

A FINITELY ADDITIVE VERSION OF THE LAW OF THE ITERATED LOGARITHM*

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Abstract. A finitely additive version of the law of the iterated logarithm (LIL) is proposed. The formulation involves only finite-dimensional distributions of a sequence of independent random variables $(X_n)_{n \geq 1}$. It is also proved that in the case where one deals with σ -additive probabilities, the given result is equivalent to the classical version of the LIL.

Key words. finitely additive probability, law of the iterated logarithm, sums of independent random variables

PII. S0040585X97977884

1. Introduction. When dealing with probabilities that are σ -additive, one is fundamentally concerned with the problem of measurability of subsets of a given space of elementary events. In this abstract mathematical approach to probability theory, the notion of event is equivalent to that of a measurable set.

More realistically, and in the spirit of de Finetti's stance, one might think of an event as something independent of measurability issues. From this viewpoint observability becomes crucial. In other words, probability assessment of an event makes sense only when it is observable.

This remark is quite relevant with respect to limit theorems in probability theory. As an introduction to our work, consider the set $\{\limsup Z_n = c\}$, where $(Z_n)_{n \geq 1}$ is a sequence of random variables (r.v.'s), and c is a real constant. If $(Z_n)_{n \geq 1}$ satisfies some suitable conditions, well known, the law of the iterated logarithm (LIL) assigns probability 1 to the previous set. But $\{\limsup Z_n = c\}$, though measurable, is not an observable property. This raises the problem of giving alternative formulations of limit theorems, usually presented in the measure-theoretic approach to probability theory.

We proceed in proving a finitistic version of the LIL when one takes into account the possibility of working with probabilities that are not necessarily σ -additive.

The ideas we try to exploit are well expounded in de Finetti [6] and in later contributions by Dubins [7], Zanoni [19], and Regazzini [16], [17], [18]. They provide finitistic versions of some classical laws of large numbers and useful suggestions for further developments in this field.

As a matter of fact, there is a wide variety of works concerning the study of limit laws for sequences of r.v.'s in the finitely additive setting. However, they follow an approach which is different from the one we resort to. Instead of providing formulations of limiting properties in finitistic terms, they aim at giving versions of these properties that are identical to the σ -additive counterparts but still valid for non- σ -additive probabilities. The key notion in this framework is that of *strategy*, and guidelines are developed in [8], [9], [7], and [14]. We will not linger on this topic,

*Received by the editors August 6, 1997.

<http://www.siam.org/journals/tvp/44-4/97788.html>

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even though we think that tracing a comparison between the two approaches would be quite a stimulating task.

In what follows, we refer to de Finetti’s ideas in order to prove a *weak* version of the LIL. In section 2 we describe the finitely additive setting with which we work. In particular we clarify the concept of stochastically independent r.v.’s. In section 3 we prove the main theorem and specify some of its relevant features. In section 4 it is shown that in the case \mathbf{P} is a σ -additive probability measure, the classical version of the LIL is equivalent to our result. Finally, in section 5 we illustrate two examples in which the LIL fails, whereas the finitistic law being proposed is applicable.

2. Preliminaries. In this section we introduce some definitions and general results to be used later. Let Ω be any set of elementary cases, \mathcal{A} an algebra of subsets of Ω , and \mathbf{P} a *probability* on (Ω, \mathcal{A}) . Namely, \mathbf{P} is a nonnegative, finitely additive set function on Ω such that $\mathbf{P}(\Omega) = 1$. A Borel-measurable function $X_n: \Omega \rightarrow \mathbf{R}$ ($n \geq 1$) will be called a *random variable* (r.v.). Moreover, $\mathcal{B}(\mathbf{R}^n)$ denotes the class of Borel subsets of \mathbf{R}^n for all $n \geq 1$. In the following sections, we mean by $\mathbf{E}X$ the Dunford–Schwartz integral (see [10, section III.2], or, alternatively, [1, section 4.4]). A relevant concept to be introduced for our results is that of *stochastic independence*. In a finitely additive setting, one may give various definitions of stochastic independence, and in general they are not equivalent (for deeper insight see [6, pp. 321–323]). For mathematical convenience, we refer to the following definition.

DEFINITION 1. $(X_n)_{n \geq 1}$ is a sequence of stochastically independent r.v.’s on $(\Omega, \mathcal{A}, \mathbf{P})$ if

$$\begin{aligned} & \mathbf{P}\left\{\omega: (X_1(\omega), \dots, X_n(\omega)) \in A_1, (X_{n+1}(\omega), \dots, X_{n+m}(\omega)) \in A_2\right\} \\ &= \mathbf{P}\left\{\omega: (X_1(\omega), \dots, X_n(\omega)) \in A_1\right\} \mathbf{P}\left\{\omega: (X_{n+1}(\omega), \dots, X_{n+m}(\omega)) \in A_2\right\} \end{aligned}$$

for all $A_1 \in \mathcal{B}(\mathbf{R}^n)$, for all $A_2 \in \mathcal{B}(\mathbf{R}^m)$, for all $n, m \geq 1$.

In the last section we give examples of sequences of r.v.’s stochastically independent according to previous definition and governed by a proper finitely additive probability.

Now take (Ω, \mathcal{A}) to be equal to $(\mathbf{R}^\infty, \mathcal{B}(\mathbf{R}^\infty))$ and $(X_n)_{n \geq 1}$ the coordinate process, i.e., $X_k(x) = x_k, x = (x_1, x_2, \dots) \in \mathbf{R}^\infty$. It is possible to define a finitely additive probability \mathbf{P} on $(\mathbf{R}^\infty, \mathcal{B}(\mathbf{R}^\infty))$ with respect to which the X_n ’s are stochastically independent according to Definition 1. The existence of such a \mathbf{P} follows from an analogue of Kolmogorov’s extension theorem valid in this more general setting. For a proof of the latter result, the reader may refer to [5, p. 41] and [1, section 3.2].

Let us give the following definition.

DEFINITION 2. Let $(Z_n)_{n \geq 1}$ be a sequence of r.v.’s. We say that $b^*(n)$ and $b_*(n)$ are upper and lower sequences, respectively, for $(Z_n)_{n \geq 1}$ if $b^*(n) > 0, b_*(n) > 0$ and

(a) for all $\eta > 0 \exists N^* = N^*(\eta)$ such that for all $k \geq 1$

$$\mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{\frac{Z_j}{b^*(j)} < 1\right\}\right\} > 1 - \eta \quad \forall N \geq N^*;$$

(b) for all $\eta > 0 \exists N_* = N_*(\eta)$ and for all $N \geq N_*$ and for all $k \geq 1 \exists p_* = p_*(\eta, N, k)$ such that

$$\mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{\max_{n \leq j \leq n+p} \frac{Z_j}{b_*(j)} > 1\right\}\right\} > 1 - \eta \quad \forall p \geq p_*.$$

We introduced Definition 2 in order to study the oscillations of a sequence of independent r.v.'s in a finitistic setting. As one may note, there is no need to specify any extension of the finite-dimensional distributions of the process $(Z_n)_{n \geq 1}$ to $(\mathbf{R}^\infty, \mathcal{B}(\mathbf{R}^\infty))$. Therefore, our problem is equivalent to the determination, under suitable conditions, of upper and lower sequences for the sequence of partial sums. We will stress this point in Theorems 1 and 2, and a simple application of these very same theorems provides a finitely additive version of the LIL in Theorem 3.

3. Main results. In this section we illustrate a version of the LIL which holds in the finitely additive setting. The formulation is strictly related to an article by Cantelli [2], in which the author provides a first extension of Kolmogorov's LIL to a sequence of unbounded r.v.'s; see [11]. Let us specify some notation. $(X_n)_{n \geq 1}$ is a sequence of stochastically independent r.v.'s defined on $(\mathbf{R}^\infty, \mathcal{B}(\mathbf{R}^\infty))$ in the sense of section 2. Let $\sigma_n^2 = \mathbf{D}X_n$, $S_n = \sum_{i=1}^n X_i$, $s_n^2 = \mathbf{D}S_n = \sum_{i=1}^n \sigma_i^2$, $t_n^2 = 2 \log \log s_n^2$, $S_n^* = S_n/(s_n t_n)$ for all $n \geq 1$.

