Macroeconomics and Public Finance:

Static and Dynamic Models of Adverse Selection

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Economic of Information and Macroeconomics

- Asymmetric Information is pervasive in economics
- Industrial Organization and Market Structure
- Financial Crisis: Adverse Selection and Moral Hazard
- Designing Institutions:
    - Firms and Markets
    - Optimal Taxation and Public Finance
- Nobel Prizes: Horowiz, Myerson, and Mirrleese
- Macroeconomics Focuses on Dynamic Aspects and Institution Design

Economic of Information and Macroeconomics II

- Agents can save and borrow
- Markets Available to Agents
- Questions of Modern (Macro) Public Finance:
  1. Capital Taxation
  2. Social Insurance and Welfare Programs
  3. Labor Income Taxes over the Life Cycle
  4. Education Taxes and Subsidies

Mechanism Design and Bayesian Games

- Bayesian Games $\Gamma =< N, T, S, X, M, U, P >$
  - $N=$ set of players
  - S_i Strategy agent i: $S = \Pi_i S_i$
  - $X=$ set of outcomes

Def. 1: A Comunication Mechanism is a a rule

$$M : S \to \Delta(X)$$

  - Nature chooses type t_i from P (common knowledge)
  - Preferences: $U_i(t, x)$

Def. 2: A pure strategy is a function:

$$\sigma_i : T_i \to S_i$$

$$\sum_{t_{-i}} P(t_{-i}|t_i) U_i(t, M(s))$$

Bayesian-Nash Equilibrium

Def. 3: σ^* is a Bayesian-Nash (B-N) Equilibrium of Γ if for each t_i and for all $\sigma_i(t_i)$: (Incentive Compatibility)

$$\sum_{t_{-i}} P(t_{-i}|t_i) U_i(t, M(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}))) \geq \sum_{t_{-i}} P(t_{-i}|t_i) U_i(t, M(\sigma_i(t_i), \sigma_{-i}^*(t_{-i})))$$

Example: Auction

- $X = NxR_+$ Allocation of goods and payments (w, m)
- $S_i = R_+$ Bids
- Mechanism=Auction: $M(s) \to (w, m)$
- $w = \arg\max_i s_i$
- $m = \max_i s_i$ (first price); $m = \max_{i \neq w} s_i$ (second price)
- $U_w(w, m) = V_w - m$ and $U_j = 0$ if $j \neq w$
- For example, V_i is independent uniform on $[0, 1]$ ($T = [0, 1]^n$)

Exercise 1: Compute optimal strategies and B-N equilibria for first price auction. [Hint: $\max_b (V - b) \Pr(b > max_{j \neq i} b^*(V_j))$]

The Revelation Principle

Def. 4: A game Γ^D is direct if $S_i = T_i$. A direct mechanism M^D is a mechanism of a direct game

Theorem: (e.g., Myerson (1979-1984))
For any B-N equilibrium σ^* of a game Γ there exists a direct game such that truthtelling: $\sigma_i^*(t_i) = t_i$ for all i

1. Is a B-N equilibrium
2. Is outcome equivalent to the equilibrium of the original (indirect) game, that is, for any $t \in T$, $M(\sigma^*(t)) = M^D(t)$

- Proof: Define $M(\sigma^*(t)) = M^D(t)$ and note that deviations in Γ^D are equivalent to deviations to $\sigma_i^*(\hat{t}_i)$ in the game Γ. Hence IC is always true. QED
- Powerful result, especially in implementation theory

Corollary: x is implementable whenever it is IC for the direct game.

Static Adverse Selection: Set-Up

- Static model of optimal income taxation: Mirrlees (1971)
- Studies the efficiency-equity tradeoff shaping an optimal redistributive system
- Identical preferences over consumption $c \geq 0$ and labor $n \geq 0$

$$\text{Individual welfare:} \qquad \mathcal{U}(c, n)$$

- Agents only differ in their productivity $\theta \in [0, \bar{\theta}]$
- Take labor supply decisions n along the intensive margin

$$\text{Labor Income:} \qquad y = f(\theta, n)$$

- Distribution of productivities $P(\cdot)$ is continuous
- Government unable to observe productivity or labor supply
- It only observes labor income y
- Sets an incentive compatible tax schedule to maximize social welfare W subject to resources feasibility

Static Adverse Selection: Incentive Compatiblity

- Assume f is invertible and $\mathcal{U}$ additive separable
- New preferences: $u(c) - v(y, \theta)$ with income y
- Assume: $v_y(y, \theta) \geq 0$ and $v_{yy}(y, \theta) > 0$, and $v_\theta(y, \theta) \leq 0$.
- $V(\hat{\theta}|\theta) := u(c(\hat{\theta})) - v(y(\hat{\theta}), \theta)$
- IC: for all θ:

$$V(\theta) := V(\theta|\theta) \geq V(\hat{\theta}|\theta) \quad \forall \hat{\theta}$$

