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Paper: “Restrictions on Risk Prices in Dynamic Term Structure Models ”

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Introduction

Policymakers and researches are interested in estimating the expectations and term premium components in long-term interest rates. Dynamic term structure models (DTSM) which impose absence of arbitrage are used for this purpose

- The no-arbitrage assumption can be powerful if it creates a link between the cross-sectional variation of interest rates and their time-series variation, but it only does so if the risk adjustment is restricted.
- This paper provides an econometric framework for estimating DTSMs under restrictions on risk prices
- Estimation of term premia amounts to estimation of expectations of future short-term rate.
  - Doing so with only time series information is extremely difficult, because the very high persistence of interest rates leads to large **statistical uncertainty** and **small-sample bias**
- The no arbitrage assumption in DTSM can alleviate these problems, because it requires that cross section of interest rates reflects forecasts of future of future short rates, allowing for a risk adjustment, the cross sectional information can help to pin down the unobserved expectations

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Introduction

- Choosing restrictions on the parameters that determine the risk adjustment is difficult:
  - **model selection** is complicated by the large number of possible restrictions
  - choice of restrictions entails **model uncertainty** → equally plausible models, which differ only little in terms of risk-price restrictions, often reveal dramatically different short-rate expectations and term premia
- This paper introduces a Bayesian econometric framework that overcomes these challenges → The framework relies on Markov chain Monte Carlo (MCMC) methods to estimate affine Gaussian DTSMs with risk price restrictions
  - **Model selection** does not require separate estimation of every single possible model specification because the MCMC samplers visit only plausible models and do not waste time in other areas of the model space.
  - **Model uncertainty** is dealt with by means of Bayesian Model Averaging (BMA).

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- Risk-price restrictions change our interpretation of the evolution of interest rates over certain historical episodes, in particular of the secular decline in long rates over the last two decades
  - **Conventional** DTSMs explain this by substantial declines in term premia and imply only a small role for short-rate expectations
  - **Restricted models** attribute a more important role to a declining expectations component.

    The finding that expectations of short-term interest rates have decreased over the 1990's and 2000's is consistent with the sizable declines in survey-based expectations of inflation and policy rates

Econometric framework: DTSM

- The models used in this paper belong to the class of affine Gaussian Dynamic term structure model (DTSM)
- X_t is the $(N \times 1)$ vector of risk factors, which represents the new information that market participants obtain at time t
  - assume that X_t follows a 1st order VAR under the **physical** (real-world) **probability** measure $\mathbb{P}$

$$X_t = \mu + \Phi X_{t-1} + \Sigma \varepsilon_t$$

where $\varepsilon_t \sim N(0, I_N)$, Σ lower triangular and $E(\varepsilon_r \varepsilon_s') = 0, r \neq s$

- The one period interest rate r_t is an affine function of the factors,

$r_t = \delta_0 + \delta_1' X_t$ (using monthly data, one period is one month)

- Assuming absence of arbitrage, there exist a **risk-neutral probability** measure $\mathbb{Q}$ which prices all financial assets

Econometric framework: DTSM

- The stochastic discount factor (SDF), which defines the change of probability measure between $\mathbb{P}$ and $\mathbb{Q}$ is specified as exponentially affine

$$-\log(M_{t+1}) = r_t + \frac{1}{2}\lambda_t'\lambda_t + \lambda_t'\varepsilon_{t+1}$$

whit the $(N\times 1)$ vector $\boldsymbol{\lambda_t}$, the *market price of risk*, being an affine function of the factors,

$$\lambda_t = \Sigma^{-1}(\lambda_0 + \lambda_1 X_t),$$

λ_t is a vector with time varying prices of risk as a function of the state

$\boldsymbol{\lambda_t}$ → is the risk prices that measure the additional expected return required for unit of risk in each of the shocks in ε_t

- Consider how the expected excess return of an *n*-period bond rate depends on risk prices:

$$E_t\left(rx_{t+1}^{(n)}\right) + \frac{1}{2}Var_t\left(rx_{t+1}^{(n)}\right) = \lambda_t' Cov(\varepsilon_{t+1}, rx_{t+1})$$

$$i.e.\ Risk\ premium = prices\ of\ risk * quantities\ of\ risk\ (covariances)$$

In a Gaussian model, the covariances are constant → only source of time-variation in term premia are changes in the market prices of risk

Paper: “Restrictions on Risk Prices in Dynamic Term Structure Models”

Econometric framework: DTSM

- Start from a model under $\mathbb{P}$ ($X_t = \mu + \Phi X_{t-1} + \Sigma\varepsilon_t$) and move toward the risk-neutral framework
- The **risk-neutral** dynamics are given by:

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} + \Sigma\varepsilon_t^{\mathbb{Q}}, \text{ where } \varepsilon_t^{\mathbb{Q}} \sim N(0, I_k), E^{\mathbb{Q}}\left(\varepsilon_r^{\mathbb{Q}} \varepsilon_s^{\mathbb{Q}\prime}\right) = 0, r \neq s$$

- The parameters describing the physical and risk-neutral dynamics are related in the following way:

$$\mu^{\mathbb{Q}} = \mu - \lambda_0, \qquad \Phi^{\mathbb{Q}} = \Phi - \lambda_1$$

Note: λ_0 and λ_1 represent the risk premia needed to move from $\mathbb{P}$ dynamics to $\mathbb{Q}$

- In this model, yield are affine in the state variables.

Denoting the J model-implied (fitted) yields by $\hat{Y}_t$, author writes

$$\hat{Y}_t = A + BX_t,$$

where the J-vector A and the $J \times N$-matrix B contain the model-implied loading of yields on risk factors. These are determined by parameters δ_0, δ_1, $\mu^{\mathbb{Q}}$, $\Phi^{\mathbb{Q}}$ and Σ

Paper: “Restrictions on Risk Prices in Dynamic Term Structure Models”

Econometric framework: DTSM

- Risk Factors are linear combination of yields:

$$X_t = W\hat{Y}_t, \text{ for a } N \times J \text{ matrix } W$$

- Since interest lies in inference about the prices of risk associated with shocks to X_t, it is convenient that X_t is a specific linear combinations of yields
  - Take the risk factors X_t as the first three principal components of the observed yields.
  - W contains the loadings of the first three PCs of observed yields: level, slope, and curvature of the yield curve (they are sufficient to capture most of the variation in the yield curve)
- The observed bond yields used for estimation are:

$$Y_t = \hat{Y}_t + e_t,$$

e_t is a vector of measurement errors that is *iid* normal

  - Measurement errors are included because an N-dimension factor model cannot perfectly price J > N yields. Assume X_t is observable

Econometric framework: restrictions on risk prices

Absence of arbitrage requires the consistency of time-series dynamics of interest rates with their cross sectional behavior, allowing for a risk adjust.

