Advanced Macroeconomics I

Problem Set 1

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Default: Recursive Contracts and the Foundation of Liquidity Constraints

The purpose of this problem set is to study the effect of commitment problems on consumption insurance. We will also see how the methodology if recursive contracts can be used to study this type of problems as well. In class we saw one example of such a model.

Consider a consumer who lives two periods and faces an exogenous stochastic income process. Income realizations are discrete $y_t \in Y = \{y^1, y^2, ..., y^N\}$, and positive $y^i > 0$ with $y^{i+1} > y^i$. The distribution over Y is summarized by a vector $\pi_i > 0, i = 1, ..., N$, with $\sum_{i=1}^N \pi_i = 1$ (hence i.i.d. over time). Her expected discounted lifetime utility takes the following form

$$\sum_{i=1}^{N} \pi_i \left[u\left(c_1^i\right) + \beta \left(\sum_{j=1}^{N} \pi_j u\left(c_2^{i,j}\right) \right) \right], \tag{1}$$

where c_1^i is consumption in period 1 when income in period 1 is equal to y^i, and $c_2^{i,j}$ is consumption in period 2 when $y_1 = y^i$ and $y_2 = y^j$. The utility function u increasing, strictly concave, and continuously differentiable.

1. *Autarchy.* Derive the allocation of consumption assuming that the consumer does not have access to neither credit nor insurance markets.

2. *Self Insurance.* Suppose now that the agent have access to only a storage technology which allows her to save (not borrow) one unit of consumption (income) in period one and obtain $R = \frac{1}{\beta} > 1$ units of consumption good in the following period.

   2.a State the agent's problem in this case (with her budget constraint) and derive the optimality conditions.

   2.b Give, briefly, an economic intuition of the optimality conditions.

3. *Complete Markets.* Assume now that the agent has unrestricted access to the insurance and credit markets. Since by the first welfare theorem the equilibrium allocation is Pareto optimal, we can simplify the analysis and model this situation by assuming that there is a risk neutral planner who aims at maximizing the consumer's welfare. We assume that

the planner discounts at the factor β. One can solve these class of problems by solving the cost minimization (profit maximization) problem of a principal-planner subject to the agent's participation constraint. An insurance insurance contract in this environment is a collection of functions $\mathcal{W}_0 := \left\{c_t(h^t)\right\}_{t=1}^2$ that map each history of income shocks $h^1 = y_1$, and $h^2 = (y_1, y_2)$ into a consumption level $c_t\left(h^t\right)$. We can, for example, write $c_1^i = c_1\left(y_1^i\right)$ and $c_2^{i,j} = c_2\left(y_1^i, y_2^j\right)$. The planner's problem in this case is

$$\max_{\mathcal{W}_0 := \left(\left\{c_1^i\right\}, \left\{c_2^{i,i}\right\}\right)} \sum_{i=1}^N \pi_i \left[\left(y_1^i - c_1^i\right) + \beta \sum_{j=1}^N \pi_j \left(y_2^j - c_2^{i,j}\right)\right] \text{ s.t.}$$

$$\sum_{i=1}^N \pi_i \left[u\left(c_1^i\right) + \beta\left(\sum_{j=1}^N \pi_j u\left(c_2^{i,j}\right)\right)\right] \geq U_0,$$

where U_0 is a given real number.

3.a Argue that there exists a U_0 such that this problem leads to the same solution as the usual agent's welfare maximization problem with the budget constraint

$$\sum_{i=1}^N \pi_i \left[\left(y_1^i - c_1^i\right) + \beta \sum_{j=1}^N \pi_j \left(y_2^j - c_2^{i,j}\right)\right] \geq 0.$$

3.b Describe the characteristics of the Pareto optimal allocation in this case, and explain the main economic intuitions behind its characteristics.

