Objective Rationality and Uncertainty Averse Preferences*

Simone Cerreia-Vioglio
Department of Decision Sciences and IGIER, Università Bocconi

Draft of January, 2015. First Draft, July, 2011

Abstract

As in Gilboa, Maccheroni, Marinacci, and Schmeidler [12], we consider a decision maker characterized by two binary relations: $\succsim^*$ and $\succsim^\wedge$. The first binary relation is a Bewley preference. It models the rankings for which the decision maker is sure. The second binary relation is an uncertainty averse preference, as defined by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4]. It models the rankings that the decision maker expresses if he has to make a choice. We assume that $\succsim^\wedge$ is a completion of $\succsim^*$. We identify axioms under which the set of probabilities and the utility index representing $\succsim^*$ are the same as those representing $\succsim^\wedge$. In this way, we show that Bewley preferences and uncertainty averse preferences, two different approaches in modelling decision making under Knightian uncertainty, are complementary. As a by-product, we extend the main result of Gilboa, Maccheroni, Marinacci, and Schmeidler [12], who restrict their attention to maxmin expected utility completions.

*I would like to thank Itzhak Gilboa, Peter Klibanoff, Efe Ok, Ben Polak, David Schmeidler, Tomasz Strzalecki, the DTEA 2012 conference audience, and, especially, Pierpaolo Battigalli, Erio Castagnoli, Alejandro Francetich, Maria Kuecken, Fabio Maccheroni, and Massimo Marinacci for very helpful suggestions. I am also very grateful to the co–editor, Faruk Gul, and an anonymous referee for very insightful comments, which vastly improved the paper. Part of this research was done while I was visiting the Cowles Foundation at Yale University which I thank for its hospitality. The financial support of ERC (advanced grant BRSCDP–TEA) is gratefully acknowledged.

1 Introduction

In this paper, we consider two different approaches in modelling decision making under Knightian uncertainty: Bewley preferences and the class of uncertainty averse preferences of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4] (henceforth, CMMM). This latter class encompasses several models that appear in the literature: Gilboa-Schmeidler preferences (Gilboa and Schmeidler [13]), multiplier preferences (Hansen and Sargent [14] and Strzalecki [20]), variational preferences (Maccheroni, Marinacci, and Rustichini [17]), and smooth ambiguity averse preferences (Klibanoff, Marinacci, and Mukerji [16]). Our goal is to show how these two different approaches are complementary. In order to achieve this, we model a decision maker (henceforth, DM) with preferences over Anscombe-Aumann acts by means of two binary relations $(\succsim^*, \succsim^\wedge)$, where the first relation, $\succsim^*$, is a Bewley preference and the second one, $\succsim^\wedge$, is an uncertainty averse preference which is a completion of $\succsim^*$. In doing so, we extend the findings of Gilboa, Maccheroni, Marinacci, and Schmeidler [12] (henceforth, GMMS).

In GMMS, the first binary relation represents the part of the DM's rankings that appear uncontroversial to him (*objective rationality*). The second binary relation models the rankings that the DM expresses if he has to make a choice (*subjective rationality*). GMMS assume that the first binary relation is a Bewley preference while the second binary relation is assumed to be complete, transitive, and monotone as well as to satisfy c-independence. The postulate of completeness justifies the role of $\succsim^\wedge$ as summarizing the rankings of the DM if he has to make a choice. Transitivity and monotonicity are basic rationality tenets. C-independence is a weakening of the standard notion of independence. It requires that, for each $\alpha \in (0,1)$ and for each *constant* act h,

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad \alpha f + (1-\alpha)\, h \succsim^\wedge \alpha g + (1-\alpha)\, h. \tag{1}$$

Its justification rests on the fact that, since h is constant, mixing reduces symmetrically the uncertainty relative to f and g. Thus, if f is weakly preferred to g then $\alpha f + (1-\alpha)\, h$ should be weakly preferred to $\alpha g + (1-\alpha)\, h$, since h does not have any hedging effect. The decision theoretic structure of GMMS is finally capped by two axioms which impose some discipline on the relationship between $\succsim^*$ and $\succsim^\wedge$. The first axiom, dubbed *consistency*, states that, given two acts f and g, $f \succsim^* g$ implies $f \succsim^\wedge g$. Formally, $\succsim^\wedge$ extends $\succsim^*$. Consistency means that the rankings for which the DM is sure are not reverted if he has to choose. Since $\succsim^\wedge$ is complete, we will also say that $\succsim^\wedge$ is a completion of $\succsim^*$. The second axiom, termed *caution*, states that, given an act f and a constant act x, if $f \not\succsim^* x$ then $x \succsim^\wedge f$. This postulate imposes that $\succsim^\wedge$ models a rather uncertainty averse DM. In fact, whenever the DM cannot confidently declare an uncertain act f better than a certain one x, then, if he has to choose, he weakly prefers the latter over the former.[1] The main result of [12] is the following representation theorem: $(\succsim^*, \succsim^\wedge)$ satisfy the aforementioned assumptions if and only if

1. $\succsim^*$ can be represented à la Bewley (i.e. with a multi-expected utility representation) with an affine utility index u^* and a nonempty, closed, and convex set of probabilities C^*.

2. $\succsim^\wedge$ can be represented à la Gilboa and Schmeidler (i.e. with a maxmin expected utility representation) with an affine utility index u and a nonempty, closed, and convex set of probabilities C.

3. u is cardinally equivalent to u^* and $C = C^*$.[2]

Point 1 readily follows from the conditions imposed on $\succsim^*$. Consistency and caution, paired with the assumptions on $\succsim^\wedge$, imply that $\succsim^\wedge$ satisfies uncertainty aversion. Thus, point 2 is a consequence of Gilboa and Schmeidler [13, Theorem 1]. Finally, in point 3, consistency implies that the utility index used to evaluate consequences can be chosen to be the same for both binary relations –namely, we can set $u = u^*$– while caution implies that the relevant probabilities characterizing $\succsim^*$ are the same as those characterizing $\succsim^\wedge$.

Intuitively, the set C summarizes the set of probabilities that the DM deems plausible, while the utility index u represents his preferences over outcomes.[3] An act f is objectively/unambiguously better than an act g if and only if the expected utility of the first dominates that of the second for each probability in C. Nevertheless, when he has to make a choice, in evaluating an act f, the DM assigns to f the worst possible evaluation induced by the set C.

Thus, GMMS show how two seemingly unrelated approaches to address Ellsberg's critique (see [8]), that of Bewley and that of Gilboa and Schmeidler, are connected and complementary, once modelled within a preference formation framework. In other words, any Gilboa-Schmeidler preference can be reinterpreted as a cautious completion of a Bewley preference.

In this paper, we extend the representation theorem of GMMS to a larger class of completions $\succsim^\wedge$ of $\succsim^*$ that goes beyond Gilboa-Schmeidler preferences. We maintain the same assumptions of GMMS on $\succsim^*$ while we assume that $\succsim^\wedge$ is an uncertainty averse preference. Finally, we also cap our decision theoretic structure with two extra

[1] In [12], $\succsim^\wedge$ is also assumed to be continuous and nontrivial.

[2] We say that the utility index u is cardinally equivalent to the utility index u^* if and only if the former is a positive affine transformation of the latter.

[3] For a similar interpretation, see also [6, Sections 1 and 6].

axioms: consistency and a weakening of caution (dubbed weak caution). Our main result, Theorem 2, shows that $(\succsim^*, \succsim^\wedge)$ satisfy our assumptions if and only if

1. $\succsim^*$ can be represented à la Bewley with an affine utility index u^* and a nonempty, closed, and convex set of probabilities C.

2. $\succsim^\wedge$ can be represented as in [4]; that is,

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad \min_{p\in\Delta} G\left(\int u(f)\,dp, p\right) \geq \min_{p\in\Delta} G\left(\int u(g)\,dp, p\right) \tag{2}$$

   where u is an affine utility index, Δ is the set of all probabilities, and G can be interpreted as an index of uncertainty aversion (as shown by [4, Proposition 6]).