- The risk-price parameters λ_0 and λ_1 determine this risk adjustment and the behavior of risk premia ($\mu^{\mathbb{Q}} = \mu - \lambda_0$, $\Phi^{\mathbb{Q}} = \Phi - \lambda_1$)
- **Weak EH** → $\lambda_1 = 0$ and term premia are constant
- **Maximally-flexible model** → all elements of λ_0 and λ_1 are unrestricted
- **Author** thinks → the truth lies somewhere between the 2 extremes
  - focus on zero restriction on λ_0 and λ_1
- The risk-price parameters λ_0 and λ_1 determine this risk adjustment and the behavior of risk premia ($\mu^{\mathbb{Q}} = \mu - \lambda_0$, $\Phi^{\mathbb{Q}} = \Phi - \lambda_1$)
- Author provides a systematic framework to select restrictions.
- Let γ be a vector of indicator variables, each of which corresponds to an element of $\lambda \equiv (\lambda_0, \mathrm{vec}(\lambda_1))$.
- If an element of γ is equal to 0, the corresponding parameter is restricted to 0, and it is unrestricted otherwise.
- γ can take on 2^{N+N^2} different values - in a 3-factor model, there are 4096 candidate specifications.

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Econometric framework: MCMC algorithms

Model estimation and model selection will be carried out using Markov chain Monte Carlo algorithms.

- For a given model specification γ, author requires an MCMC samplers that draws from the joint posterior distribution of the parameters
- Using MCMC algorithms one can instead sample jointly across models and parameters to identify a smaller set of plausible model specifications, so that there is no need to estimate all candidate models.
  - The most interesting models (the ones with high posterior probability) will be visited more frequently by such samplers
- The task of choosing zero restrictions on risk-price parameters in a DTSM closely parallels the problem of selecting variables in multivariate regressions
- The author uses different existing approaches to variable selection:
  - Gibbs variable selection (GVS)
  - Stochastic Search Variable Selection (SSVS)
  - Reversible-Jump MCMC (RJMCMC)

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Econometric framework: MCMC algorithms

- Using the draws from a model-selection sampler, one can easily account for model uncertainty using Bayesian Model Averaging (BMA).
  - In BMA, estimates of model parameters and any objects of interest—interest rate persistence, volatilities, short-rate expectations, and term premia—are calculated as averages across specifications, using posterior model probabilities as weights.
  - The resulting BMA posterior distributions naturally account for the statistical uncertainty about the model specification.
- In this way one can avoid a false sense of confidence which may result from conditioning on one specific restricted model despite the presence of model uncertainty.

Simulation study

Because of the novelty of the econometric framework, it is important to assess its reliability and effectiveness in a simulation study

- To this end, author applies the estimation method to data that is simulated from a known model

- Simulate yield data from a data-generating process (DGP) which is a 2-factor DTSM

- To determine plausible parameters and restrictions author uses maximum likelihood estimates in actual yield data, which leads to a model with only one significant risk price parameters.

- Generate 100 samples of size T = 300 and for each sample:
  - estimate the maximally-flexible model using MCMC
  - use the model-selection samplers SSVS, GVS, and RJMCMC to carry out estimation under risk-price restrictions

Simulation study

The table shows how well these different approaches fare in recovering the true model

Table 1: Simulation study: risk-price restrictions

		Element of γ						Freq. of
		(1)	(2)	(3)	(4)	(5)	(6)	corr. model
only 1 non-zero risk-price parameter → **True model**	DGP	0	0	0	0	1	0	
Unrestricted model	MCMC	0.17	0.12	0.40	0.08	0.26	0.11	13%
Model-selection samplers	SSVS	0.42	0.05	0.24	0.10	0.59	0.06	57%
	GVS	0.43	0.05	0.24	0.09	0.61	0.06	59%
	RJMCMC	0.39	0.05	0.25	0.08	0.65	0.06	66%

- (blue) reports for each parameter how often the credibility intervals do not straddle zero

- (green) average posterior probabilities of inclusion

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- The non-zero parameter is significant in only 26% of the samples, and the parameters which are zero in the DGP are often found to be significant

  If one chooses a model based on which parameters are significant, then the DGP model is correctly identified in only 13% of the simulated samples

- The posterior probability of inclusion is largest for that parameter which is non-zero in the DGP (inclusion probability is above 60%) whereas for the parameters that are truly zero this probability is always below 50%.

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➔ The percentage of samples in which the modal model (the model with the highest posterior probability) corresponds to the DGP model, reported in the last column, is near or above 60%

- In sum, model-selection samplers do quite well in recovering the true DGP model, in particular for a plausible DGP informed by estimates on actual yield data.

Simulation study

Can the estimation method suggested in this paper more accurately recover short-rate expectations and term premia than estimation of a maximally-flexible DTSM?

Table 2: Simulation study: persistence and volatilities

		Persistence		Volatilities	
		max. eigenv.	IRF(5y)	$\Delta \tilde{f}_t$	Δftp_t
DGP		0.9939	0.7018	0.1920	0.1552
MCMC	posterior mean	0.9779	0.2607	0.0853	0.2526
	CI contains DGP	87%	57%	66%	52%
SSVS	posterior mean	0.9889	0.5761	0.1937	0.1483
	CI contains DGP	98%	98%	99%	97%
GVS	posterior mean	0.9890	0.5774	0.1930	0.1432
	CI contains DGP	96%	96%	99%	98%
RJMCMC	posterior mean	0.9887	0.5704	0.1804	0.1593
	CI contains DGP	95%	95%	98%	97%

This Table compares the estimated interest rate **persistence** and **volatilities** to the true values in the DGP

- Persistence measured by:
  - the largest eigenvalue of Φ
  - the impulse-response function for the level factor in response to level shocks at the five-year horizon
- The model-implied volatilities are for changes in 5-to-10 y risk-neutral forward rates (i.e. short-rate expectations), and in the forward term premium, in annualized percentage points.

Simulation study

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- The high persistence causes long-horizon expectations of short rates to be quite volatile: The volatility of monthly changes in five-to-ten-year risk-neutral forward is higher than the volatility of the forward term premium

➡ MCMC of the **unrestricted model** leads to

- persistence that is considerably lower, reflecting the usual downward bias in estimated persistence
- implies much too stable expectations and too volatile forward term premia.

Simulation study

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Estimation under **risk-price restrictions** leads to:

- estimates of persistence that are much closer to the true values, and it accurately recovers volatilities of the expectations and term premium components in long-horizon forward rates.
- In this case, the 95%-CIs for persistence and volatilities contain the true DGP value in almost all of the simulated samples.
- the long-horizon forward term premium to be less volatile than short-rate expectations.

- Bayesian estimation under restrictions on risk prices is successful in recovering the true restrictions, the persistence of interest rates, and the volatilities of short-rate expectations and term premia.

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- Bayesian estimation under restrictions on risk prices is successful in recovering the true restrictions, the persistence of interest rates, and the volatilities of short-rate expectations and term premia.

