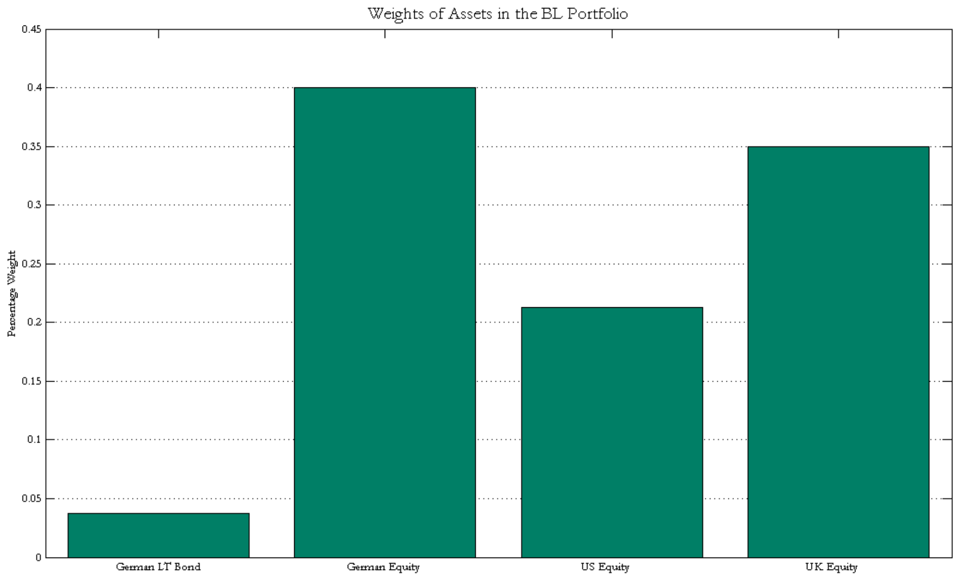
To make the weights sum to one:

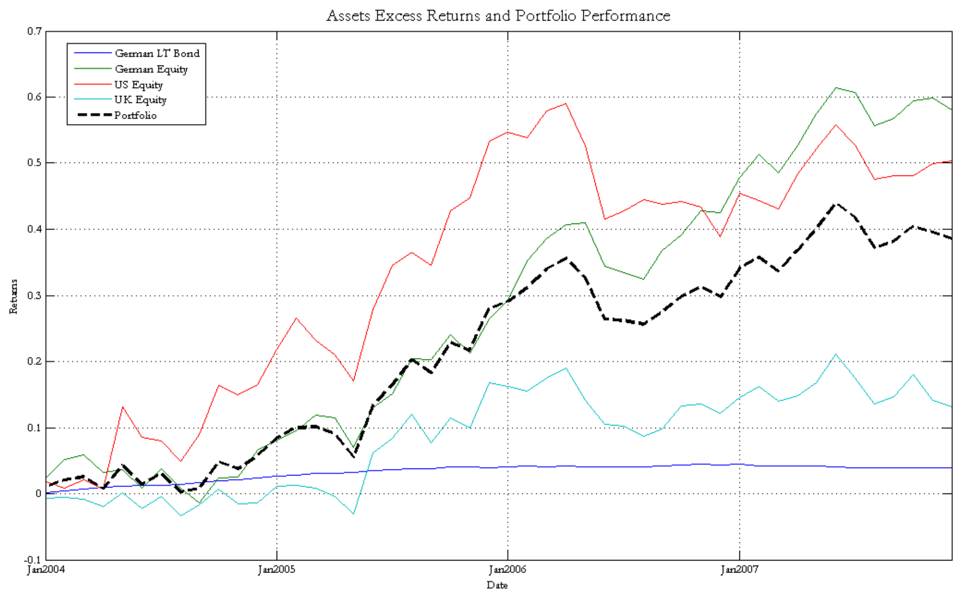
$$w_{BL} = s_{BL}/1's_{BL}$$

In the code:

```
K = tau*SigmaR*P'*(tau*P*SigmaR*P' + Gamma)^(-1);
sBL = lambda*((SigmaR^(-1))*(muMP_i) + (SigmaR^(-1))*K*(V-P*muMP_i));
wBL = sBL./(ones(4,1)'*sBL);
```

