Università Commerciale
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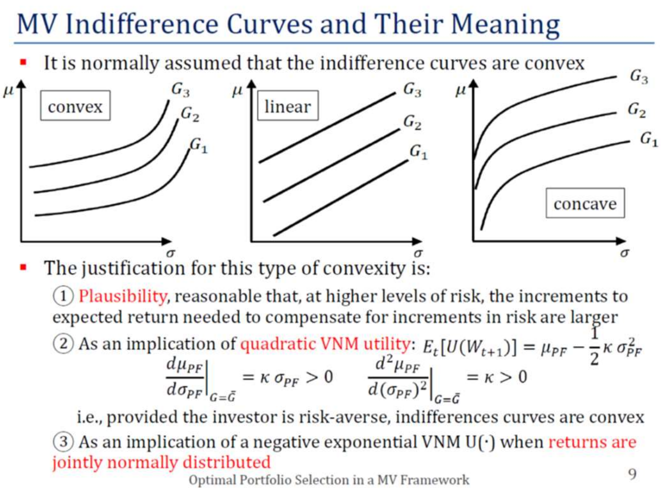
MSc. Finance/CLEFIN
2019/2020 Edition

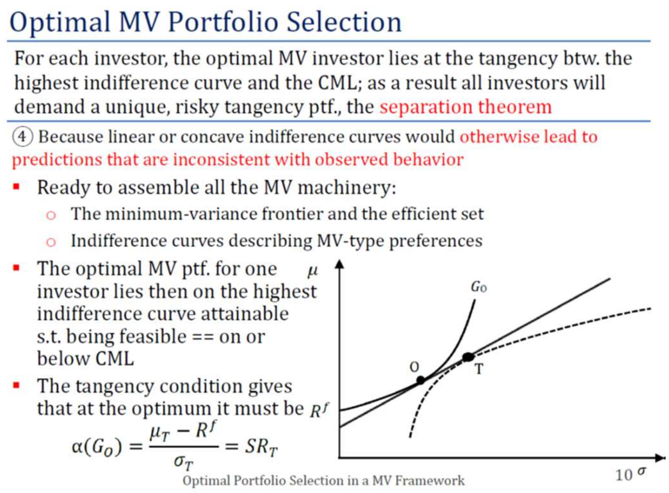
20135 Theory of Finance – Part 1

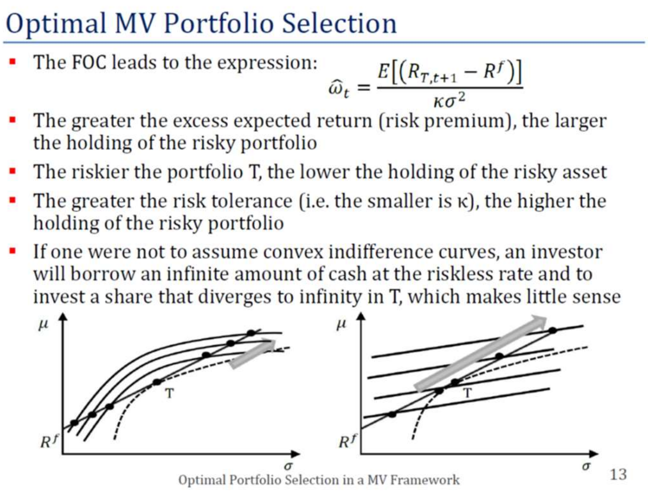
Mock Question 6 (5 points)
Time Advised: 20-22 minutes (for this questions)
Difficulty Level: MEDIUM-EASY

Question 6.A (3.75 points)
Define what is an indifference curve in a (μ, σ) Cartesian plane. What are the possible foundations/motivations for the typical monotone increasing, convex shape of standard mean-variance indifference curves? Make sure to carefully justify what you may mean by "plausible predictions" of behavior also providing, if useful, some examples.

Debriefing.







2.2 Indifference curves in mean-variance space

To make sense and represent in a useful way the trade-off between expected returns and risk, assume that—whatever its micro-foundation—the MV objective functional is monotone increasing in the mean, $\partial G(\mu_{PF}, \sigma^2_{PF})/\partial \mu_{PF} > 0$, and monotone decreasing in risk, $\partial G(\mu_{PF}, \sigma^2_{PF})/\partial \sigma^2_{PF} < 0$. Equivalently, expected return is a "good" that gives happiness and risk is a "bad" that hurts an investor's satisfaction. Consider now some small, countervailing changes in μ_{PF} and σ^2_{PF} that keep the total level of the MV satisfaction index constant at some initial level $\bar{G}$, or $dG(\mu_{PF}, \sigma^2_{PF}) = 0$:

$$0 = dG(\mu_{PF}, \sigma^2_{PF}) = \left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \mu_{PF}}\right|_{G=\bar{G}} d\mu_{PF} + \left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \sigma^2_{PF}}\right|_{G=\bar{G}} d\sigma^2_{PF} \quad (4.22)$$