Simulation study

- The econometric framework performs well in recovering the zero restrictions on risk prices and the estimated risk-price parameters
- The study shows that estimation under risk-price restrictions accurately infers the true **persistence** of interest rates and the **volatility** of short-rate expectations and term premia.
- In contrast, estimation of an unrestricted model leads to:
  - persistence that is too low,
  - short-rate expectations that are too stable
  - term premium estimates that are excessively volatile.

Economic implications

Author discusses the economic implications of restrictions on risk prices. He compares the results for the unrestricted model $\boldsymbol{\mathcal{M}_0}$, the restricted models $\boldsymbol{\mathcal{M}_1}$, $\boldsymbol{\mathcal{M}_2}$, and $\boldsymbol{\mathcal{M}_3}$ as well as the **BMA** results using the GVS sample.

- Apply the econometric framework in real-world data
- Use monthly observations of nominal zero-coupon U.S. Treasury yields, with maturities of 1 through 5, 7, and 10 years
- Sample period: Jan. 1990–Dec. 2007 (T = 216 monthly obs.)
  - $\boldsymbol{\mathcal{M}_0}$ → estimates of the unrestricted, maximally-flexible DTSM → will serve as a benchmark against which to compare subsequent results
  - $\boldsymbol{\mathcal{M}_1}$→ model with only this one element of λ non-0
  - $\boldsymbol{\mathcal{M}_2}$→ model with 2 elements of λ non-0
  - $\boldsymbol{\mathcal{M}_3}$→ model with 3 elements of λ non-0
- Since restrictions on risk prices affect mainly the time-series properties of a DTSM and leave the cross-sectional fit essentially unchanged, the focus will be on short-rate expectations and term premia
- The estimated persistence of risk factors and interest rates crucially determines the properties of short-rate expectations and term premia

Economic implications: persistence and volatilities

Author discusses the economic implications of restrictions on risk prices. He compares the results for the unrestricted model $\mathcal{M}_0$, the restricted models $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ as well as the **BMA** results using the GVS sample.

- This table reports:
  - the persistence under both probability measures, $\mathbb{Q}$ and $\mathbb{P}$, measured by the largest eigenvalues of $\Phi^{\mathbb{Q}}$ and Φ
  - model-implied volatilities of monthly changes in 5-to-10-year forward rates, in risk-neutral forward rates, and in the corresponding forward term premium.
  - For each statistic, posterior means and 95%-CIs are reported

Table 7: Persistence and volatility

Model	Max. eigenvalue		Volatilities		
	$\mathbb{Q}$	$\mathbb{P}$	$\Delta\hat{f}_t$	$\Delta\tilde{f}_t$	Δftp_t
M0	0.9987 [0.9981, 0.9994]	0.9835 [0.9639, 0.9989]	0.25 [0.24, 0.27]	0.10 [0.01, 0.31]	0.29 [0.16, 0.49]
M1	0.9985 [0.9979, 0.9992]	0.9925 [0.9870, 0.9956]	0.25 [0.24, 0.27]	0.21 [0.19, 0.24]	0.22 [0.14, 0.30]
M2	0.9986 [0.9980, 0.9992]	0.9788 [0.9708, 0.9881]	0.26 [0.25, 0.27]	0.13 [0.04, 0.29]	0.32 [0.24, 0.42]
M3	0.9986 [0.9980, 0.9992]	0.9891 [0.9869, 0.9935]	0.26 [0.25, 0.27]	0.32 [0.21, 0.47]	0.40 [0.21, 0.58]
BMA	0.9987 [0.9980, 0.9993]	0.9896 [0.9734, 0.9975]	0.25 [0.24, 0.27]	0.21 [0.07, 0.40]	0.27 [0.14, 0.50]

Economic implications: persistence and volatilities

5-to-10-y forward rates

risk-neutral forward rates: **short rate expectations**

forward term premium

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- $\mathbb{Q}$-persistence → very similar across models, since the $\mathbb{Q}$-dynamics are largely unaffected by risk-price restrictions. Consequently, the volatility of fitted forward rates does not vary across models.
- Under the $\mathbb{P}$ measure interest rates are much less persistent than under $\mathbb{Q}$, and this is true for all models. Consequently, short-rate expectations are less variable than forward rates.

Economic implications: persistence and volatilities

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M2	0.9986	0.9788	0.26	0.13	0.32
	[0.9980, 0.9992]	[0.9708, 0.9881]	[0.25, 0.27]	[0.04, 0.29]	[0.24, 0.42]
M3	0.9986	0.9891	0.26	0.32	0.40
	[0.9980, 0.9992]	[0.9869, 0.9935]	[0.25, 0.27]	[0.21, 0.47]	[0.21, 0.58]

- There are important **differences across models.**
- The restricted models (with the exception of $\mathcal{M}_2$) generally exhibit higher $\mathbb{P}$–persistence than $\mathcal{M}_0$.
- The intuition is that risk–price restrictions tighten the connection between cross section and time series, and "pull up" the $\mathbb{P}$–persistence toward to $\mathbb{Q}$–persistence.
- All restricted models imply more volatile short–rate expectations than the maximally–flexible model $\mathcal{M}_0$.

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Economic implications: persistence and volatilities

Table 7: Persistence and volatility

Model	Max. eigenvalue		Volatilities		
	$\mathbb{Q}$	$\mathbb{P}$	$\Delta \hat{f}_t$	$\Delta \tilde{f}_t$	Δftp_t
M0	0.9987 [0.9981, 0.9994]	0.9835 [0.9639, 0.9989]	0.25 [0.24, 0.27]	0.10 [0.01, 0.31]	0.29 [0.16, 0.49]
M1	0.9985 [0.9979, 0.9992]	0.9925 [0.9870, 0.9956]	0.25 [0.24, 0.27]	0.21 [0.19, 0.24]	0.22 [0.14, 0.30]
M2	0.9986 [0.9980, 0.9992]	0.9788 [0.9708, 0.9881]	0.26 [0.25, 0.27]	0.13 [0.04, 0.29]	0.32 [0.24, 0.42]
M3	0.9986 [0.9980, 0.9992]	0.9891 [0.9869, 0.9935]	0.26 [0.25, 0.27]	0.32 [0.21, 0.47]	0.40 [0.21, 0.58]
BMA	0.9987 [0.9980, 0.9993]	0.9896 [0.9734, 0.9975]	0.25 [0.24, 0.27]	0.21 [0.07, 0.40]	0.27 [0.14, 0.50]

- Since conventional DTSMs typically imply very stable long-horizon short-rate expectations, they attribute a **large role** to the **term premium** for explaining movements in long rates, which is a puzzling **short-coming** of these models.
- Risk-price restrictions often, lower the volatility of term premia. The puzzle of an implausibly large role for term premia in explaining variation in long rates is somewhat **alleviated** when plausible **restrictions** are imposed on an otherwise standard DTSM.