3. u is cardinally equivalent to u^* and the set C representing the Bewley preference $\succsim^*$ is the same one characterizing $\succsim^\wedge$ – that is, C is the smallest subset of Δ over which the min in (2) can be taken.

Point 3 is the contribution of our main result and it validates the interpretation that $(\succsim^*, \succsim^\wedge)$ capture different parts of the DM's rankings over acts. Indeed, consistency implies that the utility index used to evaluate consequences can be chosen to be the same for both binary relations while weak caution mainly implies that the relevant probabilities characterizing $\succsim^*$ are the same as those characterizing $\succsim^\wedge$.[4,5] The difference between our result and the one of GMMS is that the DM's response to the "objectively" specified ambiguity of the set C might not be as extreme as in [12]. In this way, we allow for a more permissive view on "subjective rationality" that is not prejudiced by a specific model.

Our main departures from [12] are two. (a) While we maintain the same assumptions on $\succsim^*$, we impose less stringent conditions on $\succsim^\wedge$. In particular, we still maintain that $\succsim^\wedge$ is complete, transitive, and monotone, but we assume risk independence in place of c-independence, and we *explicitly* assume that $\succsim^\wedge$ satisfies uncertainty aversion[6]. This allows us to consider preferences $\succsim^\wedge$ that are variational, as in [17], or, more generally, uncertainty averse as in [4]. (b) We weaken the assumption of caution

[4] In the appendix, we also show that C characterizes the *unambiguous preference relation* of Ghirardato, Maccheroni, and Marinacci [9] for the binary relation $\succsim^\wedge$.

[5] The paper studies completions of Bewley preferences $\succsim^*$ that use all the set of probabilities characterizing $\succsim^*$. Of course, there exist completions that use a smaller set of probabilities. Nevertheless, we can only identify when the two sets coincide, and our axioms allow us to identify when this is the case. We thank a referee for making this point.

[6] As in [12], $\succsim^\wedge$ is also assumed to be continuous and nontrivial. In addition, we assume that at least one of them satisfies unboundedness.

to *weak caution.* The axiom of weak caution states that, for each constant act x, there exists a weakly better constant act y such that, for each f,

$$f \not\succsim^* x \quad \Longrightarrow \quad y \succsim^\wedge f.$$

In words, weak caution amounts to imposing that, for any given constant act x, there exists a common bound y for *all* acts f that are not unambiguously preferred to x. In the paper of GMMS, this assumption is trivially satisfied; in fact, the bound y, for $\succsim^\wedge$, is assumed to be x itself. There are three reasons for these changes:

(i) GMMS argue that c-independence is a suitable principle for subjective rationality. Nevertheless, a priori, it is hard to argue that any particular weakening of independence for $\succsim^\wedge$ is more plausible than another. Thus, we adopt one of the weakest forms of independence available: risk independence. Risk independence is the assumption of independence restricted to constant acts, that is, (1) when acts f, g, and h are all constant.

(ii) C-independence in conjunction with weak caution implies that $(\succsim^*, \succsim^\wedge)$ satisfy caution and $\succsim^\wedge$ is a Gilboa-Schmeidler preference (see Proposition 3 and Theorem 3). Thus, to extend the result of GMMS, we need to relax both c-independence and caution.

(iii) In [12], $\succsim^\wedge$ turns out to be a Gilboa-Schmeidler preference and, in particular, it also satisfies uncertainty aversion. Since we want to relax caution and to preserve uncertainty aversion, we directly assume that $\succsim^\wedge$ satisfies the latter.

Finally, in Proposition 2, we derive the same result of GMMS under caution but with risk independence in place of c-independence. This further clarifies that a weakening of caution is needed to allow for less uncertainty averse forms of subjective rationality.

2 Preliminaries

We consider a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*: functions $f : S \to X$ that are Σ-measurable and take finitely many values. With the usual slight abuse of notation, given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. We thus identify X with the subset of constant acts in $\mathcal{F}$.

We assume additionally that X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of outcomes, as it happens in the classic setting of Anscombe and Aumann [1]. Using the linear structure of X, we define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha) g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha) g)(s) = \alpha f(s) + (1-\alpha) g(s) \in X$ for all $s \in S$.

Given a binary relation $\succsim$ on $\mathcal{F}$, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$, respectively.

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions endowed with the supnorm. Thus, we have that $u(f) \in B_0(\Sigma)$ whenever $u : X \to \mathbb{R}$ is affine and $f \in \mathcal{F}$. Given an affine function $u : X \to \mathbb{R}$, we denote by $B_0(\Sigma, u(X))$ the set of all real-valued Σ-measurable simple functions that take values in $u(X)$.

It is well known that the norm dual of $B_0(\Sigma)$ can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ, and it is a weak* compact and convex subset of $ba(\Sigma)$. The set Δ is endowed with the topology inherited from the weak* topology. The set $\mathbb{R}$ is endowed with the usual topology. The set $\mathbb{R} \times \Delta$ is endowed with the product topology. Elements of Δ are denoted by p and q.

Functions of the form $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ play a key role in the results of CMMM and ours. We denote by $dom_{\Delta} G$ the set:

$$\{p \in \Delta : G(t, p) < \infty \text{ for some } t \in \mathbb{R}\}.$$

Borrowing and modifying the notation of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [5], we denote by $\mathcal{L}_n(\mathbb{R} \times \Delta)$ the class of such functions that satisfy the following requirements:

(i) G is quasiconvex and lower semicontinuous on $\mathbb{R} \times \Delta$;

(ii) $G(\cdot, p)$ is increasing for all $p \in \Delta$;

(iii) $\min_{p \in \Delta} G(t, p) = t$ for all $t \in \mathbb{R}$.

Finally, let $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ denote the subset of $\mathcal{L}_n(\mathbb{R} \times \Delta)$ consisting of functions G that satisfy:

(iv) $\sup_{p \in dom_{\Delta} G} G(t, p) < \infty$ for all $t \in \mathbb{R}$.

3 The Axiomatic Framework

Following [12], we consider a DM characterized by two different binary relations, $\succsim^*$ and $\succsim^\wedge$. The first one captures the rankings that appear to the DM as uncontroversial, and it is potentially incomplete. The second captures the rankings of the DM if he has to make a choice or express a preference. We next list the assumptions that we impose on these two binary relations; in Section 4, we compare these axioms to the ones of [12].

We start by listing the axioms that we impose on both $\succsim^*$ and $\succsim^\wedge$. We state them for a generic binary relation $\succsim$ on $\mathcal{F}$.

Basic conditions:

Preorder: *$\succsim$ is reflexive, transitive, and nontrivial.*

Monotonicity: *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$ then $f \succsim g$.*

Mixture continuity: *If $f, g, h \in \mathcal{F}$ then the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$.*

Unboundedness: *For each x and y in X such that $x \succ y$ there are $z, z' \in X$ such that*

$$\frac{1}{2}z + \frac{1}{2}y \succsim x \succ y \succsim \frac{1}{2}x + \frac{1}{2}z'.$$

Preorder and monotonicity are standard rationality assumptions. Mixture continuity and unboundedness are technical assumptions. The latter means that there are arbitrarily good and arbitrarily bad consequences. We refer the interested reader to [12] for a more complete discussion of preorder and monotonicity as basic tenets of rationality.

Next, we list the assumptions that are specific to $\succsim^*$.

C-completeness: *If $x, y \in X$ either $x \succsim^* y$ or $y \succsim^* x$.*

Independence: *If $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$*

$$f \succsim^* g \quad \Longleftrightarrow \quad \alpha f + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h.$$

These assumptions, paired with the basic conditions, imply that the DM has complete preferences over the set of consequences, and that his preferences on X are represented by a nonconstant and affine utility index $u : X \to \mathbb{R}$. In the original Anscombe and Aumann setting, this is equivalent to saying that, when he faces objective probabilities, the DM behaves as a standard expected utility DM. At the same time, under the basic conditions, it follows that $\succsim^*$ admits a representation à la Bewley [2].