4. *Default models.* We now study a model where, although the agent faces a complete set of securities, she cannot attain the fully optimal allocation described in 3. The reason is an endogenous restriction on the trade of securities she faces, which are generated by the possibility of default. Such trade restrictions are generated by a set of default constraints of the form

$$u\left(c_1^i\right) + \beta\left(\sum_{j=1}^N \pi_j u\left(c_2^{i,j}\right)\right) \geq A^i - P$$

for all $i = 1,\ 2,\ \ldots,\ N$ and

$$u\left(c_2^{i,j}\right) \geq u\left(y^j\right) - P$$

for all i, j; where $A^i := u\left(y^i\right) + \beta \sum_{j=1}^N \pi_j u\left(y^j\right)$, and $P \geq 0$ is a default penalty.

4.a Amend the planner's problem stated above to include the default constraints.

4.b Show that when P is sufficiently large the agent is fully insured.

4.c Assume now, that the planner has limited commitment as well, with default value zero. To the previous problem, we hence add the default constraints:

$$\text{for all } i: \; y_1^i - c_1^i + \beta \sum_{j=1}^N \pi_j \left(y_2^j - c_2^{i,j} \right) \geq 0; \text{and for all } i,j: \; y^i - c^{i,j} \geq 0.$$

Show that when $P = 0$ the only feasible contract coincides with that of autarchy.

6. *Recursive formulation: a heuristic derivation.* As we will see in detail during the lectures, the optimal default problem can be written in recursive form as follows. For any given level of welfare U_0 to be delivered to the agent in period zero, we have

$$V_2(U_0) = \max_{\{c^i, U^i\}_{i=1}^N} \sum_{i=1}^N \pi_i \left[\left(y^i - c^i \right) + \beta V_1(U^i) \right] \text{ s.t.}$$

s.t.

$$u\left(c^i\right) + \beta U^i \geq A^i - P,$$

for all $i = 1, \, 2, \, ..., \, N$ and

$$\sum_{i=1}^N \pi_i \left[u\left(c^i\right) + \beta U^i \right] \geq U_0.$$

Where for all continuation utility values U^i at the beginning of the last period, we have

$$\begin{aligned} V_1(U^i) &= \max_{\{c^j\}_{j=1}^N} \sum_{j=1}^N \pi_j \left(y^j - c^j \right) \\ \text{s.t. for } j = 1, 2, \ldots, N: & \quad u\left(c^j\right) \geq u\left(y^j\right) - P, \\ \text{and} & \\ & \sum_{j=1}^N \pi_j u\left(c_1^j\right) \geq U^i. \qquad (\lambda^i) \end{aligned}$$

In the recursive formulation, for any given level of promised utility U_0, at period zero the planner chooses consumption levels c_1^i and promises future utilities U_1^i for $i = 1, \, 2, \, ..., \, N$. For the future, the planner is not choosing complicated objects such a continuation contracts any more, he is choosing the numbers U^i. Moreover, the planner knows exactly how to evaluate each U^i: $V_1(U^i)$ is the value of a relatively simple constrained maximization problem. The last constraint (with associated multiplier λ^i) is called the 'promise-keeping' constraint, and it requires the contract to deliver the promised level of utility U_1^i to the agent. It plays the role of the *law of motion* for the state variable U.

6.a Derive the properties of the consumption allocation in this case. [Hint: Guess first for which states the default constraints binds with positive associated multiplier.]

6.b Notice that the difference $a^{ij} := c_2^{ij} - y^j$ can be see as the quantity of Arrow securities the agent has to deliver in period 2, state ij. Argue that the optimal insurance contract can be seen as situation where the agent face a full set of securities but her trades are constrained so that she does not get indebted 'too much'. That is, she faces *liquidity* constraints. [Hint: Have a look at Kehoe and Levine (*Econometrica* , 2001)]

7. *The infinite horizon.* One of the key advantages of the recursive formulation is that the complexity of the problem does not change as we increase the time horizon. For example, in an infinite horizon environment, a contract is a very complicated object: $\mathcal{W}_0 = \{c_t(h^t)\}_{t=1}^{\infty}$. It is a huge collection of functions mapping histories of income shocks $h^t = (y_1, ..., y_t)$ into consumption levels c_t.

