Università Commerciale
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MSc. Finance/CLEFIN
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20135 Theory of Finance – Part 1

Mock Question 7 (5 points)

Time Advised: 20-22 minutes (for this questions)
Difficulty Level: MEDIUM-EASY

Question 7.A (3.75 points)
What are the three key models of the effects of the existence of human capital (more generally, background risks) on optimal portfolio decisions? Make sure to list and discuss the assumptions under which the three models are developed and their implications for optimal weights.

Debriefing.

Deterministic Tradable Income

- Consider the simple case in which labor income is riskless so that human capital is simply the present discounted value of such deterministic sums to be received in the future
  - When labor income is riskless, then by construction it will display zero correlation with the returns on any risky assets
- Consider an investor that maximizes power utility of terminal wealth with CRRA coefficient γ and she can only invest in the risky vs. the riskless asset, i.e. a canonical portfolio problem
- Suppose in a completely counterfactual way, that human capital were totally tradable with a value of H_t
- Investor's total wealth is then $W_t + H_t$ and the expression of the optimal asset allocation for this investor when all wealth is tradable is to sell claims against her human capital and invest $\hat{\omega}_t(W_t + H_t)$ dollars in stocks, and the remaining $(1-\hat{\omega}_t)(W_t + H_t)$ dollars in the riskless asset, where

$$\hat{\omega}_t = \frac{E_t[(r_{t+1} - r^f)] + \frac{1}{2}\sigma^2}{\gamma\sigma^2}$$

Effects of human capital on portfolio choice 7

Deterministic Non-Tradable Income

- The $0.5\sigma^2$ correction at the numerator is a Jensen's inequality factor that derives from the continuously compounded definition of returns
- Under non-tradable labor income, because labor is riskless and the investor has implicit holdings of H_t in the riskless asset, she should adjust her financial portfolio so that her total dollar holdings of each asset equal the optimal unconstrained holdings
- The share of risky assets in proportion to financial wealth is then

$$\hat{\omega}_t^{H-adj} = \frac{\hat{\omega}_t(W_t + H_t)}{W_t} = \frac{E_t[(r_{t+1} - r^f)] + \frac{1}{2}\sigma^2}{\gamma\sigma^2}\left(1 + \frac{H_t}{W_t}\right)$$

- This implies that $\hat{\omega}_t^{H-adj} > \hat{\omega}_t$: an investor endowed with riskless, non-tradable human wealth should tilt her financial portfolio toward risky assets relative to an investor who owns only tradable assets
  - When H_t is large relative to W_t, the investor may want to hold a leveraged position in stocks by borrowing at the riskless rate: the investor is trying to "undo" the endowment of riskless labor income

Effects of human capital on portfolio choice 8

Deterministic Non-Tradable Income

- The share of risky assets over financial wealth is increasing in the ratio of human to financial wealth (H_t/W_t)
- This ratio changes over the investor's life cycle: at retirement, the ratio tends to zero; early in adult life the ratio is typically large
- Therefore, a young, employed investor should invest more in risky assets (say, stocks) than an older, retiring investor with identical risk aversion and large financial wealth
  - The ratio of human to financial wealth changes with financial asset performances: If the stock market performs well, the investor's financial wealth grows relative to his human capital
  - This should lead to a reduction in the share invested in risky securities: this model predicts a "contrarian" investment strategy
- This simple model ignores some characteristics of wealth: future labor earnings are uncertain, making human capital a risky rather than a safe, non-tradable asset; investors can influence the value of human capital by varying how much they work
- The risk of human wealth will affect asset allocation

Effects of human capital on portfolio choice 9

A Formal One-Period Mean-Variance Framework

- Let's adopt a simple but formal mean-variance framework to investigates these issues
  - As in lecture 3, suppose you have MV preferences defined over your terminal wealth, with risk aversion coefficient κ

$$MV(E[W_{t+1}], Var[W_{t+1}]) = E[W_{t+1}] - \frac{1}{2}\kappa Var[W_{t+1}]$$

  - For simplicity, assume unit initial wealth
  - The asset menu is composed of N risky assets with vector of returns $\mathbf{R}_{t+1}$ and one riskless asset with return r^f_t
  - You have labor income measured by the random variable Y_{t+1}
  - This variable potentially correlated with returns from securities in the asset menu
- The portfolio optimization problem may be written as:

$$\max_{\omega_t} E[W_{t+1}] - \frac{1}{2}\kappa Var[W_{t+1}]$$

$$\begin{aligned} s.t. \quad W_{t+1} &= Y_{t+1} + (1+R^f)(1 - \iota'^{\omega_t}) + (\boldsymbol{\iota} + \mathbf{R}_{t+1})'\boldsymbol{\omega}_t \\ &= Y_{t+1} + (1+R^f) + (\mathbf{R}_{t+1} - R^f\boldsymbol{\iota})'\boldsymbol{\omega}_t \end{aligned}$$