- FOC is $V_1(\theta|\theta) = 0$; necessary SOC: $V_{11}(\theta|\theta) \leq 0$.
- By envelope FOC is equivalent to: $\dot{V}(\theta) = -v_\theta(y(\theta), \theta)$
- Since FOC true for all θ, by totally differentiating FOC, we get that SOC is equivalent to

$$V_{12} = -v_{\theta,y}(y(\theta), \theta)\dot{y}(\theta) \geq 0.$$

Single Crossing

- Single-Crossing condition: If for all y and θ, $v_{\theta,y}(y,\theta) < 0$

  Pictures:

1. $V_{12} \geq 0$ (SOC) is equivalent to $\dot{y}(\theta) \geq 0$
2. $\dot{V}(\theta) = -v_\theta(y(\theta), \theta)$ and $\dot{y}(\theta) \geq 0$ are also sufficient $\Rightarrow$

Proof of Sufficiency

- We want to show that $\dot{y} \geq 0$ and SCC imply IC.
- Fix a generic θ and a $\hat{\theta} \neq \theta$; we have

$$\begin{aligned} V(\theta|\theta) - V(\hat{\theta}|\theta) &= \int_{\hat{\theta}}^{\theta} V_1(s|\theta)ds \\ &= \int_{\hat{\theta}}^{\theta} \left[V_1(s|\theta) - V_1(s|s)\right] ds \\ &= \int_{\hat{\theta}}^{\theta} \left[\int_{s}^{\theta} V_{12}(s|t)dt\right] ds \geq 0. \end{aligned}$$

1. first equality by definition (assuming integrability of $V_1(\cdot|\theta)$)
2. second inequality holds since from FOC $V_1(s|s) \equiv 0$
3. third inequality is again by definition (assuming integrability)
4. last true because $\dot{y}(s) \geq 0$ and SCC imply:

$$V_{12}(s|t) = -v_{\theta,y}(y(s),t)\dot{y}(s) \geq 0. \quad \text{Q.E.D.}$$

Static Adverse Selection: Characterization I

- Assume: SCC and - in order to get interiority - $v_y(0, \theta) = 0$.
- From $V(\theta) := u(c(\theta)) - v(y(\theta), \theta)$, if $g = u^{-1}$ we get

$$c(\theta) = g\left(V(\theta) + v(y(\theta), \theta)\right).$$

- Resource constraint $\int_0^{\bar{\theta}} [y(\theta) - c(\theta)]\, dP = 0$.

$$\max_{y(\cdot), V(\cdot)} \int_0^{\bar{\theta}} W\left(V(\theta); \theta\right) + \lambda\left[y(\theta) - g\left(V(\theta) + v(y(\theta), \theta)\right)\right] p(\theta) d\theta$$

$$\text{s.t.} \quad \dot{V}(\theta) = -v_\theta(y(\theta), \theta) \quad \text{and} \quad \dot{y}(\theta) \geq 0.$$

- $\lambda > 0$ represents the cost of funds
- Welfare function. Eg, $W\left(V(\theta); \theta\right) = \psi(\theta) V(\theta)$, then $\psi(\theta)$ represents the relative Pareto weight given to θ.

Static Adverse Selection: Characterization II

No Bunching case, i.e., if $\dot{y}(\theta) > 0$.

- Integrating by parts the Lagrangian, FOC:

$$
\begin{aligned}
y \quad &: \qquad p(\theta)\lambda\left[1 - \frac{v_y(y(\theta),\theta)}{u'(c(\theta))}\right] = \mu(\theta)[-v_{\theta,y}(y(\theta),\theta)] \\
V \quad &: \qquad p(\theta)\left[W'\left(V(\theta);\theta\right) - \frac{\lambda}{u'(c(\theta))}\right] = \dot{\mu}(\theta)
\end{aligned}
$$

Transv. : $\quad \mu(0) = \mu(\bar{\theta}) = 0.$

Static Adverse Selection: Characterization III

1. Recall that we have $\dot{y}(\theta) > 0$, and $\dot{V}(\theta) > 0$ (Info. Rents)
2. From the transversalities $\int \dot{\mu} d\theta = \mu(\bar{\theta}) - \mu(0) = 0$, we have
$$\mathbf{E}\left[\frac{\lambda}{u'(c)}\right] = \mathbf{E}\left[W'(V, \cdot)\right].$$
Change $u(c(\theta))$ by ε to all agents and kip all $y(\theta)$ the same.
3. Recall IC: $u'(c(\theta))\dot{c}(\theta) = v_y(y(\theta), \theta)\dot{y}(\theta)$. Hence $u'(c) > 0$ and $\dot{y}(\theta) \geq 0$ implies $\dot{c}(\theta) \geq 0$.
4. Claim: If W is concave and $W_{12} \leq 0$ then $\mu(\theta) \geq 0$.
Proof: Since $\mu(0) = 0$, if $\mu(\theta) < 0$ it must be that $\dot{\mu}(s) < 0$ for $s \leq \theta$. From FOC w.r.t. V, using $\dot{c}(\theta), \dot{V}(\theta) \geq 0$, and $W_{11}, W_{12}, u'' < 0$, $\dot{\mu}(\theta)$ decreases with θ. This implies that $\mu(\theta') \leq \mu(\theta) < 0$ for all $\theta' \geq \theta$. Contradicts $\mu(\bar{\theta}) = 0$.
5. From [4.] and FOC w.r.t. y (SCC): $1 - \frac{v_y(y(\theta),\theta)}{u'(c(\theta))} > 0$, i.e., $\tau(\theta) > 0$. Non-monotone: $\tau(0) = \tau(\bar{\theta}) = 0$.