Definition 1 *Let $\succsim^*$ be a binary relation on $\mathcal{F}$. $\succsim^*$ is a Bewley preference if and only if it satisfies the basic conditions, c-completeness, and independence.*

The next three assumptions are specific to $\succsim^\wedge$.

Completeness: *If* $f, g \in \mathcal{F}$ *either* $f \succsim^\wedge g$ *or* $g \succsim^\wedge f$.

Risk independence: *If* $x, y, z \in X$ *and* $\alpha \in (0,1)$

$$x \succsim^\wedge y \iff \alpha x + (1-\alpha) z \succsim^\wedge \alpha y + (1-\alpha) z.$$

Uncertainty aversion: *If* $f, g \in \mathcal{F}$ *are such that* $f \sim^\wedge g$ *then* $\alpha f + (1-\alpha) g \succsim^\wedge f$ *for all* $\alpha \in (0,1)$.

Completeness amounts to imposing that the DM is always able to rank acts if he has to make a choice. On the other hand, in a problem of choice under Knightian uncertainty, uncertainty aversion means that hedging does not make the DM worse off.[7] Risk independence is the assumption of independence restricted to constant acts, where Knightian uncertainty has no bite.[8] Finally, given the basic conditions, completeness and risk independence, we can conclude that $\succsim^\wedge$, on X, is represented by an affine utility index $u : X \to \mathbb{R}$.

Definition 2 *Let* $\succsim^\wedge$ *be a binary relation on* $\mathcal{F}$. $\succsim^\wedge$ *is an uncertainty averse preference if and only if it satisfies the basic conditions, completeness, risk independence, and uncertainty aversion.*

Theorem 1 (CMMM, Theorems 3 and 5) *Let* $\succsim^\wedge$ *be a binary relation on* $\mathcal{F}$. $\succsim^\wedge$ *is an uncertainty averse preference that satisfies unboundedness if and only if there exist an onto and affine function* $u : X \to \mathbb{R}$ *and a linearly continuous*[9] $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ *such that* (u, G) *represent* $\succsim^\wedge$ *as in (2). Moreover,* u *is cardinally unique and, given* u, G *is unique.*

Examples of uncertainty averse preferences are variational preferences of Maccheroni, Marinacci, and Rustichini [17] and, in particular, Gilboa-Schmeidler preferences. Variational preferences are characterized by imposing weak c-independence,

[7] For a similar interpretation see also: Debreu [7], Schmeidler [19], and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [4].

[8] For the sake of generality, we could have equivalently imposed a weaker form of risk independence, as in [4]. The actual formulation allows for an easier comparison with the corresponding independence axiom imposed on $\succsim^*$.

[9] A function $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is said to be *linearly continuous* if and only if the map

$$\varphi \mapsto \inf_{p \in \Delta} G\left(\int \varphi dp, p\right)$$

from $B_0(\Sigma)$ to $[-\infty, \infty]$ is extended-valued continuous.

while Gilboa-Schmeidler preferences, by imposing c-independence.[10] Variational preferences are characterized by an additively separable function G (see Theorem 1), that is,

$$G(t,p) = t + c(p) \qquad \forall (t,p) \in \mathbb{R} \times \Delta, \tag{3}$$

where $c : \Delta \to [0,\infty]$ is the cost function of [17], which is grounded,[11] lower semicontinuous, and convex. Gilboa-Schmeidler preferences are characterized as having a function G as in (3) with c such that $c(p) = 0$ if $p \in C$ and $c(p) = \infty$ otherwise, where C is a nonempty, closed, and convex subset of Δ.

The next two assumptions connect our two binary relations.

Consistency: *If $f \succsim^* g$ then $f \succsim^\wedge g$.*

Weak caution: *For each $x \in X$ there exists $y \in X$ such that $y \succsim^\wedge x$ and for each $f \in \mathcal{F}$*

$$f \not\succsim^* x \quad \Longrightarrow \quad y \succsim^\wedge f.$$

Consistency means that $\succsim^*$ is a subrelation of $\succsim^\wedge$. Together with completeness, consistency implies that $\succsim^\wedge$ is a completion of $\succsim^*$. Intuitively, if f is clearly/objectively weakly better than g then the DM should deem f weakly better than g when he has to make a choice. Weak caution amounts to imposing that, for any given x in X, there exists a common bound y in X for *all* acts f that are not unambiguously preferred to x.

Weak caution provides the main axiomatic departure of our work from [12]. In the work of GMMS, this assumption is trivially satisfied; under caution, for each x, the bound y is x itself. In their case, consistency and caution are the two key conditions implying that $\succsim^\wedge$ satisfies

$$f \sim^\wedge \underline{x}_f \qquad \forall f \in \mathcal{F},$$

where $\underline{x}_f \sim^* \sup\{x \in X : f \succsim^* x\}$ (see Proposition 2 below). In other words, caution allows for only a very restrictive completion of $\succsim^*$, namely the one that to each act f associates the lowest possible evaluation in the interval objectively specified by $\succsim^*$. On the other hand, weak caution allows us to consider and axiomatize less restrictive completions, and therefore more general forms of subjective rationality (see also Sections 1 and 4).

As the proof of Theorem 2 shows, weak caution has bite only in a context where unboundedness from above is satisfied. In the classic Anscombe and Aumann setting, with X being the set of simple positive monetary lotteries, unboundedness from above

[10] See [17, Axiom A.2] for the definition of weak c-independence and the section below for the definition of c-independence.

[11] The function c is grounded if and only if $\min_{p\in\Delta} c(p) = 0$.

is satisfied, for example, when the DM's risk attitude is represented by a CRRA utility index.

4 Uncertainty Averse Completions

We can now state the main result of our paper. It shows that Bewley preferences and uncertainty averse preferences are connected via a completion procedure.

Theorem 2 *Let* $(\succsim^*, \succsim^\wedge)$ *be two binary relations on* $\mathcal{F}$ *and let one of them satisfy unboundedness. The following are equivalent conditions:*

(i) $\succsim^*$ *satisfies the basic conditions, c-completeness, and independence;* $\succsim^\wedge$ *satisfies the basic conditions, completeness, risk independence, and uncertainty aversion; and jointly* $(\succsim^*, \succsim^\wedge)$ *satisfy consistency and weak caution.*

(ii) There exist an onto and affine function $u : X \to \mathbb{R}$*, a linearly continuous function* $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$*, and a nonempty, closed, and convex set* $C \subseteq \Delta$ *such that* $dom_\Delta G = C$ *and for each* f *and* g

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C \tag{4}$$

and

$$f \succsim^\wedge g \iff \min_{p \in C} G\left(\int u(f)\, dp, p\right) \geq \min_{p \in C} G\left(\int u(g)\, dp, p\right). \tag{5}$$

Moreover, C *is unique,* u *is cardinally unique, and, given* u*,* G *is unique.*

Theorem 2 follows from the following arguments. The axioms on $\succsim^*$ imply that $\succsim^*$ is represented according to the unanimity rule of Bewley with a set of probabilities C and a utility index u^*. The axioms on $\succsim^\wedge$ imply that $\succsim^\wedge$ is an uncertainty averse preference. Thus, by [4], there exist a nonconstant and affine $u : X \to \mathbb{R}$ and a linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in \Delta} G\left(\int u(f)\, dp, p\right) \qquad \forall f \in \mathcal{F}, \tag{6}$$

represents $\succsim^\wedge$. Consistency delivers the fact that u^* can be chosen to be equal to u, while weak caution implies that G belongs to $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ and the min in (6) can be taken over $C = dom_\Delta G$. Given this equality, it follows that C is the smallest closed and convex set over which the min in (6) can be taken. This latter fact confirms that the set of probabilities characterizing $\succsim^\wedge$ is the same one characterizing $\succsim^*$. Moreover, in the appendix, we show that C also characterizes the *unambiguous preference relation* of Ghirardato, Maccheroni, and Marinacci [9] for $\succsim^\wedge$.