Effects of human capital on portfolio choice 11

A Formal One-Period Mean-Variance Framework

- Plugging in the budget constraint as usual, the problem becomes:

$$\max_{\boldsymbol{\omega}_t} E[Y_{t+1}] + \mathrm{E}[\boldsymbol{R}_{t+1} - R^f \boldsymbol{\iota}]'\boldsymbol{\omega}_t - \frac{1}{2}\kappa Var[Y_{t+1}] - \frac{1}{2}\kappa \boldsymbol{\omega}_t' \boldsymbol{\Sigma} \boldsymbol{\omega}_t - \kappa Cov[Y_{t+1}, \boldsymbol{R}_{t+1}]'\boldsymbol{\omega}_t,$$

  - This maximization program is now unconstrained and it is quadratic and globally concave
  - Hence the FOCs will be necessary but also sufficient
  - The variance term comes from the fact that

$$\begin{aligned} Var_t[W_{t+1}] &= Var_t[Y_{t+1} + (1 + r_t^f) + \mathbf{w}_t'(\mathbf{R}_{t+1} - r_t^f \boldsymbol{\iota}_N)] \\ &= \underbrace{Var_t[Y_{t+1}]}_{\sigma_Y^2} + Var_t[(1 + r_t^f) + \mathbf{w}_t'(\mathbf{R}_{t+1} - r_t^f \boldsymbol{\iota}_N)] + \\ &\quad +2Cov_t[Y_{t+1}, (1 + r_t^f) + \mathbf{w}_t'(\mathbf{R}_{t+1} - r_t^f \boldsymbol{\iota}_N)] \\ &= \sigma_Y^2 + \mathbf{w}_t'\Sigma_t \mathbf{w}_t + 2\mathbf{w}_t' \boldsymbol{\sigma}_{Y,\mathbf{R}} \qquad \boldsymbol{\sigma}_{Y,\mathbf{R}} \equiv Cov_t[Y_{t+1}, \mathbf{R}_{t+1}] \text{ (a } N \times 1 \text{ vector)} \end{aligned}$$

  - At this point, the FOCs are:

$$\mathrm{E}[\boldsymbol{R}_{t+1} - R^f \boldsymbol{\iota}] - \kappa \boldsymbol{\Sigma} \widehat{\boldsymbol{\omega}}_t - \kappa Cov[Y_{t+1}, \boldsymbol{R}_{t+1}] = \boldsymbol{0}$$

Effects of human capital on portfolio choice 12

A Formal One-Period Mean-Variance Framework

Labor income modifies the standard MV closed-form result iff the vector of covariances of labor income with asset returns is non-zero

- Solving/manipulating in the usual ways, we have:

$$\boxed{\hat{\mathbf{w}}_t = \frac{1}{\kappa}\boldsymbol{\Sigma}_t^{-1} \times \underbrace{E_t[\mathbf{R}_{t+1} - r_t^f \boldsymbol{\iota}_N]}_{\text{vector of risk premia}} - \boldsymbol{\Sigma}_t^{-1} \times \underbrace{\boldsymbol{\sigma}_{Y,\mathbf{R}}}_{\text{vector of covariances with asset returns}}}$$

- The interpretation is that the presence of labor income modifies the standard MV closed-form result iff the vector of covariances of labor income with asset returns is non zero
  - Does labor income reduce/increase portfolio weights and how? It all depends oh the product $\boldsymbol{\Sigma}^{-1}{}_t \boldsymbol{\sigma}_{YR}$
  - It is difficult to state in advance how a non-zero $\boldsymbol{\Sigma}^{-1}{}_t \boldsymbol{\sigma}_{YR}$ will affect portfolio weights
- Yet, notice that per se the variance of labor income, being in the background does not affect portfolio choice
- This case is also special: the scale of wealth does affect RRA

Effects of human capital on portfolio choice 13

are retired and that—more generally—the equity share ought to decline with age.