6. Bunching and Interpretation

Bunching:
If $\dot{y}(\theta) = 0$ from IC we have $\dot{c}(\theta) = 0$.

The multiplier, i.e., the value of relaxing agent θ IC constraint:
From the transversality, we have;

$$\mu(\theta) = [1 - P(\theta)]\mathbf{E}\left[\frac{\lambda}{u'(c(s))} - W'(V(s); s)|s \geq \theta\right] > 0$$

We distort agent θ's labor supply whenever we give - on average - a lower social weight $W'(V; \cdot)$ to agents with skill $s \geq \theta$ compared to their cost of funds $\frac{\lambda}{u'(c)}$ of increasing the utility to these agents. This is so since if we want to increase $y(\theta)$ we have to decrease taxes not only to agent θ we also have to increase the utility (at the margin by increasing consumption lump sum) of all agents with higher skill than him otherwise they will not tell the truth

Variational Interpretation

- If we rewrite the FOC with the expression for $\mu(\theta)$ we have

$$p(\theta)\lambda\left[1-\frac{v_y(y(\theta),\theta)}{u'(c(\theta))}\right]=\mu(\theta)\left[-v_{\theta,y}(y(\theta),\theta)\right]\qquad\Rightarrow$$

$$\Rightarrow\qquad p(\theta)\lambda\left[1-\frac{v_y(y(\theta),\theta)}{u'(c(\theta))}\right]$$
$$=[1-P(\theta)]\mathbf{E}\left[\frac{\lambda}{u'(c(s))}-W'(V(s);s)|s\geq\theta\right]\left[-v_{\theta,y}(y(\theta),\theta)\right]$$

- Consider an increase in $c(\theta)$ and $y(\theta)$ such that $V(\theta)$ is constant
- A change expressed in utils (i.e., $u\left(c(\theta)\right)$ changes by one unit)
- To keep IC with $\uparrow y(\theta)$ $\dot{V}(\theta)$ must increase by $-v_{\theta,y}(y(\theta),\theta)$
- This increases $U(s)$ for all $s>\theta$ and we can do it by increasing consumption $c(s)$ for all $s>\theta$ while keeping $y(s)$ constant
- The change increases $u\left(c(s)\right)$ uniformly, hence $\uparrow c(s)$ by $\frac{1}{u'(c(s))}$

The Taxation Principle

- In the real world, taxes are based on income, not on skill level.
- Each individual faces the same tax schedule $T : Y \to \mathbb{R}$:

$$\max_{y} \; y - T(y) - v\left(\frac{y}{\theta}\right)$$

Proposition *For each allocation $\{(c^*(\cdot), y^*(\cdot))\}$ that solves the incentive compatibility constraints, we can construct a tax schedule $T(\cdot)$ such that any individual with skill $\theta \in \Theta$ confronted with it, optimally chooses the pair $(c^*(\theta), y^*(\theta))$.*

Proof. Incentive constraints become:

$$y(\theta) - T(\theta) - v\left(\frac{y(\theta)}{\theta}\right) \geq y - T(y) - v\left(\frac{y}{\theta}\right) \text{ for all } y.$$

For all θ set $T(y^*(\theta)) = y^*(\theta) - c^*(\theta)$ and if $\nexists\theta$ such that $y = y^*(\theta)$, set $T(y) = \infty$. The taxation principle only requires the presence of a large enough punishment for incompatible choices. $\square$

Static Adverse Selection: A Formula for Taxes I

- Assume $f(\theta, n) = \theta n$; $W(V; \cdot) \equiv V$
  $\Rightarrow v(y, \theta) = v(\frac{y}{\theta})$; $-v_\theta(y, \theta) = \frac{n}{\theta} v'(n) \Rightarrow \dot{V}(\theta) = \frac{n}{\theta} v'(n)$.
- The FOCs for n are (those for V are as above with $W' \equiv 1$)

$$\left(\theta - \frac{v'(n(\theta))}{u'(c(\theta))}\right) \lambda p(\theta) = \mu(\theta) \frac{v'(n(\theta)) + n(\theta) v''(n(\theta))}{\theta}$$

- From FOC of agent with net wage w we have $\frac{v'(n)}{u'(c)} = w$ and Frisch elasticity w.r.t. w

$$\varepsilon_n(\theta) := \frac{dn}{dw}_{|\bar{c}} \frac{w}{n} = \frac{u'(c)w}{v''(n)n} = \frac{v'(n)}{v''(n)}$$

- Let $c := \theta n - T(\theta n)$, from agent's FOC, the net wage equals:

$$w(\theta) := \left(1 - T'(y(\theta))\right) \theta$$

Static Adverse Selection: A Formula for Taxes II

- Hence $\theta - \frac{v'(n(\theta))}{u'(c(\theta))} = T'(y(\theta))\theta$ and

$$\frac{v'(n(\theta)) + n(\theta)v''(n(\theta))}{\theta} = u'(c(\theta))\left(1 - T'(y(\theta))\right)\left(1 + \frac{1}{\varepsilon_n(\theta)}\right)$$