Economic implications: persistence and volatilities

Table 7: Persistence and volatility

Model	Max. eigenvalue		Volatilities		
	$\mathbb{Q}$	$\mathbb{P}$	$\Delta \hat{f}_t$	$\Delta \tilde{f}_t$	Δftp_t
M0	0.9987	0.9835	0.25	0.10	0.29
	[0.9981, 0.9994]	[0.9639, 0.9989]	[0.24, 0.27]	[0.01, 0.31]	[0.16, 0.49]
M1	0.9985	0.9925	0.25	0.21	0.22
	[0.9979, 0.9992]	[0.9870, 0.9956]	[0.24, 0.27]	[0.19, 0.24]	[0.14, 0.30]
M2	0.9986	0.9788	0.26	0.13	0.32
	[0.9980, 0.9992]	[0.9708, 0.9881]	[0.25, 0.27]	[0.04, 0.29]	[0.24, 0.42]
M3	0.9986	0.9891	0.26	0.32	0.40
	[0.9980, 0.9992]	[0.9869, 0.9935]	[0.25, 0.27]	[0.21, 0.47]	[0.21, 0.58]
BMA	0.9987	0.9896	0.25	0.21	0.27
	[0.9980, 0.9993]	[0.9734, 0.9975]	[0.24, 0.27]	[0.07, 0.40]	[0.14, 0.50]

- Large CIs for the **unrestricted model** → it is difficult to estimate the dynamic properties of interest rates using only time-series information.
- CIs generally narrower for the **restricted models** → absence of arbitrage makes information in the cross section useful for estimating the time-series properties of interest rates.
- Imposing a specific set of restrictions leads to tighter inference about $\mathbb{P}$ -dynamics, incorporation of model uncertainty naturally makes the inference less precise.

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Historical evolution: short rate exp & term premia

Long-term interest rates have declined by a significant amount over the sample period. To which extent was this due to changes in monetary policy expectations and movements in term premia?

- Table summarizes the models' implications for decomposing the decline. For both actual and risk-neutral yields, it reports the levels in 1990 and 2007, calculated as averages over each year, and the changes over this period. Also shown are 95%-CIs for levels and changes in risk-neutral forward rates

Table 8: Historical changes in long-term rates and expectations

		Sample period		
		1990	2007	Change
Ten-year yield		8.5	4.6	-3.8
Exp.	M0	4.4	3.8	-0.6
		[2.6, 6.2]	[2.0, 5.6]	[-2.4, 1.3]
	M1	4.7	2.0	-2.7
		[3.3, 6.0]	[1.0, 2.9]	[-3.1, -2.3]
	M2	4.8	4.3	-0.5
		[3.9, 5.8]	[2.9, 5.4]	[-1.8, 0.2]
	M3	5.3	3.6	-1.7
		[3.7, 6.8]	[1.6, 5.3]	[-2.8, -0.8]
	BMA	4.7	2.6	-2.1
		[3.0, 6.4]	[0.7, 5.0]	[-3.2, 0.0]

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Historical evolution: short rate exp & term premia

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Exp.	M0	4.4	3.8	-0.6
		[2.6, 6.2]	[2.0, 5.6]	[-2.4, 1.3]
	M1	4.7	2.0	-2.7
		[3.3, 6.0]	[1.0, 2.9]	[-3.1, -2.3]
	M2	4.8	4.3	-0.5
		[3.9, 5.8]	[2.9, 5.4]	[-1.8, 0.2]
	M3	5.3	3.6	-1.7
		[3.7, 6.8]	[1.6, 5.3]	[-2.8, -0.8]
	BMA	4.7	2.6	-2.1
		[3.0, 6.4]	[0.7, 5.0]	[-3.2, 0.0]

- The ten- year yield declined by 3.8 pps
  - **Unrestricted model $\mathcal{M}_0$** implies only a small share of this decline, less than 1/6, is due to declining short-rate expectations, and CI for the decline in expectations straddles zero.
  - The **restricted models**, with the exception of $\mathcal{M}_2$, imply a decline of short-rate expectations that is much more pronounced and significantly different from 0;
  - **BMA** attributes more than ½ of the yield decline to falling short-rate expectations. The decline in expectations is estimated more precisely, even though this accounts for model uncertainty.
- This suggests that the secular decline in long-term interest rates was not only caused by a lower term premium, but also to a significant extent by a downward shift in expectations of future nominal interest rates, in line with the sizable decreases in survey-based expectations of inflation and policy rates

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		[3.3, 6.0]	[1.0, 2.9]	[-3.1, -2.3]
	M2	4.8	4.3	-0.5
		[3.9, 5.8]	[2.9, 5.4]	[-1.8, 0.2]
	M3	5.3	3.6	-1.7
		[3.7, 6.8]	[1.6, 5.3]	[-2.8, -0.8]
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Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Predictability of bond returns

It is well known that returns on U.S. Treasury bonds are predictable using current interest rate and maximally flexible affine Gaussian DTSMs have been shown to successfully capture this feature of interest rate data. In the restricted models, term premia and expected return are more stable than in unrestricted models so return predictability is more limited.

Can DTSMs with tight restrictions on risk pricing still match the return predictability that we see in the data?

- Run a predictive regression for excess returns on long-term bonds and check whether the estimated R^2 are matched by those implied by the models, both in population and small samples.
  - $rx_{t,t+12}^{(n)} = \alpha^{(n)} + \beta^{(n)} X_t + v_t^{(n)}$, where
- $rx_{t,t+12}^{(n)}$ is the annual holding period returns, in excess of the 1y yield, on a bond with maturity n; $v_t^{(n)}$ is the predictive error. The predictors are the first 3 PCs of the yield curve and hence corresponds to the risk factors of the models.

- Regression is estimated for bonds with maturities of 2, 5, 7 and 10 years, using 204 monthly observations

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

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Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Predictability of bond returns

Maturity	Data
2 years	0.35
5 years	0.35
7 years	0.36
10 years	0.35

- The R^2 for the actual yield data are reported in this table. Annual excess returns are strongly predictable, with 35% of their variation explained by level, slope and curvature of the yield curve

- The model-implied population R^2 are typically below the values of the data, and the discrepancy is more pronounced for the restricted models with less variable term premia, such as $\mathcal{M}_1$.
- The **BMA** estimates imply population R^2 that are quite substantially below those in the data.