The story will unfold as follows. A household with labor income has an implicit holding of a non-tradable asset, human capital, that represents a claim to her stream of future labor income. The household adjusts explicit asset holdings to compensate for the implicit holding of human capital and reach the desired allocation of total wealth. Hence human capital may "crowd in or out" explicit asset holdings. If labor income is literally riskless (say, you are the king of your country), then riskless asset holdings are crowded out and the investor will tilt her portfolio strongly towards risky assets. Indeed, in some historical periods we did see kings hiring armies of venture soldiers to conquer the world, which tends to be considered a risky enough business. In the case where the investor is constrained from borrowing to leverage her risky investments, the solution may be a corner at which the portfolio is 100% risky assets. If labor income is risky but uncorrelated with risky financial assets, then riskless asset holdings are still crowded out but less strongly. If labor income is sufficiently positively correlated with risky financial assets, then the portfolio tilt is decreased and the household compensates for risky human capital by increasing holdings of safe financial assets. Under the assumption that labor income shocks are uncorrelated or only weakly correlated with stock returns, the results will suggest that investors who expect high future labor income—discounted at some appropriate rate and measured relative to financial wealth—should display the strongest desire to hold stocks. Hence, the standard advice that risky asset holdings ought to decline as you age increases, is upheld. In fact, a famous rule of thumb is "one hundred minus your age", which says that the proportion of your portfolio invested in equities should equal the difference between your age and one hundred.

1.1 Taking a "first stab": the case of deterministic, riskless labor income

To understand in the starkest possible way what are the forces at play when labor income appears in the financial planning of an investor, consider first the simple case in which labor income is riskless so that human capital is simply the present discounted value of such deterministic sums to be received in the future. Note that when labor income is riskless, then by construction it will display zero correlation with the returns on any risky assets. Moreover, a long-run investor maximizes power utility of terminal wealth (consumption) with CRRA coefficient γ and she can only invest in the risky vs. the riskless asset. In short, this is a canonical portfo-

lio problem. Stocks have a risky log return r_{t+1} per period, with constant mean log excess return $E_t[(r_{t+1} - r^f)]$ and constant variance σ^2. Note that in this chapter $r_{t+1} \equiv \ln(1 + R_{t+1})$, where R_{t+1} is the standard notion of return used so far in the book.

Suppose in a completely counterfactual way, that human capital were totally tradable with a value of H_t. The investor's total wealth is then $W_t + H_t$. As we have seen in chapter 4, in this case the expression for the optimal asset allocation—say, the power utility tangency portfolio—for this investor when all wealth is tradable is to sell claims against her human capital and invest $\widehat{\omega}_t(W_t + H_t)$ dollars of his total wealth in stocks, and the remaining $(1 - \widehat{\omega}_t)(W_t + H_t)$ dollars in the riskless asset, where

$$\widehat{\omega}_t = \frac{E_t[(r_{t+1} - r^f)] + \frac{1}{2}\sigma^2}{\gamma\sigma^2}. \tag{6.1}$$

The $0.5\sigma^2$ correction at the numerator is a Jensen's inequality factor that derives from the continuously compounded definition of returns. r_{t+1}. Under non-tradable labor income, once we realize that, because labor income is riskless and the investor has implicit holdings of H_t in the riskless asset, she should adjust her financial portfolio so that her total dollar holdings of each asset equal the optimal unconstrained holdings. The optimal share of risky assets in proportion to financial wealth is then

$$\widehat{\omega}_t^{H-adj} = \frac{\widehat{\omega}_t(W_t + H_t)}{W_t} = \frac{E_t[(r_{t+1} - r^f)] + \frac{1}{2}\sigma^2}{\gamma\sigma^2}\left(1 + \frac{H_t}{W_t}\right), \tag{6.2}$$

which implies that $\widehat{\omega}_t^{H-adj} > \widehat{\omega}_t$ because by construction, $H_t > 0$. Therefore, Bodie, Merton, and Samuelson (1992) observe that an investor endowed with riskless, non-tradable human wealth should tilt her financial portfolio toward risky assets relative to an investor who owns only tradable assets. In fact, even though an investor cannot borrow against her future labor income, the optimal dollar holdings of the riskless asset, $(1 - \widehat{\omega}_t^{H-adj})(W_t + H_t) - H_t$ may be negative when H_t is very large relative to W_t. In that case, the investor may want to hold a leveraged position in stocks by borrowing at the riskless rate, even though in hindsight we understand this occurs only because the investor is trying to "undo" the large endowment of riskless cash flows that derive from labor income.