The main assumptions for the validity of our results concern the variance σ_n^2 and the absolute moments of order $(2 + \delta)$ of the X_n 's, for some $\delta \in (0, 1]$:

- (I) $\sigma_n^2 \geq \beta > 0$ for some constant β and for all $n \geq 1$,
- (II) $\mathbf{E}|X_n|^{2+\delta} < \alpha_\delta$ for some $\delta \in (0, 1]$, for some $\alpha_\delta > 0$, and for all $n \geq 1$.

We are now in a position to assert the first theorem which specifies the upper sequence.

THEOREM 1. *Let $(X_n)_{n \geq 1}$ be a sequence of independent r.v.'s with zero means for which (I)–(II) hold. Then for all $\varepsilon > 0$ $(1 + \varepsilon)s_n t_n$ is an upper sequence for $(S_n)_{n \geq 1}$.*

Proof. Let $(n_i)_{i \geq 3}$ be a subsequence in \mathbf{N} defined by $n_i := [\exp\{i/\log i\}]$, where $[x]$ denotes the integer part of $x \in \mathbf{R}$. By resorting to simple probabilistic arguments, the following inequalities can be shown to hold:

$$\begin{aligned} & \mathbf{P} \left\{ \bigcup_{j=\nu}^{\nu+m} \left\{ \max_{1 \leq n \leq n_j} S_n \geq (1 + \varepsilon) s_{n_{j-1}} t_{n_{j-1}} \right\} \right\} \\ & \leq \sum_{j=\nu}^{\nu+m} \mathbf{P} \left\{ \max_{1 \leq n \leq n_j} S_n \geq (1 + \varepsilon) s_{n_{j-1}} t_{n_{j-1}} \right\} \\ & \leq 2^{1+\gamma} \sum_{j=\nu}^{\infty} \mathbf{P} \{ S_{n_j} \geq (1 + \varepsilon) s_{n_{j-1}} t_{n_{j-1}} - C \} \end{aligned}$$

for all $\gamma > 0$ and for some $C > 0$ (for the second inequality above, see Lemma 3.2 in [11]). Moreover, by Lemma 3.1 in [11], $\sum_{j=\nu}^{\infty} \mathbf{P} \{ S_{n_j} \geq (1 + \varepsilon) s_{n_{j-1}} t_{n_{j-1}} - C \}$ is the remainder of a convergent series. Therefore, for all $\varepsilon > 0$ and for all $\eta > 0$ $\exists \nu_0 = \nu_0(\varepsilon, \eta)$ such that

$$\mathbf{P} \left\{ \bigcup_{j=\nu}^{\nu+m} \left\{ \max_{1 \leq n \leq n_j} S_n \geq (1 + \varepsilon) s_{n_{j-1}} t_{n_{j-1}} \right\} \right\} < \eta \quad \forall m \geq 1,$$

or, equivalently,

$$\mathbf{P} \left\{ \bigcap_{j=\nu}^{\nu+m} \left\{ \max_{1 \leq n \leq n_j} S_n < (1 + \varepsilon) s_{n_{j-1}} t_{n_{j-1}} \right\} \right\} > 1 - \eta$$

for all $m \geq 1$. Since $s_{n_{j-1}}t_{n_{j-1}} \leq s_n t_n$ for all $n \in \{n_{j-1}, n_{j-1} + 1, \dots, n_j\}$, one has

$$\begin{aligned} \mathbf{P} \left\{ \bigcap_{j=\nu}^{\nu+m} \bigcap_{n=1}^{n_j} \{S_n < (1 + \varepsilon) s_{n_{j-1}}t_{n_{j-1}}\} \right\} &\leq \mathbf{P} \left\{ \bigcap_{j=\nu}^{\nu+m} \bigcap_{n=n_{j-1}}^{n_j} \{S_n < (1 + \varepsilon) s_n t_n\} \right\} \\ &= \mathbf{P} \left\{ \bigcap_{j=n_\nu}^{n_{\nu+m}} \{S_n < (1 + \varepsilon) s_n t_n\} \right\}. \end{aligned}$$

Take $N^* = n_{\nu_0}$ and $N = n_\nu$. For all $k \geq 1$ there exists an $m \in \mathbf{N}$ such that $n_{\nu+m} > N + k$. It follows that

$$\mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \{S_n < (1 + \varepsilon) s_n t_n\} \right\} \geq \mathbf{P} \left\{ \bigcap_{j=n_\nu}^{n_{\nu+m}} \{S_n < (1 + \varepsilon) s_n t_n\} \right\}.$$

Theorem 1 is proved.

THEOREM 2. *If $(X_n)_{n \geq 1}$ is a sequence of independent r.v.'s with zero means satisfying (I)–(II), then for all $\varepsilon, \eta > 0$, $\exists N_* = N_*(\varepsilon, \eta)$ and for all $N \geq N_* \exists p_0 = p_0(\varepsilon, \eta, N)$ such that*

$$(1) \quad \mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \{S_j^* > 1 - \varepsilon\} \right\} > 1 - \eta$$

for all $k \geq 1$, for all $p \geq \max\{p_0, e^k\}$.

It is possible to split the proof of Theorem 2 into three steps.

In the first step we need to work with independent events. Thus, we give a result for the differences of partial sums along a convenient subsequence of positive integers. The second step consists in passing from the subsequence of differences to the subsequence of partial sums. Finally, we get the assertion over the whole sequence by *reductio ad absurdum*.

The first step is characterized by the following lemma.

LEMMA 1. *Let $(n_i)_{i \geq 3}$ be a subsequence of positive integers defined by $n_i := [\exp\{i \log \log i\}]$ and $(*) := \{j_1, \dots, j_{l+1} \in \{n_{\nu+1}, n_{\nu+2}, \dots, n_{\nu+m}\} : j_1 < \dots < j_{l+1}\}$. Then for all $\varepsilon, \eta > 0 \exists \nu_0 = \nu_0(\varepsilon, \eta)$ and for all $\nu \geq \nu_0 \exists m_0 = m_0(\varepsilon, \eta, \nu)$ such that*

$$(2) \quad \mathbf{P} \left\{ \bigcup_{(*)} \bigcap_{r=2}^{l+1} \{S_{j_r} - S_{j_{r-1}} > (1 - \varepsilon) s_{j_r} t_{j_r}\} \right\} > 1 - \eta$$

for all $l \geq 1$, for all $m \geq \max\{m_0, 2^l\}$.

Proof. To simplify notation, let

$$A_{j_r}^{(\varepsilon)} := \{S_{j_r} - S_{j_{r-1}} > (1 - \varepsilon) s_{j_r} t_{j_r}\}, \quad C_{\nu, m, l}^\varepsilon := \bigcap_{(*)} \bigcup_{r=2}^{l+1} A_{j_r}^{(\varepsilon)c}.$$

We can interpret $C_{\nu, m, l}^\varepsilon$ as the event which is true if and only if at most $l - 1$ of the m events $A_{n_i}^{(\varepsilon)}$ ($i = \nu + 1, \dots, \nu + m$) are verified. Hence, if $B_{\nu, m}^{(j)}$ is the event corresponding to exactly j of the $A_{n_i}^{(\varepsilon)}$'s ($i = \nu + 1, \dots, \nu + m$) being verified,

$$\mathbf{P}(C_{\nu, m, l}^\varepsilon) = \sum_{j=0}^{l-1} \mathbf{P}(B_{\nu, m}^{(j)}).$$

We try to estimate $\mathbf{P}(B_{\nu,m}^{(j)})$. If $j = 0$,

$$\begin{aligned} \mathbf{P}(B_{\nu,m}^{(0)}) &= \mathbf{P}\left\{ \bigcap_{i=\nu+1}^{\nu+m} A_{n_i}^{(\varepsilon)c} \right\} = \prod_{i=\nu+1}^{\nu+m} (1 - \mathbf{P}(A_{n_i}^{(\varepsilon)})) \\ &\leq \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} \right\} \quad [p_{n_i} := \mathbf{P}(A_{n_i}^{(\varepsilon)})]. \end{aligned}$$

For any other $j \in \{1, \dots, l-1\}$, we define $(\square) := \{s_1, \dots, s_j \in \{n_{\nu+1}, \dots, n_{\nu+m}\}, s_1 < \dots < s_j\}$ and $q_{j+1}, \dots, q_m \in \{n_{\nu+1}, \dots, n_{\nu+m}\} \setminus \{s_1, \dots, s_j\}$, so that

$$\begin{aligned} \mathbf{P}(B_{\nu,m}^{(j)}) &= \sum_{(\square)} p_{s_1} \cdots p_{s_j} \prod_{i=j+1}^m (1 - p_{q_i}) \leq \sum_{(\square)} p_{s_1} \cdots p_{s_j} \exp\left\{ - \sum_{i=j+1}^m p_{q_i} \right\} \\ &= \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} \right\} \sum_{(\square)} p_{s_1} \cdots p_{s_j} \exp\left\{ \sum_{i=1}^j p_{s_i} \right\} \\ &= \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} \right\} \sum_{(\square)} \sum_{(**)} \frac{1}{j!} p_{\gamma_1} \cdots p_{\gamma_j} \exp\left\{ \sum_{i=1}^j p_{\gamma_i} \right\}, \end{aligned}$$

where $(**)$ stands for the set of all permutations $\gamma_1, \dots, \gamma_j$ of indices s_1, \dots, s_j .

