The Rational Core of Preference Relations*

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Abstract

We consider revealed preference relations over risky (or uncertain) prospects, and allow them to be nontransitive and/or fail the classical Independence Axiom. We identify the "rational part" of any such preference relation as its largest transitive subrelation that satisfies the Independence Axiom and that exhibits some coherence with the original relation. It is shown that this subrelation, which we call the *rational core* of the given revealed preference, exists in general, and under fairly mild conditions, it is continuous. We obtain various representation theorems for the rational core, and decompose it into other core concepts for preferences. These theoretical results are applied to compute the rational cores of a number of well-known preference models (such as Fishburn's SSB model, justifiable preferences, and variational and multiplier modes of rationalizable preferences). As for applications, we use the rational core operator to develop a theory of risk aversion for nontransitive nonexpected utility models (which may not even be complete). Finally, we show that, under a basic monotonicity hypothesis, the Preference Reversal Phenomenon cannot arise from the rational core of one's preferences.

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1 Introduction

The revealed preference relation of a decision maker over a given set of risky prospects is deduced from her choices.[1] To wit, if the agent is observed to choose a lottery p from a given feasible set, then she is declared to prefer p over all lotteries in that set. If the choices of the agent across all choice problems are "consistent" – this is captured in decision theory by a variety of postulates, such as the axioms of revealed preference and the Independence Axiom – then the resulting revealed preference relation satisfies the basic rationality tenets of transitivity and affinity.

However, it is unrealistic to presume that every choice problem is equally revealing. Depending on the context, some choices may be "easy," even "trivial," for an agent, while others may be "hard" enough that she may feel justifiably insecure about them. (For instance, most agents would choose the sure lottery that pays them \$10 over that pays them \$5 "easily," while they may find ranking two complicated lotteries with large supports "difficult.") It is only reasonable that such (subjectively) "hard" choice problems may cause the choices of the agent fail the strict requirements of rationality. This, in turn, results in the preference relation revealed through such choices fail properties like transitivity and affinity. Indeed, the literature provides ample instances of this situation in a variety of contexts. The experimental demonstrations of nontransitivity of revealed preferences in the context of risk, for instance, go back to Tversky (1969), which is to cite but one reference from a rather large literature. Besides, numerous explanations and models that accommodate nontransitivity of preferences are offered in the literature, including regret theory (Loomes and Sugden (1982)), nontransitive indifference and similarity (Luce (1956), Fishburn (1970), and Rubinstein (1988)), framing effects (Kahneman and Tversky (1979) and Salant and Rubinstein (2008)), and multi-agent decision-making (Hara, Ok and Riella (2018)), among others. Similarly, the famous Allais paradox, along with the certainty and common ratio effects, demonstrates how the revealed risk preferences of an agent may easily fail the so-called Independence Axiom.

These considerations prompt the following query: Given a (reflexive) revealed preference relation $\mathbf{R}$ on a set X of lotteries (which may fail transitivity and/or the Independence Axiom), is there a reasonable way of identifying the part of $\mathbf{R}$ that corresponds to "sure" rankings of the individual? In other words, is there a subrelation $\mathbf{S}$ of $\mathbf{R}$ such that when a lottery p is ranked over q with respect to $\mathbf{S}$, we understand that the agent is perfectly "confident" about this ranking? As this subrelation is unobservable (while $\mathbf{R}$ is), one needs to approach this query axiomatically. Insofar as the "easy" decisions of the agent would not lead to inconsistent choices, it is natural to require $\mathbf{S}$ to be transitive and satisfy the Independence Axiom, thereby not falling

[1]For concreteness, our discussion in this section is couched only in terms of preferences over lotteries. The bulk of the paper, however, applies to any context in which convex combinations of choice objects are themselves choice objects. In fact, Section 6 is devoted to preferences over Anscombe-Aumann acts.

prey to the Allais paradox and its derivatives. (It is in this sense that $\mathbf{S}$ is a rational part of $\mathbf{R}$.) Moreover, it stands to reason that $\mathbf{R}$ should act in coherence with $\mathbf{S}$. To clarify, suppose the agent declares that $p \mathbf{R} q$ and $q \mathbf{S} r$ (or $p \mathbf{S} q$ and $q \mathbf{R} r$) for some lotteries p, q and r. According to our interpretation, this says that the agent declares p superior to q (although she may not be completely confident in this judgement) while she is sure that q is better for her than r. It then seems reasonable that the agent would prefer p over r, albeit, she may be insecure about this decision (that is, $p \mathbf{R} r$ holds, but not necessarily $p \mathbf{S} r$). Consequently, in addition to transitivity and affinity, it makes sense to require $\mathbf{S}$ to satisfy the following property:

$$p \mathbf{R} q \mathbf{S} r \quad \text{or} \quad p \mathbf{S} q \mathbf{R} r \quad \text{implies} \quad p \mathbf{R} r$$

for all p, q and r in X. When this property holds, we say that $\mathbf{R}$ is *transitive with respect to* $\mathbf{S}$. In turn, we define the *rational core* of $\mathbf{R}$ – we denote this as $\mathrm{core}(\mathbf{R})$ – as the largest transitive subrelation of $\mathbf{R}$ that satisfies the Independence Axiom and with respect to which $\mathbf{R}$ is transitive.[2]

The notion of rational core is naturally motivated from the viewpoint of making welfare evaluations for individuals whose revealed preferences $\mathbf{R}$ over lotteries are either nontransitive or nonaffine (in the sense of failing the Independence Axiom). In this case, using $\mathbf{R}$ alone, we may at times not even be able to determine what a best option is for the individual in question in a finite menu of choices. Moreover, the agent may have declared that p is strictly better than q (with respect to $\mathbf{R}$), and yet we observe her choose $\frac{1}{2}p + \frac{1}{2}q$ over p. It is quite possible that this person may in time realize the incoherence of her choices, and change the latter one the next time around. But, of course, this reasoning is purely hypothetical, there is no way of using it to make a welfare evaluation on the part of the agent. In short, when $\mathbf{R}$ is nontransitive and nonaffine, one is not able to conclude comfortably that "an agent would be better off if she is given p over q" just because it is observed that she chose p over q at some point (that is, $p \mathbf{R} q$). But if $\mathrm{core}(\mathbf{R})$ applies here, that is, $p \,\mathrm{core}(\mathbf{R})\, q$, one would be far more confident in the validity of this statement. After all, $p \,\mathrm{core}(\mathbf{R})\, q$ is interpreted as saying that the agent "surely" prefers p over q; this sort of a ranking of p and q may never appear as a part of cyclic choice and/or violation of the Independence Axiom, and as such, there does not seem to be any reason for the agent to have ex-post qualms about the superiority of p over q. In short, $\mathrm{core}(\mathbf{R})$ may be used as a partial, but convincing, criterion in making welfare judgements on behalf of an economic agent, in precisely the same way the Pareto criterion is a partial, but convincing, rule of social welfare evaluation. (In fact, we will show in the body of the paper that there are formal similarities between these two preorders.) In passing, we note that this motivation also explains why we define $\mathrm{core}(\mathbf{R})$ as the *largest* transitive and affine subrelation of $\mathbf{R}$ with respect to which $\mathbf{R}$ is transitive.

[2] One may wish to dispense with this transitivity property, and focus instead on the largest transitive subrelation of $\mathbf{R}$ that satisfies the Independence Axiom. Unfortunately, even when $\mathbf{R}$ is complete and satisfies the Independence Axiom, such a subrelation of $\mathbf{R}$ need not exist.

The primary goal of the present paper is to investigate the internal structure of the concept of rational core, identify its basic properties, and compute it for various models of preferences that fail to satisfy either the transitivity property or the Independence Axiom. After introducing some basic nomenclature in Section 2, we begin our exposition by revisiting two different notions of "core" that have already received some attention in the literature. The first of these aims to identify the largest rational part of a given (revealed) preference relation $\mathbf{R}$ insofar as rationality is captured by transitivity alone. Nishimura (2018) has recently attacked this problem axiomatically by considering a class of abstract operators that map any given binary relation $\mathbf{R}$ into a transitive subrelation of it. All of Nishimura's operators agree when $\mathbf{R}$ is complete, and map $\mathbf{R}$ to a particularly interesting subrelation. In Section 3.1, we note that this subrelation exists even when $\mathbf{R}$ is not complete, and it coincides with the largest transitive subrelation of $\mathbf{R}$ with respect to which $\mathbf{R}$ is transitive.[3] Following Nishimura (2018), we call this subrelation the "transitive core" of $\mathbf{R}$, and denote it as $\mathsf{T}(\mathbf{R})$. The second core concept we borrow from the literature is the largest part of $\mathbf{R}$ that satisfies the Independence Axiom. In Section 3.2, we introduce this part of $\mathbf{R}$ as its *affine core*, and denote it by $\mathsf{A}(\mathbf{R})$. This notion has made frequent appearance in the recent literature on nonexpected utility theory under risk and uncertainty – it was introduced first by Ghirardato, Maccheroni and Marinacci (2004) in the context of uncertainty – but we develop it here in a more general setting. In particular, we provide a general characterization for it, and show that it always exists, and it is transitive if so is $\mathbf{R}$.

The first main result of the paper (Theorem 3.4) shows that the rational core of $\mathbf{R}$ decomposes into its affine and transitive cores in a pleasant way:

$$\text{core}(\mathbf{R}) = \mathsf{A}(\mathsf{T}(\mathbf{R})). \tag{1}$$

Thus, for transitive $\mathbf{R}$, the rational core and affine core coincide. Moreover, if $\mathbf{R}$ satisfies a natural strengthening of the Independence Axiom, then its rational and transitive cores coincide (Proposition 3.8).[4] Furthermore, our second main result (Theorem 3.6) uses (1) to show that the rational core of $\mathbf{R}$ is a continuous preorder, provided that $\mathbf{R}$ is continuous and satisfies a mild monotonicity condition. This allows us to utilize some well-known representation theorems of decision theory to obtain an "expected multi-utility" representation for core($\mathbf{R}$) in the context of risk (and an "expected multi-prior" representation for it in the context of uncertainty). It is in this sense that core($\mathbf{R}$) is a unanimity ordering, and hence shows formal similarities to the classical Pareto ordering.

We use these general results to compute the rational cores of a variety of preference models. In the context of risk, for instance, we compute the rational core

[3] Both in this definition and that of the rational core, the requirement of transitivity of the subrelation is redundant, but this is inconsequential as long as our heuristic discussion is concerned.

[4] It turns out that the order of applying the affine and transitive cores in the decomposition (1) is important. In Section 6.2, we provide a nontrivial example that shows that core($\mathbf{R}$) may be a proper subset of $\mathsf{T}(\mathsf{A}(\mathbf{R}))$.

for the so-called rationalizable preferences (which rank one lottery p over another iff the expected utility of p exceeds that of the latter with respect to at least one Bernoulli utility function in a given collection of utilities) as well as for Fishburn's skew-symmetric bilinear (SSB) model (which relaxes both transitivity and affinity in order to account for the famous Preference Reversal Phenomenon). In the context of uncertainty (with finitely many states), we carry out such computations for the justifiable preferences (of Lehrer and Teper (2011)), and for two interesting generalizations of that model, namely, justifiable variational preferences and justifiable multiplier preferences. In all of these models, rational core turns out to have a rather natural structure (enjoying a particularly simple type of expected multi-utility or multi-prior representation).

Finally, in the context of risk, we provide two applications to applied decision theory. First, we use the rational core operator to suggest a "new" definition of risk aversion for a preference relation $\mathbf{R}$ on a set of monetary lotteries. The standard definition would ask for the consistency of $\mathbf{R}$ with second order stochastic dominance. By contrast, we suggest being a bit more demanding than this, and say that the agent is "risk averse" provided that she is "unconflicted" in declaring that mean-preserving spreads are not desirable. The latter statement is conceptually appealing, but it is not formal. Nevertheless, we can readily use the notion of rational core of $\mathbf{R}$ to provide the required formalization, thereby declaring $\mathbf{R}$ as *risk averse* when core($\mathbf{R}$) is consistent with second order stochastic dominance. (Of course, when $\mathbf{R}$ satisfies the standard von Neumann-Morgenstern axioms, the two definitions coincide.) In fact, to apply this concept in practice, one does not even need to compute the rational core. It is quite easy to show that $\mathbf{R}$ is risk averse in this sense if, and only if, it is transitive with respect to second order stochastic dominance. Moreover, the rational core of such $\mathbf{R}$ has a particularly convenient integral representation (Section 5.2).

In our second application, we revisit the Preference Reversal (PR) Phenomenon (Grether and Plott (1979)), and ask if it is possible that this phenomenon may occur with respect to the rational core of a (potentially nontransitive and nonaffine) preference relation. We find that this is not case, provided that the preferences exhibit a mild degree of monotonicity. On one hand, this validates our interpretation of the rational core, and on the other, it shows that the preference reversal phenomenon is likely to be due to comparisons that an individual is somewhat "unsure" of making.

We conclude the paper with a few concluding remarks and an Appendix that contains the proofs of the results that are omitted in the body of the text.

2 Nomenclature

As we deal with somewhat nonstandard preference relations in this paper, which need not be either complete or transitive, we introduce here some terminology that pertains to the general theory of binary relations. We will use this nomenclature throughout the paper.

Binary Relations. Let X be a nonempty set. By a *binary relation* $\mathbf{R}$ on X, we mean any nonempty subset of $X \times X$, but, as usual, write $x \mathbf{R} y$ instead of $(x, y) \in \mathbf{R}$. Moreover, for any two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we write $x \mathbf{R} y \mathbf{S} z$ to mean $x \mathbf{R} y$ and $y \mathbf{S} z$, and so on. For any element x of X, the *upper* and *lower sets* of x with respect to $\mathbf{R}$ are defined as $x^{\uparrow,\mathbf{R}} := \{y \in X : y \mathbf{R} x\}$ and $x^{\downarrow,\mathbf{R}} := \{y \in X : x \mathbf{R} y\}$, respectively. When the relation $\mathbf{R}$ is understood from the context, we may simply write $x^{\uparrow}$ for $x^{\uparrow,\mathbf{R}}$, and $x^{\downarrow}$ for $x^{\downarrow,\mathbf{R}}$.

The *asymmetric part* of a binary relation $\mathbf{R}$ on X is defined as the binary relation $\mathbf{R}^{>}$ on X with $x \mathbf{R}^{>} y$ iff $x \mathbf{R} y$ and not $y \mathbf{R} x$, and the *symmetric part* of $\mathbf{R}$ is defined as $\mathbf{R}^{=} := \mathbf{R} \backslash \mathbf{R}^{>}$. The *composition* of two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X is defined as $\mathbf{R} \circ \mathbf{S} := \{(x, y) \in X \times X : x \mathbf{R} z \mathbf{S} y \text{ for some } z \in X\}$. We say that $\mathbf{S}$ is a *subrelation* of $\mathbf{R}$, and that $\mathbf{R}$ is a *superrelation* of $\mathbf{S}$, if $\mathbf{S} \subseteq \mathbf{R}$.

We denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X = \{(x, x) : x \in X\}$. A binary relation $\mathbf{R}$ on X is said to be *reflexive* if $\triangle_X \subseteq \mathbf{R}$, *antisymmetric* if $\mathbf{R}^{=} \subseteq \triangle_X$, *transitive* if $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$, and *complete* if either $x \mathbf{R} y$ or $y \mathbf{R} x$ holds for every x and y in X. If $\mathbf{R}$ is reflexive and transitive, we refer to it as a *preorder* on X. (Throughout the paper, a generic preorder is denoted as $\succsim$, and as usual, the asymmetric part of $\succsim$ is denoted as $\succ$.) Finally, an antisymmetric preorder on X is said to be a *partial order* on X.

All preference relations that we consider in this paper are reflexive. As reflexivity is a conceptually trivial requirement, this does not restrict the inherent content of our findings.

Continuity of a Binary Relation. Let X be a topological space and $\mathbf{R}$ a binary relation on X. There are various ways in which we can think of $\mathbf{R}$ as continuous. In particular, $\mathbf{R}$ is called **closed-continuous** if it is a closed subset of $X \times X$ (relative to the product topology), and it is called **open-continuous** if $\mathbf{R}^{>}$ is an open subset of $X \times X$. When $\mathbf{R}$ is closed-continuous, so is $\mathbf{R}^{=}$, but easy examples show that $\mathbf{R}$ need not be open-continuous. In fact, a famous result of Schmeidler (1971) says that if X is connected and $\mathbf{R}$ is a preorder on X with $\mathbf{R}^{>} \neq \varnothing$, then $\mathbf{R}$ is both closed- and open-continuous only if it is complete.[5] In this paper, we adopt closed-continuity as the primary notion of continuity, and in what follows, refer to a closed-continuous binary relation simply as **continuous**.

3 Concepts of Core for Binary Relations

3.1 The Transitive Core

Transitivity with Respect to another Binary Relation. Our main focus in this paper is on reflexive, but not necessarily transitive, binary relations. A useful concept in

[5] Which of these two continuity notions one adopts may have significant consequences in terms of the representation of a given preorder; see, for instance, Evren (2014).

the analysis of such binary relations is the notion of *transitivity with respect to a binary relation*. Put precisely, given any two reflexive binary relations $\mathbf{R}$ and $\mathbf{S}$ on a nonempty set X, we say that $\mathbf{R}$ is **S-transitive** if $\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{R}$ and $\mathbf{S} \circ \mathbf{R} \subseteq \mathbf{R}$, which means that either $x \mathbf{R} y \mathbf{S} z$ or $x \mathbf{S} y \mathbf{R} z$ implies $x \mathbf{R} z$ for any x, y and z in X. This notion generalizes the classical concept of transitivity, for, obviously, $\mathbf{R}$ is $\mathbf{R}$-transitive iff it is transitive. More generally, if $\mathbf{R}$ is $\mathbf{S}$-transitive and it is a subrelation of $\mathbf{S}$, then it must be transitive (but not conversely).