- Hence FOCs become

$$T'(y(\theta))\theta p(\theta)\lambda = \mu(\theta)u'(c(\theta))\left(1 - T'(y(\theta))\right)\left(1 + \frac{1}{\varepsilon_n(\theta)}\right)$$

- Using the definition of $\mu(\theta)$ and dividing by λ, we have:

$$\frac{T'(y(\theta))}{1 - T'(y(\theta))} = \mathbf{E}\left[\frac{1}{u'(c(s))} - \frac{1}{\lambda}|s \geq \theta\right]u'(c(\theta))\frac{1 - P(\theta)}{\theta p(\theta)}\left(1 + \frac{1}{\varepsilon_n(\theta)}\right)$$

- Note: $T' \in [0, 1)$. If $v(n) = \frac{n^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}}$ then $\varepsilon_n(\theta) = \gamma$ for all θ.

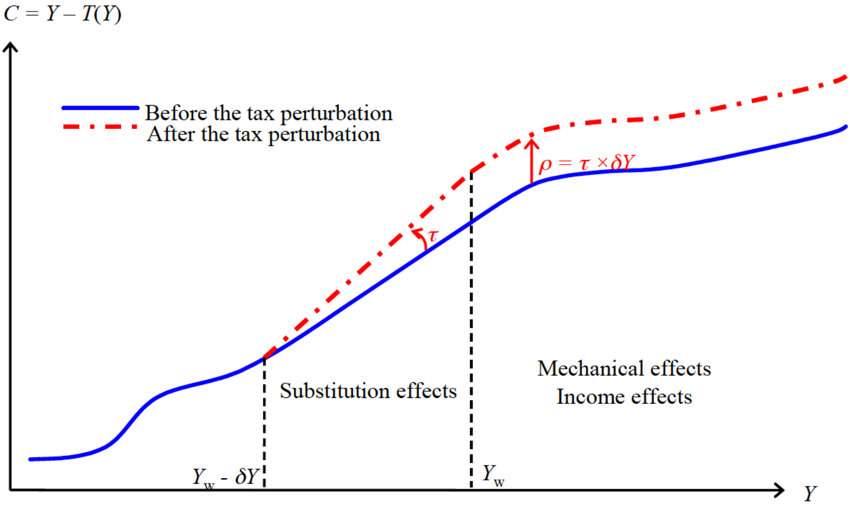

Figure 1.9: The Tax Perturbation

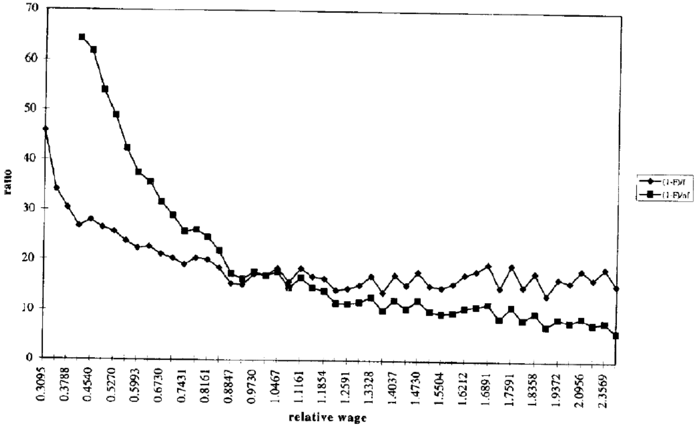

FIGURE 1. RATIOS $[1 - F(n)]/f(n)$ AND $[1 - F(n)]/[nf(n)]$ CALCULATED FROM RELATIVE WAGES

Figure 1.10: Empirical skill distribution computed by Diamond (1998)

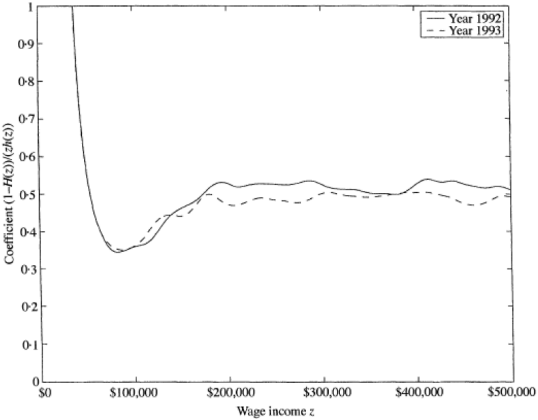

FIGURE 4
Hazard ratio $(1 - H(z))/(zh(z))$, years 1992 and 1993

Figure 1.11: Distribution of earnings as computed by Saez (2001).