Moreover, $\sum_{(\square)} \sum_{(**)} p_{\gamma_1} \cdots p_{\gamma_j} = \sum_{(+)} p_{\gamma_1} \cdots p_{\gamma_j}$, where the latter sum is extended over all j -permutations of a set of m elements. Thus

$$\begin{aligned} \mathbf{P}(B_{\nu,m}^{(j)}) &\leq \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} \right\} \frac{1}{j!} \sum_{(+)} p_{\gamma_1} e^{p_{\gamma_1}} \cdots p_{\gamma_j} e^{p_{\gamma_j}} \\ &\leq \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} \right\} \frac{1}{j!} \left(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right)^j \\ &= \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} (1 - e^{p_{n_i}}) \right\} \frac{1}{j!} \left(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right)^j \\ &\quad \times \exp\left\{ - \left(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right) \right\}, \\ \mathbf{P}(C_{\nu,m,l}^\varepsilon) &= \sum_{j=0}^{l-1} \mathbf{P}(B_{\nu,m}^{(j)}) \leq \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} (1 - e^{p_{n_i}}) \right\} \\ (3) \quad &\quad \times \sum_{j=0}^{l-1} \frac{1}{j!} \left(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right)^j \exp\left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right\}. \end{aligned}$$

In order to estimate the expression in (3), the following inequalities for p_{n_i} are valid.

With $(n_i)_{i \geq 3}$ as above, for every $\varepsilon > 0, \gamma > 0$ there exist $i_0 = i_0(\varepsilon, \gamma)$ and $C_j = C_j(\varepsilon, \gamma, \delta, \beta, \alpha_\delta)$ ($j = 1, 2$) such that for all $i \geq i_0$

$$(4) \quad C_1 \left(\frac{1}{i \log \log i} \right)^{(1+\gamma)(1-\varepsilon)^2} \leq p_{n_i} \leq C_2 \left(\frac{1}{i \log \log i} \right)^{(1-\gamma)(1-\varepsilon)^2}.$$

The required estimate of p_{n_i} is achieved by applying a large deviations type result. Proceeding in a similar fashion as in Theorem 7.1.3 in [3, p. 203], one can prove that if $(n_i)_{i \geq 1}$ is such that $n_i - n_{i-1} \sim n_i$ ($i \rightarrow \infty$), and if

$$(5) \quad z_{n_i} := \log \frac{\sum_{j=n_{i-1}+1}^{n_i} \mathbf{E}|X_j|^{2+\delta}}{s_{n_i}^{2+\delta}} + (1 + \gamma)(1 - \varepsilon)^2 \frac{s_{n_{i-1}}^2 t_{n_{i-1}}^2}{2(s_{n_i}^2 - s_{n_{i-1}}^2)} \longrightarrow -\infty$$

as $i \rightarrow \infty$ for all $\varepsilon > 0$ and for all $\gamma > 0$, then there is $i^* = i^*(\varepsilon, \gamma)$ such that

$$(6) \quad \begin{aligned} & \exp \left\{ - (1 + \gamma)(1 - \varepsilon)^2 \frac{s_{n_i}^2 t_{n_i}^2}{2(s_{n_i}^2 - s_{n_{i-1}}^2)} \right\} \\ & \leq p_{n_i} \leq \exp \left\{ - (1 - \gamma)(1 - \varepsilon)^2 \frac{s_{n_i}^2 t_{n_i}^2}{2(s_{n_i}^2 - s_{n_{i-1}}^2)} \right\} \end{aligned}$$

for all $i \geq i^*$.

When $n_i = \lceil \exp\{i \log \log i\} \rceil$, it is easily verified that $n_i - n_{i-1} \sim n_i$ and, under our assumptions (I)–(II), (5) is valid. Indeed,

$$\beta^{2+\delta} < (\sigma_n^2)^{2+\delta} \leq (\mathbf{E}|X_n|^{2+\delta})^2 < \alpha_\delta^2,$$

and

$$\begin{aligned} & \log \left\{ \frac{\sum_{j=n_{i-1}+1}^{n_i} \mathbf{E}|X_j|^{2+\delta}}{s_{n_i}^{2+\delta}} \right\} + (1 + \gamma)(1 - \varepsilon)^2 \frac{s_{n_{i-1}}^2 t_{n_{i-1}}^2}{2(s_{n_i}^2 - s_{n_{i-1}}^2)} \\ & < \log \frac{(n_i - n_{i-1})\alpha_\delta}{(n_i - n_{i-1})^{1+\delta/2} \beta^{1+\delta/2}} \\ & \quad + (1 + \gamma)(1 - \varepsilon)^2 \frac{n_i \alpha_\delta^{2/(2+\delta)} 2}{2(n_i - n_{i-1})\beta} \log \log(n_i \alpha_\delta^{2/(2+\delta)}) \\ & = \log \frac{\alpha_\delta}{(n_i - n_{i-1})^{\delta/2} \beta^{1+\delta/2}} \\ & \quad + (1 + \gamma)(1 - \varepsilon)^2 \frac{n_i \alpha_\delta^{2/(2+\delta)}}{(n_i - n_{i-1})\beta} \log \log(n_i \alpha_\delta^{2/(2+\delta)}) \\ & \sim -\frac{\delta}{2} \log(n_i - n_{i-1}) + (1 + \gamma)(1 - \varepsilon)^2 \frac{\alpha_\delta^{2/(2+\delta)}}{\beta} \log \log n_i \longrightarrow -\infty. \end{aligned}$$

By (I)–(II), for every i there is a $\kappa_i \in (\beta, \alpha_\delta^{2/(2+\delta)})$ such that $s_{n_i}^2 = \kappa_i n_i$, and, therefore,

$$\frac{s_{n_i}^2 t_{n_i}^2}{2(s_{n_i}^2 - s_{n_{i-1}}^2)} \sim \frac{1}{i \log \log i} \quad \text{as } i \rightarrow \infty$$

and this completes the proof of (4) if we choose a suitable $i_0 = i_0(\varepsilon, \gamma) > i^*$.

Now, for sufficiently small $\varepsilon > 0$ one can determine a $\gamma = \gamma(\varepsilon) > 0$ such that $(1 + \gamma)(1 - \varepsilon)^2 < 1$ and $2(1 - \gamma)(1 - \varepsilon)^2 > 1$. These two conditions imply $\sum_{i=3}^\infty p_{n_i} = \infty$ and $\sum_{i=3}^\infty p_{n_i}^2 < \infty$.

In other words, we can now write in (4) $i_0 = i_0(\varepsilon)$.

Turning back to (3), the first term in the product, $\exp\{-\sum_{i=\nu+1}^{\nu+m} p_{n_i}(1 - e^{p_{n_i}})\}$, is bounded for every $\nu, m \geq 1$. To see this, note that $-p_{n_i}(1 - e^{p_{n_i}}) \sim p_{n_i}^2$ (as $i \rightarrow \infty$).

We shall denote by K a constant such that $\exp\{-\sum_{i=\nu+1}^{\nu+m} p_{n_i}(1 - e^{p_{n_i}})\} \leq K$.