The Transitive Core of a Binary Relation. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty set X. By the **transitive core** of $\mathbf{R}$, we mean the largest binary relation $\mathbf{S}$ contained in $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, and denote this subrelation as $\mathsf{T}(\mathbf{R})$. Put explicitly, $\mathsf{T}(\mathbf{R})$ is the subrelation of $\mathbf{R}$ such that (i) $\mathbf{R} \circ \mathsf{T}(\mathbf{R}) \subseteq \mathbf{R}$ and $\mathsf{T}(\mathbf{R}) \circ \mathbf{R} \subseteq \mathbf{R}$, and (ii) if $\mathbf{S}$ is a subrelation of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, then $\mathbf{S} \subseteq \mathsf{T}(\mathbf{R})$. It is plain that $\mathbf{R}$ is transitive iff $\mathbf{R} = \mathsf{T}(\mathbf{R})$.

It is not self-evident if the transitive core of any given reflexive binary relation exists. Similarly, it is not obvious if, when it exists, the transitive core of such a relation is itself transitive. The following result settles both of these issues at one stroke, and provides a concrete characterization of the transitive core of any reflexive binary relation.

Proposition 3.1. *Let $\mathbf{R}$ be a reflexive binary relation on a nonempty set X. Then, $\mathsf{T}(\mathbf{R})$ exists, and it satisfies*

$$x \,\mathsf{T}(\mathbf{R})\, y \quad \textit{iff} \quad x^{\uparrow} \subseteq y^{\uparrow} \text{ and } y^{\downarrow} \subseteq x^{\downarrow}$$

for every x and y in X.[6] *In particular, $\mathsf{T}(\mathbf{R})$ is a preorder on X.*

Proof. Define the binary relation $\mathbf{S}$ on X as $x \mathbf{S} y$ iff $x^{\uparrow} \subseteq y^{\uparrow}$ and $y^{\downarrow} \subseteq x^{\downarrow}$. Since $\mathbf{R}$ is reflexive, $x \in x^{\uparrow}$, and it follows that $x \mathbf{S} y$ implies $x \mathbf{R} y$, for every $x, y \in X$, that is, $\mathbf{S}$ is a subrelation of $\mathbf{R}$. It is also immediate from the definition of $\mathbf{S}$ that $\mathbf{R}$ is $\mathbf{S}$-transitive. Now assume that $\mathbf{S}'$ is another subrelation of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}'$-transitive. Take any x and y in X with $x \mathbf{S}' y$. Clearly, if $z \mathbf{R} x$, then $z \mathbf{R} y$ by $\mathbf{S}'$-transitivity of $\mathbf{R}$, and it follows that $x^{\uparrow} \subseteq y^{\uparrow}$. Similarly, $y^{\downarrow} \subseteq x^{\downarrow}$ holds as well, and it follows that $x \mathbf{S} y$. Conclusion: $\mathbf{S}' \subseteq \mathbf{S}$. We conclude that $\mathbf{S}$ is the largest subrelation of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, that is, $\mathbf{S} = \mathsf{T}(\mathbf{R})$. As the second part of the proposition is a straightforward consequence of its first part, we are done. ■

Nishimura (2018) has recently introduced an axiomatic definition of the transitive core as an operator from the collection of all reflexive binary relations on a given nonempty set X to that of preorders on X, and has shown that any such operator which is consistent with his axioms is uniquely identified on the collection of all *complete* binary relations on X. For any complete binary relation $\mathbf{R}$ on X, the value

[6] Here, of course, the upper and lower sets of x and y are defined with respect to $\mathbf{R}$.

of this operator at $\mathbf{R}$ is precisely the binary operation $\mathbf{S}$ defined in the proof above. Thus, our definition of the transitive core is in full accord with the axiomatic definition of Nishimura (2018), at least for complete binary relations.

The following corollary of Proposition 3.1 simplifies the definition of the transitive core. It shows that the requirement that "$\mathsf{T}(\mathbf{R})$ be a subrelation of $\mathbf{R}$" can be replaced with "$\mathsf{T}(\mathbf{R})$ being a reflexive relation" in that definition.

Corollary 3.2. *Let $\mathbf{R}$ and $\mathbf{S}$ be two reflexive binary relations on a nonempty set X. If $\mathbf{R}$ is $\mathbf{S}$-transitive, then $\mathbf{S}$ must be a subrelation of $\mathsf{T}(\mathbf{R})$, and hence of $\mathbf{R}$. Thus, $\mathsf{T}(\mathbf{R})$ is the largest reflexive binary relation on X with respect to which $\mathbf{R}$ is transitive.*

Proof. Suppose that $\mathbf{R}$ is $\mathbf{S}$-transitive, and take any $x, y \in X$ with $x \mathbf{S} y$. If $z \in x^{\uparrow}$, then $z \mathbf{R} x \mathbf{S} y$, so $z \mathbf{R} y$, that is, $z \in y^{\uparrow}$. Thus, $x^{\uparrow} \subseteq y^{\uparrow}$, and we can similarly show that $y^{\downarrow} \subseteq x^{\downarrow}$. By Proposition 3.1, therefore, $x \mathsf{T}(\mathbf{R}) y$, establishing our claim. ∎

Remark 3.1. In mathematical order theory, especially in the context of interval orders, the relation $\mathbf{S}$ that we used in the proof of Proposition 3.1 is sometimes called the *trace* of $\mathbf{R}$ (cf. Doignon et al. (1986)). And indeed, in that literature, the trace of $\mathbf{R}$ being the largest reflexive binary relation on X with respect to which $\mathbf{R}$ is transitive appears to be a folk result.

3.2 The Affine Core

We concentrate in this paper on those binary relations that are defined on a convex subset of a linear space. For such relations, there is a natural formulation of "linearity" which leads to an alternative notion of core.

Affine Binary Relations. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. We say that $\mathbf{R}$ is **affine** if

$$x \mathbf{R} y \quad \text{implies} \quad \lambda x + (1-\lambda)z \mathbf{R} \lambda y + (1-\lambda)z$$

for every x, y and z in X and every $0 < \lambda \leq 1$. We note that if $\mathbf{R}$ is affine, so is $\mathbf{R}^{=}$, but $\mathbf{R}^{>}$ need not be affine (even when $\mathbf{R}$ is a preorder).[7] It is also evident that if $\mathbf{R}$ is a convex subset of $X \times X$ (with coordinatewise defined addition and scalar multiplication operations), then $\mathbf{R}$ is affine. The converse of this holds when $\mathbf{R}$ is transitive, so we conclude that affinity of a preorder on X is the same thing as this preorder being a convex subset of $X \times X$.

The notion of affinity that we define here in the abstract is, of course, widely used in the theory of decision making under risk and uncertainty. Indeed, affinity of a preference relation defined on a given (convex) set of lotteries (or Anscombe-Aumann

[7] To illustrate, take any positive integer $n \geq 2$, and let X stand for the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$. Consider the preorder $\succsim$ on X defined by $x \succsim y$ iff $\min\{i : x_i > 0\} \geq \min\{i : y_i > 0\}$. Then, $\succsim$ is affine, but $\succ$ is not.

acts) means simply that this relation satisfies the von Neumann-Morgenstern Independence Axiom (or its pointwise extension to the Anscombe-Aumann framework). As such, it makes due sense to consider this property as an essential rationality criterion. Similarly, in the context of social choice theory where we would consider $\mathbf{R}$ as representing the aggregated preferences of a group of individuals over, say, a consumption space, the affinity of $\mathbf{R}$ would correspond to a natural additivity property.

The Affine Core of a Binary Relation. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. By the **affine core** of $\mathbf{R}$, we mean the largest affine subrelation of $\mathbf{R}$, and denote this subrelation as $\mathsf{A}(\mathbf{R})$. It is plain that $\mathbf{R}$ is affine iff $\mathbf{R} = \mathsf{A}(\mathbf{R})$. It is also useful to note that, unlike the transitive core, the affine core is a monotonic operator, that is, if $\mathbf{S}$ is a reflexive binary relation on X with $\mathbf{S} \subseteq \mathbf{R}$, then $\mathsf{A}(\mathbf{S}) \subseteq \mathsf{A}(\mathbf{R})$. Besides, we will shortly prove that the affine core of any reflexive binary relation exists. Consequently, we may use the monotonicity of A to conclude that

$$\mathsf{A}\left(\bigcap \mathcal{R}\right) = \bigcap_{\mathbf{R}\in\mathcal{R}} \mathsf{A}(\mathbf{R}) \tag{2}$$

for any nonempty collection $\mathcal{R}$ of reflexive binary relations on X.[8]

The following observation shows that the affine core of $\mathbf{R}$ is sure to exist, and it provides a concrete characterization for it.

Proposition 3.3. *Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. Then, $\mathsf{A}(\mathbf{R})$ exists, and it satisfies*

$$x \ \mathsf{A}(\mathbf{R}) \ y \quad \textit{iff} \quad \lambda x + (1-\lambda)z \ \mathbf{R} \ \lambda y + (1-\lambda)z \ \textit{for every } z \in X \textit{ and } \lambda \in (0,1]$$

for any x and y in X. In particular, $\mathsf{A}(\mathbf{R})$ is a preorder if so is $\mathbf{R}$.

Proof. Define the binary relation $\mathbf{S}$ on X as $x \ \mathbf{S} \ y$ iff $\lambda x + (1-\lambda)z \ \mathbf{R} \ \lambda y + (1-\lambda)z$ for every $z \in X$ and $\lambda \in (0,1]$. Obviously, $\mathbf{S}$ is a reflexive subrelation of $\mathbf{R}$. It is also plain that if $\mathbf{R}'$ is an affine subrelation of $\mathbf{R}$, then $\mathbf{R}' \subseteq \mathbf{S}$. It remains to show that $\mathbf{S}$ is affine. To this end, take any x and y in X with $x \ \mathbf{S} \ y$, and fix an arbitrary (z, λ) in $X \times (0,1]$. We wish to show that $\lambda x + (1-\lambda)z \ \mathbf{S} \ \lambda y + (1-\lambda)z$, that is,

$$\alpha(\lambda x + (1-\lambda)z) + (1-\alpha)w \ \mathbf{R} \ \alpha(\lambda y + (1-\lambda)z) + (1-\alpha)w \tag{3}$$

for every $w \in X$ and $\alpha \in (0,1]$. Since our claim is trivially true for $\lambda = 1$, we assume $\lambda < 1$. Then, we may set

$$v_{\alpha,w} := \frac{\alpha(1-\lambda)}{1-\alpha\lambda} z + \left(1 - \frac{\alpha(1-\lambda)}{1-\alpha\lambda}\right) w,$$

[8] Since $\bigcap_{\mathbf{R}\in\mathcal{R}} \mathsf{A}(\mathbf{R})$ is an affine subrelation of $\bigcap \mathcal{R}$, the $\supseteq$ part of (2) is immediate. Conversely, as $\bigcap \mathcal{R} \subseteq \mathbf{R}$ for each $\mathbf{R} \in \mathcal{R}$, monotonicity of A entails that $\mathsf{A}(\bigcap \mathcal{R}) \subseteq \mathsf{A}(\mathbf{R})$ for each $\mathbf{R} \in \mathcal{R}$.

and write (3) as

$$\alpha\lambda x + (1 - \alpha\lambda)v_{\alpha,w} \mathbf{R} \alpha\lambda y + (1 - \alpha\lambda)v_{\alpha,w} \tag{4}$$

for every $w \in X$ and $\alpha \in (0, 1]$. Therefore, as $x \mathbf{S} y$, it follows from the definition of $\mathbf{S}$ that (4), and hence (3), holds for every $w \in X$ and $\alpha \in (0, 1]$. In view of the arbitrariness of (z, λ), this shows that $\mathbf{S}$ is affine. As the second part of the proposition is a straightforward consequence of its first part, we are done. ■

While our formulation here is more general, we should note that the notion of affine core was already studied in the literature on decision making under risk and uncertainty. In that literature, this concept is sometimes referred to as the "expected utility core," and is often defined through the characterization we obtained in Proposition 3.3.

Remark 3.2. In the context of uncertainty, the notion of affine core was first introduced by Ghirardato, Maccheroni and Marinacci (2004) who referred to it as the "revealed unambiguous preference." In this context, it was further explored in Cerreia-Vioglio et al. (2011), *inter alia*. In the context of risk, this notion was introduced in Cerreia-Vioglio (2009), and then further explored in Cerreia-Vioglio, Dillenberger and Ortoleva (2015), and Cerreia-Vioglio, Maccheroni and Marinacci (2017).

3.3 The Rational Core

As we have discussed at some length in the Introduction, our main interest in this paper is to identify the largest part of a given preference relation $\mathbf{R}$ on, say, a space of lotteries, which arises from the "sure" comparisons of the individual. As such, we expect the collection of all such comparisons to form a "rational" preference relation (which is, however, likely to be incomplete). So, in effect, we look for the largest "rational part" of $\mathbf{R}$. Now, two obvious requirements of being a "rational part" of $\mathbf{R}$ are being a transitive and affine subrelation of $\mathbf{R}$. And there are indeed such relations; for instance, the reflexive part of $\mathbf{R}$ is obviously a transitive and affine subrelation of $\mathbf{R}$. However, it is not difficult to give examples to show that the *largest* such subrelation of $\mathbf{R}$ need not exist. And even when it exists, this subrelation may still lack some desirable properties. After all, in general, this relation does not show any coherence with the revealed preference relation $\mathbf{R}$ (other than being a subrelation of it). By contrast, it is quite reasonable that the revealed preference relation $\mathbf{R}$ would recognize the rationality of its rational part and act coherently with it, at least in the sense of being transitive relative to it. Indeed, when we think of a subrelation $\mathbf{S}$ of the revealed preference relation $\mathbf{R}$ as a "rational part" of $\mathbf{R}$, the statement $x \mathbf{S} y$ is interpreted as saying that the agent prefers x over y in complete confidence, and the statement $y \mathbf{R} z$ as saying that it is revealed (to an outside observer) that she likes y better than z, even though she may well be somewhat insecure about this decision. But then it stands to reason that the "obvious" superiority of x over y for this agent would entail that she would like x better than z, but, of course, it is possible that she may not be secure in this judgement either.

This discussion leads one to consider the largest (transitive and) affine subrelation of $\mathbf{R}$ with respect to which $\mathbf{R}$ is transitive as a promising formalization of the intuitive notion of the "largest rational part" of $\mathbf{R}$. We shall see shortly that this formulation is free of the aforementioned existence issue.

The Rational Core of a Binary Relation. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. By the **rational core** of $\mathbf{R}$, we mean the largest affine subrelation $\mathbf{S}$ of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, and denote this subrelation by $\text{core}(\mathbf{R})$. It is plain that $\mathbf{R}$ is an affine preorder iff $\mathbf{R} = \text{core}(\mathbf{R})$. Moreover, $\text{core}(\mathbf{R})$ reduces to the affine core of $\mathbf{R}$ whenever $\mathbf{R}$ is a preorder.

The following result shows that the rational core of $\mathbf{R}$ is sure to exist, and it is a preorder on X. In addition, it clarifies exactly how $\text{core}(\mathbf{R})$ can be computed by using the two core concepts we discussed above.

Theorem 3.4. *Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. Then,*

$$\text{core}(\mathbf{R}) = \mathsf{A}(\mathsf{T}(\mathbf{R})). \tag{5}$$

In particular, $\text{core}(\mathbf{R})$ *is an affine preorder on X.*

Proof. To prove (5), define $\succsim := \mathsf{A}(\mathsf{T}(\mathbf{R}))$. Obviously, $\succsim$ is an affine subrelation of $\mathsf{T}(\mathbf{R})$, and hence of $\mathbf{R}$. Moreover, $\succsim \subseteq \mathsf{T}(\mathbf{R})$, and it follows from this that $\mathbf{R}$ is $\succsim$-transitive. On the other hand, if $\mathbf{S}$ is any affine subrelation of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, then, $\mathbf{S} \subseteq \mathsf{T}(\mathbf{R})$, and hence $\mathbf{S} = \mathsf{A}(\mathbf{S}) \subseteq \mathsf{A}(\mathsf{T}(\mathbf{R})) = \succsim$. Conclusion: $\succsim$ is the largest affine subrelation of $\mathbf{R}$ such that $\mathbf{R}$ is $\succsim$-transitive. To prove our second assertion, note that $\mathsf{T}(\mathbf{R})$ is a preorder on X (Proposition 3.1), so Proposition 3.3 entails that $\succsim$ is itself a preorder on X. ■

A natural question concerning the representation of the rational core in terms of the affine and transitive cores as in Theorem 3.4 concerns the order of application of these cores. Indeed, it is not clear if, in general, we have

$$\mathsf{A}(\mathsf{T}(\mathbf{R})) = \mathsf{T}(\mathsf{A}(\mathbf{R})) \tag{6}$$

for a binary relation $\mathbf{R}$ as in Theorem 3.4. It turns out that this equation is not true universally, but the left-hand side is always contained in the right-hand side.