Table 9: Return predictability

		M0		M1		M2		M3		BMA	
Maturity	Data	Pop.	Smpl.	Pop.	Smpl.	Pop.	Smpl.	Pop.	Smpl.	Pop.	Smpl.
2 years	0.35	0.36	0.41 (0.16)	0.14	0.26 (0.15)	0.30	0.37 (0.16)	0.36	0.37 (0.18)	0.21	0.30 (0.16)
5 years	0.35 <	0.25	0.37 (0.13)	0.11	0.27 (0.15)	0.25	0.37 (0.15)	0.31	0.38 (0.17)	0.16	0.31 (0.15)
7 years	0.36 <	0.23	0.36 (0.13)	0.11	0.29 (0.15)	0.26	0.38 (0.14)	0.32	0.39 (0.17)	0.16	0.32 (0.15)
10 years	0.35 <	0.20	0.35 (0.12)	0.11	0.29 (0.14)	0.26	0.39 (0.13)	0.32	0.40 (0.16)	0.16	0.33 (0.14)

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Predictability of bond returns

- Small-sample issues play an important role for the distribution of R^2 in predictive regressions → necessary to consider the small-sample distribution of R^2 implied by the models.
  - Author obtains it by simulating, for each model, 1000 yield data sets of the same length as the original data (T = 216), using the posterior means of the model parameters, and then running the same regressions in the simulated as in the actual data.
- Table reports means and stand. dev. for the distributions of small-sample R^2 → means are higher than the population R^2, are close to the R^2 in the data.
  - In small samples all models, including those with tightly restricted risk prices, are consistent with the empirical evidence on bond return predictability.

Table 9: Return predictability

		M0		M1		M2		M3		BMA	
Maturity	Data	Pop.	Smpl.	Pop.	Smpl.	Pop.	Smpl.	Pop.	Smpl.	Pop.	Smpl.
2 years	0.35	0.36	0.41 (0.16)	0.14	0.26 (0.15)	0.30	0.37 (0.16)	0.36	0.37 (0.18)	0.21	0.30 (0.16)
5 years	0.35	0.25	0.37 (0.13)	0.11	0.27 (0.15)	0.25	0.37 (0.15)	0.31	0.38 (0.17)	0.16	0.31 (0.15)
7 years	0.36	0.23	0.36 (0.13)	0.11	0.29 (0.15)	0.26	0.38 (0.14)	0.32	0.39 (0.17)	0.16	0.32 (0.15)
10 years	0.35	0.20	0.35 (0.12)	0.11	0.29 (0.14)	0.26	0.39 (0.13)	0.32	0.40 (0.16)	0.16	0.33 (0.14)

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

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2 years	0.35	0.36	0.41	0.14	0.26	0.30	0.37	0.36	0.37	0.21	0.30
			(0.16)		(0.15)		(0.16)		(0.18)		(0.16)
5 years	0.35	0.25	0.37	0.11	0.27	0.25	0.37	0.31	0.38	0.16	0.31
			(0.13)		(0.15)		(0.15)		(0.17)		(0.15)
7 years	0.36	0.23	0.36	0.11	0.29	0.26	0.38	0.32	0.39	0.16	0.32
			(0.13)		(0.15)		(0.14)		(0.17)		(0.15)
10 years	0.35	0.20	0.35	0.11	0.29	0.26	0.39	0.32	0.40	0.16	0.33
			(0.12)		(0.14)		(0.13)		(0.16)		(0.14)

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Conclusion

- This paper has introduced a novel econometric framework to estimate DTSMs under restrictions on risk pricing.
- It allows for a systematic model choice among a large number of restrictions and for parsimony in otherwise overparameterized models.
- Empirically, the results using U.S. Treasury yields show that the data support tight restrictions on risk prices.
  - This stands in contrast to the common practice of leaving most or all of the risk-price parameters unrestricted.
  - The restrictions change the economic implications, because they increase the estimated persistence of interest rates and therefore make short-rate expectations (i.e., risk-neutral rates) significantly more variable.
  - This resolves the puzzle of implausibly stable short-rate expectations shared by most conventional DTSM models.

APPENDIX

Likelihood and Priors

- The **conditional likelihood function** of Y_t is:

$$f(Y_t|Y_{t-1};\theta,\gamma) = f(Y_t|X_t;k^{\mathbb{Q}}_{\infty},\phi^{\mathbb{Q}},\Sigma,\sigma_e^2) \times f(X_t|X_{t-1};k^{\mathbb{Q}}_{\infty},\phi^{\mathbb{Q}},\lambda,\gamma,\Sigma)$$

"$\mathbb{Q}$-likelihood". Captures the cross sectional dependence of yields on risk factors

"$\mathbb{P}$-likelihood". Captures the time series dynamics of risk factors

- Note: risk price λ affects only the $\mathbb{P}$-likelihood
- The approach chosen follows the objective Bayesian tradition → Little prior information is imposed in order to let data speak for itself → apply largely uninformative prior formulations
- Author assumes a uniform prior distribution over models: each element of γ is independently Bernoulli distributed with success probability 0.5
- Like most other studies that use Bayesian variable selection methods, author assumes conditional prior independence of the elements of λ. This assumption substantially simplifies the model selection problem.

Estimation results

- Apply the econometric framework in real-world data
- Use monthly observations of nominal zero-coupon U.S. Treasury yields, with maturities of 1 through 5, 7, and 10 years
- Sample period: Jan. 1990–Dec. 2007 (T = 216 monthly obs.)
- $\mathcal{M}_0$ → estimates of the unrestricted, maximally-flexible DTSM → will serve as a benchmark against which to compare subsequent results
  - this comparison will reveal how risk-price restrictions change the economic implications of a typical affine Gaussian DTSMs.
- To carry out posterior inference about risk-price restrictions, author obtains model-selection results using the GVS, SSVS, and RJMCMC samplers.
- Using these data, the GVS algorithm emerges as the favoured model-selection sampler, because it converges quickly and restricts the excluded parameters to be exactly zero → The rest of the paper focuses on the GVS results

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- Using these data, the GVS algorithm emerges as the favoured model-selection sampler, because it converges quickly and restricts the excluded parameters to be exactly zero → The rest of the paper focuses on the GVS results

Estimation results

- The evidence strongly favours tightly restricted models, with only very few free risk-price parameters.
- This table shows posterior means (the posterior probabilities of inclusion for the corresponding elements of λ)

Table 4: Risk-price restrictions

	GVS Mean
1	0.349
2	0.040
3	0.071
4	0.238
5	0.071
6	0.041
7	0.987
8	0.098
9	0.050
10	0.095
11	0.052
12	0.069

Indicators restricting λ_0

Indicators restricting λ_1

- only one element of λ has a high posterior probability for inclusion, which is above 95%. For all other parameters, the inclusion probabilities are below 50%, and for most of the parameters they are near zero.

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Estimation results

- This table shows that all of the 10 most plausible models leave only one to three risk-price parameters unrestricted.
- The prior mean for the number of unrestricted parameters is 6, which contrasts with the posterior mean of only 2.2.

Model	GVS Freq.
7	0.3636
1,4,7	0.1127
1,7	0.0874
7,10	0.0327
7,8	0.0378
5,7	0.0253
3,7,12	0.0115
4,7	0.0147
7,9	0.0151
7,11	0.0157
models visited	549 / 4096 13.4 %

- Posterior model probabilities relative to the modal model for the 10 most frequently visited models.
- Models are denoted by the indices of the unrestricted elements in λ.