As far as the second term in (3) is concerned, let ξ_i ($i = \nu + 1, \dots, \nu + m$) be a random variable distributed according to a Poisson law with parameter $p_{n_i} e^{p_{n_i}}$ and let the ξ_i 's be stochastically independent. Then,

$$(7) \quad \sum_{j=0}^{l-1} \frac{1}{j!} \left(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right)^j \exp \left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}} \right\} = \mathbf{P} \left\{ \sum_{i=\nu+1}^{\nu+m} \xi_i \leq l - 1 \right\}.$$

Moreover, by the Berry–Esseen inequality (cf. [13, p. 115]), we get

$$(8) \quad \mathbf{P} \left\{ \sum \xi_i \leq l - 1 \right\} = \mathbf{P} \left\{ \sum \xi_i \leq l - 1 \right\} - \Phi \left(\frac{l - 1 - \sum p_{n_i} e^{p_{n_i}}}{\sqrt{\sum p_{n_i} e^{p_{n_i}}}} \right) + \Phi \left(\frac{l - 1 - \sum p_{n_i} e^{p_{n_i}}}{\sqrt{\sum p_{n_i} e^{p_{n_i}}}} \right) A \frac{\sum \mathbf{E}|\xi_i|^3}{b_{\nu,m}^{3/2}} + \Phi \left(\frac{l - 1 - \sum p_{n_i} e^{p_{n_i}}}{\sqrt{\sum p_{n_i} e^{p_{n_i}}}} \right),$$

where all sums are taken with i running from $\nu+1$ up to $\nu+m$, $b_{\nu,m} := \sum_{i=\nu+1}^{\nu+m} \mathbf{D}(\xi_i) = \sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}}$, $A > 0$ is an appropriate constant, and $\Phi(\cdot)$ is the standard Gaussian distribution function. Obviously,

$$\frac{\sum_{i=\nu+1}^{\nu+m} \mathbf{E}|\xi_i|^3}{b_{\nu,m}^{3/2}} = \frac{\sum_{i=\nu+1}^{\nu+m} (p_{n_i} e^{p_{n_i}} + 3p_{n_i}^2 e^{2p_{n_i}} + p_{n_i}^3 e^{3p_{n_i}})}{(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}})^{3/2}} \sim \frac{\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}}}{(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}})^{3/2}} \sim \frac{1}{(\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}})^{1/2}} \rightarrow 0, \quad m \rightarrow \infty.$$

As far as the second summand in (8) is concerned, Φ , we let $\mathcal{D} := \{(m, l) \in \mathbf{N}^2 : m \geq 2^l\}$. This implies

$$\Phi \left(\frac{l - 1 - \sum p_{n_i} e^{p_{n_i}}}{\sqrt{\sum p_{n_i} e^{p_{n_i}}}} \right) \leq \Phi \left(\frac{\log_2 m - \sum p_{n_i} e^{p_{n_i}}}{\sqrt{\sum p_{n_i} e^{p_{n_i}}}} \right) \quad \forall (m, l) \in \mathcal{D}.$$

We now try to estimate the right-hand side in the previous inequality. By Euler's summation formula (see [12, p. 521]), one obtains, with $\mu = (1 + \gamma)(1 - \varepsilon)^2$,

$$\begin{aligned} \sum_{i=\nu+1}^{\nu+m} p_{n_i} &\geq \sum_{i=\nu+1}^{\nu+m} \left(\frac{1}{i \log \log i} \right)^\mu \sim \int_{\nu+1}^{\nu+m} \left(\frac{1}{x \log \log x} \right)^\mu dx \\ &= \int_{\log(\nu+1)}^{\log(\nu+m)} \frac{e^{(1-\mu)t}}{(\log t)^\mu} dt \geq \frac{1}{1-\mu} e^{(1-\mu)\log(\nu+m)} \frac{1}{(\log \log(\nu+m))^\mu} \\ &\quad - \frac{1}{1-\mu} e^{(1-\mu)\log(\nu+1)} \frac{1}{(\log \log(\nu+1))^\mu} \quad [\text{integrate by parts}] \\ &\sim \frac{1}{1-\mu} \cdot \frac{m^{(1-\mu)}}{(\log \log m)^\mu} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Since $p_{n_i} \leq p_{n_i} e^{p_{n_i}}$ for all i

$$(9) \quad \frac{\log_2 m - \sum p_{n_i} e^{p_{n_i}}}{\sqrt{\sum p_{n_i} e^{p_{n_i}}}} \leq \frac{\log_2 m - (1-\mu)^{-1} m^{(1-\mu)} (\log \log m)^{-\mu}}{\sqrt{(1-\mu)^{-1} m^{(1-\mu)} (\log \log m)^{-\mu}}} \rightarrow -\infty$$

as $m \rightarrow \infty$ and

$$\Phi \left(\frac{l - 1 - \sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}}}{\sqrt{\sum_{i=\nu+1}^{\nu+m} p_{n_i} e^{p_{n_i}}}} \right) \rightarrow 0.$$

Finally, for all $(m, l) \in \mathcal{D}$

$$\begin{aligned} \mathbf{P}(C_{\nu, m, l}^\varepsilon) &\leq \exp \left\{ - \sum_{i=\nu+1}^{\nu+m} p_{n_i} (1 - e^{p_{n_i}}) \right\} \mathbf{P} \left\{ \sum_{i=\nu+1}^{\nu+m} \xi_i \leq l - 1 \right\} \\ &\leq K \mathbf{P} \left\{ \sum_{i=\nu+1}^{\nu+m} \xi_i \leq \log_2 m \right\}. \end{aligned}$$

Therefore, for any $\varepsilon, \eta > 0$ there is $\nu_0 = \nu_0(\varepsilon, \eta) \geq i_0(\varepsilon)$ and for all $\nu \geq \nu_0$ one can find $m_0 = m_0(\varepsilon, \eta, \nu) \in \mathbf{N}$ such that $\mathbf{P}(C_{\nu, m, l}^\varepsilon) < \eta$ for all $m \geq \max\{m_0, 2^l\}$.

Taking into account the definition of the event $C_{\nu, m, l}^\varepsilon$,

$$\mathbf{P} \left\{ \bigcup_{(*) r=2}^{l+1} \{S_{j_r} - S_{j_{r-1}} > (1 - \varepsilon) s_{j_r} t_{j_r}\} \right\} > 1 - \eta$$

and Lemma 1 is finally proved.

Next, we aim at passing from the subsequence of differences of partial sums to the subsequence of partial sums. Lemma 2 below provides the result we need.

LEMMA 2. *Let $(n_i)_{i \geq 3}$ be the subsequence of positive integers defined in Lemma 1. Then for any $\varepsilon, \eta > 0$ there is a $\nu_{00} = \nu_{00}(\varepsilon, \eta) > 0$ and for any $\nu \geq \nu_{00}$ there is a strictly positive integer $m_{00} = m_{00}(\varepsilon, \eta, \nu)$ such that*

$$(10) \quad \mathbf{P} \left\{ \bigcap_{h=\nu+1}^{\nu+l} \left\{ \max_{h \leq j \leq h+m} \frac{S_{n_j}}{s_{n_j} t_{n_j}} > 1 - \varepsilon \right\} \right\} > 1 - 2\eta,$$

for all $m \geq \max\{m_{00}, 2^l\}$.

Proof. Since

$$\bigcap_{h=\nu+1}^{\nu+l} \bigcup_{j=h}^{h+m} \{S_{n_j} > (1 - \varepsilon) s_{n_j} t_{n_j}\} \supset \bigcup_{(*) r=2}^{l+1} \{S_{j_r} > (1 - \varepsilon) s_{j_r} t_{j_r}\} =: D_{\nu, m, l}^\varepsilon,$$

it is sufficient to prove the assertion for $D_{\nu, m, l}^\varepsilon$. One may note that

$$\begin{aligned} &\mathbf{P} \left\{ \bigcup_{(*) r=2}^{l+1} \{S_{j_r} > (1 - \varepsilon) s_{j_r} t_{j_r} - (1 + \varepsilon) s_{j_{r-1}} t_{j_{r-1}}\} \right\} \\ &\geq \mathbf{P} \left\{ \bigcup_{(*) r=2}^{l+1} \{S_{j_r} - S_{j_{r-1}} > (1 - \varepsilon) s_{j_r} t_{j_r}\} \cap \{S_{j_{r-1}} > -(1 + \varepsilon) s_{j_{r-1}} t_{j_{r-1}}\} \right\} \\ &\geq \mathbf{P} \left\{ \bigcup_{(*) r=2}^{l+1} \{S_{j_r} - S_{j_{r-1}} > (1 - \varepsilon) s_{j_r} t_{j_r}\} \right. \\ &\quad \left. \bigcap_{i=1}^m \{-S_{n_{\nu+i}} < (1 + \varepsilon) s_{n_{\nu+i}} t_{n_{\nu+i}}\} \right\} > 1 - 2\eta, \end{aligned}$$

because of (2) and Theorem 1 applied to the sequence $(-X_n)_{n \geq 1}$. Since for a sufficiently large r there exists $\varepsilon' > \varepsilon$ such that $(1 - \varepsilon) s_{j_r} t_{j_r} - (1 + \varepsilon) s_{j_{r-1}} t_{j_{r-1}} > (1 - \varepsilon') s_{j_r} t_{j_r}$, then

$$\mathbf{P} \left\{ \bigcup_{(*)} \bigcap_{r=2}^{l+1} \{S_{j_r} > (1 - \varepsilon) s_{j_r} t_{j_r}\} \right\} > 1 - 2\eta.$$

Lemma 2 is proved.