(4.22) is just a total differential function equated to zero in correspondence to some target level $\bar{G}$. This means that the loci of combinations of expected returns and that keep MV unchanged is characterized by the following slope:

$$\alpha(\bar{G}) \equiv \left.\frac{d\mu_{PF}}{d\sigma^2_{PF}}\right|_{G=\bar{G}} = -\frac{\left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \sigma^2_{PF}}\right|_{G=\bar{G}}}{\left.\frac{\partial G(\mu_{PF}, \sigma^2_{PF})}{\partial \mu_{PF}}\right|_{G=\bar{G}}} > 0 \quad (4.23)$$

The positive sign derives from our earlier assumptions. So such a loci, when drawn in the standard expected return/variance space will have a positive slope: increasing risk must be rewarded by increasing expected return to keep an investor equally satisfied, at an arbitrary level $\bar{G}$. Because for positive σ, σ^2_{PF} is a monotone increasing function of standard deviation, if the slope of the loci is increasing as σ^2_{PF} increases, the same must be true of increase in standard deviation, σ_{PF}. The loci of such combinations of means and standard deviation also gets a very specific name:

MV indifference curve: The loci in the mean-standard deviation of space of the infinite combinations (μ_{PF}, σ_{PF}) that yield some fixed level of identical MV (expected) utility as measured by the function $G(\mu_{PF}, \sigma^2_{PF}) = \bar{G}$ is called a MV indifference curve.

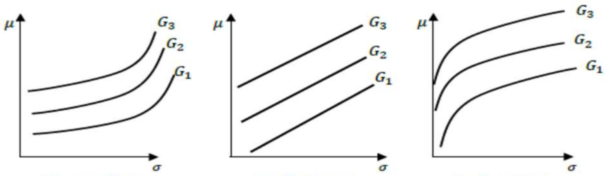


Figure 4.2(a) Figure 4.2(b) Figure 4.2(c)

Figure 4.2 shows a few such indifference curves. In the plots, as the index of G increases, we face increasing levels of satisfaction/utility. Clearly, the issue one faces concerns the type of concavity of the indifference curves, because the earlier definition fails to rule out any case and simply implies that the curves only need to be monotonically increasing. Yet, it is normally assumed that the indifference curves are convex as in figure 4.2(a). The justification for this type of convexity is advanced on several grounds:

1. Intuitive plausibility, because it seems reasonable that, at higher levels of risk, the greater are the increments to expected return needed to compensate for increments in risk if the decision maker's utility is kept constant; of course this sounds a like the DARA property stated in section 1.
2. As an implication of a quadratic von Neumann–Morgenstern utility function that is increasing in wealth, because as we know, in that case we have
$$E_t[U(W_{t+1})] = \mu_{PF} - \frac{1}{2}\kappa\, \sigma^2_{PF} \quad (4.24)$$
and clearly $\partial E_t[U(W_{t+1})]/\partial \mu_{PF} = 1$ while $\partial E_t[U(W_{t+1})]/\partial \sigma_{PF} = -\kappa\, \sigma_{PF} < 0$, so that
$$\left.\frac{d\mu_{PF}}{d\sigma_{PF}}\right|_{G=\bar{G}} = \kappa\, \sigma_{PF} > 0 \qquad \left.\frac{d^2\mu_{PF}}{d(\sigma_{PF})^2}\right|_{G=\bar{G}} = \kappa > 0, \quad (4.25)$$
i.e., provided the investor is risk-averse, the indifferences curves are also convex.
3. As an implication of a negative exponential von Neumann–Morgenstern utility function when asset returns are jointly normal-

ly distributed.

4. Because linear or concave indifference curves would otherwise lead to predictions that are inconsistent with commonly observed behavior, as we are about to show in section 2.3.

$$\hat{\omega}_t = \frac{E[R_{PF,t+1}] - R^f}{\kappa Var[R_{PF,t+1}]} \quad (4.32)$$

confirms the standard intuition that: (a) the greater the excess expected return (risk premium), the larger the holding of the risky portfolio; (b) the riskier the risky portfolio, the lower the holding of the risky asset; and (c) the greater the risk tolerance (i.e. the smaller is κ), the higher the holding of the risky portfolio.