Our DM acts as if, in forming his preferences, he first identifies the set of relevant and plausible probabilities C and a utility index u. These two objects characterize the rankings, as in (4), that the DM deems uncontroversial. For example, in the classic Ellsberg two-color urn experiment, C could be the convex hull of all possible urn compositions of the unknown urn, and u, a utility index over all the objective urns. Nevertheless, C and u are not enough for the DM to be able to always rank acts. For this reason, to complete his preferences, he then selects an index of uncertainty aversion G that is also bounded on C. This allows him to consider certain probabilistic scenarios in C more plausible than others. Finally, he uses these three objects to consistently form his preferences $\succsim^\wedge$ according to the cautious rule in (5) and thus he only uses the probabilities in C.

The condition that G belongs to $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ amounts to imposing that, in completing his preferences $\succsim^*$, the DM might not be willing to consider all the probabilistic scenarios in C to be equivalent, as in the Gilboa-Schmeidler case. But, at the same time, he does not want to penalize these alternative probabilities in an arbitrarily unbounded fashion.

The class of functions $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ characterizes a subset of uncertainty averse preferences that we call *effectively bounded* uncertainty averse preferences:

Definition 3 *Let $\succsim^\wedge$ be a binary relation on $\mathcal{F}$. $\succsim^\wedge$ is an effectively bounded uncertainty averse preference if and only if there exist an onto and affine function $u : X \to \mathbb{R}$ and a linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that (u, G) represent $\succsim^\wedge$ as in (2).*

The intersection between effectively bounded uncertainty averse preferences and variational preferences is easily characterized and contains Gilboa-Schmeidler preferences. If the DM's preferences $\succsim^\wedge$ are variational,[12] then G is as in (3). If we further impose that $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, the condition

$$\infty > \sup_{p \in dom_\Delta G} G(t,p) = \sup_{p \in dom(c)} \{t + c(p)\} \qquad \forall t \in \mathbb{R} \tag{7}$$

is equivalent to c being bounded over $dom_\Delta G = dom(c)$, where

$$dom(c) = \{p \in \Delta : c(p) < \infty\}$$

is the effective domain of c. Thus, it is also immediate to verify that the function G characterizing a Gilboa-Schmeidler preference satisfies condition (7) and is an element of $\mathcal{L}_{bd}(\mathbb{R} \times \Delta)$.

[12] This is equivalent to imposing that $\succsim^\wedge$ also satisfies weak c-independence as in [17, Axiom A.2].

It follows that our main result provides a foundation for the larger class of effectively bounded uncertainty averse preferences (Gilboa-Schmeidler preferences being a subclass of the latter).[13]

As already mentioned, in Theorem 2, $\succsim^{\wedge}$ turns out to belong to the special class of effectively bounded uncertainty averse preferences. The next result shows that this latter class is "dense" in the class of uncertainty averse preferences, proving that effectively bounded uncertainty averse preferences are a "topologically" large subset of the set formed by uncertainty averse preferences.

Proposition 1 *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies unboundedness. If $\succsim$ is an uncertainty averse preference then there exists a sequence of effectively bounded uncertainty averse preferences $\{\succsim_n\}_{n\in\mathbb{N}}$ such that*

$$\lim_n V_n(f) = V(f) \qquad \forall f \in \mathcal{F},$$

where $V, V_n : \mathcal{F} \to \mathbb{R}$ represent $\succsim$ and $\succsim_n$ as in (6) and for all $n \in \mathbb{N}$.

We conclude by formally discussing the relationship between our main result and the work of GMMS. We start by listing two assumptions that play a major role in [12]:

C-independence: *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$*

$$f \succsim^{\wedge} g \iff \alpha f + (1-\alpha)x \succsim^{\wedge} \alpha g + (1-\alpha)x.$$

Caution: *If $f \in \mathcal{F}$ and $x \in X$*

$$f \not\succsim^{*} x \implies x \succsim^{\wedge} f.$$

Theorem 3 (GMMS, Theorem 3) *Let $(\succsim^{*}, \succsim^{\wedge})$ be two binary relations on $\mathcal{F}$. The following are equivalent conditions:*

(i) $\succsim^{*}$ satisfies the basic conditions, c-completeness, and independence; $\succsim^{\wedge}$ satisfies the basic conditions, completeness, c-independence; and jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy consistency and caution.

(ii) There exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a nonempty, closed, and convex set $C \subseteq \Delta$ such that for each f and g

$$f \succsim^{*} g \iff \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C$$

and

$$f \succsim^{\wedge} g \iff \min_{p\in C} \int u(f)\,dp \geq \min_{p\in C} \int u(g)\,dp.$$

[13]It is also possible, within the single preference framework adopted by [4], to provide a foundation for the class of effectively bounded uncertainty averse preferences. This can be achieved by requiring the unambiguous preference relation of [9] to satisfy weak caution.

Moreover, C is unique and u is cardinally unique.

Our work departs from [12] in three ways. The first departure consists in restricting attention to binary relations that satisfy unboundedness, that is, preferences for which there are arbitrarily good and arbitrarily bad consequences. For example, this is the case if $X = \mathbb{R}$, and the DM satisfies the basic conditions as well as c-completeness and risk independence.[14] The second departure consists in weakening c-independence to risk independence and in explicitly assuming uncertainty aversion for $\succsim^\wedge$.[15] This allows us to consider preferences $\succsim^\wedge$ that are variational, as in [17], or, more generally, uncertainty averse as in [4].[16] Finally, we impose on $(\succsim^*, \succsim^\wedge)$ a weaker form of caution, termed weak caution. It is immediate to see that weak caution is a weakening of caution.

In order to explore the extent of the assumption of caution, we first propose a stronger version of [12, Theorem 3] (see also [12, Theorem 4]). In comparison to [12, Theorem 3], here we only weaken c-independence to risk independence in (i) but we still obtain the same functional characterization in (ii).

Proposition 2 *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$. The following are equivalent conditions:*

(i) $\succsim^*$ satisfies the basic conditions, c-completeness, and independence; $\succsim^\wedge$ satisfies the basic conditions, completeness, and risk independence; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency and caution.

(ii) There exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a nonempty, closed, and convex set $C \subseteq \Delta$ such that for each f and g

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C$$

and

$$f \succsim^\wedge g \iff \min_{p \in C} \int u(f)\, dp \geq \min_{p \in C} \int u(g)\, dp.$$

Moreover, C is unique and u is cardinally unique.

[14] In this case, the DM can also be interpreted as being risk neutral.

[15] Binary relations that satisfy the basic conditions, completeness, and c-independence are studied and defined as invariant biseparable preferences by Ghirardato, Maccheroni, and Marinacci [9]; see also [10] and [11].

[16] See also Maccheroni, Marinacci, and Rustichini [17, p. 1454] for a positive/normative discussion justifying an axiomatic departure from c-independence.

Thus, in GMMS, weakening c-independence to risk independence has no effect. Note also that, in the previous proposition, we did not make any assumption on $\succsim^{\wedge}$ regarding attitudes toward uncertainty or independence involving uncertain acts.[17] Thus, as it emerges also from the proof of Proposition 2, it is primarily caution that drives the representation of the completion $\succsim^{\wedge}$ of $\succsim^{*}$ to be maxmin expected utility. The next proposition shows that, *under unboundedness*, weakening caution to weak caution also has no effect.

Proposition 3 *Let $(\succsim^{*}, \succsim^{\wedge})$ be two binary relations on $\mathcal{F}$ and let one of them satisfy unboundedness. Moreover, let $\succsim^{*}$ satisfy the basic conditions, c-completeness, and independence; $\succsim^{\wedge}$ satisfy the basic conditions, completeness, and c-independence; and jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy consistency. The following conditions are equivalent:*

(i) jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy weak caution;

(ii) jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy caution.

As a corollary to this result, we can prove [12, Theorem 3] again. In fact, we can replace caution with weak caution and retain c-independence and still obtain the same representation result via Proposition 3 and Theorem 3.