Proof of Theorem 2. By virtue of Lemma 2, we need only to extend (10) to the whole sequence. We proceed by *reductio ad absurdum*. Suppose (1) does not hold. Then for some $\varepsilon, \eta > 0$, for every $N_*, p_0 \in \mathbf{N}$ there are $\tilde{N} \geq N_*, \tilde{k} = \tilde{k}(p_0, N_*)$, and $p^* \geq \max\{p_0, e^{\tilde{k}}\}$ such that

$$1 - \eta > \mathbf{P} \left\{ \bigcap_{n=\tilde{N}}^{\tilde{N}+\tilde{k}} \bigcup_{j=n}^{n+p^*} \{S_j^* > 1 - \varepsilon\} \right\} \geq \mathbf{P} \left\{ \bigcup_{j=\tilde{N}+\tilde{k}}^{\tilde{N}+p^*} \{S_j^* > 1 - \varepsilon\} \right\}.$$

Choose η^* such that $2\eta^* < \eta$. Then we can find a $\nu_{00} = \nu_{00}(\varepsilon, \eta^*)$ and for any $\nu \geq \nu_{00}$ there is a $m_{00} = m_{00}(\varepsilon, \eta^*, \nu)$ such that for every $l \in \mathbf{N}$, and $m \geq \max\{m_{00}, 2^l\}$, relation (10) holds with η^* in place of η . Observe that

$$\begin{aligned} \bigcap_{r=\nu+1}^{\nu+l} \bigcup_{j=r}^{r+m} \{S_{n_j}^* > 1 - \varepsilon\} &\subset \bigcap_{r=1}^{l-1} \bigcap_{j=n_{\nu+r}}^{n_{\nu+r+1}} \bigcup_{i=j}^{n_{\nu+m+r+1}} \{S_i^* > 1 - \varepsilon\} \\ &\subset \bigcap_{j=n_{\nu+1}}^{n_{\nu+2}} \bigcup_{i=j}^{n_{\nu+m+2}} \{S_i^* > 1 - \varepsilon\} \subset \bigcup_{i=n_{\nu+2}}^{n_{\nu+m+2}} \{S_i^* > 1 - \varepsilon\}. \end{aligned}$$

For \tilde{N} and \tilde{k} fixed above, it is possible to find a $\nu \in \mathbf{N}$ ($\nu > \nu_{00}$) such that $\tilde{N} + \tilde{k} \leq n_{\nu+2} = \lceil \exp\{(\nu + 2) \log \log(\nu + 2)\} \rceil$, i.e., $\tilde{k} \leq \exp\{(\nu + 2) \log \log(\nu + 2)\} - \tilde{N} - 1$. If we also find out that $\tilde{N} + p^* \geq n_{\nu+m+2} = \lceil \exp\{(\nu + m + 2) \log \log(\nu + m + 2)\} \rceil$, then we would have

$$(11) \quad \bigcup_{n=\tilde{N}+\tilde{k}}^{\tilde{N}+p^*} \{S_n^* > 1 - \varepsilon\} \supset \bigcup_{j=n_{\nu+2}}^{n_{\nu+m+2}} \{S_j^* > 1 - \varepsilon\}.$$

Since $p^* > e^{\tilde{k}}$, it is sufficient that

$$\tilde{N} + e^{\tilde{k}} \geq \exp\{(\nu + m + 2) \log \log(\nu + m + 2)\},$$

which means $\tilde{k} > \log(\exp\{(\nu + m + 2) \log \log(\nu + m + 2)\} - \tilde{N}) = (\nu + m + 2) \log \log(\nu + m + 2) + \log(1 - \tilde{N} e^{-(\nu+m+2) \log \log(\nu+m+2)})$. Hence, (11) holds if

$$\begin{aligned} z_\nu &= (\nu + m + 2) \log \log(\nu + m + 2) + \log(1 - \tilde{N} e^{-(\nu+m+2) \log \log(\nu+m+2)}) \\ &\leq \tilde{k} \leq \exp\{(\nu + 2) \log \log(\nu + 2)\} - \tilde{N} - 1 = y_\nu, \end{aligned}$$

and these inequalities are consistent for a sufficiently large $\nu \geq \nu_{00}$. In the case $z_\nu > \tilde{k}$ we have

$$\bigcup_{n=\tilde{N}+\tilde{k}}^{\tilde{N}+p^*} \{S_n^* > 1 - \varepsilon\} \supset \bigcup_{n=\tilde{N}+[z_\nu]}^{\tilde{N}+p^*} \{S_n^* > 1 - \varepsilon\} \supset \bigcup_{j=n_{\nu+2}}^{n_{\nu+m+2}} \{S_j^* > 1 - \varepsilon\}.$$

Finally, as a consequence of our hypotheses we would have

$$1 - \eta > \mathbf{P} \left\{ \bigcup_{n=\tilde{N}+k}^{\tilde{N}+p^*} \{S_n^* > 1 - \varepsilon\} \right\} \geq \mathbf{P} \left\{ \bigcup_{j=n_\nu+2}^{n_\nu+m+2} \{S_j^* > 1 - \varepsilon\} \right\} > 1 - 2\eta^*$$

and this is absurd because $2\eta^* < \eta$. Theorem 2 is proved.

Joint application of the results in Theorems 1 and 2 provides a finitistic characterization of the oscillations problem for the sequence $(S_n)_{n \geq 1}$.

THEOREM 3. *Let $(X_n)_{n \geq 1}$ be a sequence of independent r.v.'s with zero means. If assumptions (I)–(II) are satisfied, then for all $\varepsilon, \eta > 0$, $\exists N_0 = N_0(\varepsilon, \eta) \in \mathbf{N}$, and for all $N \geq N_0$ $\exists p_0 = p_0(\varepsilon, \eta, N)$ such that*

$$(12) \quad \mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \left\{ \left| \max_{n \leq j \leq n+p} \frac{S_j}{s_j t_j} - 1 \right| < \varepsilon \right\} \right\} > 1 - \eta$$

for all $k \geq 1$, for all $p \geq \max\{p_0, e^k\}$.

Proof. Let us put

$$A_1^{(\varepsilon)}(N, p, k) := \bigcap_{n=N}^{N+k} \left\{ \max_{n \leq j \leq n+p} S_j^* < 1 + \varepsilon \right\},$$

$$A_2^{(\varepsilon)}(N, p, k) := \bigcap_{n=N}^{N+k} \left\{ \max_{n \leq j \leq n+p} S_j^* > 1 - \varepsilon \right\}.$$

In order to prove (12) one can show that for every $\varepsilon > 0$ and $\eta > 0$ there is a strictly positive integer $N_0 = N_0(\varepsilon, \eta)$ and for every $N \geq N_0$ there exists a $p_0 = p_0(\varepsilon, \eta, N) \in \mathbf{N}$ such that, if $p^* = \max\{p_0, e^k\}$,

$$(13) \quad \mathbf{P}(A_1^{(\varepsilon)}(N, p, k)) > 1 - \frac{\eta}{2} \quad \forall N \geq N_0, \forall k \geq 1, \forall p \geq p^*,$$

$$(14) \quad \mathbf{P}(A_2^{(\varepsilon)}(N, p, k)) > 1 - \frac{\eta}{2} \quad \forall N \geq N_0, \forall k \geq 1, \forall p \geq p^*.$$

But, clearly, if $N_0 = \max\{N^*, N_*\}$ (see Theorems 1 and 2), then Theorem 1 entails (13), whereas (1) implies (14). Theorem 3 is proved.