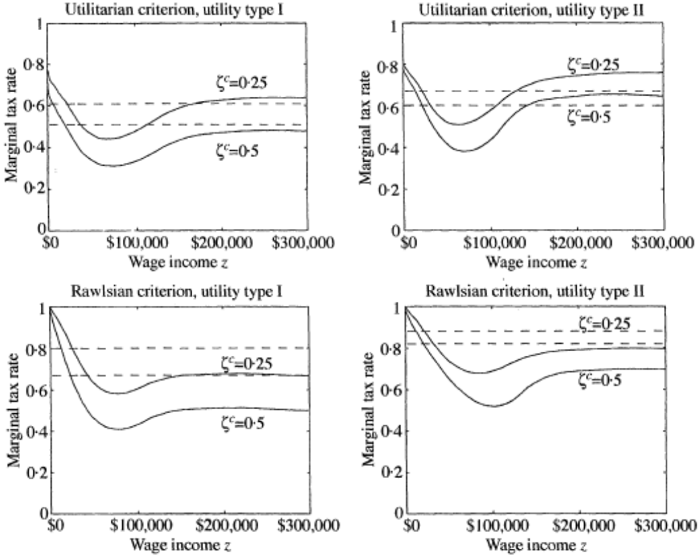

FIGURE 5

Optimal tax simulations

Figure 1.12: The numerical simulations of Saez (2001).

The Simplest Dynamic Adverse Selection Model

- Add payment and consumption in period zero and θ_0 known

$$\max_{c_0, y_0, y(\cdot), V(\cdot)} u(c_0) - v(y_0, \theta_0) + \beta \int_0^{\bar{\theta}} V(\theta) p(\theta) d\theta$$

$$+\lambda \left[y_0 - c_0 + q \int_0^{\bar{\theta}} \left[y(\theta) - g\left(V(\theta) + v(y(\theta), \theta) \right) \right] p(\theta) d\theta \right]$$

$$\text{s.t.} \quad \dot{V}(\theta) = -v_\theta(y(\theta), \theta) \quad \text{and} \quad \dot{y}(\theta) \geq 0.$$

- $q =$ price of the bond and $\beta =$ agent's discount factor.
- θ only revealed to the agent at $t = 1$. Full Commitment
- Obviously, y_0 such that $1 = \frac{v_y(y_0, \theta_0)}{u'(c_0)}$.
- Euler variation: $$\frac{\beta}{u'(c_0)} = \frac{1}{\lambda} = \mathbf{E}\left[\frac{q}{u'(c)} \right].$$
- Using Jensen's Inequality (positive capital tax?):

$$q u'(c_0) < \mathbf{E}\left[\beta u'(c) \right].$$

Intuitions for the Previous Result

1. Bond is a bad asset for incentives as it increases utility in an unbalanced way, detrimental for incentives
2. Joint deviations
3. Frontloaded consumption desirable: Since $\dot{c}_1 > 0$ when $qu'(c_0) = \beta \mathbf{E}\left[u'(c_1(\theta))\right]$ the planner can increase c_0 and reduce all $c_1(\theta)$ keeping IC. If the agent is contemplating to lie, s/he will reduce next period expected returns and consumption. As a consequence he would like a relatively low c_0. To discourage this deviation the planner keeps c_0 relatively high at the expenses of future payments.
4. Social intertemporal margin differs form the private: A bond perturbation $\downarrow c_0$ by $\beta\epsilon$ and $\uparrow c_1(\theta)$ by ϵ makes lying more attractive, hence it induces an additional cost for the planner.

Implementation Results

- Golosov and Tsyvinski (2006): DI with absorbing shocks
  - Means tested transfers on income: Upperbound on wealth or k
- Albanesi and Sleet (2006): iid shocks
  - One-to-one mapping between $k(\theta)$ and $U(\theta)$, with $\dot{y}(\theta) \geq 0$
  - $\tau(k, y)$ with $\tau_{y,k}(k, y) \neq 0$
- Kocherlakota (2005): general income process

$$1 - \tau^k(\theta) := \frac{qu'(c_0)}{\beta u'(c(\theta))}$$

  hence, by construction zero expected tax on capital:

$$\mathbf{E}\left[\left(1 - \tau^k(\theta)\right)\right] = 1 \quad \Rightarrow \mathbf{E}\left[\tau^k(\theta)\right] = 0.$$

- Pavoni and Violante (2005): UI discrete effort
  Tax for joint deviations and leave all agents at their liquidity constraint, say $k \geq 0$. Positive capital income tax
- Gottardi Pavoni (2011): Moral Hazard
  Uncontingent tax $\Rightarrow$ optimal tax on bond price is positive

The Value of Enduring Relationships I

- Pioneers: Townsend (1982), Green (1987), and Thomas and Worrall (1991)
- Endowment Economy
  - The model can be sees as a special case of our Mirrlees
  - $\mathcal{U}(c, n) = u(c - n)$ and $y = f(\theta, n) = \theta + n$
  - $U(\theta) := u(c(\theta) - y(\theta) + \theta) = u(\tau(\theta) + \theta)$
  - Only relevant the insurance aspect: $\tau(\theta) := y(\theta) - c(\theta)$
- Start with the Static Model

Dynamic Model: Heuristic Intro to Recursive Constracts

Define first of all, the following value function

$$\begin{aligned} V(w) := \max_{\tau(\cdot)} \quad & -\int_0^{\bar{\theta}} \tau(\theta)p(\theta)d\theta \\ s.t. \quad & \text{for all } \theta,\ \hat{\theta} \\ u(\tau(\theta)+\theta) \quad \geq \quad & u(\tau(\hat{\theta})+\theta); \qquad (IC_{\theta,\hat{\theta}}) \\ w \quad = \quad & \int_0^{\bar{\theta}} u(\tau(\theta)+\theta)p(\theta)d\theta. \qquad (PK) \end{aligned}$$

- The IC implies $\dot{\tau}(\theta) = 0$ so no insurance

$$u(\tau(\theta)+\theta) \geq u(\tau(\hat{\theta})+\theta)$$

$\Rightarrow$ FOC $u'(\tau+\theta)\dot{\tau}(\theta) = 0 \Rightarrow$ Autarchy!

