Università Commerciale
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MSc. Finance/CLEFIN
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THEORY OF FINANCE – PART 1

Mock Question 4 (total 5 points) + Sample Optional Question (1 point)

Time Advised: 20-21 + 8 minutes (for these questions)

Difficulty Level: MEDIUM-HIGH

Question 4.A (3.75 points)

Provide a heuristic (it means that it matters to understand key conditions and implications not the full list of technical conditions) statement of Cass-Stiglitz theorem. Make sure to discuss its implications for the architecture and development of modern financial systems.

Debriefing:

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all investors have power utility, then their starting wealth levels will be irrelevant to asset allocation and fund management strategies.

As noted in chapter 2, a risk neutral investor is one who does not care about risk and ranks investments solely on the basis of their expected returns. As we know, the utility of money function of such an agent is necessarily of the form $U(W) = a + bW$ with $b > 0$. What proportion of her wealth will such a decision maker invest in the risky asset? It is easy to show that provided the premium characterizing the risky fund in excess over the riskfree rate is positive, all of her wealth will be invested in the risky asset. This is clearly seen from the following. Consider the portfolio problem when $U(W) = k + bW$:

$$\max_a E[k + bW_0(1 + R^f) + ba(R - R^f)] \Leftrightarrow \max_a ba\, E[R - R^f]. \quad (4.13)$$

When $E[R - R^f] > 0$, this expression is increasing in a. This means that if the risk neutral investor is unconstrained, she will attempt to borrow as much as possible at R^f and re-invest the proceeds in the risky portfolio. Therefore, she is willing, and without any bounds, to exchange riskless payments for uncertain claims of greater expected value. If we instead specify that the investor is prevented from borrowing then the maximum will occur at $a = W_0$.

1.3 A second two-fund separation result: Cass-Stiglitz's theorem

The interesting comparative statics results in Section 1.2 depend upon the fact that there are only two assets, one risky and one riskless. When there is more than one risky asset, in general we cannot say, for example, that the wealth elasticity of the demands for risky assets are greater than unity when an individual exhibits decreasing relative risk aversion. When an investor's initial wealth increases, he may want to change his portfolio composition of the risky assets such that the investment in one risky asset increases while the investment in another asset decreases. As we have already commented in Section 1.1, such shifts in demands may also be motivated by hedging purposes and it is hard to tell on a purely theoretical basis—i.e., without performing any numerical calculations—how asset demands shift when wealth changes.

Obviously, only if an individual always chooses to hold the same portfolio of risky assets and hence simply changes the mix between that portfolio and the riskless asset for differing levels of initial wealth, then the comparative statics for the two-asset case will be valid in a multi-asset world. In such event, the individual's optimal portfolios for differing levels of initial wealth are always linear combinations of the riskless asset and a risky asset mutual fund. This important property of optimal choices in a multi-asset word is commonly called *two fund monetary separation*. Once again, it is not just a curiosity (to some economists, cute mathematical results may also be attractive): the ability to extend a number of results from Section 1.2 to the real, multi-asset world is exactly what we are looking for!

110 ESSENTIALS OF APPLIED PORTFOLIO MANAGEMENT

Cass and Stiglitz (1970) have demonstrated that a necessary and sufficient condition on utility functions for two fund monetary separation is that marginal utility satisfies one of two properties:

Result 4.4 (Cass-Stiglitz): An individual who is risk averse and who strictly prefers more to less will exhibit *two-fund separation* if and only if either

$$U'(W) = (A + BW)^C \text{ or } U'(W) = A\exp(BW), \quad (4.14)$$

where in the former case, $B > 0, C < 0$, and $W \geq \max[0, -(A/B)]$, or $A > 0$, $B < 0$, $C > 0$ and $0 \leq z < -(A/B)$; $A > 0$, $B < 0$ and $W \geq 0$ in the latter case.[5]

The proof can be found in the original article published in 1970 and it is not particularly enlightening. However, it is difficult to understate the importance of this result. Cass-Stiglitz's theorem implies that for a number of standard VNM felicity functions, an investor always holds the same risky portfolio independently of her initial wealth, in the sense that the composition of such a portfolio is constant and the very portfolio may be treated as if it is a large, market portfolio to be demanded. The fact that the utility functions are of standard types comes from realizing that

$$U(W) = const + \int U'(W)dW$$
$$= \text{const} + \int (A + BW)^C dW = \text{const} + \frac{1}{B}\frac{(A+BW)^{C+1}}{C+1} \quad (4.15)$$

Clearly, when we set A = 0, B = 1, and C = $-\gamma < 0$ (when $\gamma \neq 1$), the first marginal utility function in Result 4.4 has a CRRA, power utility function as

[5] These hardly memorable restrictions on the parameters A, B, and C simply guarantee that the relevant, underlying utility function is strictly concave and increasing.