Remark 1. As a matter of fact, (12) does not completely characterize oscillations of partial sums on finitary sets. We have to look at the behavior of $\min S_n^*$ as well.

COROLLARY 1. *Suppose hypotheses (I)–(II) are satisfied. Then for all $\varepsilon, \eta > 0$ $\exists N_0 = N_0(\varepsilon, \eta) \in \mathbf{N}$ and for all $N \geq N_0$ $\exists p_0 = p_0(\varepsilon, \eta, N) \in \mathbf{N}$ such that*

$$(15) \quad \mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \left\{ \left| \min_{n \leq j \leq n+p} \frac{S_j}{s_j t_j} + 1 \right| < \varepsilon \right\} \right\} > 1 - \eta$$

for all $k \geq 1$, for all $p \geq \max\{p_0, e^k\}$.

Proof. One may proceed exactly as in the proof of (12) using the sequence $(-X_n)_{n \geq 1}$, instead of $(X_n)_{n \geq 1}$, and observing that $\min S_j^* = -\max(-S_j^*)$.

Remark 2. A natural question to be raised in our framework regards uniqueness of the upper and lower sequences obtained in Theorem 3. More precisely, it is worth investigating the possibility of existence of an r.v. Y (defined on the same probability

space as the X_n 's) such that $\bar{b}(\varepsilon, n) = (Y + \varepsilon) s_n t_n$ and $\underline{b}(\varepsilon, n) = (Y - \varepsilon) s_n t_n$ are also upper and lower sequences. The answer requires the notion of *equivalence*.

DEFINITION 3. *Two r.v.'s, X and Y , are called equivalent; write $X \simeq Y$ if for all $\varepsilon > 0$ $\mathbf{P}\{|X - Y| < \varepsilon\} = 1$.*

Notice that, unlike the σ -additive setting, the previous definition does not imply $\mathbf{P}\{X = Y\} = 1$ (see [19] for some illustrative examples).

PROPOSITION 1. *Suppose (12) holds. Then for all $\varepsilon, \eta > 0$, $\exists N_0 = N_0(\varepsilon, \eta)$, and for all $N \geq N_0$ $\exists p_0 = p_0(\varepsilon, \eta, N)$ such that*

$$(16) \quad \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{\left|\max_{n \leq j \leq n+p} S_j^* - Y\right| < \varepsilon\right\}\right\} > 1 - \eta \quad \forall k \geq 1, \forall p \geq \max\{p_0, e^k\}$$

if and only if $Y \simeq 1$.

Proof. Suppose Y is an r.v. for which (16) holds. We need to show (i) $\mathbf{P}\{Y > 1 + \varepsilon\} = 0$ and (ii) $\mathbf{P}\{Y < 1 - \varepsilon\} = 0$. Observe that trivially

$$\begin{aligned} \mathbf{P}\{Y > 1 + \varepsilon\} &= \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \left\{S_j^* > Y - \frac{\varepsilon}{2}\right\} \cap \{Y > 1 + \varepsilon\}\right\} \\ &\quad + \mathbf{P}\left\{\bigcup_{n=N}^{N+k} \bigcap_{j=n}^{n+p} \left\{S_j^* \leq Y - \frac{\varepsilon}{2}\right\} \cap \{Y > 1 + \varepsilon\}\right\} \end{aligned}$$

and $\{Y > 1 + \varepsilon\} \equiv \{Y - \varepsilon/2 > 1 + \varepsilon/2\}$. Thus, by (16),

$$\mathbf{P}\{Y > 1 + \varepsilon\} < \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \left\{S_j^* > 1 + \frac{\varepsilon}{2}\right\}\right\} + \frac{\eta}{2} < \eta.$$

Proceeding in a similar fashion

$$\mathbf{P}\{Y < 1 - \varepsilon\} < \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{S_n^* < 1 - \frac{\varepsilon}{2}\right\}\right\} + \frac{\eta}{2} < \eta.$$

The arbitrariness of $\eta > 0$ provides the result we were to prove.

Conversely, suppose $Y \simeq 1$. Let us choose $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$. If we fix $\eta > 0$, then there exists $N_0 = N_0(\varepsilon, \eta)$ such that for all $N \geq N_0$ and for all $k \geq 1$

$$\begin{aligned} 1 - \eta &< \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{S_n^* < 1 + (\varepsilon - \delta)\right\}\right\} \\ &= \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{S_n^* < 1 + (\varepsilon - \delta)\right\} \cap \{Y > 1 - \delta\}\right\} \leq \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{S_n^* < Y + \varepsilon\right\}\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} 1 - \eta &< \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \left\{S_j^* > 1 - (\varepsilon - \delta)\right\}\right\} \\ &= \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \left\{S_j^* > 1 - (\varepsilon - \delta)\right\} \cap \{Y < 1 + \delta\}\right\} \\ &\leq \mathbf{P}\left\{\bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \left\{S_j^* > Y - \varepsilon\right\}\right\}. \end{aligned}$$

Proposition 1 is proved.

4. Connections with the σ -additive case. We shall look more carefully at (12) in order to explain why it can be considered a finitistic version of the LIL. As is well known, in the σ -additive setting lower and upper sequences for the partial sums of independent r.v.'s are provided by the LIL. In the case where \mathbf{P} is σ -additive, our result (12) implies the LIL. Conversely, it is also possible to prove that if the LIL holds, then an *analogue* to (12) is easily deduced. We will better specify this point below.

PROPOSITION 2. *Let \mathbf{P} be a probability measure. Then*

$$(17) \quad \mathbf{P}\{\limsup S_n^* = 1\} = 1$$

if and only if for all $\varepsilon, \eta > 0$, for all $k \geq 1 \exists N_0 = N_0(\varepsilon, \eta)$, and for all $N \geq N_0 \exists p_0 = p_0(\varepsilon, \eta, N, k)$ such that

$$\mathbf{P}\left\{\bigcap_{n=N}^{N+k} \left\{\left|\max_{n \leq j \leq n+p} S_j^* - 1\right| < \varepsilon\right\}\right\} > 1 - \eta \quad \forall p \geq p_0.$$

Proof. If \mathbf{P} is σ -additive, (17) holds if and only if

$$(18a) \quad \mathbf{P}\{S_n^* < 1 + \varepsilon \text{ for all but finitely many } n\} = 1 \quad \forall \varepsilon > 0$$

and

$$(18b) \quad \mathbf{P}\{S_n^* > 1 - \varepsilon \text{ infinitely often}\} = 1 \quad \forall \varepsilon > 0.$$

Now, (18a) can be written as $\mathbf{P}\{\bigcap_{n=1}^\infty \bigcap_{j=n}^\infty \{S_j^* < 1 + \varepsilon\}\} = 1$, that is,

$$\mathbf{P}\left\{\lim_{n \rightarrow \infty} \bigcap_{j=n}^\infty \{S_j^* < 1 + \varepsilon\}\right\} = 1.$$

By the continuity of \mathbf{P} , we have $\lim_n \mathbf{P}\{\bigcap_{j=n}^\infty \{S_j^* < 1 + \varepsilon\}\} = 1$. This entails that for all $\varepsilon, \eta > 0, \exists N_0 = N_0(\varepsilon, \eta)$ such that

$$\mathbf{P}\left\{\bigcap_{n=N}^{N+k} \{S_n^* < 1 + \varepsilon\}\right\} > 1 - \eta \quad \forall N \geq N_0, \forall k \geq 1.$$

On the other hand, (18b) is tantamount to $\mathbf{P}\{\bigcap_{n=1}^\infty \bigcup_{j=n}^\infty \{S_j^* > 1 - \varepsilon\}\} = 1$. Let us denote by $D_j^{(n)}$ ($j > n$) the event

$$\{S_m^* < 1 - \varepsilon, m = n, \dots, j - 1, \text{ and } S_j^* > 1 - \varepsilon\},$$

whereas $D_n^{(n)} = \{S_n^* > 1 - \varepsilon\}$. For a given n , the events $\{D_j^{(n)} : j = n, n + 1, \dots\}$ are disjoint and $\bigcup_{j=n}^\infty D_j^{(n)} = \bigcup_{j=n}^\infty \{S_j^* > 1 - \varepsilon\}$. As usual, \mathbf{I}_A is the indicator function of A ; (18b) is rewritten as follows:

$$\mathbf{E}\left(\prod_{n=1}^\infty \sum_{j=n}^\infty \mathbf{I}_{D_j^{(n)}}\right) = 1.$$