A Appendix A

Given a binary relation $\succsim^{\wedge}$ on $\mathcal{F}$, we define $\succsim^{\circ}$ by

$$f \succsim^{\circ} g \iff \lambda f + (1-\lambda) h \succsim^{\wedge} \lambda g + (1-\lambda) h \qquad \forall \lambda \in (0,1], \forall h \in \mathcal{F}.$$

The binary relation $\succsim^{\circ}$ is the revealed unambiguous preference relation of Ghirardato, Maccheroni, and Marinacci [9]. In the sequel, with a small abuse of notation, given $k \in \mathbb{R}$, we will denote by k both the real number and the constant function on S that takes value k.

In the rest of the paper, we will invoke some of the results of GMMS. Even though all the results in [12] were derived under the hypothesis that X is the set of all simple lotteries over a generic outcome space, their extension to the case when X is a generic convex set is straightforward.

Before proving the main results, we need some extra notation and an ancillary proposition. Given a functional $I : B_0(\Sigma) \to \mathbb{R}$, we define $\succcurlyeq$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq \psi \iff I(\varphi) \geq I(\psi).$$

[17] Binary relations that satisfy the basic conditions, completeness, and risk independence are called rational preferences and studied in [3].

We define $\succcurlyeq^\circ$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq^\circ \psi \Longleftrightarrow I(\lambda\varphi + (1-\lambda)\phi) \geq I(\lambda\psi + (1-\lambda)\phi) \quad \forall\lambda \in (0,1], \forall\phi \in B_0(\Sigma). \quad (8)$$

Given $C \subseteq \Delta$, we define $\succcurlyeq_C$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq_C \psi \quad \Longleftrightarrow \quad \int \varphi dp \geq \int \psi dp \qquad \forall p \in C.$$

Given C and I, we say that I is consistent with C if and only if

$$\varphi \succcurlyeq_C \psi \quad \Longrightarrow \quad I(\varphi) \geq I(\psi).$$

Finally, a function $G : \mathbb{R}\times\Delta \to (-\infty, \infty]$ is said to be *linearly continuous* if and only if the map

$$\varphi \mapsto \inf_{p\in\Delta} G\left(\int \varphi dp, p\right)$$

from $B_0(\Sigma)$ to $[-\infty, \infty]$ is extended-valued continuous. For example, the function G defined in (3) is linearly continuous.

Proposition 4 *Let I be a functional from $B_0(\Sigma)$ to $\mathbb{R}$ and let C be a nonempty, closed, and convex subset of Δ. The following conditions are equivalent:*

(i) I is normalized, monotone, continuous, quasiconcave, consistent with C, and such that for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\succcurlyeq_C k \quad \Longrightarrow \quad h \geq I(\varphi). \quad (9)$$

(ii) There exists a unique linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R}\times\Delta)$ such that

$$I(\varphi) = \min_{p\in\Delta} G\left(\int \varphi dp, p\right) \qquad \forall\varphi \in B_0(\Sigma)$$

and $dom_\Delta G = C$.

Proof. (i) implies (ii). We proceed by steps. But, first, by construction, observe that $\succcurlyeq$, $\succcurlyeq_C$, and $\succcurlyeq^\circ$ are binary relations over acts in an Anscombe and Aumann setting where S is the state space, Σ is the algebra, and $X = \mathbb{R}$.

Step 1. $\succcurlyeq$ satisfies the basic conditions, completeness, risk independence, and uncertainty aversion. Moreover, $\succcurlyeq$ restricted to $\mathbb{R}$ is represented by the identity.

Proof of the Step.

Since I is normalized, $\succcurlyeq$ restricted to $\mathbb{R}$ is represented by the identity. By [4, Lemma 57] and since I is normalized, monotone, continuous, and quasiconcave, the statement follows. □

Step 2. There exists a nonempty, closed, and convex set $C^\circ \subseteq \Delta$ such that for each φ and ψ in $B_0(\Sigma)$

$$\varphi \succsim^\circ \psi \iff \int \varphi dp \geq \int \psi dp \qquad \forall p \in C^\circ \tag{10}$$

and

$$\varphi \succsim^\circ \psi \implies I(\varphi) \geq I(\psi).$$

Moreover, C° is unique and $\succsim^\circ = \succsim_{C^\circ}$.

Proof of the Step.

By definition of $\succsim^\circ$ and $\succsim$, we have that

$$\varphi \succsim^\circ \psi \iff \lambda\varphi + (1-\lambda)\phi \succsim \lambda\psi + (1-\lambda)\phi \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma). \tag{11}$$

By Step 1 and [3, Proposition 2], the first part of the statement follows as well as the uniqueness of C° and $\succsim^\circ = \succsim_{C^\circ}$. By taking $\lambda = 1$ in (11) and by definition of $\succsim$, the second part follows as well. □

Step 3. $C^\circ \subseteq C$.

Proof of the Step.

By the definition of $\succsim_C$ and $\succsim^\circ$ and Step 2 and since I is consistent with C, we have that

$$\begin{aligned}\varphi \succsim_C \psi &\implies \lambda\varphi + (1-\lambda)\phi \succsim_C \lambda\psi + (1-\lambda)\phi \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma) \\ &\implies I(\lambda\varphi + (1-\lambda)\phi) \geq I(\lambda\psi + (1-\lambda)\phi) \qquad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma) \\ &\implies \varphi \succsim^\circ \psi.\end{aligned}$$

By Step 2 and [9, Proposition A.1.], this implies that $C^\circ \subseteq C$. □

Step 4. There exists a unique linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma). \tag{12}$$

Moreover, for each $(t,p) \in \mathbb{R} \times \Delta$

$$G(t,p) = \sup\left\{I(\varphi) : \int \varphi dp \leq t\right\} \tag{13}$$

and $cl(dom_\Delta G) = C^\circ$.

Proof of the Step.

By [4] (see also [5]) and Step 1 and since I is normalized, monotone, continuous, and quasiconcave, there exists a unique linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that (12) and (13) hold. By [4, Theorem 10], we also have that $cl(dom_\Delta G) = C^\circ$.

Step 5. $C^{\circ} = C$.

Proof of the Step.

We start by giving a definition. Given $\phi \in B_0(\Sigma)$, we define $k_\phi = \min_{p\in C} \int \phi dp$. By contradiction, suppose that $C^{\circ} \neq C$. By Steps 3 and 4, we know that this implies that there exists $q \in C \backslash C^{\circ}$ and $q \notin dom_{\Delta} G$. By [18, Theorem 3.4] and since C° is closed and convex, there exists $\psi \in B_0(\Sigma)$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\min_{p\in C} \int \psi dp \leq \int \psi dq \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \min_{p\in C^{\circ}} \int \psi dp. \tag{14}$$

Without loss of generality, we can assume that ψ is such that $k_\psi \leq -\varepsilon < 0$ and $\min_{p\in C^{\circ}} \int \psi dp \geq \varepsilon > 0$. By (14), if we define the sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma)$ to be such that $\varphi_n = n\psi$ for all $n \in \mathbb{N}$ then it follows that

$$k_{\varphi_n} < 0 \quad \text{and} \quad \min_{p\in C^{\circ}} \int \varphi_n dp = \min_{p\in C^{\circ}} \int n\psi dp = n \min_{p\in C^{\circ}} \int \psi dp \geq n\varepsilon > 0 \qquad \forall n \in \mathbb{N}. \tag{15}$$

Recall that I satisfies (9), that is, for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\geq_C k \quad \Longrightarrow \quad h \geq I(\varphi).$$

Take $k = 0$ and h as in (9). By (15), it follows that there exists $\bar{n} \in \mathbb{N}$ such that

$$k_{\varphi_{\bar{n}}} < 0 = k \quad \text{and} \quad \int \varphi_{\bar{n}} dp' \geq \min_{p\in C^{\circ}} \int \varphi_{\bar{n}} dp > h + 1 \qquad \forall p' \in C^{\circ}. \tag{16}$$

By Step 2, I is consistent with C°. Thus, the first part of (16) yields that $\varphi_{\bar{n}} \not\geq_C k$ while the second part delivers that $I(\varphi_{\bar{n}}) > h$, a contradiction. □

Step 6. $\sup_{p\in dom_\Delta G} G(t,p) < \infty$ *for all* $t \in \mathbb{R}$, *that is,* $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$.