Interestingly, as a result of expression (6.2), the share of risky assets over financial wealth is increasing in the ratio of human to financial wealth (H_t/W_t). This ratio changes over the investor's life cycle. At retirement, the ratio H_t/W_t tends to zero; early in adult life the ratio is typically large—many of the Readers of this book have no financial wealth of their own (they may actually be in debt, i.e., have negative net financial worth) but they are hopefully sitting on a huge human capital value. First, they expect to receive labor incomes for many years; second, they have had little time to accumulate financial wealth. Therefore, a young, employed investor should invest more in risky asset (say, stocks) than an older, retiring investor with identical risk aversion and large financial wealth.

The ratio of human to financial wealth also changes with financial asset performances. If the stock market performs well, the investor's financial wealth grows relative to his human capital. This should lead to a reduction in the share of financial wealth invested in risky securities. Thus this model predicts a "contrarian" investment strategy that not only rebalances the portfolio regularly, but goes further to reduce the risky portfolio share after this has performed well.

This simple model with riskless labor income reveals the basic mechanisms that link together human capital and the optimal allocation to financial assets. However, it ignores some important characteristics of wealth that will complicate the analysis.[2] In particular, future labor earnings are uncertain for most investors, making human capital a risky rather than a safe, non-tradable asset. The risk characteristics of human wealth should affect the allocation of financial assets. This is what section 1.2 models, although still in a simplified format.

[2] A second important characteristic of human capital is that investors can influence its value by varying how much they work. The ability to vary work effort allows individuals to hold riskier portfolios because they can work harder if they need extra labor income to compensate for losses in their financial portfolios. In this chapter we do not even attempt to model a few additional characteristics of human capital, for instance that individuals may deliberately choose to invest in their own human wealth, increasing its value through education, as the Reader is hopefully seeking to accomplish. They may also have some control over the risk characteristics of their human wealth through their choice of career paths: for instance, jobs in investment banking will increase the investments in financial risky assets much less than (successful) careers in academic research can.

1.2 A first formal, one-period stochastic model

Let's adopt a simple but formal static mean-variance framework to investigate these issues. As in chapter 4, suppose an investor is characterized by MV preferences defined over her terminal wealth, with risk aversion coefficient κ:

$$MV(E[W_{t+1}], Var[W_{t+1}]) = E[W_{t+1}] - \frac{1}{2}\kappa Var[W_{t+1}] \quad (6.3)$$

Of course, the same objective may be re-written in terms of overall portfolio returns (say, R^P_{t+1}) and exploiting the fact that $W_{t+1} = (1 + R^P_{t+1})W_t$. For simplicity, assume unit initial wealth. Note that κ is assumed to be constant and not to depend on labor income, which will hold only locally, for a labor income process that is not very volatile.[3,4] The asset menu is composed of N risky assets with vector of returns $\boldsymbol{R}_{t+1}$ and one riskless asset with return R^f, assumed to be constant without loss of generality, given our static, one-period framework. The investor receives labor income measured by the random variable Y_{t+1}. There are no labor supply decisions, so income is exogenous. Labor income is a non-tradable asset, i.e., as already said, you cannot borrow against future labor income (as the commitment of an individual to work is not legally enforceable). This variable is potentially correlated with returns from securities in the asset menu, $Cov[Y_{t+1}, \boldsymbol{R}_{t+1}] \neq \boldsymbol{0}$, where $Cov[Y_{t+1}, \boldsymbol{R}_{t+1}]$ is a $N \times 1$ vector that collects the covariance of labor income with the returns paid by each of the assets in the menu. The portfolio optimization problem may be written as:

$$\begin{aligned} &\max_{\omega_t} E[W_{t+1}] - \frac{1}{2}\kappa Var[W_{t+1}] \\ s.t. \quad W_{t+1} &= Y_{t+1} + (1 + R^f)(1 - \boldsymbol{\iota}'^{\omega_t}) + (\boldsymbol{\iota} + \boldsymbol{R}_{t+1})'\boldsymbol{\omega}_t \\ &= Y_{t+1} + (1 + R^f) + (\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota})'\boldsymbol{\omega}_t \quad (6.4) \end{aligned}$$

Plugging in the budget constraint in the objective function, the problem becomes:

[3] Human capital is often also understood as to broadly include the value of privately owned firms.