- Notation:
  - $\mathcal{M}_1 \rightarrow$ model with only this one element of λ non-0
  - $\mathcal{M}_2 \rightarrow$ model with 2 elements of λ non-0
  - $\mathcal{M}_3 \rightarrow$ model with 3 elements of λ non-0

Estimation results

- The evidence is also quite clear on which restrictions are favoured by data.
- $\mathcal{M}_1$ has by far the highest posterior model probability → data call for **tight restrictions** on the market prices of risk and favour a model in which only one out of twelve risk-price parameters is unrestricted
- As a reality check for the model-selection results, look at SBIC → SBIC is consistent with the ranking based on Bayesian model selection → results are actually driven by information in the data, and not by the choice of priors or some feature of the sampling algorithms.

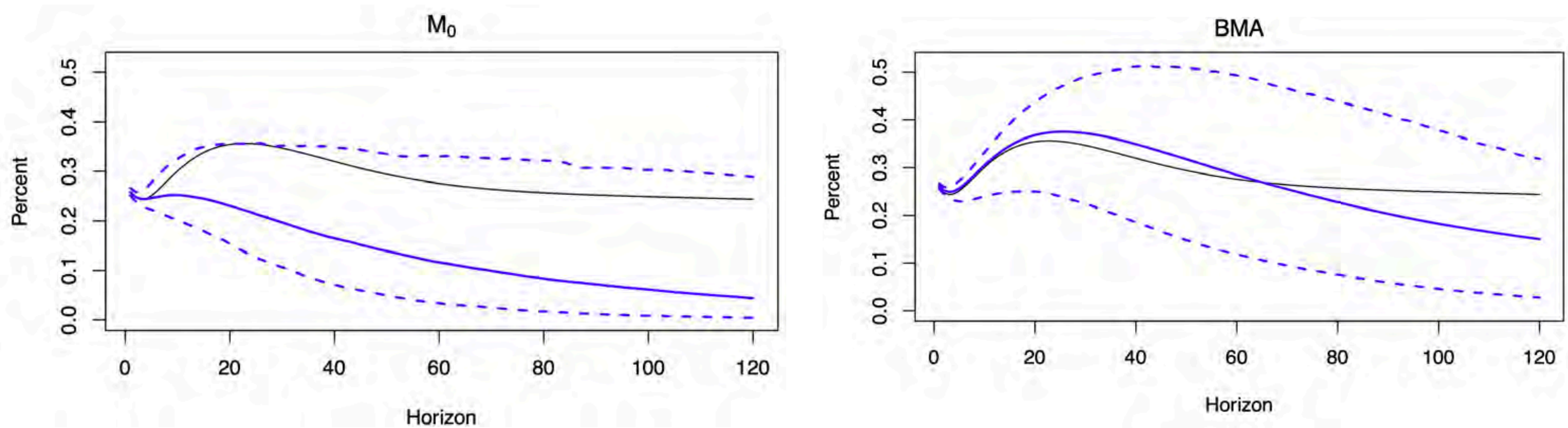
Model	GVS Freq.	SBIC
7	0.3636	-25280.1
1,4,7	0.1127	-25279.0
1,7	0.0874	-25278.8
7,10	0.0327	-25276.1
7,8	0.0378	-25277.0
5,7	0.0253	-25276.6
3,7,12	0.0115	-25277.0
4,7	0.0147	-25274.8
7,9	0.0151	-25275.1
7,11	0.0157	-25275.7
models visited	549 / 4096 13.4 %	

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Economic implications: persistence and volatilities

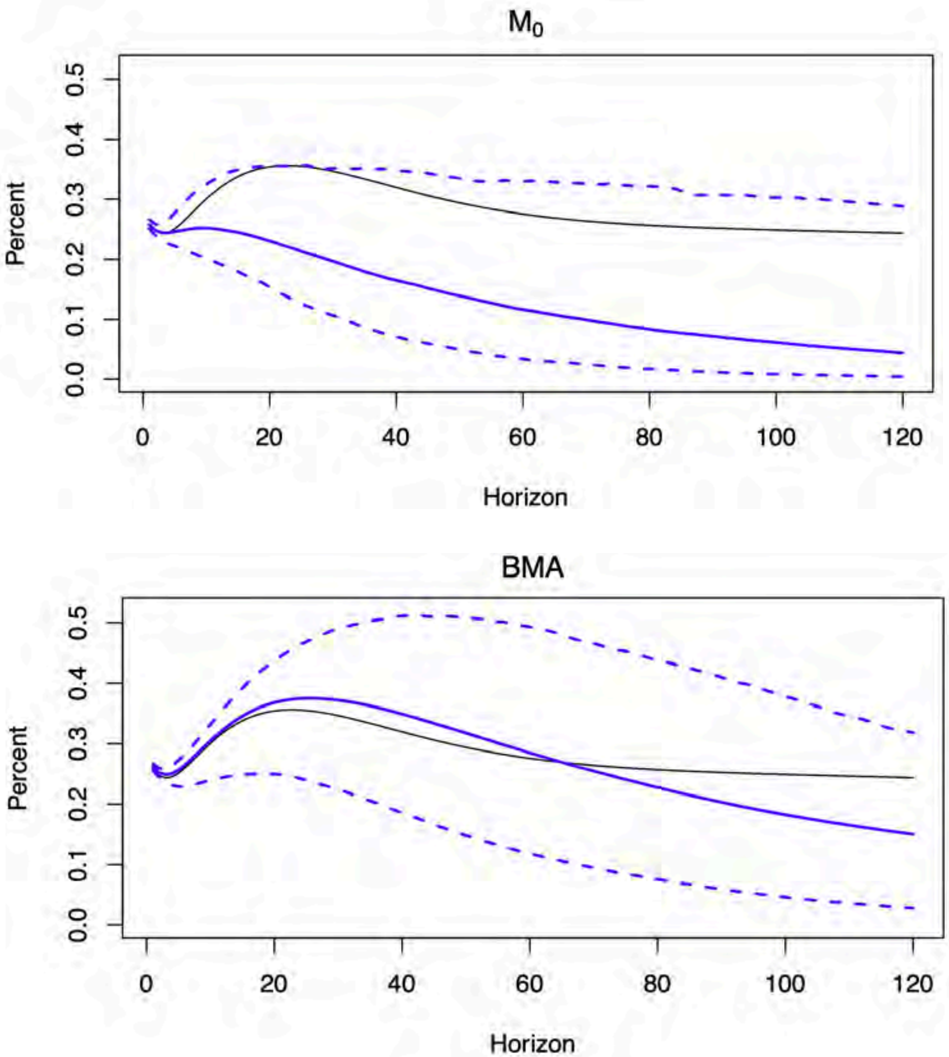
To understand the dynamic properties of a DTSM, it is instructive to consider volatilities across maturities. Focusing on volatilities of forward rates helps to isolate the behaviour of expectations at specific horizons.

- The figure displays the **term structure of volatility.**
- Posterior means of volatilities of changes in fitted forward rates (thin solid line) and risk-neutral forward rates (thick solid line), as well as 95%-credibility intervals for risk-neutral volatilities (dashed lines).