- Envelope: $V'(w) = \frac{1}{\int_0^{\bar{\theta}} u'(\tau(\theta)+\theta)p(\theta)d\theta}$

Dynamic Model: Heuristic Intro to Recursive Contracts

- Consider the two period repetition
- In $t = 1$ first shock θ; in $t = 2$ second shock θ'
- Planner Problem

$$\max_{\tau_1(\cdot), w(\cdot)} \quad \int_0^{\bar{\theta}} \left[-\tau_1(\theta) + qV(w(\theta)) \right] p(\theta) d\theta$$

$$\text{s.t.} \quad \int_0^{\bar{\theta}} \left[u(\tau_1(\theta) + \theta) + \beta w(\theta) \right] p(\theta) d\theta \geq U_0; \qquad (IR)$$

$$u(\tau_1(\theta) + \theta) + \beta w(\theta) \geq u(\tau_1(\hat{\theta}) + \theta) + \beta w(\hat{\theta}). \quad (IC)$$

- What matters for the IC and IR is the promized utility w
- IC uses $\tau_1(\theta)$ and $w(\theta) \iff \tau_2(\theta, \theta')$
- From previous argument, $\tau_2(\theta, \theta') \equiv \tau_2(\theta)$.

Recursive contracts: Formal Derivation

Sequential Problem

$$\max_{\tau_1(\theta),\tau_2(\theta,\theta')} \int_0^{\bar{\theta}} \left[-\tau_1(\theta) + q \int_0^{\bar{\theta}} -\tau_2(\theta,\theta') p(\theta') d\theta' \right] p(\theta) d\theta \quad \text{s.t.}$$

$$\int_0^{\bar{\theta}} \left[u(\tau_1(\theta) + \theta) + \beta \left(\int_0^{\bar{\theta}} u\left(\theta' + \tau_2(\theta,\theta')\right) p(\theta') d\theta' \right) \right] p(\theta) d\theta \geq U_0; \; (IR)$$

$$u(\tau_1(\theta) + \theta) + \beta \int_0^{\bar{\theta}} u\left(\theta' + \tau_2(\theta,\theta')\right) p(\theta') d\theta' \qquad (IC)$$

$$\geq u(\tau_1(\hat{\theta}) + \theta) + \beta \int_0^{\bar{\theta}} u\left(\theta' + \tau_2(\hat{\theta},\theta')\right) p(\theta') d\theta' \; \forall \hat{\theta} \in \Theta.$$

Step 1: useful redefinition

The transfer scheme (contract) can be written as
$\mathcal{T} := \{\tau_1(\theta), \mathcal{T}_2(\theta)\}_{\theta\in\Theta}$, where $\forall\theta$ $\mathcal{T}_2(\theta) := \{\tau_2(\theta,\theta')\}_{\theta'\in\Theta}$
Let

$$\mathbf{V}(\mathcal{T}_2(\theta)) := \int_0^{\bar{\theta}} -\tau_2(\theta,\theta')p(\theta')d\theta'$$

be the planner value in period 2 from contract after θ occurred in period 1.
Similarly, for the agent, the equilibrium expected value

$$\mathbf{U}(\mathcal{T}_2(\theta)) := \int_0^{\bar{\theta}} u\left(\theta' + \tau_2(\theta,\theta')\right) p(\theta')d\theta'.$$

Note, these are values from any contract

Planner's Problem

Then planner problem can be written as

$$\max_{\tau_1(\cdot),\mathcal{T}_2(\cdot)} \quad \int_0^{\bar{\theta}} \left[-\tau_1(\theta) + q\mathbf{V}(\mathcal{T}_2(\theta))\ \right] p(\theta) d\theta$$

$$\text{s.t.} \quad \int_0^{\bar{\theta}} \left[u(\tau_1(\theta) + \theta) + \beta\mathbf{U}(\mathcal{T}_2(\theta))\right] p(\theta) d\theta \geq U_0;$$

$$\forall\theta \in \Theta : \quad u(\tau_1(\theta) + \theta) + \beta\mathbf{U}(\mathcal{T}_2(\theta)) \geq u(\tau_1(\hat{\theta}) + \theta) + \beta\mathbf{U}(\mathcal{T}_2(\hat{\theta})) \ \ \forall\hat{\theta}$$

$$\forall(\theta, \theta') \in \Theta^2 : \quad u(\theta' + \tau_2(\theta, \theta')) \geq u(\theta' + \tau_2(\theta, \hat{\theta}')) \ \ \forall\hat{\theta}' \in \Theta$$

NB: Since shocks are iid from the last IC, any agent who lied in period one still has incentive to tell the truth in period 2.