[4] As we have stressed already, under quadratic utility function, the investor's ARA coefficient will be increasing in wealth and—even though the mapping between one and the other is far from trivial—to expect κ to decline in the (mean of the) labor income process may be realistic. In the following, we rule these effects out by assumption.

$$\begin{aligned} &\max_{\omega_t}(1 + R^f) + E[Y_{t+1} + (\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota})'\boldsymbol{\omega}_t] - \frac{1}{2}\kappa Var[Y_{t+1} + (\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota})'\boldsymbol{\omega}_t] \\ &\Leftrightarrow \max_{\omega_t} E[Y_{t+1}] + \mathrm{E}[\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota}]'\boldsymbol{\omega}_t - \frac{1}{2}\kappa Var[Y_{t+1}] - \frac{1}{2}\kappa\boldsymbol{\omega}_t'\boldsymbol{\Sigma}\boldsymbol{\omega}_t \\ &\quad - \kappa Cov[Y_{t+1}, \boldsymbol{R}_{t+1}]'\boldsymbol{\omega}_t, \quad (6.5) \end{aligned}$$

where $\boldsymbol{\Sigma} \equiv Var[\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota}] = Var[\boldsymbol{R}_{t+1}]$. This maximization program is now unconstrained and it is quadratic and globally concave. Therefore, the FOCs will be necessary and also sufficient The variance term comes from the fact that in $W_{t+1} = Y_{t+1} + (1 + R^f) + (\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota})'\boldsymbol{\omega}_t$, $(1 + R^f)$ is a constant and $R^f\boldsymbol{\iota}$ is a vector of constants. At this point, and recalling that $\partial(\boldsymbol{y}'\boldsymbol{x})/\partial\boldsymbol{x} = \boldsymbol{y}$ and $\partial(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})/\partial\boldsymbol{x} = 2\boldsymbol{A}\boldsymbol{x}$, the FOCs are:

$$\mathrm{E}[\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota}] - \kappa\boldsymbol{\Sigma}\widehat{\boldsymbol{\omega}}_t - \kappa Cov[Y_{t+1}, \boldsymbol{R}_{t+1}] = \boldsymbol{0} \quad (6.6)$$

or

$$\kappa\boldsymbol{\Sigma}\widehat{\boldsymbol{\omega}}_t = \mathrm{E}[\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota}] - \kappa Cov[Y_{t+1}, \boldsymbol{R}_{t+1}] \quad (6.7)$$

which can be solved by pre-multiplying both sides by $(1/\kappa)\boldsymbol{\Sigma}^{-1}$, to give:[5]

$$\begin{aligned} \widehat{\boldsymbol{\omega}}_t &= \boldsymbol{\Sigma}^{-1}\frac{E[\boldsymbol{R}_{t+1} - R^f\boldsymbol{\iota}]}{\kappa} - \boldsymbol{\Sigma}^{-1}Cov[Y_{t+1}, \boldsymbol{R}_{t+1}] \\ &= Inverse\ of\ \text{"}matrix\ of\ risks\text{"} \times \frac{Risk\ premia}{Risk\ aversion} \\ &\quad - Inverse\ of\ \text{"}matrix\ of\ risks\text{"} \\ &\quad \times background\ risk\ covariances \quad (6.8) \end{aligned}$$

The interpretation is that the presence of labor income modifies the standard MV closed-form result if and only if the vector of covariances of labor income with asset returns is non-zero:

$$\widehat{\boldsymbol{\omega}}_t = Myopic\ MV\ asset\ demand + Hedging\ background\ risk \quad (6.9)$$

[5] Because $\boldsymbol{\Sigma}$ is a covariance matrix and hence positive definite (by construction, in the absence of redundant assets), $\boldsymbol{\Sigma}^{-1}$ will exist.