Proof of the Step.

Before starting recall that, by Step 4, G satisfies (13), that is,

$$G(t,p) = \sup \left\{ I(\varphi) : \int \varphi dp \leq t \right\} \qquad \forall (t,p) \in \mathbb{R} \times \Delta.$$

By contradiction, suppose that $\sup_{p\in dom_\Delta G} G(\bar{t},p) = \infty$ for some $\bar{t}$ in $\mathbb{R}$. By working hypothesis, there exists a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq dom_\Delta G$ such that $G(\bar{t},p_n) \geq n$ for all $n \in \mathbb{N}$. By (13) and since $C = C^{\circ} = cl(dom_\Delta G)$, this implies that for each $n \in \mathbb{N}$ there exists $\varphi_n \in B_0(\Sigma)$ such that

$$\min_{p\in C} \int \varphi_n dp \leq \int \varphi_n dp_n \leq \bar{t} < \bar{t} + 1 \quad \text{and} \quad I(\varphi_n) \geq \frac{n}{2}. \tag{17}$$

Since I satisfies (9), consider $k = \bar{t} + 1$ and fix $h \geq k$ to satisfy (9). From the first part of (17), we have that $\varphi_n \not\geq_C k$ for all $n \in \mathbb{N}$. At the same time, by the second part of (17), it is immediate to see that there exists $\bar{n} \in \mathbb{N}$ such that $I(\varphi_{\bar{n}}) \geq \frac{\bar{n}}{2} \geq h$, a contradiction with I satisfying (9). $\square$

Step 7. $cl\,(dom_\Delta G) = dom_\Delta G$.

Proof of the Step.

It is enough to prove that given a generic net $\{p_\alpha\}_{\alpha \in A} \subseteq dom_\Delta G$ such that $p_\alpha \to \bar{p}$ then $\bar{p} \in dom_\Delta G$. Fix a generic $t \in \mathbb{R}$. By Step 6, it follows that $G(t, p_\alpha) \leq \sup_{p \in dom_\Delta G} G(t,p) < \infty$ for all $\alpha \in A$. Since $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, we have that

$$\infty > \sup_{p \in dom_\Delta G} G(t,p) \geq \liminf_\alpha G(t, p_\alpha) \geq G(t, \bar{p}).$$

Hence, $\bar{p} \in dom_\Delta G$. $\square$

Steps 4 and 6 imply the first part of (ii) while Steps 4, 5, and 7 imply that $C = C^\circ = cl\,(dom_\Delta G) = dom_\Delta G$.

(ii) implies (i). By assumption, we have that there exists a linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma).$$

By [5], it follows that I is normalized, monotone, and quasiconcave. Since G is linearly continuous, I is continuous. Next, by definition of $dom_\Delta G$, we have that

$$I(\varphi) = \min_{p \in dom_\Delta G} G\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma). \tag{18}$$

Since G is increasing in the first component and $dom_\Delta G = C$, it follows that I is consistent with C. Finally, we show that I satisfies (9). We proceed by arguing by contradiction. Suppose that there exists $k \in \mathbb{R}$ such that for each $h \geq k$ in $\mathbb{R}$ we can find $\varphi_h \in B_0(\Sigma)$ such that $\varphi_h \not\succsim_C k$ and $I(\varphi_h) > h$. It follows that for each $n \in \{\lfloor k \rfloor + 1, ..., \lfloor k \rfloor + m, ...\}$ there exists $\varphi_n \in B_0(\Sigma)$ such that $\varphi_n \not\succsim_C k$ and $I(\varphi_n) > n$.[18] Thus, for each $n \in \{\lfloor k \rfloor + 1, ..., \lfloor k \rfloor + m, ...\}$ there exists $p_n \in C = dom_\Delta G$ such that $\int \varphi_n dp_n < k$. By (18) and since $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, it follows that

$$\sup_{p \in dom_\Delta G} G(k,p) \geq G(k, p_n) \geq G\left(\int \varphi_n dp_n, p_n\right) \geq I(\varphi_n) > n,$$

for all $n \in \{\lfloor k \rfloor + 1, ..., \lfloor k \rfloor + m, ...\}$, a contradiction with $\sup_{p \in dom_\Delta G} G(t,p) < \infty$ for all $t \in \mathbb{R}$. ■

Proof of Theorem 2. (i) implies (ii). We again proceed by steps.

Step 1. $\succsim^\wedge$ *coincides to* $\succsim^*$ *on* X.

[18] Given $k \in \mathbb{R}$, we denote by $\lfloor k \rfloor$ the floor of k, that is, the largest integer not greater than k.

Proof of the Step.

Notice that $\succsim^{*}$ and $\succsim^{\wedge}$ restricted to X satisfy c-completeness, mixture continuity, and risk independence. By [15] and since $\succsim^{*}$ and $\succsim^{\wedge}$ satisfy the basic conditions, it follows that there exist two nonconstant and affine functions, u^{*} and $u^{\wedge}$, from X to $\mathbb{R}$ that represent $\succsim^{*}$ and $\succsim^{\wedge}$ on X respectively. Since jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy consistency, it follows that for each $x, y \in X$

$$u^{*}(x) \geq u^{*}(y) \quad \Longrightarrow \quad u^{\wedge}(x) \geq u^{\wedge}(y).$$

By [9, Corollary B.3.], it follows that u^{*} and $u^{\wedge}$ are equal up to an affine and positive transformation, hence the statement. □

Step 2. There exist an onto and affine function $u^{*} : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that

$$f \succsim^{*} g \quad \Longleftrightarrow \quad \int u^{*}(f)\, dp \geq \int u^{*}(g)\, dp \qquad \forall p \in C. \tag{19}$$

Moreover, C is unique.

Proof of the Step.

By assumption, $\succsim^{*}$ satisfies the basic conditions, c-completeness, and independence. By [12, Theorem 1] and since, by Step 1 and the premises of Theorem 2, $\succsim^{*}$ satisfies unboundedness, the statement follows. □

Step 3. There exist an onto and affine function $u^{\wedge} : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim^{\wedge} g \quad \Longleftrightarrow \quad I\left(u^{\wedge}(f)\right) \geq I\left(u^{\wedge}(g)\right).$$

Moreover, $u^{\wedge}$ is cardinally unique and, given $u^{\wedge}$, I is unique.

Proof of the Step.

By assumption, Step 1, and the premises of Theorem 2, $\succsim^{\wedge}$ satisfies the basic conditions, completeness, risk independence, uncertainty aversion, and unboundedness. By [4, Lemma 57 and Lemma 59], the statement follows. □

Notice that, by Step 1, we can assume without loss of generality that $u^{*} = u^{\wedge} = u$.

Step 4. I is consistent with C.

Proof of the Step.

Consider $\varphi, \psi \in B_0(\Sigma)$ and assume that $\varphi \succcurlyeq_C \psi$. It is immediate to see that there exist $f, g \in \mathcal{F}$ such that $\varphi = u(f)$, $\psi = u(g)$, and $f \succsim^{*} g$. By Steps 2 and 3 and since jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy consistency, we have that

$$\begin{aligned} \varphi \succcurlyeq_C \psi \quad &\Longrightarrow \quad f \succsim^{*} g \quad \Longrightarrow \quad f \succsim^{\wedge} g \quad \Longrightarrow \quad I(u(f)) \geq I(u(g)) \\ &\Longrightarrow \quad I(\varphi) \geq I(\psi), \end{aligned}$$

proving the statement. $\square$

Step 5. I satisfies (9).

Proof of the Step.