The recursive formulation is however very similar to that described above for the two period model. When the time horizon is infinite, the value function $V = V_\infty$ becomes time invariant, and solves the following functional equation:

$$V(U) = \max_{\{c^i, U^i\}_{i=1}^N} \sum_{i=1}^N \pi_i \left[\left(y^i - c^i \right) + \beta V(U^i) \right]$$

s.t.: for $i = 1, 2, ..., N$

$$u\left(c^i\right) + \beta U^i \geq A^i - P, \qquad (\pi_i \mu_i)$$

and

$$\sum_{i=1}^N \pi_i \left[u\left(c^i\right) + \beta U^i \right] \geq U. \quad (\lambda)$$

where the autarchy value can be written as $A^i = u\left(y^i\right) + \beta A$ with $A = \sum_i \pi_i A^i$. [Check it!].

7.a Argue that V is a concave function [Hint: By the change in variable $z^i = u\left(c^i\right)$ one obtains a concave objective function (in z^i and U^i) with linear constraints. Now use the argument of Theorem 4.8 in Stokey, Lucas and Prescott (1989). If you assume u^{-1} is bounded you can make the argument rigorous.]

7.b Show that the envelope and first order conditions are

$$\begin{aligned} -V'(U) &= \lambda \\ -V'\left(U^i\right) &= \lambda + \mu^i = \frac{1}{u'\left(c^i\right)} \end{aligned}$$

7.c Show that if $\mu_i > 0$ then $\mu_{i+1} > 0$. Hence, for all U we would have a set of indexes where for all for all $i \leq \underline{i}(U)$ $\lambda = \frac{1}{u'(c^i)} = \frac{1}{u'(\underline{c})}$, and $V'(U) = V'\left(U^i\right)$ (hence $U^i = U$).

And for $i > \underline{i}(U)$ consumption and utility will be higher as $-V'\left(U^i\right) = \frac{1}{u'(c^i)} > \frac{1}{u'(\underline{c})} = -V'(U)$.

7.d Note that when $\mu_i > 0$ we have $u\left(c^i\right) + \beta U^i = A^i - P$. Show that consumption and utilities increase through time: $c^i_{t+1} \geq c_t$ and $U^i_{t+1} \geq U_t$. This is a key property of the optimal contract under limited commitment. Moreover show that when payments are at the level that satisfy the incentive constraint for the maximal income y^N, i.e., when

$$u\left(c^N\right) + \beta U^N = A^N - P,$$

they will not change any more.

7.e *Steady state distribution and two sided lack of commitment.* Now assume there is continuum of identical agents, each agent living infinitely many periods, and facing the same stochastic process of income. By the law of large numbers, in such economy π_i will also denote the fraction of agents getting shock y^i. It is possible to show that, since $\pi_N > 0$, in the long run each agent will eventually get this high income shock, and from there onward she will be fully insured. The steady state distribution of consumption in this economy will hence degenerate at one point: c^N. Argue that the steady state distribution is not unique as it depends on the initial condition U_0.

7.f *Two-sided commitment.* Assume that the planner has limited commitment as well, with zero default value. The optimal contract has the the following set of additional incentive constraints

$$\text{for all } i: \quad y^i - c^i + \beta V(U^i) \geq 0.$$

Derive the optimal contract in this case. Would it be very different from that with one-sided commitment? And the steady state distribution, how would it look like? Finally, if $P = 0$, would the optimal contract be autarchy as in the two periods case? [Hint: Refer to Ljungqvist and Sargent, second edition, Chapter 20; and Kocherlakota's article in the *Review of Economic Studies*, 1996)