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its primitive.[6] Alternatively, when A = 1, B = κ, and C = 1, the first type of marginal utility function becomes a simple quadratic utility. Finally,

$$U(W) = \text{const} + \int A\exp(BW)dW = const + \frac{A}{B}\exp(BW), \quad (4.16)$$

which delivers a negative exponential utility when the constant is set to 1, B = $-\theta < 0$, and A = $\theta > 0$. As noted earlier, the CARA class of preferences has the property that the total amount invested in risky assets is invariant to the level of wealth. It is not surprising that the proportionate allocation among the available risky assets is similarly invariant as in Result 4.4. Cass-Stiglitz's result is implicitly at the heart of big portions of the modern financial architecture in which standardized investment products (think of mutual and pension funds, and wealth management services that are tailored not to individual needs but to the inferred targets of clusters of investors) seem to be routinely offered and may imply that:

1. The composition of the risky portfolio is homogeneous across different investors and the latter differ in a cross-sectional dimension simply because they invest in different proportions of such a risky mutual fund and the riskless asset;
2. identical products, indeed structured as fixed proportion combinations of the risky mutual fund and of cash, are offered to investors with very different wealth levels.

Of course, the latter, stronger characterization requires that Result 4.4 be applied in the presence of a VNM utility of wealth that implies constant relative risk aversion, such as log- or power utility. Such two properties justify the existence of identical equity mutual and pension funds offered as if rather heterogeneous investors (in terms of age and demographic features) may actually desire to buy such standardized products instead of demanding a personalized wealth management. The following example closes this section exploring in some greater depth the implications of C ass-Stiglitz's result.

[6] Moreover, in the case A=0, B = 1, and C = -1, it is possible to check that $U(W) = const + \int(W)^{-1}dW = const + lnW$. In the power utility case, because relative risk aversion is constant, the proportions invested in the riskless asset a and in the risky asset mutual fund $(1 - a)$ are also invariant to different levels of initial wealth.

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	Stock A		Stock B		Stock C		Cash		Tot. risky
Initial wealth	Total	% risky	Total	% risky	Total	% risky	Total	% total	
100	30	60	10	20	10	20	50	50	50
50	15	60	5	20	5	20	25	50	25
150	45	60	15	20	15	20	75	50	75

The routine usage of asset and country/sector allocation "grids" by all major financial institutions, tailored to the risk profile of different clients, but independent of their wealth levels (and of changes in their wealth), is predicated on the hypothesis that differences in wealth (across clients) and changes in their wealth do not require adjustments in portfolio composition provided risk tolerance is either unchanged or controlled for. Result 4.4 under specific assumptions on the preferences required for it to hold, provide support to such practice.

Question 4.B (0.75 points)

John is characterized by quadratic Von Neumann-Morgenstern felicity function but he is initially non-satiated, i.e., his wealth is currently below his bliss point. Does Cass-Stiglitz theorem apply to John's portfolio of risky assets, i.e., will we able to describe how he selects his optimal portfolio using a two-fund monetary separation theorem? Make sure to justify your answer. Complete the following table with *plausible* numbers in the light of John's preferences (Hint: there are infinite sets of plausible configurations, but no exact, precise answer)

	Stock A		Stock B		Stock C		Cash		Tot. risky
Initial wealth	Total	% risky	Total	% risky	Total	% risky	Total	% total	
100	20	50	10	25	10	25	60	60	40
50	25								
150	15								

Debriefing:

On the one hand, also from 4A above, we know that Cass-Stiglitz theorem applies for all wealth levels only under either negative exponential or generalized power utility functions. Therefore no, because a quadratic VNM utility is neither negative exponential nor power, a two-fund monetary separation result will not always apply which means that the structure of John's risky portfolio will depend on his wealth. To be precise: this depends on whether after investing in his optimal portfolio, John's wealth remains or not below the bliss point. On the other hand, we also know that in the case of quadratic utility, $U(W) = W - \frac{1}{2}\kappa W^2$ with $\kappa > 0$, for $W < 1/\kappa$, we have $U'(W) = 1 - \kappa W$, $U''(W) = -\kappa$, so that

$$ARA(W) = -\frac{-\kappa}{1 - \kappa W} = \frac{\kappa}{1 - \kappa W}.$$

Because $\kappa W < 1$ below the bliss point but as wealth increases, the denominator declines, this implies that ARA(W) is increasing as wealth increases (this was also discussed when justifying the typical convex shape of indifference curves in MV space). As a result, because a IARA investor increases the holdings of cash as wealth increases and the composition of his risky portfolio is not constant because no two-fund theorem applies, then one possible set of numbers for the table are as follows:

Initial wealth	Stock A Total	Stock A % risky	Stock B Total	Stock B % risky	Stock C Total	Stock C % risky	Cash Total	Cash % total	Tot. risky
100	20	50	10	25	10	25	60	60	40
50	25	55.6	15	33.3	5	11.1	5	10	45
150	15	60	5	20	5	20	125	83.3	25

It must be stressed: provided you have cash increasing in absolute amount as wealth increases, and increases in absolute amount more than the increase in wealth, any numbers in the table will work. This also includes applying Cass-Stiglitz (constant shares for the risky assets) provided you emphasize that you assume that for all the involved wealth levels (note you have insufficient information to say that, but let's pretend it is okay) John remains below his bliss point.

Question 4.C (0.5 points)

You know that Mary has preferences such that: (i) when her wealth increases, she invests at least some portion of the increase in her wealth in risky assets; (ii) when her wealth increases, she keeps the structure of her risky portfolio constant, i.e., her optimal risky portfolio weights are independent of her wealth. Among the Von Neumann-Morgenstern (VNM) utility functions that were covered in the lectures, what it the VNM function that is most likely to characterize Mary's behavior? Make sure to carefully justify your answer.

Debriefing:

Comparative Statics in the Canonical Problem

- We know that $E\left[U'\left(W_0(1+R^f) + \hat{a}(R_j - R^f)\right)(R_j - R^f)\right] = 0$ for the case of a single risky asset
- Implicit differentiation of this expression makes it possible to prove that when the risk on the risky mutual fund is small, then

$$\hat{a} \cong \frac{E[R - R^f]}{Var[R]ARA(W_0)}$$

- An individual who is risk averse and who strictly prefers more to less will demand a growing (decreasing/constant) amount of the unique risky asset as her wealth increases, if and only if her ARA declines (grows/is constant) as function of initial wealth:

$$\frac{d\hat{a}}{dW_0} \gtreqless 0 \Leftrightarrow \frac{dARA(W_0)}{dW_0} \lesseqgtr 0$$

  - There is also a relationship involving the risk premium for any given lottery H: $\frac{d\Pi(W,H)}{dW_0} \lesseqgtr 0 \Leftrightarrow \frac{dARA(W_0)}{dW_0} \lesseqgtr 0$
    i.e., the risk premium grows/declines/is constant with wealth when the absolute risk aversion coefficient grows/declines/is constant

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Comparative Statics in the Canonical Problem

- When dARA(W_0)/$dW_0 < 0$, we write about decreasing absolute risk aversion (DARA); when dARA(W_0)/$dW_0 = 0$, we write of constant absolute risk aversion (CARA); finally, when dARA(W_0)/$dW_0 > 0$, we have the case of increasing absolute risk aversion (IARA)
- Under negative exponential utility, dARA(W_0)/$dW_0 = 0$ and this is equivalent to $d\hat{a}/dW_0 = 0$, when an investor's wealth increases, the weight invested in the risky asset declines
  - Correspondingly, the weight invested in cash will increase
  - Riskless borrowing and lending absorb all changes in initial wealth
  - These facts cast doubts on the plausibility of CARA case
  - IARA utility functions are usually deemed rather implausible too, because they imply that as an individual gets wealthier, she will sell risky assets to hoard cash in a more-than-proportional fashion
- Only DARA utility functions enjoy adequate plausibility, e.g., power utility such that ARA(W)=RRA(W)/W=γ/W
- Arrow-Pratt's measure RRA(W_0) may also reveal important information when the investor's wealth undergoes a change