We need to show that for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\geq_C k \quad \Longrightarrow \quad h \geq I(\varphi).$$

Fix a generic $k \in \mathbb{R}$. Since $\succsim^{\wedge}$ satisfies unboundedness, there exists $x \in X$ such that $k = u(x)$. Define $h = u(y)$ where $y \in X$ is such that $y \succsim^{\wedge} x$ and

$$f \not\succsim^{*} x \quad \Longrightarrow \quad y \succsim^{\wedge} f.$$

Next, consider $\varphi \in B_0(\Sigma)$ such that $\varphi \not\geq_C k$. Given (19), it is immediate to see that there exists $f \in \mathcal{F}$ such that $\varphi = u(f)$ and $f \not\succsim^{*} x$. Since jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy weak caution, it follows that $y \succsim^{\wedge} f$. By Step 3, this implies that $h = u(y) = I(u(y)) \geq I(u(f)) = I(\varphi)$, hence the statement. $\square$

Step 6. There exist an onto and affine function $u : X \to \mathbb{R}$, a linearly continuous function $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$, and a nonempty, closed, and convex set $C \subseteq \Delta$ such that $dom_{\Delta} G = C$ and for each f and g

$$f \succsim^{*} g \quad \Longleftrightarrow \quad \int u(f)\,dp \geq \int u(g)\,dp \qquad \forall p \in C \tag{20}$$

and

$$f \succsim^{\wedge} g \quad \Longleftrightarrow \quad \min_{p \in C} G\left(\int u(f)\,dp, p\right) \geq \min_{p \in C} G\left(\int u(g)\,dp, p\right). \tag{21}$$

Proof of the Step.

Define $V : \mathcal{F} \to \mathbb{R}$ by $V(f) = I(u(f))$ for all $f \in \mathcal{F}$ where $u = u^{\wedge}$ and I are as in Step 3. It is immediate to see that V represents $\succsim^{\wedge}$. By Steps 2, 3, 4, and 5 and Proposition 4, it follows that there exists a linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that $dom_{\Delta} G = C$ where C is nonempty, closed, and convex and

$$V(f) = \min_{p \in \Delta} G\left(\int u(f)\,dp, p\right) = \min_{p \in C} G\left(\int u(f)\,dp, p\right) \qquad \forall f \in \mathcal{F},$$

proving that (21) holds. By Step 2 and since $u^{*} = u$, (20) holds. $\square$

(ii) implies (i). Consider a nonempty, closed, and convex set $C \subseteq \Delta$, an onto and affine function $u : X \to \mathbb{R}$, and a linearly continuous $G \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that $C = dom_{\Delta} G$. Suppose also that C and (u, G) satisfy (4) and (5). By [12, Theorem 1], it follows that $\succsim^{*}$ satisfies the basic conditions, c-completeness, and independence. By [4, Theorem 3], $\succsim^{\wedge}$ satisfies the basic conditions, completeness, risk independence, and uncertainty aversion (as well as unboundedness). Define $I : B_0(\Sigma) \to \mathbb{R}$ by

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi\,dp, p\right) \qquad \forall \varphi \in B_0(\Sigma).$$

By Proposition 4, it follows that I is consistent with C and satisfies (9). Since I composed with u represents $\succsim^{\wedge}$, this implies that jointly $(\succsim^{*}, \succsim^{\wedge})$ satisfy consistency and weak caution.

The uniqueness part of the statement follows from routine arguments (see [12] and [4]). ■

Proof of Proposition 1. Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies unboundedness and assume $\succsim$ is an uncertainty averse preference. By [4, Lemma 57 and Lemma 59], there exist an onto and affine function $u : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\Sigma) \to \mathbb{R}$ such that $f \succsim g$ if and only if $V(f) \geq V(g)$ where $V(f) = I(u(f))$ for all $f \in \mathcal{F}$. For each $n \in \mathbb{N}$ define $J_n : B_0(\Sigma) \to \mathbb{R}$ by $\varphi \mapsto \min_{s\in S} \varphi(s) + n$ and $I_n : B_0(\Sigma) \to \mathbb{R}$ by

$$I_n(\varphi) = \min\{I(\varphi), J_n(\varphi)\} \qquad \forall \varphi \in B_0(\Sigma).$$

It is immediate to verify that I_n is a normalized, monotone, continuous, and quasiconcave functional for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define $\succcurlyeq_n^{\circ}$ as in (8). It follows that there exists a nonempty, closed, and convex set C_n of Δ such that

$$\varphi \succcurlyeq_n^{\circ} \psi \iff \varphi \succcurlyeq_{C_n} \psi$$
and
$$\varphi \succcurlyeq_{C_n} \psi \implies I_n(\varphi) \geq I_n(\psi).$$

We next show that I_n satisfies (9) for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. First, given $k \in \mathbb{R}$ define $h_k = k + n$. Next, consider $\varphi \in B_0(\Sigma)$ such that $\varphi \not\succcurlyeq_{C_n} k$. This implies that $\min_{s\in S} \varphi(s) < k$. It follows that

$$I_n(\varphi) = \min\{I(\varphi), J_n(\varphi)\} \leq J_n(\varphi) < k + n = h_k,$$

proving that I_n satisfies (9). By Proposition 4, it follows that for each $n \in \mathbb{N}$ there exists a unique linearly continuous $G_n \in \mathcal{L}_{bd}(\mathbb{R} \times \Delta)$ such that

$$I_n(\varphi) = \min_{p\in\Delta} G_n\left(\int \varphi dp, p\right) \qquad \forall \varphi \in B_0(\Sigma).$$

Moreover, note that

$$\lim_n I_n(\varphi) = I(\varphi) \qquad \forall \varphi \in B_0(\Sigma). \tag{22}$$

For each $n \in \mathbb{N}$ define $V_n : \mathcal{F} \to \mathbb{R}$ and $\succsim_n$ to be such that

$$V_n(f) = \min_{p\in\Delta} G_n\left(\int u(f)\, dp, p\right) \qquad \forall f \in \mathcal{F}$$
and
$$f \succsim_n g \iff V_n(f) \geq V_n(g).$$

It follows that $\succsim_n$ is an effectively bounded uncertainty averse preference for all $n \in \mathbb{N}$. By (22), we also have that

$$\lim_n V_n(f) = V(f) \qquad \forall f \in \mathcal{F},$$

proving the statement. ■

Proof of Proposition 2. (i) implies (ii). By [12, Theorem 1] and since $\succsim^*$ satisfies the basic conditions, c-completeness, and independence, there exist a nonconstant and affine function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that

$$f \succsim^* g \iff \int u^*(f)\,dp \geq \int u^*(g)\,dp \qquad \forall p \in C. \tag{23}$$

By [3, Proposition 1] and since $\succsim^\wedge$ satisfies the basic conditions, completeness, and risk independence, there exist a nonconstant and affine function $u^\wedge : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, u^\wedge(X)) \to \mathbb{R}$ such that

$$f \succsim^\wedge g \iff I(u^\wedge(f)) \geq I(u^\wedge(g)).$$

Moreover, by [3, Proposition 2], it follows that there exists a nonempty, closed, and convex set C° such that

$$f \succsim^\circ g \iff \int u^\wedge(f)\,dp \geq \int u^\wedge(g)\,dp \qquad \forall p \in C^\circ.$$

Since $(\succsim^*, \succsim^\wedge)$ jointly satisfy consistency, it follows that for each $x, y \in X$

$$u^*(x) \geq u^*(y) \implies u^\wedge(x) \geq u^\wedge(y).$$

By [9, Corollary B.3.], it follows that u^* is a positive affine transformation of $u^\wedge$. Wlog, we can assume that $u^\wedge = u^* = u$. By (23), we have that if $f \succsim^* g$ then $\lambda f + (1-\lambda)h \succsim^* \lambda g + (1-\lambda)h$ for all $\lambda \in (0,1]$ and all $h \in \mathcal{F}$. Since $(\succsim^*, \succsim^\wedge)$ jointly satisfy consistency, it follows that

$$\lambda f + (1-\lambda)h \succsim^\wedge \lambda g + (1-\lambda)h \qquad \forall \lambda \in (0,1], \forall h \in \mathcal{F}$$

which in turn yields $f \succsim^\circ g$. In other words, we have that if $f \succsim^* g$ then $f \succsim^\circ g$. Since $B_0(\Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$ and by [9, Proposition A.1.], this implies that $C^\circ \subseteq C$. By [3, Corollary 3], we have that

$$\min_{p \in C} \int u(f)\,dp \leq \min_{p \in C^\circ} \int u(f)\,dp \leq I(u(f)) \qquad \forall f \in \mathcal{F}. \tag{24}$$