Step 2: A New Variable

Now, for each $\mathcal{T}$ let, for each $\theta \in \Theta$ $w(\theta) := \mathbf{U}(\mathcal{T}_2(\theta))$. The above problem is equivalent to

$$\max_{\tau_1(\cdot), w(\theta), \mathcal{T}_2(\cdot)} \quad \int_0^{\bar{\theta}} \left[-\tau_1(\theta) + q\mathbf{V}(\mathcal{T}_2(\theta)) \right] p(\theta) d\theta$$

$$\text{s.t.} \quad \int_0^{\bar{\theta}} \left[u(\tau_1(\theta) + \theta) + \beta \mathbf{U}(\mathcal{T}_2(\theta)) \right] p(\theta) d\theta \geq U_0;$$

$$\forall \theta \in \Theta : \quad u(\tau_1(\theta) + \theta) + \beta \mathbf{U}(\mathcal{T}_2(\theta)) \geq u(\tau_1(\hat{\theta}) + \theta) + \beta \mathbf{U}(\mathcal{T}_2(\hat{\theta})) \ \ \forall \hat{\theta}$$

$$\forall (\theta, \theta') \in \Theta^2 : \qquad u(\theta' + \tau_2(\theta, \theta')) \geq u(\theta' + \tau_2(\theta, \hat{\theta}')) \ \ \forall \hat{\theta}' \in \Theta$$

$$\forall \theta \in \Theta : \quad w(\theta) = \mathbf{U}(\mathcal{T}_2(\theta))$$

Step 3: The Bellman Principle

- We are now able to give a form to the value $\mathbf{V}(\mathcal{T}_2(\theta))$
- Notice that when $w(\theta)$ is chosen, $\mathcal{T}_2(\cdot)$ does not affect period zero choices of $\tau_1(\cdot)$.
  We can hence 'pass the max over'

$$\max_{\tau_1(\cdot), w(\theta)} \int_0^{\bar{\theta}} \left[-\tau_1(\theta) + q \max_{\substack{\mathcal{T}_2(\theta) s.t. \\ w(\theta) = \mathbf{U}(\mathcal{T}_2(\theta)) \\ u(\theta' + \tau_2(\theta, \theta')) \geq u(\theta' + \tau_2(\theta, \hat{\theta}')) \forall \hat{\theta}'}} \mathbf{V}(\mathcal{T}_2(\theta)) \right] p(\theta) d\theta$$

$$\text{s.t.} \quad \int_0^{\bar{\theta}} \left[u(\tau_1(\theta) + \theta) + \beta w(\theta) \right] p(\theta) d\theta \geq U_0;$$

$$\forall \theta \in \Theta : \quad u(\tau_1(\theta) + \theta) + \beta w(\theta) \geq u(\tau_1(\hat{\theta}) + \theta) + \beta w(\hat{\theta}) \;\; \forall \hat{\theta} \in \Theta.$$

The 'handy' recursive formulation

$$V_1(w_0) = \max_{\tau_1, w} \int_0^{\bar{\theta}} \left[-\tau_1(\theta) + qV(w(\theta)) \right] p(\theta) d\theta$$

$$\text{s.t.} \quad \int_0^{\bar{\theta}} \left[u(\tau_1(\theta) + \theta) + \beta w(\theta) \right] p(\theta) d\theta = w_0; \quad \text{(PK)}$$

$$\forall \theta \in \Theta : \quad u(\tau_1(\theta) + \theta) + \beta w(\theta) \geq u(\tau_1(\hat{\theta}) + \theta) + \beta w(\hat{\theta}) \;\; \forall \hat{\theta} \in \Theta.$$

- MULTIPERIOD: In the recursive formulation, for a given level of promised utility w_t the government chooses a transfer scheme and promises for future utility $w_{t+1}(\theta)$, and so on.
- The constraint (PK) is the 'promise-keeping' constraint that requires the contract to deliver the promised level of utility. It plays the role of a law of motion for the state variable w_0.

Observations

1. Note the we have an equality above. One must be careful in distinguishing between 'utility possibility frontier' and 'Pareto frontier'
2. Because of the Bellman Principle, all that matters to reconstruct the optimal contract continuation is a particular 'statistic' $w(\theta) = \mathbf{U}(\mathcal{T}_2^*(\theta))$, one number for each θ in this case.
3. This number induces a constraint on the next period problem. We then let the planner 're-maximize' subject to this constraint.
4. Optimality requires that $\mathbf{V}(\mathcal{T}_2^*(\theta)) = V(w(\theta))$. That is, the planner always goes on the frontier of the utility possibility set.