The desirability of the risky asset in the investor's portfolio will therefore depend not only upon its excess return (above the risk free rate) relative to its variance (risk), but also the extent to which it can be used to hedge variations in the investor's labor income. An investor will depart from the standard, tangency portfolio MV weights found in chapter 3 only because she aims at tilting her portfolio positions to buy (self-) insurance against labor income risks. Interestingly, the size of such a tilt caused by correlations between labor income and security returns shall not depend on the coefficient of risk aversion, κ.

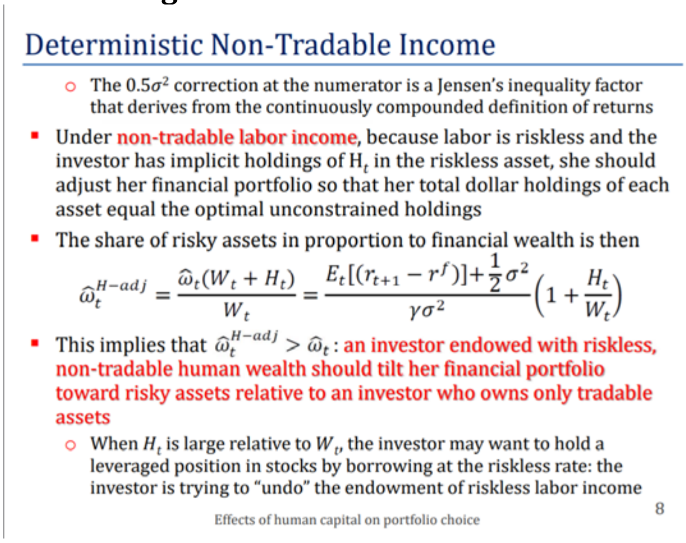
Does labor income reduce/increase portfolio weights and how? Clearly, equation (6.8) shows that it all depends on the product $\boldsymbol{\Sigma}^{-1}Cov[Y_{t+1}, \boldsymbol{R}_{t+1}]$.[6] Therefore, as the result of this product depends on the data of the problem, it is difficult to state a-priori how a non-zero $Cov[Y_{t+1}, \boldsymbol{R}_{t+1}]$ will affect the portfolio weights, for all the assets, i.e., all N portfolio weights. Yet, notice that per se the variance of labor income, being in the background, does not affect portfolio choice: in equation (6.8), $Var[Y_{t+1}]$ fails to appear because it does not interact with portfolio weights and only the correlations between labor income and asset returns turns out to matter. Finally, also $E[Y_{t+1}]$ eventually does not affect optimal asset allocation: this comes from the fact that background choices—such as employment or retirement—are not affected by portfolio decisions and as such they do not interact with portfolio selections.

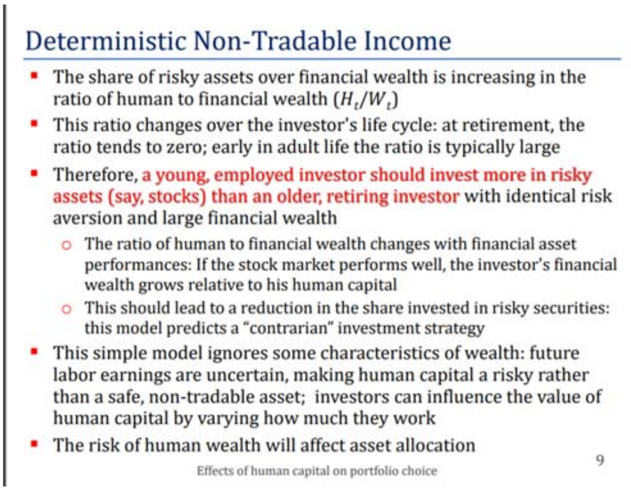
The following example illustrates these features and leads to dramatic conclusions: the same equity index may go from a benign neglect in one investor's portfolio to be in very aggressive demand just because the same investor experiences a change in the "structure" and/or outlook of her human capital returns, for fixed risk-aversion and available current wealth.

Question 7.B (0.75 points)

Professor Max Fifty is a risk-averse investor that maximizes power utility of terminal wealth with a given CRRA coefficient, γ. When he was unemployed, he used to invest 50% of his wealth in the market portfolio and the rest in the riskless asset. However, he has recently found a new job in a top five university that will pay him an essentially riskless, non-tradable salary. Without any additional information, how would you expect Prof. Fifty to change his optimal weights after he starts serving as a professor? Do you think that the age of Prof. Fifty may be related to the percentage of wealth that he will now invest in the riskless asset, and why? Make sure to carefully justify your answer in the light of one or more analytical frameworks of optimal portfolio decision.