- It shows model-implied volatilities of monthly changes in forward rates and risk-neutral forward rates for maturities from one month to ten years

Economic implications: persistence and volatilities



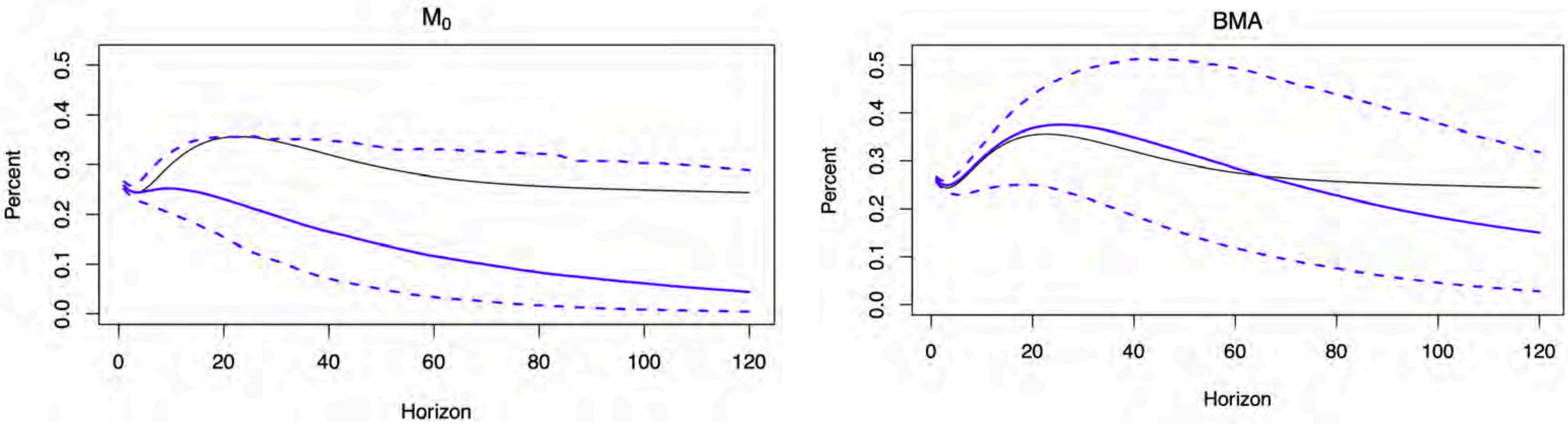
- The forward rate volatilities are similar for the two models, declining only slowly and almost levelling out for horizons longer than 5 years.

- The risk-neutral volatility curves differ substantially.
  - $\boldsymbol{\mathcal{M}_0}$ → Except for the very shortest maturities, risk-neutral volatilities are much lower than forward rate volatilities, implying only a limited role for changes in expectations to account for movements in interest rates.
  - **BMA** → risk-neutral volatilities stay much closer to forward rate volatilities for horizons up to five years, and only for longer maturities do they drop below

Paper: "Restrictions on Risk Prices in Dynamic Term Structure Models"

Economic implications: persistence and volatilities

Overall, **BMA** attributes a larger role to short-rate expectations for explaining interest rate volatilities across maturities, due to the restrictions on risk prices.

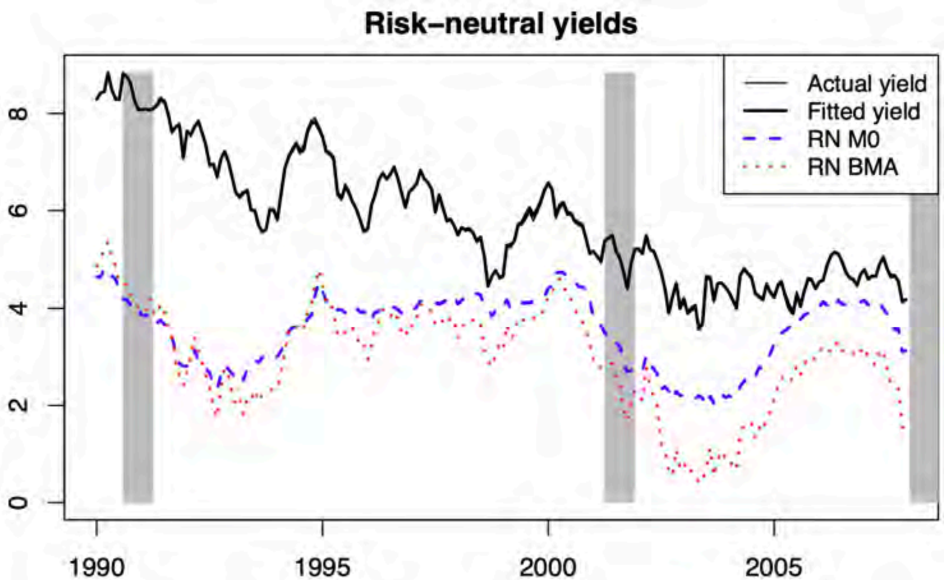


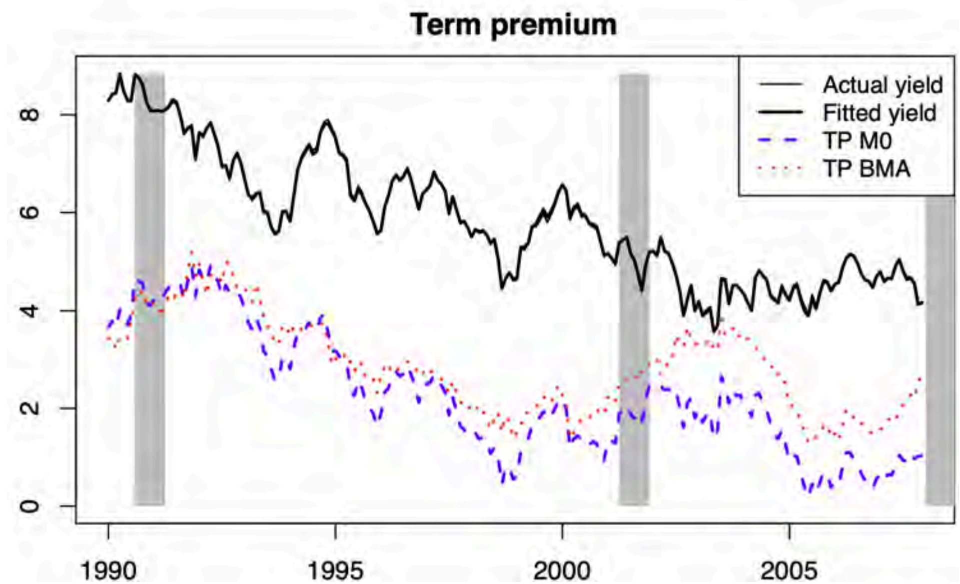
- Figure also shows that it is hard to estimate risk-neutral volatilities -the CIs are quite wide in both cases. While for any individual restricted model, these are much narrower (not shown) than for the maximally flexible model, it is shown that taking into account the model uncertainty significantly widens the range of plausible volatility estimates.