Conversely, fix $f \in \mathcal{F}$ and define $k = \min_{p \in C} \int u(f)\,dp$. Since u is affine and $C \subseteq \Delta$, we have that $k \in u(X)$. Thus, there exists $x \in X$ such that $u(x) = k$. We have two cases:

1. $x \succsim^\wedge y$ for all $y \in X$. By monotonicity, this implies that $x \succsim^\wedge f$, that is,

$$I(u(f)) \leq I(u(x)) = u(x) = \min_{p \in C} \int u(f)\, dp.$$

2. There exists $y \in X$ such that $y \succ^\wedge x$. Define $x_\varepsilon = \varepsilon y + (1-\varepsilon) x$ for all $\varepsilon \in (0,1)$. Since u is affine and represents $\succsim^\wedge$ on X, we have that

$$u(x_\varepsilon) > u(x) \qquad \forall \varepsilon \in (0,1).$$

This implies that $f \not\succsim^* x_\varepsilon$ for all $\varepsilon \in (0,1)$. Since $(\succsim^*, \succsim^\wedge)$ jointly satisfy caution, it follows that $x_\varepsilon \succsim^\wedge f$ for all $\varepsilon \in (0,1)$, that is,

$$I(u(f)) \leq I(u(x_\varepsilon)) = u(x_\varepsilon) = \varepsilon u(y) + (1-\varepsilon) u(x) \qquad \forall \varepsilon \in (0,1).$$

This implies that $I(u(f)) \leq u(x) = \min_{p \in C} \int u(f)\, dp$.

In both cases and by (24), we obtain that $I(u(f)) = \min_{p \in C} \int u(f)\, dp$, proving the statement since f was chosen to be generic.

(ii) implies (i). It follows from [12, Theorem 3].

The uniqueness part of the statement follows from routine arguments. ∎

Proof of Proposition 3. (i) implies (ii). By contradiction, suppose that jointly $(\succsim^*, \succsim^\wedge)$ do not satisfy caution. Therefore, there exist $\bar{x} \in X$ and $\bar{f} \in \mathcal{F}$ such that $\bar{f} \not\succsim^* \bar{x}$ and $\bar{f} \succ^\wedge \bar{x}$. By the premises and [12, Theorem 1], it follows that there exist an affine and nonconstant function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that

$$f \succsim^* g \iff \int u^*(f)\, dp \geq \int u^*(g)\, dp \qquad \forall p \in C.$$

By the premises and [15], it follows that there exists an affine and nonconstant function $u : X \to \mathbb{R}$ such that

$$x \succsim^\wedge y \iff u(x) \geq u(y).$$

By [9, Corollary B.3.] and since jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency and one binary relation between $\succsim^*$ and $\succsim^\wedge$ satisfies unboundedness, we can assume that $u^* = u$, $u(\bar{x}) = 0$, and $u(X) = \mathbb{R}$.

By the premises and [9, Lemma 1] there exists a normalized and positively homogeneous functional $I : B_0(\Sigma) \to \mathbb{R}$ such that

$$f \succsim^\wedge g \iff I(u(f)) \geq I(u(g)).$$

Moreover, since jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency, we have that $f \succsim^* g$ implies $I(u(f)) \geq I(u(g))$. Define $x_a, x_b \in X$ to be such that

$$u(x_a) = I\left(u\left(\bar{f}\right)\right) \text{ and } u(x_b) = \min_{p \in C} \int u\left(\bar{f}\right) dp.$$

Since $\bar{f} \not\succsim^* \bar{x}$ and $\bar{f} \succ^\wedge \bar{x}$, it follows that $u(x_a) > 0$ and $u(x_b) < 0$. Define now $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ to be such that for each $n \in \mathbb{N}$

$$u(f_n) = nu(\bar{f}) \text{ and } u(x_n) = nu(x_a).$$

This implies that for each $n \in \mathbb{N}$

$$\min_{p\in C} \int u(f_n)\, dp = \min_{p\in C} \int nu(\bar{f})\, dp = n \min_{p\in C} \int u(\bar{f})\, dp = nu(x_b) < 0 = u(\bar{x})$$
$$\text{and}$$
$$I(u(f_n)) = I(nu(\bar{f})) = nI(u(\bar{f})) = nu(x_a) = u(x_n).$$

That is, we have that $f_n \not\succsim^* \bar{x}$ and $f_n \succsim^\wedge x_n$ for all $n \in \mathbb{N}$. Finally, observe that jointly $(\succsim^*, \succsim^\wedge)$ satisfy weak caution. Therefore, it follows that there exists $\bar{y} \succsim^\wedge \bar{x}$ such that

$$f \not\succsim^* \bar{x} \quad \Longrightarrow \quad \bar{y} \succsim^\wedge f.$$

Consider then $\bar{n} \in \mathbb{N}$ such that $u(x_{\bar{n}}) = \bar{n}u(x_a) > u(\bar{y})$. By construction, it follows that $f_{\bar{n}} \not\succsim^* \bar{x}$ but $f_{\bar{n}} \succsim^\wedge x_{\bar{n}} \succ^\wedge \bar{y}$, a contradiction.

(ii) implies (i). It is trivial. ■

References

[1] F. J. Anscombe and R. J. Aumann, A definition of subjective probability, *The Annals of Mathematical Statistics*, 34, 199–205, 1963.

[2] T. Bewley, Knightian decision theory: Part I, *Decisions in Economics and Finance*, 25, 79–110, 2002.

[3] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 341–375, 2011.

[4] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, *Journal of Economic Theory*, 146, 1275–1330, 2011.

[5] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Complete monotone quasiconcave duality, *Mathematics of Operations Research*, 36, 321–339, 2011.

[6] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Ambiguity and robust statistics, *Journal of Economic Theory*, 148, 974–1049, 2013.

[7] G. Debreu, *Theory of value*, Yale University Press, New Haven, 1972.

[8] D. Ellsberg, Risk, ambiguity, and the Savage axioms, *Quarterly Journal of Economics*, 75, 643–669, 1961.

[9] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.

[10] P. Ghirardato and M. Marinacci, Risk, ambiguity, and the separation of utility and beliefs, *Mathematics of Operations Research*, 26, 864–890, 2001.

[11] P. Ghirardato and M. Marinacci, Ambiguity made precise: a comparative foundation, *Journal of Economic Theory*, 102, 251–289, 2002.

[12] I. Gilboa, F. Maccheroni, M. Marinacci, and D. Schmeidler, Objective and subjective rationality in a multiple prior model, *Econometrica*, 78, 755–770, 2010.

[13] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 141–153, 1989.

[14] L. P. Hansen and T. J. Sargent, Robust control and model uncertainty, *American Economic Review*, 91, 60–66, 2001.

[15] I. N. Herstein and J. Milnor, An axiomatic approach to measurable utility, *Econometrica*, 21, 291–297, 1953.

[16] P. Klibanoff, M. Marinacci, S. Mukerji, A smooth model of decision making under ambiguity, *Econometrica*, 73, 1849–1892, 2005.

[17] F. Maccheroni, M. Marinacci, and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 1447–1498, 2006.

[18] W. Rudin, *Functional analysis*, 2nd ed., McGraw-Hill, Boston, 1991.

[19] D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 571–587, 1989.

[20] T. Strzalecki, Axiomatic foundations of multiplier preferences, *Econometrica*, 79, 47–73, 2011.