Proposition 3.5. *Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. Then,*

$$\text{core}(\mathbf{R}) \subseteq \mathsf{T}(\mathsf{A}(\mathbf{R})).$$

Proof. As $\text{core}(\mathbf{R})$ is an affine subrelation of $\mathbf{R}$, we have $\text{core}(\mathbf{R}) \subseteq \mathsf{A}(\mathbf{R})$. Consequently, by definition of the transitive core, our assertion will be proved if we can show that $\mathsf{A}(\mathbf{R})$ is core($\mathbf{R}$)-transitive. To this end, take any x, y and z in X, and assume that x core($\mathbf{R}$) y $\mathsf{A}(\mathbf{R})$ z. Then, for any $w \in X$ and $\lambda \in (0,1]$, the affinity of core($\mathbf{R}$) and $\mathsf{A}(\mathbf{R})$ implies

$$\lambda x + (1-\lambda)w \text{ core}(\mathbf{R}) \ \lambda y + (1-\lambda)w \ \mathsf{A}(\mathbf{R}) \ \lambda z + (1-\lambda)w,$$

so, as $\mathsf{A}(\mathbf{R}) \subseteq \mathbf{R}$ and $\mathbf{R}$ is core($\mathbf{R}$)-transitive, we find $\lambda x + (1-\lambda)w \ \mathbf{R} \ \lambda z + (1-\lambda)w$, thereby establishing that $x \ \mathsf{A}(\mathbf{R}) \ z$ (Proposition 3.3). As one can similarly show that $x \ \mathsf{A}(\mathbf{R}) \ y$ core($\mathbf{R}$) z implies $x \ \mathsf{A}(\mathbf{R}) \ z$, we are done. ■

The fact that the containment in Proposition 3.5 may hold properly will be proved in Section 6.2.

Continuity of the Rational Core of a Binary Relation. In our applications to decision theory, it will be important to know when the rational core of a continuous binary relation defined on a nonempty convex subset of a topological linear space is itself continuous. Unfortunately, this is not true universally, precisely because the transitive core of such a relation need not be continuous.[9] However, the difficulty disappears under a relatively weak monotonicity condition.

Let $\mathbf{R}$ be a reflexive binary relation on a topological space X. We say that $\mathbf{R}$ is **locally non-saturated** if for any distinct x and y in X with $x \ \mathbf{R} \ y$, the following hold:

(i) For every open neighborhood O of x in X, there are an $x_O \in O$ and an open neighborhood U of y in X such that $x_O \ \mathbf{R} \ z$ for every $z \in U$;

(ii) For every open neighborhood U of y in X, there are a $y_U \in U$ and an open neighborhood O of x in X such that $z \ \mathbf{R} \ y_U$ for every $z \in O$.

In words, this property means that if $x \ \mathbf{R} \ y$, then we can perturb x marginally to obtain an alternative better than all the alternatives in a neighborhood of y, and similarly, we can perturb y marginally to obtain an alternative worse than all the alternatives in a neighborhood of x. As we shall see, many binary relations encountered in decision theory satisfy this property.[10] Furthermore, as we prove in the Appendix, this property warrants that the transitive core of a reflexive binary relation inherits the continuity of that relation. But we also prove in the Appendix that the affine core of a continuous binary relation (on a nonempty convex subset of a topological linear space) is automatically continuous. Putting these two observations together, and invoking Theorem 3.4, we obtain the following:

[9] Let $X := \{0,1\} \times \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$, and consider the preorder $\succcurlyeq$ on X defined by $(x_1, x_2) \succcurlyeq (y_1, y_2)$ iff $x_2 \geq y_2$, and put $\mathbf{R} := \succcurlyeq \cup \{((0,0),(0,1)),((1,0),(1,1))\}$. As it is the union of two closed subsets of $X \times X$, $\mathbf{R}$ is a continuous reflexive binary relation on X. But we have $(0, \frac{1}{m}) \ \mathsf{T}(\mathbf{R}) \ (1, \frac{1}{m})$ for every integer $m \geq 2$, while $(0,0)$ and $(1,0)$ are not comparable with respect to $\mathsf{T}(\mathbf{R})$.

[10] As an immediate example, we note that the usual (coordinatewise) ordering of $\mathbb{R}^n$ is certainly locally non-saturated. Similarly, any preorder on a topological space X that admits a continuous and injective utility representation is locally non-saturated.

Theorem 3.6. *Let* $\mathbf{R}$ *be a reflexive binary relation on a nonempty convex subset* X *of a topological linear space. If* $\mathbf{R}$ *is continuous and locally non-saturated, then* core$(\mathbf{R})$ *is continuous.*

This result will streamline some of the computations we will perform in the following sections.

3.4 The Strongly Rational Core

Strongly Affine Binary Relations. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. We say that $\mathbf{R}$ is **strongly affine** provided that

$$x \;\mathbf{R}\; y \quad \text{if and only if} \quad \lambda x + (1-\lambda)z \;\mathbf{R}\; \lambda y + (1-\lambda)z$$

for every x, y and z in X and every $0 < \lambda \leq 1$. Obviously, if $\mathbf{R}$ is strongly affine, it is affine, but the converse of this is not true in general (even when $\mathbf{R}$ is a preorder).[11]

This stronger notion of affinity is also used in the theory of decision making under risk and uncertainty, where it is sometimes referred to as the *Strong Independence Axiom*. Indeed, the normative motivation behind the von Neumann-Morgenstern Independence Axiom as a rationality trait applies to the Strong Independence Axiom without modification. (In that theory, one often posits only the Independence Axiom at the outset, only to derive the Strong Independence Axiom by using some form of continuity.) Consequently, one may wish to require a "rational part" of a preference over lotteries (or acts) to conform with the Strong Independence Axiom (instead of merely the Independence Axiom). This leads to a strengthening of our rational core concept in the following way.

The Strongly Rational Core of a Binary Relation. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a linear space. By the **strongly rational core** of $\mathbf{R}$, we mean the largest strongly affine subrelation $\mathbf{S}$ of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, and denote this subrelation as core$^*(\mathbf{R})$. It is plain that, when it exists, core$^*(\mathbf{R})$ is a subrelation of core$(\mathbf{R})$.

The motivation behind core$^*(\mathbf{R})$ is surely on a par with that of core$(\mathbf{R})$, if not superior. However, mathematically speaking, the strongly rational core operator is not as well-behaved as the rational core. Indeed, it is not even clear if core$^*(\mathbf{R})$ exists for every reflexive binary relation $\mathbf{R}$ on X. Fortunately, in a variety of cases of interest, we get the best of the two worlds, because the strongly rational core and rational core coincide, as our next result demonstrates.

Proposition 3.7. *Let* $\mathbf{R}$ *be a reflexive, continuous, and locally non-saturated binary relation on a nonempty convex subset* X *of a topological linear space.*[12] *Then,* core$^*(\mathbf{R}) =$ core$(\mathbf{R})$.

[11] For an example, see footnote 7.

[12] More generally, it is enough to assume that $\mathbf{R}$ is a reflexive binary relation such that $\mathsf{T}(\mathbf{R})$ is continuous.

In all of the applications we consider in Sections 5 and 6, the strongly rational core and rational core coincide. This is proved either by directly verifying that the rational core is strongly affine or by using Proposition 3.7. Thus, we will henceforth focus only on the rational core of preference relations.

3.5 A "Decomposition" of Rationality

We have motivated the rational core as corresponding to the largest rational part of a (revealed) preference relation $\mathbf{R}$, insofar as rationality is captured by (strong) affinity and transitivity of preferences (and transitivity of $\mathbf{R}$ with respect to this part). From this vantage point, Theorem 3.4 may be seen as "decomposing" the largest rational part of $\mathbf{R}$ into these two rationality traits. Moreover, as we show next, if any one of these traits is already present in one's preferences, the rational core would indeed coincide with the remaining rationality trait.

Proposition 3.8. *Let $\mathbf{R}$ be a reflexive binary relation on a linear space X. If $\mathbf{R}$ is transitive, then* $\text{core}(\mathbf{R}) = \mathsf{A}(\mathbf{R})$, *and if $\mathbf{R}$ is strongly affine, then* $\text{core}(\mathbf{R}) = \mathsf{T}(\mathbf{R})$.

Proof. If $\mathbf{R}$ is transitive, then $\mathbf{R} = \mathsf{T}(\mathbf{R})$, and hence $\text{core}(\mathbf{R}) = \mathsf{A}(\mathbf{R})$ by Theorem 3.4. Next, suppose that $\mathbf{R}$ is strongly affine. We wish to show that $\mathsf{T}(\mathbf{R})$ is affine, which will complete our proof, because Theorem 3.4 would then yield $\text{core}(\mathbf{R}) = \mathsf{A}(\mathsf{T}(\mathbf{R})) = \mathsf{T}(\mathbf{R})$. Take any $x, y, z \in X$ and $0 < \lambda \leq 1$, and assume that $x \ \mathsf{T}(\mathbf{R}) \ y$. Now take any $w \in X$ with $w \ \mathbf{R} \ \lambda x + (1-\lambda)z$, and define $w' := \frac{1}{\lambda}(w - (1-\lambda)z)$. Then, $w = \lambda w' + (1-\lambda)z$, so by strong affinity of $\mathbf{R}$, we find $w' \ \mathbf{R} \ x$. Since $\mathbf{R}$ is $\mathsf{T}(\mathbf{R})$-transitive, therefore, $w' \ \mathbf{R} \ y$, and hence, by affinity of $\mathbf{R}$, $w = \lambda w' + (1-\lambda)z$ $\mathbf{R} \ \lambda y + (1-\lambda)z$. Thus:

$$(\lambda x + (1-\lambda)z)^{\uparrow} \subseteq (\lambda y + (1-\lambda)z)^{\uparrow}.$$

Since we can similarly show that $(\lambda x + (1-\lambda)z)^{\downarrow} \supseteq (\lambda y + (1-\lambda)z)^{\downarrow}$, we may invoke Proposition 3.1 to find $\lambda x + (1-\lambda)z \ \mathsf{T}(\mathbf{R}) \ \lambda y + (1-\lambda)z$. In view of the arbitrary choice of x, y, z and λ, we conclude that $\mathsf{T}(\mathbf{R})$ is affine. ■

4 Rationalizable Preferences

The literature on choice theory and multi-criteria decision-making provides various models of choice on the basis of a collection of (rational) preference relations. The recent work of Cherepanov, Feddersen and Sandroni (2013), for instance, characterizes those choice functions for which there is a collection of complete preorders such that a choice from a given set is the maximizer of at least one of those preorders. Put differently, such a choice function obtains by maximizing a (potentially) nontransitive reflexive binary relation, namely, the *union* of a collection of complete and transitive preference relations. In this section we show that the rational core of this sort of a binary relation (on a suitable space) corresponds precisely to the common agreement between all of the preferences in the collection, provided that there is at least some agreement between these preferences to begin with. We will derive this result here in quite an abstract setup, and in Sections 5 and 6, apply it in the context of decision making under risk and uncertainty.

Rationalizable Preferences. Let X be a nonempty convex subset of a topological linear space. Let $\mathcal{P}$ stand for a nonempty collection of continuous and affine complete preorders on X. We may think of $\mathcal{P}$ as representing the set of preferences of a group of rational individuals. Alternatively, we may think of each element $\succsim$ in $\mathcal{P}$ as a (rational) preference relation of a different "self" of the same individual. (For instance, the agent may not know which of these relations will be the relevant one at the time of consumption, so entertains them all before making her choice.) Still another interpretation is that each $\succsim$ in $\mathcal{P}$ tells us how good the elements of X are with respect to some rationale (or criterion, or attribute). Borrowing the terminology used by Cherepanov, Feddersen and Sandroni (2013), therefore, we refer to the binary relation $\bigcup\mathcal{P}$ on X as a **rationalizable preference** on X.

The Rational Core of Rationalizable Preferences. Our main goal in this section is to show that if there is a minimal agreement between the strict parts of the preference relations in $\mathcal{P}$, then the rational core of $\bigcup\mathcal{P}$ corresponds exactly to $\bigcap\mathcal{P}$.

We will actually prove something a bit more general at first. In what follows, we refer to a preorder $\succsim$ on X as **convex** if $\lambda x + (1-\lambda)y \succ y$ for every $x, y \in X$ with $x \succ y$ and $\lambda \in (0, 1]$.

Proposition 4.1. *Let $\mathcal{P}$ be a nonempty collection of continuous and convex complete preorders on a nonempty convex subset X of a topological linear space. Suppose that for every $x \in X$, either x is a maximum of X with respect to each $\succsim \in \mathcal{P}$, or there is an $x^* \in X$ with $x^* \succ x$ for each $\succsim \in \mathcal{P}$. Then, $\mathsf{T}(\bigcup\mathcal{P}) = \bigcap\mathcal{P}$, and*

$$x \text{ core}(\textstyle\bigcup\mathcal{P})\ y \quad \textit{iff} \quad x\ \mathsf{A}(\succsim)\ y \textit{ for every } \succsim \in \mathcal{P}$$

for any x and y in X.

Proof. Let us put $\mathbf{R} := \bigcup\mathcal{P}$ and $\trianglerighteq := \bigcap\mathcal{P}$ to simplify the notation. Note that $\mathbf{R}$ is $\trianglerighteq$-transitive, so we have $\mathsf{T}(\mathbf{R}) \supseteq \trianglerighteq$. Conversely, take any $x, y \in X$ with $x\ \mathsf{T}(\mathbf{R})\ y$, but, to derive a contradiction, assume that $x \trianglerighteq y$ is false, that is, $y \succ' x$ for some $\succsim'$ in $\mathcal{P}$. By hypothesis, there is an $x^* \in X$ such that $x^* \succ x$ for each $\succsim \in \mathcal{P}$. Clearly, by continuity of $\succsim'$, we can choose a large enough λ in $(0, 1)$ such that $y \succ' \lambda x + (1-\lambda)x^*$. On the other hand, as $x^* \succ x$ for each $\succsim$ in $\mathcal{P}$, and because every $\succsim$ in $\mathcal{P}$ is convex, $\lambda x + (1-\lambda)x^* \succ x$ for each $\succsim$ in $\mathcal{P}$. Thus, $x\ \mathsf{T}(\mathbf{R})\ y\ \mathbf{R}\ \lambda x + (1-\lambda)x^*$ but not $x\ \mathbf{R}\ \lambda x + (1-\lambda)x^*$, which means that $\mathbf{R}$ is not $\mathsf{T}(\mathbf{R})$-transitive, a contradiction. Conclusion: $\mathsf{T}(\mathbf{R}) = \trianglerighteq$. Moreover, combining this finding with Theorem 3.4 yields $\text{core}(\mathbf{R}) = \mathsf{A}(\trianglerighteq)$. But, by (2), we have $\mathsf{A}(\trianglerighteq) = \bigcap\{\mathsf{A}(\succsim) : \succsim \in \mathcal{P}\}$, and we are done. ■

As an immediate corollary of this result we obtain the following generalization of Proposition 4 of Nishimura (2018).

Corollary 4.2. *Let $\mathcal{P}$ be a nonempty collection of continuous and affine complete preorders on a nonempty convex subset X of a topological linear space. Suppose that for every $x \in X$, either x is a maximum of X with respect to each $\succsim \in \mathcal{P}$, or there is an $x^* \in X$ with $x^* \succ x$ for each $\succsim \in \mathcal{P}$. Then,*

$$\mathsf{T}(\bigcup \mathcal{P}) = \bigcap \mathcal{P} = \text{core}(\bigcup \mathcal{P}).$$

When all members of the collection $\mathcal{P}$ are affine (as in Corollary 4.2), we can actually ensure that $\text{core}(\bigcup \mathcal{P}) = \bigcap \mathcal{P}$ with less restrictive hypotheses, as we show next.

Proposition 4.3. *Let X and $\mathcal{P}$ be as in Corollary 4.2. Suppose that there exist two elements x^* and x_* of X such that $x^* \succ x_*$ for every $\succsim$ in $\mathcal{P}$. Then,*

$$\bigcap \mathcal{P} = \text{core}(\bigcup \mathcal{P}).$$

Proof. Let us again put $\mathbf{R} := \bigcup \mathcal{P}$ and $\trianglerighteq := \bigcap \mathcal{P}$ to simplify the notation, and note that $\mathsf{T}(\mathbf{R}) \supseteq \trianglerighteq$ (because $\mathbf{R}$ is $\trianglerighteq$-transitive). As it is plain that $\trianglerighteq$ is affine, it follows from the definition of $\text{core}(\mathbf{R})$ and Corollary 3.2 that $\trianglerighteq \subseteq \text{core}(\mathbf{R})$. To prove the converse containment, suppose that $x \trianglerighteq y$ is false. By definition of $\trianglerighteq$, then, there is a preorder $\succsim$ in $\mathcal{P}$ such that $y \succ x$. As $\succsim$ is a continuous and complete preorder on X, the binary relation $\succ$ is open in $X \times X$, and hence, thanks to the continuity of the addition and scalar multiplication operations on X, there exists a large enough λ in the interval $(0, 1)$ such that $\lambda y + (1 - \lambda)x_* \succ \lambda x + (1 - \lambda)x^*$. On the other hand, as every element of $\mathcal{P}$ is affine, we have $\lambda y + (1 - \lambda)x^* \succ' \lambda y + (1 - \lambda)x_*$ for every $\succ'$ in $\mathcal{P}$, which means that $\lambda y + (1 - \lambda)x_* \ \mathbf{R} \ \lambda y + (1 - \lambda)x^*$ is false. It follows from Proposition 3.1 that $\lambda x + (1 - \lambda)x^* \ \mathsf{T}(\mathbf{R}) \ \lambda y + (1 - \lambda)x^*$ is false. But then, by Proposition 3.3, x is not ranked higher than y by the affine core of $\mathsf{T}(\mathbf{R})$, which, by Theorem 3.4, means that $x \ \text{core}(\mathbf{R}) \ y$ is false. Thus: $\text{core}(\mathbf{R}) \subseteq \trianglerighteq$. ∎

Remark 4.1. The use of the convex structure of preferences is essential in Proposition 4.1. For, in general, the transitive core of the union of a set $\mathcal{P}$ of complete preorders need not equal the intersection of these orders (although we always have $\mathsf{T}(\bigcup \mathcal{P}) \supseteq \bigcap \mathcal{P}$). Indeed, let $X := \{x, y, z\}$, $\succsim_1 := \{(x, y), (x, z), (y, z)\} \cup \triangle_X$ and $\succsim_2 := \{(y, x), (x, z), (y, z)\} \cup \triangle_X$. Then, $\succsim_1 \cup \succsim_2$ is transitive, so it equals its transitive core. In particular, this preorder includes both (x, y) and (y, x), but $\succsim_1 \cap \succsim_2$ does not contain either (x, y) or (y, x).