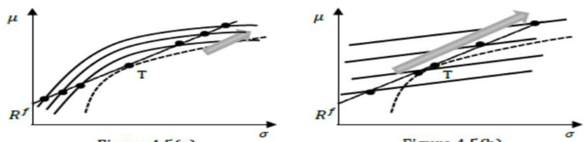


Figure 4.5(a) Figure 4.5(b)

Finally, one may wonder what would happen to these results if one were not to assume convex indifference curves. Figures 4.5(a)-(b) show such cases. In figure 4.5(a), the indifference curves are concave and as we travel up in the map of the indifference curves, new optima can be found as intersections between higher curves and the CML, both to the right and to the left of the tangency portfolio. This means that an investor will have the incentive to borrow an infinite amount of cash at the riskless rate and to invest a share that diverges to infinity in T.[10] In figure 4.5(b), the same occurs with linear indifference curves, but in this case the intersection points only occur to the right of T. Also in this case, the investor has an incentive to borrow an infinite amount of cash at the riskless rate and to invest a share that diverges to infinity in T.[11] Of course, both these solutions to the

[10] It may appear that in figure 4.5(a), the optimal point of intersection between the CML and the map of indifference curves also travels to the left, in direction of portfolios which invest more than 100% in cash. However, such portfolios would imply negative risk if we prolong the CML below the point R^f and this makes no sense.

[11] If we had drawn the linear indifferences curves to be steeper than the CML (as opposed to flatter as in 4.5(b)), then the intersection point would have apparently moved towards the south west, down and to the left. Yet, as explained in the previous

Question 6.B (0.75 points)

Mary is characterized by a power utility function, such that $U_{Mary}(W) = \frac{W^{1-\gamma}}{1-\gamma}$, with γ. Her initial wealth is $W = \$1000$. Mary has told to her friend John, that she would accept to enter in a bet in which she may gain or lose \$10 she would require a probability of winning the lottery higher or equal than 0.525. Compute Mary's coefficient of risk aversion, γ, making sure to show your work. John claims that he has the same preferences as Mary, but he would accept a bet in which he may gain or lose \$10 if the probability of winning was higher or equal than 0.51. Are these two claims compatible? Carefully explain your reasoning.

Debriefing.
We know that for small bets, the following approximation holds:

$$\pi_{Mary}(W;h) \cong \frac{1}{2} + \frac{1}{4}\frac{\gamma}{W}h$$

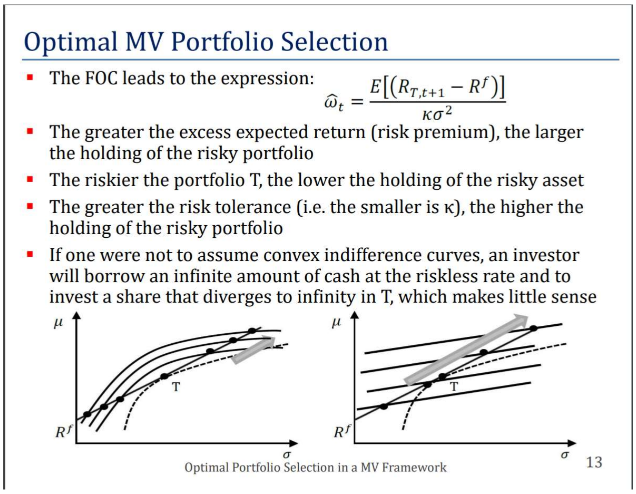
Therefore,

$$\gamma = (0.525 - 0.5)\frac{4 \times 1000}{10} = 10.$$

The two claims about John are not necessarily incompatible. Indeed, we do not have information about John's wealth. Indeed, power utility investors are characterized by a constant relative risk aversion, but a decreasing absolute risk aversion: if John is richer than Mary, then $\pi_{John}(W_J;h) < \pi_{Mary}(W_M;h)$ for the same size of the bet.

Question 6.C (0.5 points)
Mary is a non-satiated, risk-averse mean-variance optimizer characterized by a MV risk-aversion coefficient κ = 0.5. You know that Mary's optimal portfolio is characterized by a standard deviation of 25% and that she is investing 20% of her wealth in the risk-free asset. Her brother, John, is also a non-satiated, risk-averse mean-variance optimizer, but he is characterized by a risk aversion coefficient κ = 1. What can you say about the composition—in terms of weight assigned to riskless cash investments and the risky tangency portfolio—of John's optimal portfolio?

Debriefing



It is possible to obtain the risk premium based on what we know about Mary's asset allocation. Indeed, we have that

$$0.8 = \frac{0.8 \times E[R_{T,t+1} - R_f]}{0.5 \times 0.25^2} \Rightarrow E[R_{T,t+1} - R_f] = 0.5 \times 0.25^2 = 0.0025$$

The 0.80 at the numerator derives from the fact that the data in the problem concern the mean of Mary's portfolio, not the entire tangency portfolio. This can be then used to compute ω_t for John:

$$\omega_t = \frac{0.0025}{0.25^2} = 0.04$$