Debriefing.





Question 7.C (0.5 points)

Consider the monthly statistics concerning the following two equity indices:

$$E\left[R_{t+1}^{banks}\ R_{t+1}^{industrials}\right]' = [2.7\%\ \ 0.5\%]'$$

$$\boldsymbol{\Sigma} \equiv Var[\boldsymbol{R}_{t+1}] = \begin{bmatrix} (4.0)^2 & 0 \\ 0 & (3.0)^2 \end{bmatrix}$$

The risk-free rate is 0.2%. Mr. Baldtree is a risk averse, mean-variance optimizer. He was originally unemployed and characterized by a coefficient of risk aversion $\kappa = 0.1$. Compute Mr. Baldtree's optimal risky portfolio. At this point, Mr. Baldtree finds a high-ranking job with a bank that will pay an income characterized by a rate of growth of 0.3% per month and a variance of 2.5% per month; furthermore, Mr. Baldtree's labor income implies the following covariances with the equity returns:

$$Cov[\tilde{Y}_{t+1}^{Bald}, R_{t+1}^{banks}]=5 \quad Cov[\tilde{Y}_{t+1}^{Bald}, R_{t+1}^{ind}]=2.$$

Compute his new optimal portfolio that takes into account the effect of labor income. Will Mr. Baldtree tilt his portfolio towards or away from bank stocks? (*Hint*: note that the inverse of a diagonal matrix is the matrix of the inverses of the elements on the diagonal).

Debriefing.

Considering the monthly statistics concerning the two equity indices, the corresponding Sharpe ratios are

$$SR^{banks} = \frac{2.7-0.2}{4} = 0.625 \quad SR^{industrials} = \frac{0.5-0.2}{3} = 0.1.$$

Because the inverse of a diagonal matrix is the matrix of the inverses of the elements on the diagonal, note that

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.1111 \end{bmatrix}$$

Baldy's optimal risky portfolio will be as follows:

$$\begin{bmatrix} \hat{\omega}_{t+1}^{banks} \\ \hat{\omega}_{t+1}^{industrials} \end{bmatrix} = \frac{1}{0.1}\begin{bmatrix} 0.0625 & 0 \\ 0 & 0.1111 \end{bmatrix}\begin{bmatrix} 2.7-0.2 \\ 0.5-0.2 \end{bmatrix} = \begin{bmatrix} 0.625 & 0 \\ 0 & 1.111 \end{bmatrix}\begin{bmatrix} 2.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 1.5625 \\ 0.3333 \end{bmatrix}$$

and as a result the weight in the riskless asset is: 1-1.5625-0.333= -0.8958, i.e., he is actually leveraging to buy stocks.

At this point, Mr. Baldtree finds a high-ranking job with a bank that pays his an income characterized by a rate of growth of 0.3% per month and a variance of 2.5% per month. Mr. Baldtree's new optimal portfolio will now be:

$$\begin{bmatrix} \tilde{\omega}_{t+1}^{banks} \\ \tilde{\omega}_{t+1}^{industrials} \end{bmatrix} = \frac{1}{0.1}\begin{bmatrix} 0.0625 & 0 \\ 0 & 0.1111 \end{bmatrix}\begin{bmatrix} 2.7-0.2 \\ 0.5-0.2 \end{bmatrix} - \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.1111 \end{bmatrix}\begin{bmatrix} 5.0 \\ 2.0 \end{bmatrix}$$
$$= \begin{bmatrix} 1.5625 \\ 0.3333 \end{bmatrix} - \begin{bmatrix} 0.3125 \\ 0.2222 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 0.1111 \end{bmatrix},$$

which implies a percentage investment in the riskless asset of -0.3611. Interestingly, the leverage decreases considerably, also as a reaction to the riskiness of the labor income flow. Moreover, in this case and with these specific numbers (not to mention the fact that the covariance matrix of portfolio returns has been assumed to be diagonal), the share in banks greatly decreases from 156 to 125 percent, which is obviously caused by the high correlation between Mr. Baldtree's labor income and bank stock returns. Given the diagonal structure of the covariance matrix, also the share of industrial stocks decline