A fortiori, $\mathbf{E}(\prod_{n=N}^{N+k} \sum_{j=n}^\infty \mathbf{I}_{D_j^{(n)}}) = 1$ for all $N \geq 1$, for all $k \geq 1$ and, consequently,

$$\mathbf{E}\left(\lim_{p \rightarrow \infty} \prod_{n=N}^{N+k} \sum_{j=n}^{n+p} \mathbf{I}_{D_j^{(n)}}\right) = 1.$$

Since $\prod_{n=N}^{N+k} \sum_{j=n}^{n+p} \mathbf{I}_{D_j^{(n)}}$ is a nondecreasing sequence in p , by the monotone convergence theorem

$$\lim_{p \rightarrow \infty} \mathbf{E} \left(\prod_{n=N}^{N+k} \sum_{j=n}^{n+p} \mathbf{I}_{D_j^{(n)}} \right) = 1,$$

which means that for all $\varepsilon, \eta > 0$, for all $k \geq 1$, for all $N \geq 1$, $\exists p_0 = p_0(\varepsilon, \eta, N, k)$ such that

$$\mathbf{E} \left(\prod_{n=N}^{N+k} \sum_{j=n}^{n+p} \mathbf{I}_{D_j^{(n)}} \right) > 1 - \eta \quad \forall p \geq p_0.$$

If we express the latter in terms of the probability \mathbf{P} we reach the result. Indeed

$$\begin{aligned} \mathbf{E} \left(\prod_{n=N}^{N+k} \sum_{j=n}^{n+p} \mathbf{I}_{D_j^{(n)}} \right) &= \mathbf{E} \left(\mathbf{I}_{\bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} D_j^{(n)}} \right) = \mathbf{E} \left(\mathbf{I}_{\bigcap \{S_j^* > 1 - \varepsilon\}} \right) \\ &= \mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \bigcup_{j=n}^{n+p} \{S_j^* > 1 - \varepsilon\} \right\}. \end{aligned}$$

Conversely, suppose that for all $\varepsilon, \eta > 0$, $\exists N_0 = N_0(\varepsilon, \eta)$ such that

$$\mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \{S_j^* < 1 + \varepsilon\} \right\} > 1 - \eta \quad \forall N \geq N_0, \forall k \geq 1.$$

By continuity of \mathbf{P} we get $\mathbf{P} \{ \bigcap_{n=N}^{\infty} \{S_n^* < 1 + \varepsilon\} \} > 1 - \eta$ which clearly implies (18a).

Moreover, if for all $\varepsilon, \eta > 0 \exists N_0 = N_0(\varepsilon, \eta)$ and for all $N \geq N_0 \exists p_0 = p_0(\varepsilon, \eta, N)$ such that $\mathbf{P} \{ \bigcap_{n=N}^{N+k} \{ \max_{n \leq j \leq n+p} S_j^* > 1 - \varepsilon \} \} > 1 - \eta$ for all $k \geq 1$ and for all $p \geq \max\{p_0, e^k\}$, a fortiori

$$\mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \left\{ \max_{j \geq n} S_j^* > 1 - \varepsilon \right\} \right\} > 1 - \eta.$$

Again, by continuity of \mathbf{P} , if we take the limit as $k \rightarrow \infty$ we have $\mathbf{P} \{ \bigcap_{n=N}^{\infty} \bigcup_{j=n}^{\infty} \{S_j^* > 1 - \varepsilon\} \} > 1 - \eta$. But the latter implies, by virtue of the arbitrariness in η , (18b). Proposition 2 is proved.

This very same proposition allows us to say that (12) implies, in a σ -additive setting, the LIL. Actually (12) contains an even stronger information, in the sense it provides an estimate of $p_0(\varepsilon, \eta, N, k)$ in k , since we found that $p_0(\varepsilon, \eta, N, k) > e^k$.

5. Two illustrative examples. In this section we illustrate a couple of examples which do not allow the application of the σ -additive version of the LIL. On the other hand, it is meaningful to study the oscillations of partial sums in a finitistic setting according to Theorem 3 and Corollary 1. This also supports the fact that (12) and (17) are equivalent only if \mathbf{P} is completely additive.

Example 1. Let $\Omega := \{1, 2, \dots\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$, the power set of Ω . We define on $(\Omega, \mathcal{P}(\Omega))$ the following probability measure, for every $n \geq 1$:

$$\mathbf{P}_n(E) := \frac{1}{n} \#(\omega \in \Omega: 1 \leq \omega \leq n, \omega \in E)$$

for all $E \in \mathcal{P}(\Omega)$ and $\#B$ denotes the *cardinality* of the set B . A finitely additive probability, \mathbf{P} , is obtained via the rule

$$\mathbf{P}(E) = \lim_{n \rightarrow \infty} \mathbf{P}_n(E)$$

for all E such that the previous limit exists. Let \mathcal{L} be the class of events for which this limit exists. Now consider the set of prime numbers and its ordering in p_0, p_1, p_2, \dots ($p_0 = 1, p_1 = 2, p_2 = 3, \dots$) and associate to each $\omega \in \Omega$ its factorization in prime numbers $\omega = \prod_{r=1}^{\infty} p_r^{\alpha_r(\omega)}$, where $\alpha_r(\omega)$ stands for the exact power of p_r in the factorization of ω . If ω is not divisible by p_r , then $\alpha_r(\omega) = 0$. As a consequence, only a finite number of α_r 's in the previous representation is different from zero. In this framework, let $(X_n)_{n \geq 1}$ be a sequence of r.v.'s on $(\Omega, \mathcal{L}, \mathbf{P})$ defined by

$$X_i(\omega) = \begin{cases} 1, & \omega \in [0]_{p_i}, \\ 0, & \omega \notin [0]_{p_i}, \end{cases}$$

and $[0]_{p_i}$ is the residue class of 0 modulo p_i . It is easy to prove that

$$\mathbf{P}_n\{\omega: X_i(\omega) = 1\} = \frac{1}{n} \left[\frac{n}{p_i} \right] \quad \forall i \geq 1 \quad \forall n \geq 1,$$

and immediately one obtains $\mathbf{P}\{\omega: X_i(\omega) = 1\} = 1/p_i$ for all $i \geq 1$. By virtue of the additivity of the \mathbf{P}_n 's

$$\mathbf{P}_n\{\omega: X_i(\omega) = 0\} = 1 - \mathbf{P}_n\{\omega: X_i(\omega) = 1\} \xrightarrow{n \rightarrow \infty} 1 - \frac{1}{p_i}.$$

Hence, X_i is an r.v. distributed according to a Bernoulli law with parameter $1/p_i$. Now take a pair of r.v.'s X_i, X_j (for all $i \neq j$) and notice that

$$\begin{aligned} \mathbf{P}_n\{\omega: X_i(\omega) = 1, X_j(\omega) = 1\} &= \frac{1}{n} \left[\frac{n}{p_i p_j} \right] \xrightarrow{n \rightarrow \infty} \frac{1}{p_i p_j} \\ &= \mathbf{P}\{\omega: X_i(\omega) = 1, X_j(\omega) = 1\}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbf{P}_n\{\omega: X_i(\omega) = 0, X_j(\omega) = 1\} \\ &= \mathbf{P}_n\{\omega: X_j(\omega) = 1\} - \mathbf{P}_n\{\omega: X_i(\omega) = 1, X_j(\omega) = 1\} \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{p_j} \left(1 - \frac{1}{p_i} \right) = \mathbf{P}\{\omega: X_i(\omega) = 0, X_j(\omega) = 1\} \end{aligned}$$

and

$$\begin{aligned} &\mathbf{P}_n\{\omega: X_i(\omega) = 0, X_j(\omega) = 0\} \\ &= \mathbf{P}_n\{\omega: X_i(\omega) = 0\} - \mathbf{P}_n\{\omega: X_i(\omega) = 0, X_j(\omega) = 1\} \\ &\xrightarrow{n \rightarrow \infty} \left(1 - \frac{1}{p_i} \right) \left(1 - \frac{1}{p_j} \right) = \mathbf{P}\{\omega: X_i(\omega) = 0, X_j(\omega) = 0\}. \end{aligned}$$