The Optimal Allocation I

- The Relaxed planner's problem

$$\max_{\tau_1, w} \quad \int_0^{\bar\theta} \left[-\tau_1(\theta) + qV(w(\theta)) \right] p(\theta) d\theta$$

$$\text{s.t.} \quad \int_0^{\bar\theta} \left[u(\tau_1(\theta) + \theta) + \beta w(\theta) \right] p(\theta) d\theta \geq U_0; \qquad (IR)$$

$$\dot\tau_1(\theta) u'(\tau_1(\theta) + \theta) + \dot w(\theta) = 0. \qquad (IC)$$

It can be shown that

$$\dot w(\theta) = \dot\tau_2(\theta) \mathbf{E} \left[u'(\tau_2(\theta) + \theta') \right]$$

The Optimal Allocation II

- From IC we immediately have: if $\dot{\tau}_1(\theta) < 0$ then $\dot{\tau}_2(\theta) > 0$
- Intuition: Recall the intertemporal utility of agent $\theta_1 = \theta$:

$$u(\tau^1(\theta) + \theta) + \beta \mathbf{E} u(\tau^2(\theta) + \theta').$$

  In $t = 1$ low θ's expect an improvement of the situation, hence willing to give up on the future for a today's subsidy $\tau_1 > 0$. High θ's expect a deterioration hence are willing to accept $\tau_1 < 0$ for a deterministic increase in future payments $\tau_2 > 0$.

- Endogenous return to savings

The Optimal Allocation III

- If we integrate by parts in the Lagrangian:

$$\int \mu(\theta)\dot{\tau}_2(\theta)\mathbf{E}\left[u'(\tau_2(\theta)+\theta')\right] d\theta = \int \dot{\mu}(\theta)\mathbf{E}\left[u(\tau_2(\theta)+\theta')\right]$$

$$\text{and} \int \mu(\theta)\dot{\tau}_1(\theta)u'(\tau_1(\theta)+\theta)d\theta$$

$$= \int \mu(\theta)\left[1+\dot{\tau}_1(\theta)\right]u'(\tau_1(\theta)+\theta)d\theta - \int \mu(\theta)u'(\tau_1(\theta)+\theta)d\theta$$

$$= \int \dot{\mu}(\theta)u(\tau_1(\theta)+\theta)d\theta - \int \mu(\theta)u'(\tau_1(\theta)+\theta)d\theta$$

- The associated Lagrangian becomes:

$$\begin{aligned}\mathcal{L} &= \int_0^{\bar{\theta}} \left[-\tau_1(\theta) - q\tau_2(\theta)\right] p(\theta)d\theta - \int \mu(\theta)u'(\tau_1(\theta)+\theta)d\theta \\ &\quad + \int_0^{\bar{\theta}} \left[\lambda p(\theta) + \dot{\mu}(\theta)\right]\left\{u(\tau_1(\theta)+\theta) + \beta\mathbf{E}\left[u(\tau_2(\theta)+\theta')\right]\right\} d\theta\end{aligned}$$

The Optimal Allocation IV

- The FOCs :

$$\begin{aligned} \left[\frac{1}{u'(\tau_1(\theta)+\theta)} - \lambda\right] p(\theta) &= \dot{\mu}(\theta) - \mu(\theta)\frac{u''(\tau_1(\theta)+\theta)}{u'(\tau_1(\theta)+\theta)} \\ \left[\frac{q}{\beta \mathbf{E}\left[u'(\tau_2(\theta)+\theta')\right]} - \lambda\right] p(\theta) &= \dot{\mu}(\theta) \end{aligned}$$

- Rearranging terms and using IC we have

$$p(\theta)\frac{\dot{\tau}_1(\theta)+q\dot{\tau}_2(\theta)}{\dot{\tau}_1(\theta)u'(\tau_1(\theta)+\theta)} = \mu(\theta)a(\theta) > 0,$$

where $a(\theta) = -\frac{u''(\tau_1(\theta)+\theta)}{u'(\tau_1(\theta)+\theta)}$

- Since $\dot{\tau}_1(\theta) < 0$ it must be that (NPV decreases with θ)

$$\dot{\tau}_1(\theta) + q\dot{\tau}_2(\theta) < 0.$$

Endogenizing Market Incompleteness

- Consider again the IC

$$\dot{\tau}_1(\theta) + \dot{\tau}_2(\theta)\beta \int_0^{\bar{\theta}} \frac{u'(c_2(\theta,\theta'))}{u'(c_1(\theta))} p(\theta') d\theta' = 0$$

- Assume now the agent has access to credit market as the planner
- Bond price $= q$
- Standard EEq.: Let $c_t = \tau_t + \theta_t$

$$\beta \int_0^{\bar{\theta}} \frac{u'(c_2(\theta,\theta'))}{u'(c_1(\theta))} p(\theta') d\theta' = q \quad \forall \theta$$

$\Rightarrow$ $\dot{\tau}_1(\theta) + q\dot{\tau}_2 = 0$

- The planner can only mimic the bond, no insurance on top of self-insurance: Bond economy
- Informational frictions can be used to explain why insurance markets are incomplete

$\Rightarrow$ Macro Literature on Endogenous incomplete markets

Dynamic Adverse Selection and Commitment

1. Full Commitment and no skill shocks: Identical to the static model (with randomizations)

2. No Commitment: Neither the principal nor the agent can commit after θ) is realized and it has been announced.
   - Take the Money and run
   - Ratchet effect
   - $\Rightarrow$ with continuum of types the contract is pooling

3. Partial commitment
   - Only commitment on the planner: Ex-post incentives (Rawlsian)
   - Only commitment on the agent: Restricted Revelation principle (Bester and Strausz, Econometrica, 2001)
     - Many agents vs 1 agent
     - Incentive feasible versus Constrained Efficient
     - True-telling with $P > 0$ (random mechanisms)