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Because of the fact (i), Mary is clearly DARA as she invests at least some of her additional wealth in risky assets and not only in cash. But fact (ii) is even more revealing because Mary clearly satisfies the statement of Cass-Stiglitz theorem so that Mary must have a power-type utility function, among those seen in class:

$$U(W) = \begin{cases} \dfrac{W^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ lnW & \gamma = 1 \end{cases}$$

Optional Question (1 point)

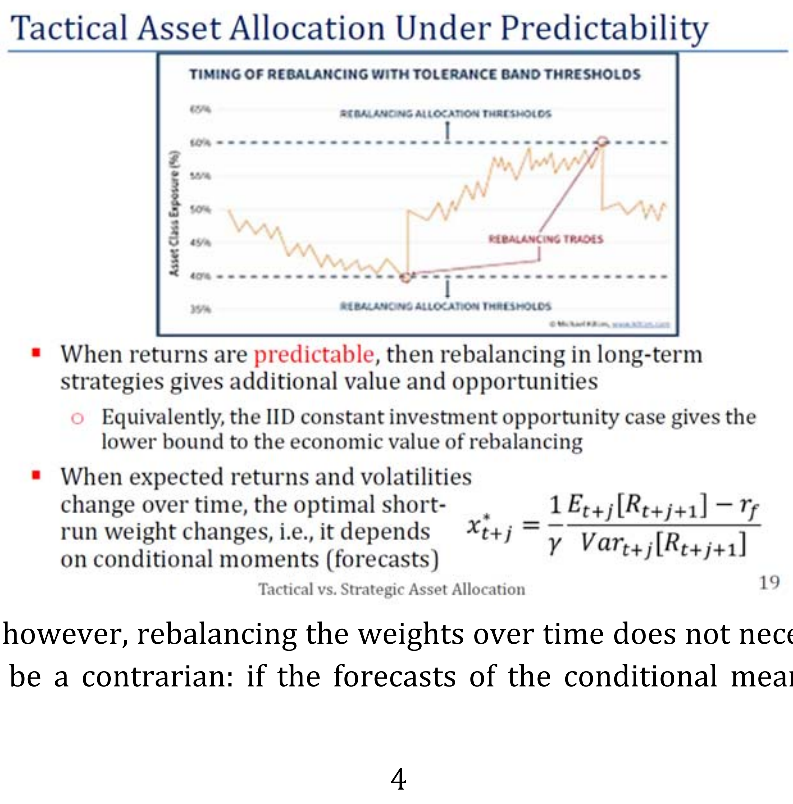
Explain the difference between the rebalancing optimally implied by a fixed-proportion investment rule and the rebalancing optimal as a result of timing strategies. In the former case, discuss why rebalancing implicitly makes an investor a contrarian (i.e., “sell high, buy low”). In the latter case, briefly discuss why timing strategies may be compatible with a momentum strategy (simply buy as prices get higher, hoping they keep going higher).

Debriefing:

The example of fixed-proportion rule mentioned in the lectures is the classical 60-40 rule.

In this case, as the prices of the securities randomly move over time, the securities whose price is increasing (decreasing), come to exceed (to fall below) the optimally assigned wealth; this therefore require rebalancing, that however will consist of selling the securities whose price has increased (has grown more than the average, in value-weighted terms), and of buying the securities whose price has declined (has grown less than the average, in value-weighted terms). This is clearly a contrarian strategy.

We have also seen that when an investor receives information allowing her to change over time her forecasts of the mean and variance of asset returns (more generally, her forecasts of the predictive joint density of asset returns), this may also lead to rebalancing of the portfolio weights over time.



In this latter case however, rebalancing the weights over time does not necessarily imply that an investor must be a contrarian: if the forecasts of the conditional mean of asset returns increase (decrease) after positive (negative) returns and/or the forecasts of the conditional variance of asset returns decline (increase) after positive (negative) returns (or low/high risk), then it is possible for rebalancing to track a momentum, not a contrarian logic.