Paper: “Restrictions on Risk Prices in Dynamic Term Structure Models”

Historical evolution: short rate exp & term premia

Figure shows how $\mathcal{M}_0$ and **BMA** differ in their decomposition of the ten-year yield into expectations and term premium components.





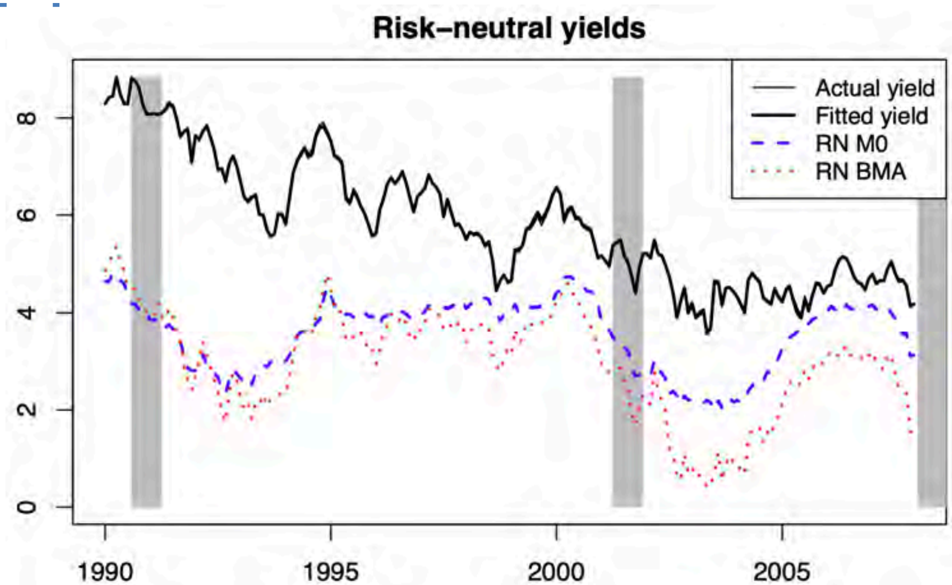
- This shows forward rates and estimates of **risk-neutral forward rates** (RN) across models
- RN are estimates of the risk-neutral yield, i.e., of the expectations component of the ten-year yield

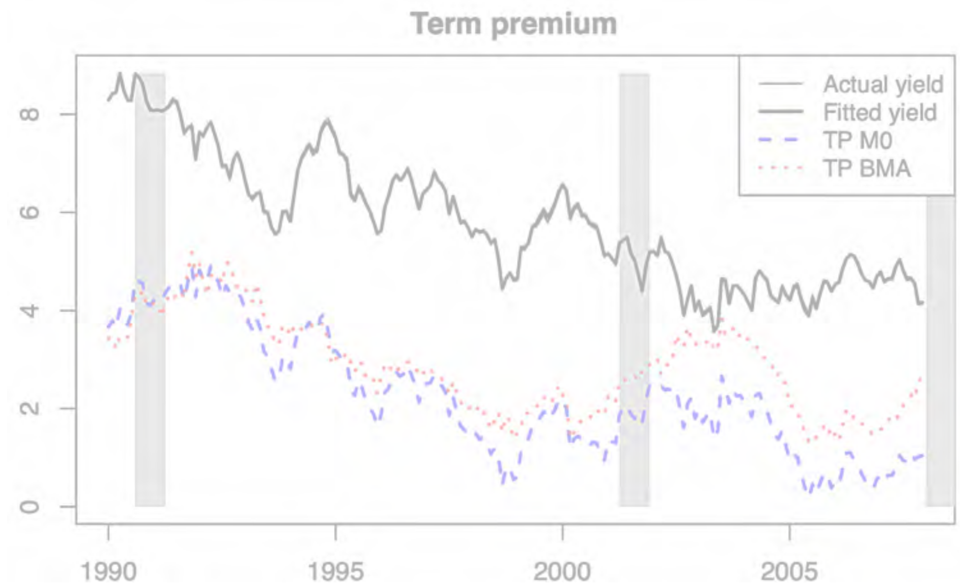
- This shows forward rates and **estimates of forward term premium** (TP) across models.
- TP are calculated as the difference between fitted and risk-neutral yield.

- The fitted yields is obtained from $\mathcal{M}_0$

Paper: “Restrictions on Risk Prices in Dynamic Term Structure Models”

Historical evolution: short rate exp & term premia





- For **BMA**, the estimate of the risk-neutral yield exhibits pronounced variation, falling very significantly around the 2001 recession, and with the onset of the Great Recession (2007–2009).
- The expectations component estimated from model $\mathcal{M}_0$ is very stable, and the movements around the recessions are more muted.
- The yield term premium is noticeably more stable for **BMA** relative to $\mathcal{M}_0$, and more counter-cyclical as it rises before and during recessions and falls during expansions.
- This is appealing in light of much theoretical and empirical work suggesting that term premia are slow-moving and behave in a counter-cyclical fashion

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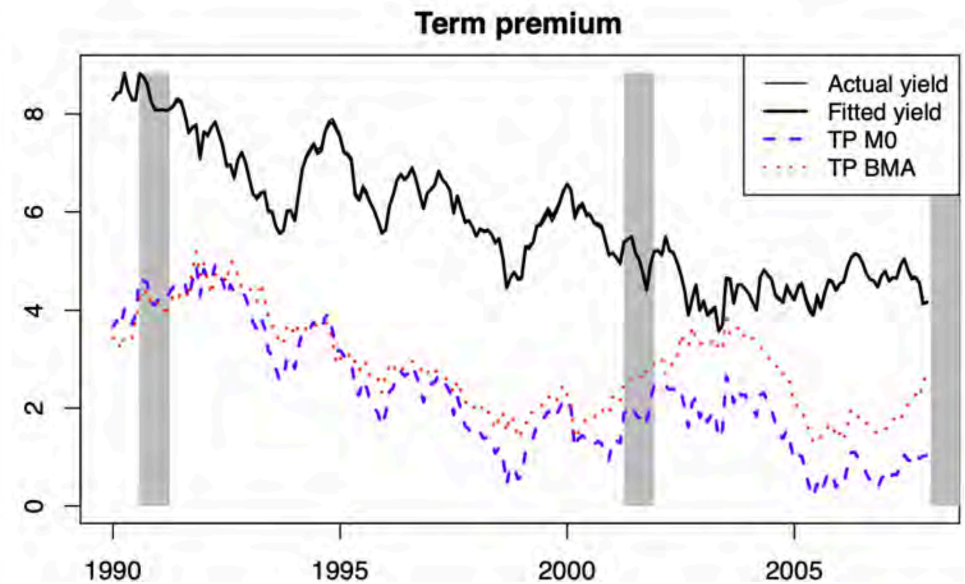

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