Remark 4.2. Giarlotta and Greco (2013) study an ordered pair of binary relations $(\mathbf{R}_1, \mathbf{R}_2)$ on a given nonempty set in which $\mathbf{R}_1$ is a preorder and $\mathbf{R}_2$ is a complete superrelation of $\mathbf{R}_1$. The interpretation is that the agent would always act according to $\mathbf{R}_1$ so long as this relation applies, and that her choices would never disagree with $\mathbf{R}_2$. In accord with this interpretation, if $\mathbf{R}_2$ is $\mathbf{R}_1$-transitive and, for any two alternatives x and y, either $x\ \mathbf{R}_1\ y$ or $y\ \mathbf{R}_2\ x$, Giarlotta and Greco (2013) call $(\mathbf{R}_1, \mathbf{R}_2)$ a *necessary and possible* (NaP) *preference* on X, and show that for any such $(\mathbf{R}_1, \mathbf{R}_2)$ there is a collection $\mathcal{P}$ of complete preorders on X such that $\mathbf{R}_1 = \bigcap \mathcal{P}$ and $\mathbf{R}_2 = \bigcup \mathcal{P}$. In the context of Proposition 4.3, therefore, we find here that $(\mathrm{core}(\mathbf{R}), \mathbf{R})$ is a NaP preference on X for any rationalizable preference $\mathbf{R}$ on X.

An Example. Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset of a linear space. If $\mathbf{R}$ is transitive, Theorem 3.4 says that $\mathrm{core}(\mathbf{R}) = \mathsf{A}(\mathbf{R})$. If, on the other hand, $\mathbf{R}$ is such that $\mathsf{T}(\mathbf{R})$ is affine (as in Corollary 4.2), we have $\mathrm{core}(\mathbf{R}) = \mathsf{T}(\mathbf{R})$. In any one of these cases, therefore, one does not need the combined powers of the transitive and affine cores of $\mathbf{R}$, and the characterization we obtained in Theorem 3.4 becomes somewhat trivial. We now use Proposition 4.1 to provide a concrete example that shows that this is not a general phenomenon.

Let n be an integer with $n \geq 2$, and let Δ^{n-1} stand for the unit simplex in $\mathbb{R}^n$.[13] Fix any $\sigma \in \Delta^{n-1}$ and any nonempty closed and convex subset $\mathcal{M}$ of Δ^{n-1}, and consider the preorders $\succsim_1$ and $\succsim_2$ on $\mathbb{R}^n$ defined by

$$x \succsim_1 y \quad \text{iff} \quad \sigma \cdot x \geq \sigma \cdot y$$

and

$$x \succsim_2 y \quad \text{iff} \quad \min_{\mu \in \mathcal{M}} \mu \cdot x \geq \min_{\mu \in \mathcal{M}} \mu \cdot y,$$

for every $x, y \in \mathbb{R}^n$, where we use the usual inner product notation on $\mathbb{R}^n$.[14] To avoid trivialities, we assume that $\mathcal{M}$ is not a singleton and it does not contain σ. Finally, we define $\mathbf{R} := \succsim_1 \cup \succsim_2$. We wish to compute the rational core of $\mathbf{R}$, and show that it is distinct from the transitive and affine cores of $\mathbf{R}$.

First, we apply Proposition 4.1 to find that $x\ \mathrm{core}(\mathbf{R})\ y$ iff $x \succsim_1 y$ and $x\ \mathsf{A}(\succsim_2)\ y$, for any $x, y \in \mathbb{R}^n$. Put more explicitly,

$$x\ \mathrm{core}(\mathbf{R})\ y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for every } \mu \in \mathcal{M} \cup \{\sigma\}.$$

On the other hand, again by Proposition 4.1, $x\ \mathsf{T}(\mathbf{R})\ y$ iff $x \succsim_1 y$ and $x \succsim_2 y$, for any $x, y \in \mathbb{R}^n$. Since $\mathcal{M}$ is not a singleton, $\succsim_2$ is not affine, and this implies that $\mathsf{T}(\mathbf{R})$ is not affine. Thus: $\mathrm{core}(\mathbf{R}) \neq \mathsf{T}(\mathbf{R})$.

Now, let us put $\mathbf{R}^* := \succsim_1 \cup \mathsf{A}(\succsim_2)$. Then, $\mathbf{R}^*$ is an affine subrelation of $\mathbf{R}$, so we have $\mathbf{R}^* \subseteq \mathsf{A}(\mathbf{R})$ while, in view of the computation above, $\mathrm{core}(\mathbf{R}) \subseteq \mathbf{R}^*$. Therefore, if we can show that $\mathrm{core}(\mathbf{R})$ is a proper subset of $\mathbf{R}^*$, we may conclude that it is a proper subset of $\mathsf{A}(\mathbf{R})$ as well, establishing that $\mathrm{core}(\mathbf{R}) \neq \mathsf{A}(\mathbf{R})$. In other words, we

[13] That is, $\Delta^{n-1} := \{\mu \in \mathbb{R}^n_+ : \mu_1 + \cdots + \mu_n = 1\}$.

[14] One could interpret $\succsim_2$ as corresponding to a Rawlsian aggregation that is made up of a collection of weighted utilitarian social welfare functions. Alternatively, we can think of $\succsim_2$ as Gilboa-Schmeidler maxmin preferences over monetary Anscombe-Aumann acts.

wish to show that the intersection of the sets $\succsim_1$ and $\mathsf{A}(\succsim_2)$ does not equal their union, that is, $\succsim_1$ and $\mathsf{A}(\succsim_2)$ are distinct preorders. To this end, we apply the Separating Hyperplane Theorem to find a nonzero n-vector x and a real number α such that $\mu \cdot x \geq \alpha > \sigma \cdot x$ for every $\mu \in \mathcal{M}$. Then, where $\succsim^*$ is the preorder on $\mathbb{R}^n$ defined by $y \succsim^* z$ iff $\mu \cdot y \geq \mu \cdot z$ for every $\mu \in \mathcal{M}$, and $\hat{\alpha}$ stands for the n-vector $(\alpha, ..., \alpha)$, we have $x \succsim^* \hat{\alpha}$. But it is plain that $\succsim^*$ is an affine subrelation of $\succsim_2$. Thus, $\succsim^* \subseteq \mathsf{A}(\succsim_2)$, and it follows that $x\ \mathsf{A}(\succsim_2)\ \hat{\alpha}$ while $\hat{\alpha} \succ_1 x$. Conclusion: $\succsim_1$ and $\mathsf{A}(\succsim_2)$ are distinct, and hence, $\text{core}(\mathbf{R}) \neq \mathsf{A}(\mathbf{R})$.

5 Rational Core and Choice under Risk

In this section, we investigate the rational core operator in the setting of decision-making under risk (leaving the case of uncertainty to the next section). In particular, our goal is to "compute" the rational core of a number of risk-preference models.

Throughout this section, Z stands for a metric space, which we view as the space of all riskless alternatives. As is standard, by a **lottery** on Z, we mean a Borel probability measure on Z, and denote the collection of all such lotteries by $\Delta(Z)$. The (risk) preferences of an individual are defined over $\Delta(Z)$. As usual, we think of this set as a topological space relative to the topology of weak convergence; as such, when Z is separable, this space is metrizable.

In what follows, by an **expected utility preference** on $\Delta(Z)$, we mean any complete preorder $\succsim$ on $\Delta(Z)$ for which there is a continuous and bounded (Bernoulli utility) function $u : Z \to \mathbb{R}$ such that

$$p \succsim q \quad \text{if and only if} \quad \int_Z u \, \mathrm{d}p \geq \int_Z u \, \mathrm{d}q$$

for any p and q in $\Delta(Z)$. It is well-known that if Z is separable, and $\succsim$ is a continuous and affine complete preorder on $\Delta(Z)$, then $\succsim$ is an expected utility preference. More generally, a preorder $\succsim$ on $\Delta(Z)$ is said to be an **expected multi-utility preference** on $\Delta(Z)$ if there is a nonempty collection $\mathcal{U}$ of continuous and bounded (Bernoulli utility) functions on Z such that

$$p \succsim q \quad \text{if and only if} \quad \int_Z u \, \mathrm{d}p \geq \int_Z u \, \mathrm{d}q \text{ for every } u \in \mathcal{U} \tag{7}$$

for any p and q in $\Delta(Z)$. (In this case, we say that "*$\mathcal{U}$ is an expected multi-utility representation for $\succsim$.*") Such preferences satisfy all axioms of the von Neumann-Morgenstern theory with the potential exception of the completeness property.

5.1 Representation of the Rational Core under Risk

Let $\mathbf{R}$ be a reflexive binary relation on $\Delta(Z)$, representing the (observed) pairwise ranking of lotteries by a decision maker. We allow this ranking procedure to lead to indecisiveness and/or cyclic choices, as well as failing the classical Independence Axiom, thereby allowing for, say, the Allais paradox. That is, $\mathbf{R}$ need not be complete, transitive and/or affine. Loosely speaking, Theorem 3.4 says that we can think of the rational core of $\mathbf{R}$ in this setting as an incomplete, but transitive, preference relation on the lottery space $\Delta(Z)$ that satisfies the Independence Axiom. In turn, these properties entail that the rational core of $\mathbf{R}$ has a fairly special structure which we can identify without knowing anything about the concrete description of $\mathbf{R}$. Indeed, under some basic regularity conditions, this preference relation admits an expected multi-utility representation.

Theorem 5.1. *Let Z be a compact metric space, and $\mathbf{R}$ a reflexive, continuous and locally non-saturated binary relation on $\Delta(Z)$. Then, core$(\mathbf{R})$ is an expected multi-utility preference on $\Delta(Z)$.*

Proof. As $\Delta(Z)$ is a convex subset of the topological linear space ca(Z),[15] we can apply Theorems 3.4 and 3.6 to conclude that core$(\mathbf{R})$ is a continuous affine preorder on $\Delta(Z)$. Given that Z is compact, therefore, we may apply the Expected Multi-Utility Theorem of Dubra, Maccheroni and Ok (2004) to complete the proof. ■

Continuity and local non-saturation are fairly standard hypotheses for preferences over risky prospects. Theorem 5.1 thus says that we may think of the rational core of essentially any type of binary relation $\mathbf{R}$ encountered in the theory of decision making under risk (with a compact prize space) as a *unanimity* ordering, where the unanimity is required of a nonempty collection of expected utility preferences. Exactly which collection is to be used for this purpose depends on the particular structure of $\mathbf{R}$.

5.2 Applications to the Theory of Risk Aversion

While the theory of risk aversion is fairly well-developed in the context of expected and nonexpected utility theory, there is hardly any work in the literature that investigates the risk attitudes of nontransitive risk preferences. In this section, our goal is to demonstrate that one can use the notion of rational core to extend the classical theory of risk aversion to the context of such preferences.

5.2.1 Monotonic Risk Preferences

Let $\mathbf{R}$ be a reflexive binary relation on $\Delta := \Delta([0, 1])$, which we interpret as the risk preferences of an agent over lotteries with monetary payoffs. Consider the following

[15] Here ca(Z) stands for the normed linear space of all signed finite Borel measures on Z relative to the total variation norm. Since Z is compact, this space is isometrically isomorphic to the topological dual of $\mathbf{C}(Z)$ – this is the Radon-Riesz Representation Theorem – and we use this duality to topologize ca(Z) with the weak*-topology. (Here $\mathbf{C}(Z)$ is the normed linear space of all continuous real maps on Z relative to the sup-norm.) This makes ca(Z) a Hausdorff topological linear space such that the subspace topology on $\Delta(Z)$ is exactly the topology of weak convergence.

question: What does it mean for $\mathbf{R}$ to be *monotonic*? This is a meaningful question because monotonicity is often (and rightly) seen as a trait of rationality that is distinct from both transitivity and affinity; there is no reason for a nontransitive and nonaffine $\mathbf{R}$ not to rank a monetary lottery over another one if the former is "unambiguously" better from the payout standpoint. And there seems to be an immediate answer; just declare $\mathbf{R}$ as "monotonic" when $\geq_{\text{FSD}} \subseteq \mathbf{R}$, where $\geq_{\text{FSD}}$ is the *first order stochastic dominance* ordering on Δ. (By definition, $\geq_{\text{FSD}}$ is the partial order on Δ for which the collection of all continuous and increasing real maps on $[0,1]$ is an expected multi-utility representation.)

On the other hand, intuitively speaking, it makes sense to ask an individual to view the comparison of two $\geq_{\text{FSD}}$-comparable lotteries as an "easy" one (barring, of course, computational difficulties which we do not model in this paper). This suggests that we say $\mathbf{R}$ is **monotonic** if $\geq_{\text{FSD}} \subseteq \text{core}(\mathbf{R})$, and **strictly monotonic** if $>_{\text{FSD}} \subseteq \text{core}(\mathbf{R})^{>}$ (where $>_{\text{FSD}}$ stands for the asymmetric part of $\geq_{\text{FSD}}$). The following observation provides a characterization of this property that sidesteps the computation of $\text{core}(\mathbf{R})$.

Proposition 5.2. *A reflexive binary relation* $\mathbf{R}$ *on* $\Delta([0,1])$ *is monotonic if, and only if, it is* $\geq_{\text{FSD}}$*-transitive.*

Proof. The "if" part follows immediately from the fact that $\geq_{\text{FSD}}$ is an affine binary relation on Δ and the definition of $\text{core}(\mathbf{R})$. Since $\mathbf{R}$ is $\text{core}(\mathbf{R})$-transitive, the "only if" part follows from the fact that $\mathbf{R}$ is $\mathbf{S}$-transitive for any subrelation $\mathbf{S}$ of $\text{core}(\mathbf{R})$. ■

While it is trivial, this proposition has conceptual power. Arguably, it shows that the "correct" generalization of *monotonicity* to the context of an arbitrary preference relation on Δ (which need not satisfy any property other than reflexivity) is not requiring that $\mathbf{R}$ be a superrelation of $\geq_{\text{FSD}}$, but more stringently, that $\mathbf{R}$ be $\geq_{\text{FSD}}$-transitive.

The following proposition, which characterizes the rational core of a (strictly) monotonic preference relation that satisfies the conditions of Theorem 5.1, gives further credence to this argument.

Proposition 5.3. *Let* $\mathbf{R}$ *be a reflexive, continuous and locally non-saturated binary relation on* $\Delta([0,1])$*. Then,* $\mathbf{R}$ *is monotonic if, and only if, there is a nonempty collection* $\mathcal{U}$ *of continuous and increasing real functions on* $[0,1]$ *such that*

$$p \text{ core}(\mathbf{R})\, q \quad \textit{if and only if} \quad \int_{[0,1]} u \,\mathrm{d}p \geq \int_{[0,1]} u \,\mathrm{d}q \text{ for every } u \in \mathcal{U} \qquad (8)$$

for any p *and* q *in* $\Delta([0,1])$*. Moreover, if* $\mathbf{R}$ is $\geq_{\text{FSD}}$*-transitive, and* $\delta_a\ \mathbf{R}^{>}\ \delta_b$ *holds whenever* $a > b$*, then* $\mathbf{R}$ *is strictly monotonic, and in this case, we can choose* $\mathcal{U}$ *here as strictly increasing.*[16]

[16] For any $c \in [0,1]$, by δ_c we denote the lottery on $[0,1]$ that pays c with probability 1. Besides, by

5.2.2 Risk Aversion

With $\mathbf{R}$ being any reflexive binary relation on Δ, we now turn to the following question: What does it mean for $\mathbf{R}$ to be *risk averse*? Since $\mathbf{R}$ need not satisfy the von Neumann-Morgenstern Independence Axiom, one needs to make a distinction between the notions of *weak risk aversion* and *strong risk aversion*.[17] While these concepts are identical for monotonic expected utility preferences on Δ, they are distinct in the context of more general preference structures. Moreover, it is well-known that strong risk aversion is much better behaved for nonexpected utility preferences. (See, for instance, Cohen (1995).) Consequently, it seems desirable that the answer to our question is phrased by means of a property to be imposed on $\mathbf{R}$ that would reduce to strong risk aversion when $\mathbf{R}$ is a preorder on Δ. In turn, this suggests that we should declare $\mathbf{R}$ as "risk averse" when $\geq_{\text{SSD}} \subseteq \mathbf{R}$, where $\geq_{\text{SSD}}$ is the *second order stochastic dominance* ordering on Δ. (By definition, $\geq_{\text{SSD}}$ is the partial order on Δ for which the collection of all continuous, increasing and concave real maps on $[0,1]$ is an expected multi-utility representation.)