In a similar fashion, one may prove that

$$\mathbf{P}\{\omega: X_1(\omega) = x_1, \dots, X_k(\omega) = x_k\} = \prod_{i=1}^k \left(\frac{1}{p_i} \right)^{x_i} \left(1 - \frac{1}{p_i} \right)^{1-x_i}$$

for every $(x_1, \dots, x_k) \in \{0, 1\}^k$ and $k \geq 1$. Therefore, $(X_n)_{n \geq 1}$ is a sequence of stochastically independent r.v.'s, in accordance with Definition 2. It is clear that $S_n(\omega) = \sum_{j=1}^n X_j(\omega)$ is bounded for every $\omega \in \Omega$ and $n \geq 1$. Furthermore, since it is known that $p_j \asymp j \log j$, $\mu_n := \mathbf{E}S_n = \sum_{j=1}^n (1/p_j) \rightarrow \infty$, $n \rightarrow \infty$, $s_n^2 = \sum_{j=1}^n (1/p_j)(1 - (1/p_j))$, and

$$\left\{ \omega : \lim_{n \rightarrow +\infty} \frac{S_n(\omega) - \mu_n}{\sqrt{2s_n^2 \log \log s_n^2}} = -\infty \right\} = \Omega.$$

Consequently, the classical version of the LIL does not hold. On the other hand, since assumptions of Theorem 3 are satisfied, it is possible to characterize the oscillating behavior of S_n on finitary sets in the following terms: for all $\varepsilon, \eta > 0 \exists N_0 = N_0(\varepsilon, \eta)$ and for all $N \geq N_0 \exists p_0 = p_0(\varepsilon, \eta, N)$ such that for all $k \geq 1$ for all $p \geq \max\{p_0, e^k\}$

$$\mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \left\{ \left| \max_{n \leq j \leq n+p} \frac{S_j - \mu_j}{\sqrt{2s_n^2 \log \log s_n^2}} - 1 \right| < \varepsilon \right\} \right\} > 1 - \eta,$$

$$\mathbf{P} \left\{ \bigcap_{n=N}^{N+k} \left\{ \left| \min_{n \leq j \leq n+p} \frac{S_j - \mu_j}{\sqrt{2s_n^2 \log \log s_n^2}} + 1 \right| < \varepsilon \right\} \right\} > 1 - \eta.$$

An example, analogous to the one just illustrated, was given by de Finetti [4] and it was later rediscovered, independently, by Ramakrishnan and Sudderth [15]. They aim at showing that the σ -additive law of large numbers may fail in the finitely additive setting.

Example 2. Let $\pi(\cdot)$ be a probability measure defined by $\pi\{X = x\} = e^{-1}/x!$, $x = 0, 1, 2, \dots$, and take

$$Q_1\{X_1 = x_1, \dots, X_m = x_m\} = \frac{e^{-m}}{\prod_{j=1}^m (x_j!)}.$$

Let us now introduce, for some strictly positive integer M , the probability

$$Q\{X_1 = x_1, \dots, X_m = x_m\} := \frac{Q_1\{X_1 = x_1, \dots, X_m = x_m, X_1 \leq M, \dots, X_m \leq M\}}{Q_1\{X_1 \leq M, \dots, X_m \leq M\}}$$

$$= \prod_{j=1}^m \frac{\pi\{X_j = x_j; X_j \leq M\}}{\pi\{X_j \leq M\}} =: \prod_{j=1}^m Q\{X_j = x_j\},$$

where $Q\{X = x\} = (1/x!) (\sum_{y=0}^M 1/y!)^{-1}$, $x = 0, 1, \dots, M$. Let \mathcal{C} be the algebra generated by the cylinders of \mathbf{N}^∞ with measurable and finite-dimensional basis. Let $\sigma(\mathcal{C})$ be the σ -algebra generated by \mathcal{C} and P_1 the σ -additive extension of Q_1 to $\sigma(\mathcal{C})$. If

$$\mathbf{P}(E) := \lim_{m \rightarrow +\infty} \frac{P_1\{E \cap \bigcap_{j=1}^m \{X_j \leq M\}\}}{Q_1\{\bigcap_{j=1}^m \{X_j \leq M\}\}}$$

is defined on the class $\mathcal{L} = \{E : \mathbf{P}(E) \text{ exists}\}$, then $\mathcal{L} \subset \sigma(\mathcal{C})$ and \mathbf{P} is a probability on \mathcal{L} (see [15]). Moreover, $\mathbf{P}(E) = Q(E)$ for all $E \in \mathcal{C}$. Indeed, one observes that $\mathcal{C} \subset \mathcal{L}$, since for all $E \in \mathcal{C}$

$$\mathbf{P}(E) = \lim_{m \rightarrow +\infty} \frac{Q_1\{E \cap \bigcap_{j=1}^m \{X_j \leq M\}\}}{Q_1\{\bigcap_{j=1}^m \{X_j \leq M\}\}} = \lim_{m \rightarrow \infty} Q(E) = Q(E).$$

The r.v.'s X_1, \dots, X_n, \dots are independent with respect to \mathbf{P} . In fact, for every k -tuple n_1, \dots, n_k of integers, one has

$$\begin{aligned} \mathbf{P}\{X_{n_1} = x_1, \dots, X_{n_k} = x_k\} &= Q\{X_{n_1} = x_1, \dots, X_{n_k} = x_k\} \\ &= \prod_{j=1}^k Q\{X_{n_j} = x_j\} = \prod_{j=1}^k \mathbf{P}\{X_{n_j} = x_j\}. \end{aligned}$$

By straightforward algebra, we obtain the following expressions for the mean and the variance of the X_n 's:

$$\begin{aligned} \mathbf{E}X &= 1 - \frac{1}{M! \sum_{x=0}^M 1/x!} =: \mu_M, \\ \mathbf{D}X &= 1 - \frac{1}{M! \sum_{x=0}^M 1/x!} \left(M - \frac{1}{M! \sum_{x=0}^M 1/x!} \right) =: \sigma_M^2. \end{aligned}$$

It is clear that $0 < \mu_M < 1$. We shall verify $\{\limsup(S_n - n\mu_M)/\sqrt{2n\sigma_M^2 \log \log \sigma_M^2} = 1\} \in \mathcal{L}$. In order to show it, write

$$\begin{aligned} \frac{S_n - n\mu_M}{\sqrt{2n\sigma_M^2 \log \log \sigma_M^2}} &= \frac{S_n - n}{\sqrt{2n \log \log n}} \cdot \sqrt{\frac{\log \log n}{\log \log n\sigma_M^2}} \cdot \frac{1}{\sigma_M} \\ &\quad + \frac{n(1 - \mu_M)}{\sqrt{2n\sigma_M^2 \log \log \sigma_M^2}} = A_n + B_n. \end{aligned}$$

Trivially, $B_n \rightarrow +\infty$ ($n \rightarrow \infty$). Moreover, $\liminf(S_n - n)/\sqrt{2n \log \log n} = -1$ P_1 -a.s. But $\limsup(A_n + B_n) \geq \liminf A_n + \lim B_n = +\infty$ P_1 -a.s. Therefore,

$$P_1 \left\{ \limsup \frac{S_n - n\mu_M}{\sqrt{2n\sigma_M^2 \log \log \sigma_M^2}} = 1 \right\} = 0.$$

By taking into account the definition of \mathbf{P} ,

$$\mathbf{P} \left\{ \limsup \frac{S_n - n\mu_M}{\sqrt{2n\sigma_M^2 \log \log \sigma_M^2}} = 1 \right\} = 0.$$

Thus $\{\limsup(S_n - n\mu_M)/\sqrt{2n\sigma_M^2 \log \log \sigma_M^2} = 1\} \in \mathcal{L}$ and the LIL does not hold.

If we instead consider the problem in a finitistic setting, we notice that the hypotheses of Theorem 3 are verified with respect to \mathbf{P} . Again, by means of (12) we can describe, in finitistic terms, the oscillations of the S_n 's.

Acknowledgment. We are grateful to Professor E. Regazzini for his enlightening suggestions which helped us in developing this research.

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