The plots of weights and performance are reported below:


Weights of Assets in the BL Portfolio
Percentage Weight
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
German LT Bond
German Equity
US Equity
UK Equity


Assets Excess Returns and Portfolio Performance
German LT Bond
German Equity
US Equity
UK Equity
Portfolio
Returns
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
Jan2004
Jan2005
Jan2006
Jan2007
Date

4.1 Box. Yield to maturity, Duration and Holding Period returns

4.1.1 Zero-Coupon Bonds

Define as follows the relationship between price and yield to maturity of a zero-coupon bond:

$$P_{t,T} = \frac{1}{(1+Y_{t,T})^{T-t}} \tag{2}$$

Where $P_{t,T}$ is the price at time t of a bond maturing at time T , ed $Y_{t,T}$ is th yield to maturity. Taking logs of the left and the right-hand side of **??** and defining the continously compounded return $\mathrm{y}_{t,T}$ as $\log(1+Y_{t,T})$, we have the following relationship:

$$p_{t,T} = -(T-t)\, y_{t,T} \tag{3}$$

which clearly illustrates that the elasticity of the yield to maturity to the price of the zero-coupon bond is the maturity of the security. The one period holding-period retunr on the bond $\mathrm{r}_{t,t+1}^{T}$ is then defined as follows :

$$\begin{aligned} r_{t,t+1}^{T} &= p_{t+1,T} - p_{t,T} = -(T-t-1)\, E_t y_{t+1,T} + (T-t)\, y_{t,T} \\ &= y_{t,T} - (T-t-1)\,(E_t y_{t+1,T} - y_{t,T}) \end{aligned} \tag{4}$$

Note that the duration of the bond equals maturity as no coupons are paid. Note that $r_{t,t+1}^{T}$ is uncertan at time t.

4.1.2 Coupon Bonds

The relationship between the price and the yield to maturity of a coupon bond is defined as:

$$P_{t,T}^{c} = \frac{C}{\left(1+Y_{t,T}^{c}\right)} + \frac{C}{\left(1+Y_{t,T}^{c}\right)^{2}} + ... + \frac{1+C}{(1+Y_{t,T})^{T-t}} \tag{5}$$

Note that when the bond is selling at par, the YTM is equal to the coupon rate. To measure the length of time that a bondholder has invested money we need to introduce the concept of duration:

$$D_{t,T}^{c} = \frac{\frac{C}{\left(1+Y_{t,T}^{c}\right)} + 2\frac{C}{\left(1+Y_{t,T}^{c}\right)^{2}} + ... + (T-t)\frac{1+C}{\left(1+Y_{t,T}\right)^{T-t}}}{P_{t,T}^{c}} \tag{6}$$

$$= \frac{C\sum_{i=1}^{T-t}\frac{i}{\left(1+Y_{t,T}^{c}\right)^{i}} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}}}{P_{t,T}^{c}}$$

Note that when a bond is floating at par we have :

$$D_{t,T}^{c} = Y_{t,T}^{c}\sum_{i=1}^{T-t}\frac{i}{\left(1+Y_{t,T}^{c}\right)^{i}} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}} \tag{7}$$

$$= Y_{t,T}^{c}\frac{\left((T-t)\frac{1}{1+Y_{t,T}^{c}} - (T-t) - 1\right)\frac{1}{\left(1+Y_{t,T}^{c}\right)^{T-t+1}} + \frac{1}{1+Y_{t,T}^{c}}}{\left(1-\frac{1}{1+Y_{t,T}^{c}}\right)^{2}} + \frac{(T-t)}{\left(1+Y_{t,T}\right)^{T-t}}$$

$$= \frac{1-\left(1+Y_{t,T}^{c}\right)^{-(T-t)}}{1-\left(1+Y_{t,T}^{c}\right)^{-1}}$$

as, when $|x| < 1$

$$\sum_{k=0}^{n} kx^{k} = \frac{(nx-n-1)\,x^{n+1}+x}{(1-x)^{2}}$$

Duration can be used to find approximate linear relationships between log-coupon yields and holding period returns:

Applying the log-linearization of one-period returns to a coupon bond we have:

$$p_{c,t,T} - c = -r_{t+1}^{c} + k + \rho\left(p_{c,t+1,T} - c\right)$$
$$r_{t+1}^{c} = k + \rho p_{c,t+1,T} + (1-\rho)\,c - p_{c,t,T}$$

When the bond is selling at par $\rho = (1+C)^{-1} = \left(1+Y_{t,T}^{c}\right)^{-1}$
Solving this expression forward to maturity delivers:

$$p_{c,t,T} = \sum_{i=0}^{T-t-1}\rho^{i}\left(k + (1-\rho)\,c - r_{t+1+i}^{c}\right)$$

he log yield to maturity $y_{t,T}^{c}$ satisfies an expression of the same form

$$
\begin{aligned}
p_{c,t,T} &= \sum_{i=0}^{T-t-1} \rho^i \left(k + (1-\rho)\, c - y_{t,T}^c\right) \\
&= \frac{1-\rho^n}{1-\rho} \left(k + (1-\rho)\, c - y_{t,T}^c\right) \\
&= D_{t,T}^c \left(k + (1-\rho)\, c - y_{t,T}^c\right)
\end{aligned}
$$

By substituting this expression back in the equation for linearized returns we have:

$$
r_{t+1}^c = D_{t,T}^c y_{t,T}^c - \left(D_{t,T}^c - 1\right) y_{t+1,T}^c
$$

References

[1] Boudoukh, J., Michaely, R., Richardson, M. and M. Roberts, 2007, On the Importance of Measuring Payout Yield: Implications for Empirical Asset Pricing, Journal of Finance.

[2] Campbell, John Y., and Robert Shiller, 1988, Stock Prices,Earnings, and Expected Dividends, Journal of Finance, 43, 661-676.

[3] Campbell, John Y., and Robert Shiller, 1988, The Dividend-Price Ratio and Expectations of future Dividends and Discount Factors, Review of Financial Studies, 1:195-228

[4] Campbell, John Y., and Robert Shiller, 1998 Valuation Ratios and The Long-Run Stock Market Outlook, 1998, Journal of Portfolio Management

[5] Campbell, John Y., and Robert Shiller, 2001 Valuation Ratios and The Long-Run Stock Market Outlook, an update, Cowles Foundation DP 1295

[6] John Y. Campbell and Tuomo Vuolteenaho (2004), "Inflation Illusion and Stock Prices" (Cambridge: NBER Working Paper 10263)

[7] Cochrane John, 2006, The dog that did Not Bark: a Defense of return Predictability, NBER WP 12026

[8] Cochrane J.(1999) New Facts in Finance

[9] Fama, Eugene and Kenneth R. French, 1988, Dividend Yields and Expected Stock Returns, Journal of Financial Economics, 22, 3-26.

[10] Lander J., Orphanides A. and M.Douvogiannis(1997) "Earning forecasts and the predictability of stock returns: evidence from trading the S&P" Board of Governors of the Federal Reserve System, http:www.bog.frb.fed.org

[11] Lettau, Martin, and Sydney Ludvigson, 2005, Expected Returns and Expected Dividend Growth, Journal of Financial Economics, 76, 583-626

[12] Lettau, Martin, and Stijn Van Nieuwerburgh, 2008, Reconciling the Return Predictability Evidence, Review of Financial Studies, 21, 4, 1607-1652

[13] Franco Modigliani and Richard Cohn (1979), "Inflation, Rational Valuation, and the Market," Financial Analysts' Journal.

[14] Littermann B.(2003), "Modern Investment Management. An Equilibrium Approach", Wiley Finance

[15] Robert J. Shiller. "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?" American Economic Review 71 (June 1981), 421-436. 21