On the other hand, as in the discussion of monotonicity above, it makes sense to qualify an individual as "risk averse" if there is no doubt in her rational mind that she prefers p to q for any two lotteries p and q on $[0,1]$ with $p \geq_{\text{SSD}} q$. In other words, it may be desirable that we insist her preferences be "unambiguous" about the ranking of two lotteries that are comparable by $\geq_{\text{SSD}}$; the "irrational" part of the agent, if any, should not interfere with how such two lotteries are ranked. Of course, when the preferences in question are transitive and affine, this issue does not arise, because every ranking of lotteries is then "unambiguous," insofar as rationality is captured by transitivity and the Independence Axiom. But in the absence of transitivity, this argument demands that $\geq_{\text{SSD}}$ be contained not only in $\mathbf{R}$, but in the rational core of $\mathbf{R}$. This leads us to the following definition: $\mathbf{R}$ is **risk averse** if $\geq_{\text{SSD}} \subseteq \text{core}(\mathbf{R})$.

Obviously, this definition reduces to asking merely for $\geq_{\text{SSD}} \subseteq \mathbf{R}$ when $\mathbf{R}$ is transitive (because in that case $\text{core}(\mathbf{R}) = \mathsf{A}(\mathbf{R})$ while $\geq_{\text{SSD}}$ is an affine preorder on $[0,1]$). Thus, in particular, our definition is in concert with how (strong) risk aversion is defined for transitive nonexpected utility preferences. Our next proposition, which obtains by replacing $\geq_{\text{FSD}}$ with $\geq_{\text{SSD}}$ in the proof of Proposition 5.2, shows that our definition is in general more demanding.

Proposition 5.4. *A reflexive binary relation* $\mathbf{R}$ *on* $\Delta([0,1])$ *is risk averse if, and only if, it is* $\geq_{\text{SSD}}$*-transitive.*

Again, the upshot of this proposition is conceptual. It suggests that the "correct" generalization of *strong risk aversion* to the context of an arbitrary preference relation

$\mathcal{U}$ being (*strictly*) *increasing* here, we mean simply that every member of $\mathcal{U}$ is (strictly) increasing.

[17] A preorder $\succsim$ on Δ is said to be *weakly risk averse* if $\delta_{e(p)} \succsim p$ for every $p \in \Delta$. (Here $e(p)$ stands for the expectation of p, and $\delta_{e(p)}$ is the degenerate lottery on $[0,1]$ that pays $e(p)$ dollars with probability 1.) By contrast, $\succsim$ is said to be *strongly risk averse* if it is a superrelation of the second order stochastic dominance ordering on Δ.

on Δ is not requiring that $\mathbf{R}$ be a superrelation of $\geq_{\text{SSD}}$, but more stringently, that $\mathbf{R}$ be $\geq_{\text{SSD}}$-transitive.

The following result extends Proposition 5.3 to the context of risk averse preferences.

Proposition 5.5. *Let* $\mathbf{R}$ *be a reflexive, continuous and locally non-saturated binary relation on* $\Delta([0,1])$. *Then,* $\mathbf{R}$ *is risk averse if, and only if, there is a nonempty collection* $\mathcal{U}$ *of continuous, increasing and concave real functions on* $[0,1]$ *such that* (8) *holds for any* p *and* q *in* $\Delta([0,1])$.

As their proofs in the Appendix demonstrate, Propositions 5.3 and 5.5 remain valid for any reflexive $\mathbf{R}$ on Δ such that core($\mathbf{R}$) admits an expected multi-utility representation. (In that case, one does not need to verify that $\mathbf{R}$ is continuous and locally non-saturated.) This fact may be useful in determining whether or not a given relation on Δ is risk averse, even though local non-saturation of the relation may be difficult to check (or may even fail). The following example illustrates.

Example 5.1. (*Justifiable Risk Preferences*) By a **justifiable risk preference** on Δ, we mean a binary relation $\mathbf{R}_{\mathcal{U}}$ on Δ such that

$$p \; \mathbf{R}_{\mathcal{U}} \; q \quad \text{if and only if} \quad \int_{[0,1]} u \, \mathrm{d}p \geq \int_{[0,1]} u \, \mathrm{d}q \text{ for some } u \in \mathcal{U},$$

where $\mathcal{U}$ is a nonempty collection of continuous real maps on $[0,1]$ such that there exist some $z^*, z_* \in [0,1]$ with $u(z^*) > u(z_*)$ for all $u \in \mathcal{U}$. Any such relation is affine, but it need not be transitive. The upshot here is that, by Proposition 4.3, $\mathcal{U}$ is an expected multi-utility representation for core($\mathbf{R}_{\mathcal{U}}$). Therefore, not by Propositions 5.3 and 5.5, but by the remark above, we may conclude that $\mathbf{R}_{\mathcal{U}}$ is monotonic iff $\mathcal{U}$ is increasing, and it is risk averse iff every member of $\mathcal{U}$ is increasing and concave. □

5.2.3 Comparative Risk Aversion

The rational core operator provides a natural method of making comparative assessments of risk attitudes of two individuals with possibly nontransitive and nonaffine preferences over monetary lotteries. For any reflexive and strictly monotonic binary relations $\mathbf{R}_1$ and $\mathbf{R}_2$ on Δ, we say that $\mathbf{R}_1$ is **more risk averse than** $\mathbf{R}_2$ if

$$p \text{ core}(\mathbf{R}_1) \; \delta_a \quad \text{implies} \quad p \text{ core}(\mathbf{R}_2) \; \delta_a \tag{9}$$

and

$$\delta_a \text{ core}(\mathbf{R}_2) \; p \quad \text{implies} \quad \delta_a \text{ core}(\mathbf{R}_1) \; p \tag{10}$$

for every $p \in \Delta$ and $a \in [0,1]$. In words, this means that if a person "undoubtedly" prefers a risky lottery to a riskless lottery, then a less risk averse person would also do so, and dually, if a person "undoubtedly" prefers a degenerate lottery to a lottery, then a more risk averse person would also do so.[18] It is plain that this definition reduces to the standard one in the case where both $\mathbf{R}_1$ and $\mathbf{R}_2$ admit expected utility representations.

As in the classical theory, while intuitively appealing, this definition of comparative risk aversion is difficult to work with in practice. For this reason, we develop below a sufficient condition for our comparative risk aversion ordering to apply. First, let us agree to say that a continuous real map u on $[0,1]$ is **more (less**, resp.) **concave than** another such map v if u is a strictly increasing, continuous and concave (convex, resp.) transformation of v (that is, there is a strictly increasing, continuous and concave (convex, resp.) map $f : v([0,1]) \to \mathbb{R}$ with $u = f \circ v$). In turn, we say that a nonempty subset $\mathcal{U}$ of $\mathbf{C}[0,1]$ is **more concave than** another such set $\mathcal{V}$ if (i) for every $v \in \mathcal{V}$, there is a $u \in \mathcal{U}$ that is more concave than v; and (ii) for every $u \in \mathcal{U}$, there is a $v \in \mathcal{V}$ that is less concave than u.[19]

The following proposition shows that if two reflexive preferences over Δ have rational cores which admit expected multi-utility representations – in particular, when these preferences are continuous and locally non-saturated – then comparability of the concavity of these representations is sufficient to compare the involved risk attitudes.

Proposition 5.6. *Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two reflexive binary relations on $\Delta([0,1])$. If there exist expected multi-utility representations for $\text{core}(\mathbf{R}_1)$ and $\text{core}(\mathbf{R}_2)$ such that the one for $\text{core}(\mathbf{R}_1)$ is more concave than that for $\text{core}(\mathbf{R}_2)$, then $\mathbf{R}_1$ is more risk averse than $\mathbf{R}_2$.*

This proposition does not yield a characterization of the "more risk averse than" relation even when we know that the rational cores of the involved preferences have expected multi-utility representations. However, in the interesting case where both preferences are strictly monotonic, and one of them is a standard expected utility preference, then the result becomes a characterization.

Corollary 5.7. *Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two reflexive binary relations on $\Delta([0,1])$ such that $\mathbf{R}_2$ is a (continuous) strictly monotonic expected utility preference. If there exists a*

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[18] Even when $\text{core}(\mathbf{R}_1)$ and $\text{core}(\mathbf{R}_2)$ do possess expected multi-utility representations, easy examples would show that (9) does not imply (10), and conversely, unless, of course, both of these preordes are complete (in which case the situation reduces to the comparison of two expected utility preferences).

[19] It may be useful to give an order-theoretic formulation of this definition to highlight its connection to the standard notion of concave transformations. For any preorder $\succsim$ on a nonempty set X, the **upper set-ordering** $\succsim^{\mathrm{u}}$ induced by $\succsim$ on 2^X is defined as $A \succsim^{\mathrm{u}} B$ iff for every $a \in A$, there is a $b \in B$ with $a \succsim b$. Similarly, the **lower set-ordering** $\succsim^{\mathrm{l}}$ induced by $\succsim$ on 2^X is defined as $A \succsim^{\mathrm{l}} B$ iff for every $b \in B$, there is an $a \in A$ with $a \succsim b$. (Easy examples would show that these preorders are, in general, distinct.) In turn, we define the **set-ordering** induced by $\succsim$ as $\succsim^{*} := \succsim^{\mathrm{u}} \cap \succsim^{\mathrm{l}}$.

Now consider the preorder $\succsim_{\text{con}}$ on $\mathbf{C}[0,1]$ defined by $u \succsim_{\text{con}} v$ iff u is an increasing, continuous and concave transformation of v. Then, the definition above maintains that $\mathcal{U}$ is more concave than $\mathcal{V}$ iff $\mathcal{U} \succsim^{*}_{\text{con}} \mathcal{V}$.

strictly increasing expected multi-utility representation $\mathcal{U}$ for core($\mathbf{R}_1$)*, then* $\mathbf{R}_1$ *is more risk averse than* $\mathbf{R}_2$ *if and only if each* $u \in \mathcal{U}$ *is more concave than any Bernoulli utility for* $\mathbf{R}_2$.[20]

Example 5.2. (*Justifiable Risk Preferences, Again*) Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty collections of strictly increasing and continuous real maps on $[0,1]$, and consider the justifiable risk preferences $\mathbf{R}_{\mathcal{U}}$ and $\mathbf{R}_{\mathcal{V}}$ on Δ. Again, by Proposition 4.3, $\mathcal{U}$ is an expected multi-utility representation for core($\mathbf{R}_{\mathcal{U}}$), and $\mathcal{V}$ for core($\mathbf{R}_{\mathcal{V}}$). Therefore, by Proposition 5.6, if $\mathcal{U}$ is more concave than $\mathcal{V}$, then $\mathbf{R}_{\mathcal{U}}$ is more risk averse than $\mathbf{R}_{\mathcal{V}}$. And, provided that either $\mathcal{U}$ or $\mathcal{V}$ is a singleton, this is a complete characterization: $\mathcal{U}$ is more concave than $\mathcal{V}$ iff $\mathbf{R}_{\mathcal{U}}$ is more risk averse than $\mathbf{R}_{\mathcal{V}}$.

We can easily illustrate these observations by using the CARA utility functions. To wit, define $u_\lambda \in \mathbf{C}[0,1]$ by $u_\lambda(t) := -e^{-\lambda t}/\lambda$ for any $\lambda > 0$. Next, take two nonempty closed subsets I and J of $\mathbb{R}_{++}$, and put $\mathcal{U} := \{u_\lambda : \lambda \in I\}$ and $\mathcal{V} := \{u_\lambda : \lambda \in J\}$. If $\sup I \geq \sup J$ and $\inf I \geq \inf J$, then $\mathbf{R}_{\mathcal{U}}$ is more risk averse than $\mathbf{R}_{\mathcal{V}}$. And if J is a singleton, say, $J := \{\alpha\}$, then $\mathbf{R}_{\mathcal{U}}$ is more risk averse than $\mathbf{R}_{\mathcal{V}}$ iff $\inf I \geq \alpha$. $\square$

Our final result in this section provides a consistency check between how we defined above the notions of risk aversion and comparative risk aversion. In its statement, by the **risk neutral preference** $\succsim$ on Δ, we mean the complete preorder $\succsim$ with $p \succsim q$ iff $e(p) \geq e(q)$, where $e(p)$ and $e(q)$ stand for the expected values of the lotteries p and q, respectively.

Corollary 5.8. *Let* $\mathbf{R}$ *be a reflexive, strictly monotonic, continuous and locally non-saturated binary relation on* $\Delta([0,1])$*. Then,* $\mathbf{R}$ *is risk averse if, and only if, it is more risk averse than the risk neutral preference on* Δ.

5.3 Application: The Preference Reversal Phenomenon, Revisited

Among the many experimental observations that refute the basic premises of expected utility theory, a particularly striking one is the so-called *preference reversal* (*PR*) *phenomenon*. This phenomenon was first demonstrated by Slovic and Lichtenstein (1968), and then explored by Grether and Plott (1979) in meticulous detail. Let us formalize the PR-phenomenon in terms of lotteries on $[0,1]$. Let $\mathbf{R}$ be a binary relation on $\Delta := \Delta([0,1])$ which models the (observable) preferences of an individual over such lotteries. For any $p \in \Delta$, we define

$$S_{\mathbf{R}}(p) := \inf\{a \in [0,1] : \delta_a \ \mathbf{R} \ p\},$$

[20] We can replace the roles of $\mathbf{R}_1$ and $\mathbf{R}_2$ in this result. That is, if $\mathbf{R}_1$ has an expected utility representation with a strictly increasing and continuous Bernoulli utility function u, and if there exists a strictly increasing expected multi-utility representation $\mathcal{V}$ for core($\mathbf{R}_2$), then $\mathbf{R}_1$ is more risk averse than $\mathbf{R}_2$ iff u is more concave than each $v \in \mathcal{V}$.

the **minimum selling price of** p for the individual. We say that **R exhibits preference reversals** if there exist two lotteries p and q in Δ such that $p\ \mathbf{R}\ q$ and yet $S_{\mathbf{R}}(p) < S_{\mathbf{R}}(q)$.[21]

It is obvious that a preference relation that exhibits the PR-phenomenon must depart from rationality.[22] Since we would like to think of the rational core of a preference relation as representing the rankings that an agent can make with complete confidence, a natural question here is if this core itself may exhibit the PR-phenomenon. A positive answer would either make our interpretation of the rational core suspect or it would suggest that the PR-phenomenon is truly fundamental in that it may arise from the "sure" rankings of a decision maker. Fortunately, under a mild monotonicity hypothesis, this situation does not arise.

Let us first note that no continuous preorder (affine or not) may possibly exhibit the PR-phenomenon, provided that the minimum selling price with respect to this preorder is well-defined on Δ.

Lemma 5.9. *Let $\succsim$ be a continuous preorder on $\Delta([0,1])$ such that $\delta_1 \succsim p$ for every $p \in \Delta([0,1])$. Then, $\succsim$ does not exhibit preference reversals.*

Proof. The hypothesis that $\delta_1 \succsim p$ ensures $S_{\succsim}(p)$ being well-defined, for every $p \in \Delta$. In turn, continuity of $\succsim$ guarantees that $S_{\succsim}(p) = \min\{a \in [0,1] : \delta_a \succsim p\}$, and hence $\delta_{S_{\succsim}(p)} \succsim p$, for every $p \in \Delta$. Consequently, for any $p, q \in \Delta$ with $p \succsim q$, we have $\delta_{S_{\succsim}(p)} \succsim q$, and hence $S_{\succsim}(p) \geq S_{\succsim}(q)$, by transitivity of $\succsim$. ■

We now show that monotonicity of the preference relation $\mathbf{R}$ of a person on Δ, along with the primitive axioms of reflexivity, continuity and local non-saturation, rules out the possibility of the rational core of $\mathbf{R}$ being subject to the PR-phenomenon, while, of course, allowing the observed preferences of the agent, that is, $\mathbf{R}$, to exhibit preference reversals.

Proposition 5.10. *Let $\mathbf{R}$ be a reflexive, monotonic, continuous and locally non-saturated binary relation on $\Delta([0,1])$. Then,* $\text{core}(\mathbf{R})$ *does not exhibit preference reversals.*

Proof. By Theorems 3.4 and 3.6, $\text{core}(\mathbf{R})$ is a continuous preorder on Δ. Moreover, since $\mathbf{R}$ is monotonic, $\geq_{\text{FSD}} \subseteq \text{core}(\mathbf{R})$. But then, since $\delta_1 \geq_{\text{FSD}} p$ for every $p \in \Delta$, we find $\delta_1\ \text{core}(\mathbf{R})\ p$ for every $p \in \Delta$. Our claim thus follows from Lemma 5.9. ■

We conclude that, as long as the preferences of an individual are monotonic enough to be transitive with respect to first order stochastic dominance, then preference

[21] The classical preference reversal phenomenon observed in experiments takes p and q as two-outcome lotteries with p yielding a "small" positive return with high probability, and q a "large" return with small probability. We will not need to specialize to this case here, however.

[22] In passing, we note that this departure may be less excessive than one might initially suspect. Hara, Ok and Riella (2018) have recently shown that the preference reversal phenomenon is consistent with preferences that are continuous, complete, affine, monotone (in the sense of being consistent with first-order stochastic dominance) and acyclic.

reversals that these preferences may exhibit cannot arise from their rational part. On one hand, this validates our interpretation of the rational core, and on the other, it shows that the preference reversal phenomenon is likely to be due to comparisons about which the decision maker is somewhat "conflicted."

5.4 The Rational Core of Rationalizable Risk Preferences

We now go back to our general treatment where lotteries need not be monetary. Let $\mathcal{U}$ be a nonempty collection of non-constant continuous and bounded real functions on the metric space Z. Suppose that each element of $\mathcal{U}$ corresponds to a Bernoulli utility function on Z, and that there exist two riskless prizes z^* and z_* in Z such that $u(z^*) > u(z_*)$ for every $u \in \mathcal{U}$. This amounts to asking only for an extremely weak agreement among the members of $\mathcal{U}$. In particular, this assumption is much weaker than requiring that there are a "best" and a "worst" element in Z with respect to each $u \in \mathcal{U}$.

We say that a binary relation $\mathbf{R}$ on $\Delta(Z)$ is a **rationalizable risk preference** on $\Delta(Z)$ if there is such a collection $\mathcal{U}$ such that

$$p \, \mathbf{R} \, q \quad \text{iff} \quad \int_Z u \, \mathrm{d}p \geq \int_Z u \, \mathrm{d}q \quad \text{for some } u \in \mathcal{U}$$

for any p and q in $\Delta(Z)$. (This definition generalizes the one we gave in Example 5.1.) Obviously, this is a particular type of rationalizable preference on $\Delta(Z)$, and we can apply Proposition 4.3 to readily compute its rational core:

Proposition 5.11. *For any metric space Z, the rational core of a rationalizable risk preference on $\Delta(Z)$ is an expected multi-utility preference* (*with the same set of utilities*).

An individual with rationalizable risk preferences on $\Delta(Z)$ may be thought of as saying that a lottery p is better than lottery q if there is at least one rationale for her (that is, there exists some $u \in \mathcal{U}$) with respect to which the expected utility of p is higher than that of q. In turn, by Proposition 5.11, we see that this individual prefers p over q "unambiguously" (that is, relative to the rational core of her preferences) iff p is better for her than q with respect to *all* of her rationales, an apparently plausible conclusion.

5.5 The SSB Model

The *skew-symmetric bilinear* (*SSB*) model is a generalization of the classical von Neumann-Morgenstern expected utility model that allows for nontransitive preferences over risk. This model was first introduced by Kreweras (1961) but did not make much of an impact on decision theory until the experiments on the preference reversal phenomenon were performed. These experiments have pointed to the nontransitive evaluation of lotteries by the subjects, and has led to the development of the SSB theory.[23]

The SSB Model. Let Z be a metric space and $\varphi : Z \times Z \to \mathbb{R}$ a continuous, bounded and skew-symmetric function. (The *skew-symmetry* of φ means that $\varphi(x, y) = -\varphi(y, x)$ for every x and y in Z.) The **SSB model induced by** φ is defined as the binary relation $\mathbf{R}_\varphi$ on $\Delta(Z)$ with

$$p \, \mathbf{R}_\varphi \, q \quad \text{iff} \quad \int_Z \int_Z \varphi \, \mathrm{d}p \, \mathrm{d}q \geq 0.$$

It is plain that this relation is complete, but it does not have to be transitive. (When nontransitive, this relation fails affinity as well; indeed, an affine SSB model is sure to be an expected utility preference on $\Delta(Z)$.[24])

The Rational Core of the SSB Model. We have seen in Corollary 4.2 that the rational core of a rationalizable preference reduces to the transitive core of that relation under fairly general circumstances. By contrast, we find here that the rational core of any SSB model reduces to its affine core.

Proposition 5.12. *Let Z be a metric space and $\varphi : Z \times Z \to \mathbb{R}$ a continuous, bounded and skew-symmetric function. Then,* $\mathrm{core}(\mathbf{R}_\varphi) = \mathsf{A}(\mathbf{R}_\varphi)$, *and we have*

$$p \, \mathrm{core}(\mathbf{R}_\varphi) \, q \quad \textit{iff} \quad \int_Z \varphi(\cdot, y) \, \mathrm{d}p \geq \int_Z \varphi(\cdot, y) \, \mathrm{d}q \text{ for every } y \in Z \tag{11}$$

for any p and q in $\Delta(Z)$.

In the context of the SSB model $\mathbf{R}_\varphi$, we can think of $\varphi(x, y)$ as the "relative utility" of receiving the prize x instead of the prize y. (In other words, $\varphi(\cdot, y)$ acts as a reference-dependent Bernoulli utility function in this model, where y is the reference alternative.) Given this interpretation, $\mathbf{R}_\varphi$ says that lottery p is preferred to lottery q iff the expectation of the relative utility of every prize in the support of p over every prize in that of q with respect to the product measure $p \times q$ is nonnegative. In turn, Proposition 5.12 says that the rational core of $\mathbf{R}_\varphi$ ranks p higher than q – that is, we may think that the subject individual declares p "unambiguously" better than q – iff the expected utility of p is higher than that of q with respect to all of the (reference-dependent) Bernoulli utility functions of the individual. In particular,

[23] This model was first axiomatized by Fishburn (1982). For excellent overviews of the SSB theory, see Fishburn (1984a, 1991).

[24] Indeed, by Proposition 5.12, we have $\mathrm{core}(\mathbf{R}) = \mathsf{A}(\mathbf{R})$. If $\mathbf{R}$ is affine, then, $\mathrm{core}(\mathbf{R}) = \mathbf{R}$, that is, $\mathbf{R}$ is a complete, continuous and affine preorder on $\Delta(Z)$. Conclusion: Every affine SSB model induced by ϕ is an expected utility preference.

we find here that the rational core of the SSB model is also an expected multi-utility preference on $\Delta(Z)$, even though, being nonaffine, this model does not belong to the class of rationalizable risk preferences.

Risk Aversion in the SSB Model. Despite the considerable interest the SSB model has received in the 1980s, the literature, surprisingly, provides little in the way of characterizing risk attitudes in the context of this model.[25] By contrast, we may readily appeal to the results of Section 5.2 and obtain such a characterization. To wit, let φ and ψ be two continuous and skew-symmetric real maps on $[0,1]^2$. Then, by Proposition 5.6, $\mathbf{R}_\varphi$ is more risk averse than $\mathbf{R}_\psi$ if $\{\varphi(\cdot,y) : y \in [0,1]\}$ is more concave than $\{\psi(\cdot,y) : y \in [0,1]\}$. On the other hand, Corollary 5.7 and Proposition 5.12 say that if φ is strictly increasing in the first component and $\succsim$ is an expected utility preference on $\Delta([0,1])$, that is, if there is a strictly increasing $u \in \mathbf{C}[0,1]$ with $p \succsim q$ iff the expectation of u relative to p exceeds that relative to q, then $\mathbf{R}_\varphi$ is more risk averse than $\succsim$ iff $\varphi(\cdot,y)$ is a strictly increasing, continuous and concave transformation of u *for every* $y \in [0,1]$. In particular, by Proposition 5.5 and the remark that follows it, $\mathbf{R}_\varphi$ is risk averse iff $\varphi(\cdot,y)$ is increasing and concave *for every* $y \in [0,1]$.

6 Rational Core and Choice under Uncertainty

In this section we continue "computing" the rational cores of various types of decision-making models, but this time we work in the context of uncertainty. To simplify the exposition, however, we concentrate only on models with a finite state space and monetary consequences.

Let n be a positive integer, which will remain fixed throughout this section. We consider a (Savagean) environment with n many states, labelled as $1, ..., n$. Then, any element x of $\mathbb{R}^n$ can be considered as a **monetary act** which yields a payoff of x_i dollars at state i. Given this interpretation, we may consider $\mathbb{R}^n$ as the space of all Savagean acts on the state space $\{1, ..., n\}$ with real-valued consequences.[26] In turn, for any $x \in \mathbb{R}^n$ and $\mu \in \Delta^{n-1}$, we think of the inner product $\mu \cdot x$ as the expectation of the act x with respect to the prior μ. Thus, any complete preorder on $\mathbb{R}^n$ that is represented by a utility function of the form $x \mapsto \mu \cdot x$ corresponds to a preference relation on the monetary act space $\mathbb{R}^n$ which admits a (subjective) expected utility representation with prior μ.[27]

[25] Fishburn (1984b) is an exception to this, but that paper is particularly focused on the effect of cross partials of φ on the risk attitudes of $\mathbf{R}_\varphi$, and does not say when $\mathbf{R}_\varphi$ would be considered risk averse, nor does it discuss comparative risk aversion in the context of the SSB model.

[26] Alternatively, we can view x as a utility profile as in the Anscombe-Aumann framework (with n states). Or, in the context of social welfare, we may think of the ith component of an n-vector as the cardinal utility of person i in an n-person society. In that interpretation, $\mathbb{R}^n$ would be regarded as the set of all (cardinal) utility profiles in the society.

[27] We restrict our attention to the case of finite state space mainly for expositional purposes. To be

On the other hand, we refer to a preorder $\succsim$ on $\mathbb{R}^n$ for which there is a nonempty, closed and convex subset $\mathcal{M}$ of Δ^{n-1} such that

$$x \succsim y \quad \text{if and only if} \quad \mu \cdot x \geq \mu \cdot y \text{ for every } \mu \in \mathcal{M} \tag{12}$$

for every $x, y \in \mathbb{R}^n$, as a **Bewley preference** (with prior set $\mathcal{M}$) on $\mathbb{R}^n$. Such preferences were introduced by Bewley (1986) in his formulation of Knightian uncertainty theory, and are characterized by the set $\mathcal{M}$, which is interpreted as a set of priors on the state space $\{1, ..., n\}$. They correspond to the (incomplete) preferences of a decision-maker who is unable to pin down his subjective assessment of the relative likelihoods of the states of nature, and who, therefore, possesses a set $\mathcal{M}$ of priors on the state space. This individual ranks an act x over another act y only when the expectation of x is higher than that of y with respect to each of her priors.

6.1 Representation of the Rational Core under Uncertainty

We have seen in Theorem 5.1 that the rational core of any reflexive, continuous and locally non-saturated preference over a lottery space admits an expected multi-utility representation. In the present context, we can show that the rational core of such a preference relation enjoys an expected utility representation with multiple priors, provided that it is monotonic in a suitable sense.

Theorem 6.1. *Let* $\mathbf{R}$ *be a reflexive binary relation on* $\mathbb{R}^n$*. If* $\mathbf{R}$ *is continuous, locally non-saturated and* $\geq$*-transitive, then* $\text{core}(\mathbf{R})$ *is a Bewley preference.*

Our proof is based on the following result which relates closely to the main representation theorem of Bewley (1986). It is proved, in a more general setting, by Ghirardato, Maccheroni and Marinacci (2004, Proposition A.1).

Theorem. *A preorder* $\succsim$ *on* $\mathbb{R}^n$ *is continuous, affine and monotonic if, and only if, it is a Bewley preference.*

Theorem 6.1 is readily proved by combining our earlier findings on the rational core with this theorem.

Proof of Theorem 6.1. By Theorem 3.6, $\text{core}(\mathbf{R})$ is a continuous binary relation on $\mathbb{R}^n$. Moreover, by Corollary 3.2, $\geq$ is an affine subrelation of $\mathsf{T}(\mathbf{R})$. By Theorem 3.4, therefore, $\geq = \mathsf{A}(\geq) \subseteq \mathsf{A}(\mathsf{T}(\mathbf{R})) = \text{core}(\mathbf{R})$. Thus, $\text{core}(\mathbf{R})$ is monotonic. Since $\text{core}(\mathbf{R})$ is an affine preorder on $\mathbb{R}^n$, applying the theorem above completes our proof. ■

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clear, let (Ω, Σ) be a measurable space, and denote by $B_0(\Sigma)$ the set of all Σ-measurable real maps x on Ω with $|x(\Omega)| < \infty$. We view $B_0(\Sigma)$ as a normed linear space relative to the sup-norm, and recall that the norm-dual of $B_0(\Sigma)$ is $\text{ba}(\Omega, \Sigma)$, the set of all bounded and finitely additive set functions on Σ, which we view as a Hausdorff topological linear space relative to the weak*-topology. If we replace $\mathbb{R}^n$ with $B_0(\Sigma)$ and Δ^{n-1} with $\{\mu \in \text{ba}(\Omega, \Sigma) : \mu \geq 0 \text{ and } \mu(\Omega) = 1\}$, and define $\mu \cdot x := \int_X x \, \mathrm{d}\mu$, then all of the results of this section remain valid.

6.2 Justifiable Preferences

Justifiable Preferences. Lehrer and Teper (2011) have recently proposed a closely related model to Bewley's Knightian uncertainty model. Following that paper, we refer to a binary relation $\mathbf{R}$ on $\mathbb{R}^n$ as a **justifiable preference** (with prior set $\mathcal{M}$) if there is a nonempty, closed and convex collection $\mathcal{M}$ of probability n-vectors such that

$$x \mathbf{R} y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for some } \mu \in \mathcal{M}$$

for any x and y in $\mathbb{R}^n$. By contrast to a Bewley preference, a justifiable preference $\mathbf{R}$ is complete, but it need not be transitive (although $\mathbf{R}^>$ is transitive). Both models envisage that an agent may not be able to have a precise assessment of the likelihoods of states of nature, but behaviorally speaking, they are quite different. In particular, an agent with justifiable preferences is always decisive (because all she needs is to justify choosing an act over another with respect to one of her priors), but this decisiveness may lead her to make cyclic choices.[28]

The Rational Core of Justifiable Preferences. It is plain that Bewley preferences and justifiable preferences are closely linked, but fleshing out this connection formally is not a trivial matter. The notion of rational core turns out to be very useful in this regard. Indeed, as the justifiable preference $\mathbf{R}$ is strongly affine, it readily follows from Proposition 3.8 that the rational core of $\mathbf{R}$ coincides with its transitive core. Moreover, obviously, this binary relation is a rationalizable preference on $\mathbb{R}^n$ (as we have defined the term above). Indeed, if, for each μ in $\mathcal{M}$, we write $\succsim_\mu$ for the preference relation on $\mathbb{R}^n$ represented by the map $x \mapsto \mu \cdot x$, then $\mathbf{R} = \bigcup\{\succsim_\mu : \mu \in \mathcal{M}\}$. As each preorder $\succsim_\mu$ is affine here, we may then apply Proposition 4.3 to conclude that

$$x \text{ core}(\mathbf{R})\, y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for all } \mu \in \mathcal{M}$$

for any x and y in $\mathbb{R}^n$. But this means that the Bewley preferences on $\mathbb{R}^n$ induced by the prior set $\mathcal{M}$ is none other than the rational core of the justifiable preferences induced by $\mathcal{M}$. In other words:

Proposition 6.2. *The rational core of a justifiable preference is a Bewley preference* (*with the same set of priors*)*.*

This result is the counterpart of Proposition 5.11 in the context of uncertainty. It says that an individual with a justifiable preference on $\mathbb{R}^n$ prefers an act x over another act y in an "unconflicted" manner (that is, relative to the rational core of her preferences) iff x is better for her than y with respect to *all* of her priors.

The Rational Core of Justifiable Variational Preferences. By a **cost function** on Δ^{n-1}, we mean any map $c : \Delta^{n-1} \to [0, \infty]$ which is grounded, lower semicontinuous

[28] See Cerreia-Vioglio et al. (2016) for a two-preference model in which the justifiable preferences and the Knightian uncertainty model are jointly characterized by means of a completion procedure.

and convex.[29] We denote the *effective domain* of a cost function c on Δ^{n-1} by $\{c < \infty\}$; this is the set of all $\mu \in \Delta^{n-1}$ with $c(\mu) < \infty$.

For any cost function c on Δ^{n-1}, by a **justifiable variational preference** on $\mathbb{R}^n$ **with the cost function** c, we mean the binary relation $\mathbf{R}$ on $\mathbb{R}^n$ with

$$x \mathbf{R} y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y + c(\mu) \text{ for some } \mu \in \Delta^{n-1} \tag{13}$$

for any x and y in $\mathbb{R}^n$. In general, by a **justifiable variational preference**, we mean a justifiable variational preference on $\mathbb{R}^n$ with some cost function c. Such a binary relation is sure to be complete, but in general, it does not have to be either transitive or affine.

This model is patterned after the formulation of *variational preferences* by Maccheroni, Marinacci and Rustichini (2006), and it has recently been axiomatized by Cerreia-Vioglio, et al. (2016). The interpretation is that an agent whose preference relation $\mathbf{R}$ on $\mathbb{R}^n$ is represented as in (13) entertains essentially all priors on the state space. (Put more precisely, the agents uses all priors in $\text{cl}\{\mu < \infty\}$.) But she does not exhibit the same "trust" on every prior. Intuitively speaking, the model allows the agent to act on the basis of any prior μ in the evaluation of two acts x and y, but for her to declare that x is at least as good as y, the expected value of x with respect to μ must be "sufficiently higher than" that of y, that is, it must exceed $\mu \cdot y$ by an amount that depends on μ (namely, $c(\mu)$). We may thus think of $c(\mu)$ as the "cost" of using the prior μ in the evaluation of any two acts.

We note that the model of justifiable variational preferences is a rather substantive generalization of the model of justifiable preferences we have looked at above. Indeed, any justifiable preference on $\mathbb{R}^n$ with prior set $\mathcal{M}$ is a justifiable variational preference with the cost function c, where $c(\mu) := 0$ if $\mu \in \mathcal{M}$, and $c(\mu) := \infty$, otherwise. (We will look at another important special case below.)

Our main result in this section characterizes the rational core of justifiable variational preferences.

Proposition 6.3. *Let $\mathbf{R}$ be a justifiable variational preference on $\mathbb{R}^n$ with the cost function c. Then,*

$$x \text{ core}(\mathbf{R})\, y \quad \textit{iff} \quad \mu \cdot x \geq \mu \cdot y \textit{ for every } \mu \in \text{cl}\{c < \infty\} \tag{14}$$

for any x and y in $\mathbb{R}^n$.

Theorem 6.1 says that the rational core of a justifiable variational preference with a cost function c must be a Bewley preference with some prior set $\mathcal{M}$ on $\mathbb{R}^n$. In turn, Proposition 6.3 tells us exactly what that prior set is in terms of the cost function c. It turns out that $\mathcal{M}$ is precisely the closure of the effective domain of c (which is a convex set due to convexity of c). We will use this fact to obtain an important insight about the rational core operator below.

[29] By *groundedness* of c, we mean that $c(\mu) = 0$ for at least one $\mu \in \Delta^{n-1}$.

Remark 6.1. There are alternative ways of representing the rational core of a justifiable variational preference. In particular, where $\mathbf{R}$ and c are as in Proposition 6.3, and where $I:\mathbb{R}^n \to \mathbb{R}$ is defined as $I(z) := \max\{\mu \cdot z - c(\mu) : \mu \in \Delta^{n-1}\}$, we have x $\mathbf{R}$ y iff $I(x-y) \geq 0$, and

$$x \text{ core}(\mathbf{R})\ y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for every } \mu \in \bigcup_{z \in I^{-1}(0)} \partial I(z)$$

for any x and y in $\mathbb{R}^n$. (Here, $\partial I(z)$ stands for the subdifferential of the convex map I at z.) In turn, using this representation, one can show that the rational core of $\mathbf{R}$ reduces to the transitive core of $\mathbf{R}$ in this instance, that is, $\text{core}(\mathbf{R}) = \mathsf{T}(\mathbf{R})$. For brevity, we omit the proofs of these assertions (which are, of course, available from the authors upon request).

Remark 6.2. It is easy to compute the affine core of a justifiable variational preference. As we will need it shortly, we provide this computation here. Where $\mathbf{R}$ and c are as in Proposition 6.3, our claim is:

$$x\ \mathsf{A}(\mathbf{R})\ y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for some } \mu \in c^{-1}(0)$$

for any x and y in $\mathbb{R}^n$. To prove this, define the binary relation $\trianglerighteq$ on $\mathbb{R}^n$ by $x \trianglerighteq y$ iff $\mu \cdot x \geq \mu \cdot y$ for some $\mu \in c^{-1}(0)$. Since $c^{-1}(0)$ is nonempty (because c is grounded), $\trianglerighteq$ is a subrelation of $\mathbf{R}$. As it is readily checked to be affine, therefore, $\trianglerighteq$ must be a subrelation of $\mathsf{A}(\mathbf{R})$. Conversely, take any two n-vectors x and y with $x\ \mathsf{A}(\mathbf{R})\ y$. Then, by Proposition 3.3, for every positive integer m, there is a $\mu_m \in \Delta^{n-1}$ such that $\mu_m \cdot \left(\frac{1}{m}x + \left(1 - \frac{1}{m}\right) y\right) \geq \mu_m \cdot y + c(\mu_m)$, that is,

$$\mu_m \cdot x \geq \mu_m \cdot y + mc(\mu_m) \quad \text{for each } m = 1, 2, ... \tag{15}$$

Since Δ^{n-1} is a compact subset of $\mathbb{R}^n$, there is a subsequence of (μ_m) that converges to a point $\mu \in \Delta^{n-1}$; relabelling if necessary, we denote this subsequence also as (μ_m), and thus write $\mu_m \to \mu$. Since (15) implies $\mu_m \cdot x \geq \mu_m \cdot y$ for each m, we obviously have $\mu \cdot x \geq \mu \cdot y$. On the other hand, (15) also implies that $\max\{x_1, ..., x_n\} - \min\{y_1, ..., y_n\} \geq mc(\mu_m)$ for every positive integer m, which is possible only if $\inf\{c(\mu_m) : m \in \mathbb{N}\} = 0$. Thus, there is a sequence (m_k) in $\mathbb{N}$ with $c(\mu_{m_k}) \downarrow 0$ (as $k \uparrow \infty$). Then, since c is lower semicontinuous, $c(\mu) \leq \liminf c(\mu_{m_k}) = 0$, and we find that $c(\mu) = 0$, that is, $\mu \in c^{-1}(0)$. This proves that $\mathsf{A}(\mathbf{R})$ is a subrelation of $\trianglerighteq$, and we are done.

The Rational Core of Justifiable Multiplier Preferences. We now turn to an important special case of justifiable variational preferences. For any (probability) vectors $\mu, \nu \in \Delta^{n-1}$ such that $\nu_i > 0$ for each $i = 1, ..., n$, we recall that the *Kullback-Leibler divergence from* ν *to* μ (also known as the *relative entropy of* μ *with respect to* ν) is defined as

$$D(\mu \| \nu) := \sum_{i=1}^{n} \mu_i \log \left(\frac{\mu_i}{\nu_i}\right)$$

(with the convention that $0(-\infty) = 0$). By Gibbs' Inequality, we have $D(\mu \| \nu) \geq 0$ for every $\mu \in \Delta^{n-1}$, and $D(\mu \| \nu) = 0$ iff $\mu = \nu$. Moreover, $D(\cdot \| \nu)$ is grounded (since $D(\nu \| \nu) = 0$), continuous and convex. Thus, $D(\cdot \| \nu)$ is a (real-valued) cost function on Δ^{n-1}.

Take any $\nu \in \Delta^{n-1}$ with $\nu_i > 0$ for each $i = 1, ..., n$. We define the **justifiable multiplier preference** on $\mathbb{R}^n$ **with the reference probability** ν as the justifiable variational preference on $\mathbb{R}^n$ with the cost function $D(\cdot \| \nu)$. This is the binary relation $\mathbf{R}$ on $\mathbb{R}^n$ with

$$x \mathbf{R} y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y + D(\mu \| \nu) \text{ for some } \mu \in \Delta^{n-1}$$

for any x and y in $\mathbb{R}^n$. The interpretation of this relation parallels that of the *multiplier preferences* introduced by Hansen and Sargent (2001). In comparing any two acts, the individual with this preference relation $\mathbf{R}$ has some best guess about the true likelihoods of states of nature; this is captured by ν. But she does not have full confidence in ν; instead, she considers all other priors as possible, but regards a prior less and less plausible as this prior diverges more and more from ν. Thus, for this agent to view an act x at least as good as another act y, the required gain in expectation of x over that of y with respect to a prior μ is increasing in the "distance" of μ from ν.

Now, since $D(\cdot \| \nu)$ is real-valued, Proposition 6.3 says that x core($\mathbf{R}$) y iff $\mu \cdot x \geq \mu \cdot y$ for every $\mu \in \Delta^{n-1}$. But, of course, the latter statement is equivalent to say that $x \geq y$. Thus, we find that the rational core of any justifiable multiplier preference on $\mathbb{R}^n$ (with a strictly positive reference probability vector) is none other than the usual (coordinatewise) ordering of $\mathbb{R}^n$:

$$x \text{ core}(\mathbf{R}) \ y \quad \text{iff} \quad x \geq y$$

for any x and y in $\mathbb{R}^n$.

This observation allows us to demonstrate that, in general, $\mathsf{A}(\mathsf{T}(\mathbf{R})) \neq \mathsf{T}(\mathsf{A}(\mathbf{R}))$. Indeed, in view of Remark 6.2, and because $D(\mu \| \nu) = 0$ iff $\mu = \nu$, we have $x \ \mathsf{A}(\mathbf{R}) \ y$ iff $\nu \cdot x \geq \nu \cdot y$, in the present case. It follows that $\mathsf{A}(\mathbf{R})$ is transitive, and hence, $x \ \mathsf{T}(\mathsf{A}(\mathbf{R})) \ y$ iff $\nu \cdot x \geq \nu \cdot y$. For instance, where $\nu := (\frac{1}{n}, ..., \frac{1}{n})$, we have $x \ \mathsf{T}(\mathsf{A}(\mathbf{R})) \ y$ iff $\frac{1}{n} \sum x_i \geq \frac{1}{n} \sum y_i$. Thus, not only is $\mathsf{A}(\mathsf{T}(\mathbf{R}))$ is a proper subrelation of $\mathsf{T}(\mathsf{A}(\mathbf{R}))$ in this case, but there is a rather serious gap between these two preorders. While $\mathsf{A}(\mathsf{T}(\mathbf{R}))$ is the coordinatewise ordering of $\mathbb{R}^n$ (which is highly incomplete), $\mathsf{T}(\mathsf{A}(\mathbf{R}))$ is a complete preorder that ranks any n-vector over any other n-vector with smaller arithmetic mean.

7 Conclusion

In this paper, we have defined, axiomatically, an operator that maps any given reflexive binary relation $\mathbf{R}$ on a nonempty convex subset of a linear space X to the largest transitive and affine subrelation of that relation with respect to which $\mathbf{R}$ is transitive. The latter preorder is called the *rational core* of $\mathbf{R}$, and is denoted by core($\mathbf{R}$). We interpret X as a space of alternatives (e.g. a space of risky or uncertain prospects), and $\mathbf{R}$ as a revealed (observed) preference relation which may violate rationality traits of transitivity and/or affinity. In turn, we are interested in identifying the largest "rational part" of $\mathbf{R}$ which, presumably, arises from the comparisons of alternatives that the decision maker feels particularly confident about. Of course, when an individual is "sure" of her preferential rankings, and when she is "conflicted" about them, is unobservable (unlike $\mathbf{R}$). We thus approach the notion of the largest "rational part" of $\mathbf{R}$ axiomatically, and define it through behavioral properties that such a subrelation of $\mathbf{R}$ would reasonably satisfy. This leads us to propose core($\mathbf{R}$), which is observable, as the binary relation that may well capture the essence of the largest "rational part" of $\mathbf{R}$.

As all axiomatically defined concepts, the use of the notion of rational core would eventually be determined by its structural properties and amenability for applications. Our main results in this paper show that in most cases of interest, the rational core of a binary relation has an appealing structure. Moreover, this concept decomposes into two other core concepts developed in the earlier literature on decision theory. We have applied these results to compute the rational cores of a number of well-known preference models, and in the context of risk, looked at two simple, conceptual applications.

There are many avenues for further research on the rational core (or its potential variations). Perhaps most important of these is to develop an optimization theory for reflexive binary relations through their rational core. Given our interpretation of things here, it is natural to define an element in a nonempty subset S of X as a *rational maximal* in S with respect to $\mathbf{R}$ if this element is maximal in S with respect to core($\mathbf{R}$). Investigating this optimality notion, and its game-theoretic generalizations (such as *rational Nash equilibrium*) looks like an inviting project. Second, it will be interesting to see how the rational core operator would work in applications. The general thrust of this kind of research would be to take a result that is known to hold under the expected utility hypothesis (such as the willingness to pay of a risk averse agent for actuarially fair risk, and/or portfolio diversification), and examine if this result continues to hold with respect to the rational core of a given preference relation which may fail the expected utility hypothesis in a variety of ways. We hope that our theoretical work here will prove useful in the context of such applications.

APPENDIX: Proofs

This appendix contains the proofs of the results that were omitted in the body of the text.

Proof of Theorem 3.6

The crux of the argument is that the transitive core of a reflexive binary relation on a topological space inherits the continuity of that relation.

Lemma A.1. *Let $\mathbf{R}$ be a reflexive binary relation on a topological space X. If $\mathbf{R}$ is continuous and locally non-saturated, then $\mathsf{T}(\mathbf{R})$ is continuous.*

Proof. Assume that $\mathbf{R}$ is continuous and locally non-saturated, and take any net (x_τ, y_τ) in $X \times X$ with $x_\tau \ \mathsf{T}(\mathbf{R}) \ y_\tau$ for each τ. Suppose $x_\tau \to x$ and $y_\tau \to y$ for some x and y in X. We wish to show that $x \ \mathsf{T}(\mathbf{R}) \ y$. To this end, take any $z \in x^\uparrow$. If $z = x$, then, since $x_\tau \ \mathbf{R} \ y_\tau$ for each τ (because $\mathsf{T}(\mathbf{R}) \subseteq \mathbf{R}$), we get $z \ \mathbf{R} \ y$ by continuity of $\mathbf{R}$, and we find $z \in y^\uparrow$. We thus assume that z and x are distinct. Let $\mathcal{O}$ stand for the set of all open neighborhoods of z, consider it as a (directed) partially ordered set relative to set inclusion. As $\mathbf{R}$ is locally non-saturated, for every $O \in \mathcal{O}$, there are a $z_O \in O$ and an open neighborhood U of x such that $z_O \ \mathbf{R} \ w$ for every $w \in U$. Then, (z_O) is a net in X with the index set $\mathcal{O}$, and clearly, $z_O \to z$. On the other hand, for each $O \in \mathcal{O}$, we have $z_O \ \mathbf{R} \ x_\tau$ eventually for all τ (because $x_\tau \to x$). Therefore, for each $O \in \mathcal{O}$, we have $z_O \in x_\tau^\updownarrow$ eventually for all τ, while, by Proposition 3.1, $x_\tau^\updownarrow \subseteq y_\tau^\updownarrow$ for all τ, and it follows that $z_O \ \mathbf{R} \ y_\tau$ eventually for all τ. Since $\mathbf{R}$ is continuous, therefore, $z_O \ \mathbf{R} \ y$ for each $O \in \mathcal{O}$, and hence using the continuity of $\mathbf{R}$ one more time, we get $z \ \mathbf{R} \ y$, that is, $z \in y^\uparrow$. In view of the arbitrary choice of z, we may thus conclude that $x^\uparrow \subseteq y^\uparrow$. As we can similarly verify that $y^\downarrow \subseteq x^\downarrow$, we can use Proposition 3.1 to conclude that $x \ \mathsf{T}(\mathbf{R}) \ y$, as we sought. ■

Continuity of a reflexive binary relation on a topological linear space is readily inherited by the affine core of that relation.

Lemma A.2. *Let $\mathbf{R}$ be a reflexive binary relation on a nonempty convex subset X of a topological linear space. If $\mathbf{R}$ is continuous, then $\mathsf{A}(\mathbf{R})$ is continuous.*

Proof. Assume that $\mathbf{R}$ is continuous, and take any net (x_τ, y_τ) in $X \times X$ such that $x_\tau \ \mathsf{A}(\mathbf{R}) \ y_\tau$ for each τ. Suppose that $x_\tau \to x$ and $y_\tau \to y$ for some x and y in X. We wish to show that $x \ \mathsf{A}(\mathbf{R}) \ y$. To this end, take any $z \in X$ and $\lambda \in (0, 1]$, and note that $\lambda x_\tau + (1 - \lambda) z \ \mathbf{R} \ \lambda y_\tau + (1 - \lambda) z$ for every τ by Proposition 3.3. Therefore, by continuity of $\mathbf{R}$ and that of the addition and scalar multiplication operations on X, we have $\lambda x + (1 - \lambda) z \ \mathbf{R} \ \lambda y + (1 - \lambda) z$. In view of the arbitrariness of z and λ, and Proposition 3.3, we conclude that $x \ \mathsf{A}(\mathbf{R}) \ y$, as we sought. ■

Proof of Theorem 3.6. Assume that $\mathbf{R}$ is a continuous and locally non-saturated reflexive binary relation on a nonempty convex subset X of a topological linear space. Then, by Lemma A.1 and Proposition 3.1, $\mathsf{T}(\mathbf{R})$ is a continuous preorder on X. We may thus apply Lemma A.2 and Theorem 3.4 to conclude that $\text{core}(\mathbf{R})$ is continuous. ■

Proof of Proposition 3.7

The argument is based on the following observation.

Lemma A.3. *Let $\succsim$ be a continuous preorder on a nonempty convex subset X of a topological linear space. Then, $\succsim$ is affine if, and only if, it is strongly affine.*

While its setting is slightly different, the proof of this result is identical to that of Lemma 1 of Dubra, Maccheroni and Ok (2004), so we omit it here.

Proof of Proposition 3.7. By definition, $\text{core}(\mathbf{R})$ contains every strongly affine subrelation $\mathbf{S}$ of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive. But, combining Theorem 3.6 and Lemma A.3 shows that $\text{core}(\mathbf{R})$ is itself a strongly affine subrelation of $\mathbf{R}$ such that $\mathbf{R}$ is $\text{core}(\mathbf{R})$-transitive. It thus follows from the definition of $\text{core}^*(\mathbf{R})$ that $\text{core}(\mathbf{R}) = \text{core}^*(\mathbf{R})$. ■

Proof of Proposition 5.3

The proof of the "if" part of the assertion here is routine, so we focus only on its "only if" part. We know from Theorem 5.1 that there is a nonempty $\mathcal{U} \subseteq \mathbf{C}[0,1]$ such that

$$p \text{ core}(\mathbf{R})\ q \quad \text{if and only if} \quad \int_{[0,1]} u\,\mathrm{d}p \geq \int_{[0,1]} u\,\mathrm{d}q \text{ for every } u \in \mathcal{U} \tag{16}$$

for any p and q in $\Delta := \Delta([0,1])$. For any binary relation $\mathbf{S}$ on Δ, let $\mathcal{U}(\mathbf{S})$ stand for the set of all $u \in \mathbf{C}[0,1]$ such that

$$p\ \mathbf{S}\ q \text{ implies } \int_{[0,1]} u\,\mathrm{d}p \geq \int_{[0,1]} u\,\mathrm{d}q$$

for every p and q in Δ. By (16), we have $\varnothing \neq \mathcal{U} \subseteq \mathcal{U}(\text{core}(\mathbf{R}))$. This implies readily that $\mathcal{U}(\text{core}(\mathbf{R}))$ is an expected multi-utility representation for core($\mathbf{R}$). On the other hand, $\mathcal{U}(\geq_{\text{FSD}})$ coincides with the set of all continuous and increasing real maps on $[0,1]$. But, since $\mathbf{R}$ is monotonic,

$$p \geq_{\text{FSD}} q \quad \text{implies} \quad \int_{[0,1]} u\,\mathrm{d}p \geq \int_{[0,1]} u\,\mathrm{d}q$$

for any p and q in Δ and any $u \in \mathcal{U}(\text{core}(\mathbf{R}))$. Thus: $\mathcal{U} \subseteq \mathcal{U}(\text{core}(\mathbf{R})) \subseteq \mathcal{U}(\geq_{\text{FSD}})$. We conclude that core($\mathbf{R}$) admits an expected multi-utility representation whose elements are continuous and increasing real maps on $[0,1]$.

Now assume that $\mathbf{R}$ is $\geq_{\text{FSD}}$-transitive, and we have $\delta_a\ \mathbf{R}^{>}\ \delta_b$ whenever $a > b$. By Proposition 5.2, $\mathbf{R}$ is monotonic, that is, $\geq_{\text{FSD}} \subseteq \text{core}(\mathbf{R})$. Thus, $a > b$ implies $\delta_a \text{ core}(\mathbf{R})\ \delta_b$. Moreover, if $\delta_b \text{ core}(\mathbf{R})\ \delta_a$ held here, we would have $\delta_b\ \mathbf{R}\ \delta_a$, which, by hypothesis, can hold only if $b \geq a$. Conclusion: If either $\geq_{\text{FSD}}$-transitive and $\delta_a\ \mathbf{R}^{>}\ \delta_b$ whenever $a > b$, or $\mathbf{R}$ is strictly monotonic, then $a > b$ implies $\delta_a \text{ core}(\mathbf{R})^{>}\ \delta_b$. In what follows, we use just this latter property to show that core($\mathbf{R}$) admits a strictly increasing multi-utility representation.

Finally, let $\mathcal{U} \subseteq \mathbf{C}[0,1]$ be found as in the first part of the proposition. Combining Theorem 3.6 with Proposition 3 of Dubra, Maccheroni, and Ok (2004), we find a map $f \in \mathbf{C}[0,1]$ such that

$$p \text{ core}(\mathbf{R})\ q \quad \text{implies} \quad \int_{[0,1]} f\,\mathrm{d}p \geq \int_{[0,1]} f\,\mathrm{d}q$$

and

$$p \text{ core}(\mathbf{R})^{>}\ q \quad \text{implies} \quad \int_{[0,1]} f\,\mathrm{d}p > \int_{[0,1]} f\,\mathrm{d}q.$$

Since $a > b$ implies $\delta_a \text{ core}(\mathbf{R})^{>}\ \delta_b$, it is clear that f is strictly increasing. To complete the proof, then, define $\mathcal{V} := \{\lambda f + (1-\lambda)u : 0 < \lambda < 1 \text{ and } u \in \mathcal{U}\}$, and note that $\mathcal{V}$ is an expected multi-utility representation of core($\mathbf{R}$) whose elements are all continuous and strictly increasing. It is also plain that $>_{\text{FSD}} \subseteq \text{core}(\mathbf{R})^{>}$. In the case where $\mathbf{R}$ is $\geq_{\text{FSD}}$-transitive and $\delta_a\ \mathbf{R}^{>}\ \delta_b$ whenever $a > b$, this implies that $\mathbf{R}$ is strictly monotonic.

Proof of Proposition 5.5

The proof obtains by exactly the same argument we gave in the first paragraph of the proof of Proposition 5.3, and thus omitted.

Proof of Proposition 5.6

For each $i \in \{1,2\}$, put $\succsim_i := \text{core}(\mathbf{R}_i)$, and let $\mathcal{U}_i$ be a nonempty subset of $\mathbf{C}[0,1]$ such that $p \succsim_i q$ iff $\int_{[0,1]} u\,\mathrm{d}p \geq \int_{[0,1]} u\,\mathrm{d}q$ for every $u \in \mathcal{U}_i$. Assume that $\mathcal{U}_1$ is more concave than $\mathcal{U}_2$, and note that we will be done if we can show that $\succsim_1$ is more risk averse than $\succsim_2$.

To this end, take an arbitrary $p \in \Delta$ and $a \in [0,1]$ such that $p \succsim_1 \delta_a$. It follows that $\int_{[0,1]} u \,\mathrm{d}p \geq u(a)$ for every $u \in \mathcal{U}_1$. Next, take any $v \in \mathcal{U}_2$. By assumption, there exist a $u \in \mathcal{U}_1$ and a strictly increasing, continuous, and concave $f : v([0,1]) \to u([0,1])$ such that $u = f \circ v$. By Jensen's Inequality, it follows that

$$f\left(\int_{[0,1]} v \,\mathrm{d}p\right) \geq \int_{[0,1]} f \circ v \,\mathrm{d}p = \int_{[0,1]} u \,\mathrm{d}p \geq u(a) = f(v(a)),$$

and hence

$$\int_{[0,1]} v \,\mathrm{d}p \geq v(a)$$

because f is strictly increasing. In view of the arbitrary choice of v in $\mathcal{U}_2$, therefore, we find $p \succsim_2 \delta_a$. As the analogous argument would show that $\delta_a \succsim_2 p$ implies $\delta_a \succsim_1 p$, we conclude that $\mathbf{R}_1$ is more risk averse than $\mathbf{R}_2$.

Proof of Corollary 5.7

Let $\mathcal{U}$ be a strictly increasing expected multi-utility representation for $\mathbf{R}_1$, and v a strictly increasing Bernoulli utility the expectation of which represents $\mathbf{R}_2$. The "if" part of the corollary follows from Proposition 5.6. To prove the "only if" part, for any strictly increasing $w \in \mathbf{C}[0,1]$, define the *certainty equivalent* map $\mathrm{ce}_w : \Delta \to [0,1]$ by $\mathrm{ce}_w(p) := w^{-1}\left(\int_{[0,1]} w \,\mathrm{d}p\right)$, which is well-defined and continuous. Now, for any $p \in \Delta$, we have $v(\mathrm{ce}_v(p)) = \int_{[0,1]} v \,\mathrm{d}p$, that is, $\delta_{\mathrm{ce}_v(p)}$ $\mathbf{R}_2$ p. Since $\mathbf{R}_2$ is an expected utility preference, it is equal to its own rational core, and hence $\delta_{\mathrm{ce}_v(p)}$ $\mathrm{core}(\mathbf{R}_2)$ p for any $p \in \Delta$. Since $\mathbf{R}_1$ is more risk averse than $\mathbf{R}_2$, therefore, $\delta_{\mathrm{ce}_v(p)}$ $\mathrm{core}(\mathbf{R}_1)$ p for any $p \in \Delta$. In other words, $u(\mathrm{ce}_v(p)) \geq \int_{[0,1]} u \,\mathrm{d}p$ for every $p \in \Delta$ and $u \in \mathcal{U}$. Since u is strictly increasing, and $\int_{[0,1]} u \,\mathrm{d}p = u(\mathrm{ce}_u(p))$ for every $p \in \Delta$, therefore, we conclude that $\mathrm{ce}_v(\cdot) \geq \mathrm{ce}_u(\cdot)$ for each $u \in \mathcal{U}$. It is well-known that this is the same thing as saying that u is more concave than v, for each $u \in \mathcal{U}$.

Proof of Corollary 5.8

By Theorem 5.1 and Proposition 5.3, there is a nonempty and strictly increasing subset $\mathcal{U}$ of $\mathbf{C}[0,1]$ such that (8) holds for every $p, q \in \Delta$. Recall that the risk neutral binary relation $\succsim$ is the expected utility preference whose Bernoulli utility function is $\mathrm{id}_{[0,1]}$, the identity map on $[0,1]$. By Corollary 5.7, therefore, $\mathbf{R}$ is more risk averse than $\succsim$ iff $\mathcal{U}$ is more concave than the set $\{\mathrm{id}_{[0,1]}\}$ which is the same thing as saying that each element of $\mathcal{U}$ is concave. Thus, by Proposition 5.5, $\mathbf{R}$ is more risk averse than $\succsim$ iff $\mathbf{R}$ is risk averse.

Proof of Proposition 5.12

Define $\Phi : \Delta(Z) \times \Delta(Z) \to \mathbb{R}$ by

$$\Phi(p,q) := \int_Z \int_Z \varphi \,\mathrm{d}p \,\mathrm{d}q,$$

and notice that an immediate application of Fubini's Theorem ensures that Φ is skew-symmetric, that is, $\Phi(p,q) = -\Phi(q,p)$ for every $p, q \in \Delta(Z)$. Next, define the binary relation $\succsim$ on $\Delta(Z)$ by $p \succsim q$ iff the right-hand side of (11) holds. Obviously, $\succsim$ is an affine preorder. Furthermore, $p \succsim q$ implies that $\Phi(p,q) \geq \Phi(q,q) = 0$ for any p and q in $\Delta(Z)$, so $\succsim$ is a subrelation of $\mathbf{R}_\varphi$. Now notice that, for every p and q in $\Delta(Z)$,

$$p \succsim q \quad \text{iff} \quad \Phi(p,r) \geq \Phi(q,r) \text{ for every } r \in \Delta(Z). \tag{17}$$

Therefore, $p \succsim q \;\mathbf{R}_\varphi\; r$ implies $\Phi(p,r) \geq \Phi(q,r) \geq 0$, that is, $p \;\mathbf{R}_\varphi\; r$. Similarly, if $p \;\mathbf{R}_\varphi\; q \succsim r$, we may use (17) (with q playing the role of p, and r that of q) to find $\Phi(q,p) \geq \Phi(r,p)$. Then, using the skew-symmetry of Φ and $p\ \mathbf{R}_\varphi\ q$, we obtain $0 \geq -\Phi(p,q) \geq -\Phi(p,r)$, that is, $p\ \mathbf{R}_\varphi\ r$. Conclusion: $\succsim$ is an affine subrelation of $\mathbf{R}_\varphi$ such that $\mathbf{R}_\varphi$ is $\succsim$-transitive. Thus: $\succsim \subseteq \text{core}(\mathbf{R}_\varphi)$.

Now, to prove that $\text{core}(\mathbf{R}_\varphi) = \mathsf{A}(\mathbf{R}_\varphi)$, we fix any p and q in $\Delta(Z)$, and using the affinity of Φ in its first and second arguments, observe that, for any $r \in \Delta(Z)$ and $\lambda \in (0,1]$,

$$\begin{aligned}\Phi(\lambda p + (1-\lambda)r, \lambda q + (1-\lambda)r) &= \lambda^2\Phi(p,q) + \lambda(1-\lambda)\Phi(p,r) + (1-\lambda)\lambda\Phi(r,q) \\ &= \lambda\left(\lambda\Phi(p,q) + (1-\lambda)(\Phi(p,r) - \Phi(q,r)\right),\end{aligned}$$

because $\Phi(r,r) = 0$ and $\Phi(r,q) = -\Phi(q,r)$. Notice that the right-most part of these equations can be nonnegative for all $\lambda \in (0,1]$ iff both $\Phi(p,q) \geq 0$ and $\Phi(p,r) - \Phi(q,r)$ are nonnegative, whereas, by (17), $\Phi(p,r) - \Phi(q,r) \geq 0$ for every $r \in \Delta(Z)$ iff $p \succsim q$. It follows from Proposition 3.3 that $p\ \mathsf{A}(\mathbf{R}_\varphi)\ q$ iff $p\ \mathbf{R}_\varphi\ q$ and $p \succsim q$. As $\succsim$ is a subrelation of $\mathbf{R}_\varphi$, therefore, $p\ \mathsf{A}(\mathbf{R}_\varphi)\ q$ iff $p \succsim q$. Thus: $\mathsf{A}(\mathbf{R}_\varphi) = \succsim \subseteq \text{core}(\mathbf{R}_\varphi)$. But as $\text{core}(\mathbf{R}_\varphi)$ is an affine subrelation of $\mathbf{R}_\varphi$, we obviously have $\text{core}(\mathbf{R}_\varphi) \subseteq \mathsf{A}(\mathbf{R}_\varphi)$. Conclusion: $\mathsf{A}(\mathbf{R}_\varphi) = \succsim = \text{core}(\mathbf{R}_\varphi)$.

Proof of Proposition 6.3

Let us define the preorder $\succsim$ on $\mathbb{R}^n$ by

$$x \succsim y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for every } \mu \in \text{cl}\{c < \infty\}.$$

Now take any $x, y, z \in \mathbb{R}^n$ such that $x \succsim y\ \mathbf{R}\ z$. Then, $\mu \cdot y \geq \mu \cdot z + c(\mu)$ for some $\mu \in \Delta^{n-1}$. Obviously, $c(\mu) < \infty$, so we have $\mu \cdot x \geq \mu \cdot y$ by definition of $\succsim$. It follows that $\mu \cdot x \geq \mu \cdot z + c(\mu)$, that is, $x\ \mathbf{R}\ z$. As we can similarly verify that $x\ \mathbf{R}\ y \succsim z$ implies $x\ \mathbf{R}\ z$, we conclude that $\mathbf{R}$ is $\succsim$-transitive. It then follows from Corollary 3.2 that $\succsim$ is a subrelation of $\mathsf{T}(\mathbf{R})$. By Theorem 3.4, and since $\succsim$ is affine, we then have $\succsim \subseteq \mathsf{A}(\mathsf{T}(\mathbf{R})) = \text{core}(\mathbf{R})$. The rest of the proof is geared toward establishing the converse of this fact.

We begin by noting that $\mathbf{R}$ is reflexive (because $c^{-1}(0) \neq \varnothing$), and it is easily checked to be a closed subset of $X \times X$. Moreover, $\mathbf{R}$ is $\geq$-transitive, because if $x \geq y\ \mathbf{R}\ z$, then $\mu \cdot x \geq \mu \cdot y \geq \mu \cdot z + c(\mu)$ for some $\mu \in \Delta^{n-1}$, and hence $x\ \mathbf{R}\ z$. (Similarly, $x\ \mathbf{R}\ y \geq z$ implies $x\ \mathbf{R}\ z$.) Finally, we verify that $\mathbf{R}$ is locally non-saturated. To this end, take any $x, y \in \mathbb{R}^n$ with $x\ \mathbf{R}\ y$ so that $\mu \cdot x \geq \mu \cdot y + c(\mu)$ for some $\mu \in \Delta^{n-1}$, and pick any open neighborhood O of x in $\mathbb{R}^n$. Then, there exists an $\varepsilon > 0$ small enough that $x + \varepsilon\mathbf{1} \in O$, where $\mathbf{1}$ is the n-vector of 1s. Choose U to be the open ball around y with radius ε relative to the $\|\cdot\|_\infty$ norm. As $z \mapsto \mu \cdot z$ is a 1-Lipschitz real map on $\mathbb{R}^n$ (relative to the $\|\cdot\|_\infty$ norm), we have $\mu \cdot y \geq \mu \cdot z - \varepsilon$ for every $z \in U$. But then $\mu \cdot (x + \varepsilon\mathbf{1}) = \mu \cdot x + \varepsilon \geq \mu \cdot y + \varepsilon + c(\mu) \geq \mu \cdot z + c(\mu)$, so $x + \varepsilon\mathbf{1}\ \mathbf{R}\ z$, for every $z \in U$. As requirement (ii) of being locally non-saturated is similarly verified, we conclude that $\mathbf{R}$ is locally non-saturated.

In view of what we have shown in the previous paragraph, we may apply Theorem 6.1 to find a nonempty closed and convex subset $\mathcal{M}$ of Δ^{n-1} such that

$$x \text{ core}(\mathbf{R})\ y \quad \text{iff} \quad \mu \cdot x \geq \mu \cdot y \text{ for every } \mu \in \mathcal{M}$$

for any $x, y \in \mathbb{R}^n$. Since $\succsim \subseteq \text{core}(\mathbf{R})$, we have $\mathcal{M} \subseteq \text{cl}\{c < \infty\}$. Then, clearly, our proof will be complete if we can show that $\{c < \infty\} \subseteq \mathcal{M}$. To derive a contradiction, let us suppose that this is false, and take any $\mu \in \Delta^{n-1} \backslash \mathcal{M}$ with $c(\mu) < \infty$. Then, given that $\mathcal{M}$ is a compact subset of $\mathbb{R}^n$, we may apply the Separating Hyperplane Theorem to find an n-vector x such that $\mu \cdot x < 0 < \min_{\sigma \in \mathcal{M}} \sigma \cdot x$. We then choose an $\varepsilon > 0$ small enough so that

$$\mu \cdot x < -\varepsilon < \varepsilon < \min_{\sigma \in \mathcal{M}} \sigma \cdot x.$$

On the other hand, since $c(\mu) < \infty$, there is a real number $K > 0$ large enough that $c(\mu) < K\varepsilon$. Clearly, for the n-vector $y := K\varepsilon\mathbf{1}$, we have $\sigma \cdot y = K\varepsilon < K(\sigma \cdot x)$ for every $\sigma \in \mathcal{M}$, so Kx core$(\mathbf{R})$ y, while

$$\mu \cdot Kx + c(\mu) < K(-\varepsilon) + K\varepsilon = 0 = \mu \cdot \mathbf{0},$$

so $\mathbf{0}$ $\mathbf{R}$ Kx. (Here $\mathbf{0}$ stands for the n-vector of 0s.) Since $\mathbf{R}$ is core($\mathbf{R}$)-transitive, therefore, we find $\mathbf{0}$ $\mathbf{R}$ y. But this is impossible, because $\mathbf{0}$ $\mathbf{R}$ y implies $0 = \nu \cdot \mathbf{0} \geq \nu \cdot y + c(\nu) \geq \nu \cdot y = K\varepsilon > 0$ for some $\nu \in \Delta^{n-1}$, a contradiction.

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