The Structure of Variational Preferences*

S. Cerreia-Vioglio,[a] F. Maccheroni,[a] M. Marinacci,[a] and A. Rustichini[a,b]
[a]IGIER and Università Bocconi and [b]University of Minnesota

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Abstract

Maccheroni, Marinacci, and Rustichini [17], in an Anscombe-Aumann framework, axiomatically characterize preferences that are represented by the variational utility functional

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c(p) \right\} \qquad \forall f \in \mathcal{F},$$

where u is a utility function on outcomes and c is an index of uncertainty aversion. In this paper, for a given variational preference, we study the class $\mathcal{C}$ of functions c that represent V. Inter alia, we show that this set is fully characterized by a minimal and a maximal element, $c^\star$ and $d^\star$. The function $c^\star$, also identified by Maccheroni, Marinacci, and Rustichini [17], fully characterizes the decision maker's attitude toward uncertainty, while the novel function $d^\star$ characterizes the uncertainty perceived by the decision maker.

1 Introduction

In this paper we study the functional structure of variational preferences, a class of binary relations introduced by Maccheroni, Marinacci, and Rustichini [17] (henceforth, MMR). In an Anscombe and Aumann framework, a binary relation $\succsim$ over the set of acts $\mathcal{F}$ is a variational preference if and only if it admits the following representation

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c(p) \right\} \qquad \forall f \in \mathcal{F}, \tag{1}$$

where u is an affine utility index, Δ is the set of probabilities, and $c : \Delta \to [0, \infty]$ is a grounded, lower semicontinuous, and convex function. In other words, each variational preference is characterized by a pair (u, c), where u is a utility index over consequences and c is an index of uncertainty aversion.

For a given variational preference $\succsim$ and a given u, we study the set of all functions $c : \Delta \to [0, \infty]$ which are grounded, lower semicontinuous, convex, and such that the corresponding V, given by (1), represents $\succsim$. We denote this set by $\mathcal{C}$. MMR showed that if $\succsim$ also satisfies an unboundedness axiom, then the function c in (1) is unique; that is, $\mathcal{C}$ is a singleton. Without such an axiom, $\mathcal{C}$ is no longer a singleton. Our analysis sheds light on the structure of $\mathcal{C}$ when it contains more than one element. In Theorem 1 we show that $\mathcal{C}$ is a convex set and a complete lattice. In particular, $\mathcal{C}$ admits a minimum and a maximum element, denoted by $c^\star$ and $d^\star$.

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From a decision theoretic point of view, the function $c^\star$ is the function identified by MMR, which captures the decision maker's uncertainty attitudes (see [17, Proposition 8]). The function $d^\star$ is a novel object; we show it characterizes the revealed unambiguous preference as defined by Ghirardato, Maccheroni, and Marinacci [10].[1]

As a consequence of our main result, we show that each lower semicontinuous and convex function c such that $c^\star \leq c \leq d^\star$ also satisfies (1), and thus represents $\succsim$ (Corollary 1). From a conceptual and formal point of view, these observations suggest that variational representations of preferences are characterized by a triple $(u, c^\star, d^\star)$, which reduces to a pair (u, c) in the unbounded case, a case more thoroughly studied by MMR.

2 Preliminaries

2.1 Decision Theoretic Set Up

We consider a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*, that is, of Σ-measurable functions $f : S \to X$ that take finitely many values.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. With the usual slight abuse of notation, we thus identify X with the subset of constant acts in $\mathcal{F}$.

We assume that X is a convex subset of a vector space. This is the case, for instance, if X is the set of all lotteries on a set of outcomes, as in the classic setting of Anscombe and Aumann [1]. Using the linear structure of X, we define a mixture operation over $\mathcal{F}$ as follows: For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha) g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha) g)(s) = \alpha f(s) + (1-\alpha) g(s) \in X$ for all $s \in S$.

We model a decision maker's *preferences* on $\mathcal{F}$ by a binary relation $\succsim$. Given such a binary relation $\succsim$, $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of $\succsim$. Finally, we denote by $\mathcal{F}_{\text{int}}$ the set of acts

$$\{f \in \mathcal{F} : \exists x, y \in X \text{ s.t. } x \succ f(s) \succ y \qquad \forall s \in S\}.$$

2.2 Mathematical Preliminaries

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions, so that $u(f) \in B_0(\Sigma)$ whenever $u : X \to \mathbb{R}$ is affine and $f \in \mathcal{F}$. Given an interval $K \subseteq \mathbb{R}$, we denote by $B_0(\Sigma, K)$ the set of all real-valued Σ-measurable simple functions that take values in the interval K. Note that, if $K = \mathbb{R}$, then $B_0(\Sigma, \mathbb{R}) = B_0(\Sigma)$.

When $B_0(\Sigma)$ is endowed with the supnorm, its norm dual can be identified with the set $ba(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $ba(\Sigma)$ is denoted by Δ; it is a (weak*) compact and convex subset of $ba(\Sigma)$. The set Δ is endowed with the relative weak* topology.

Given a function $c : \Delta \to [0, \infty]$, we say that c is grounded if and only if $\min_{p \in \Delta} c(p) = 0$. We denote the effective domain of c by

$$\operatorname{dom} c = \{p \in \Delta : c(p) < \infty\}.$$

[1] This result is also based on an equivalence between Greenberg–Pierskalla differentials and Clarke's differentials, established in Theorem 2.

2.3 Variational Preferences

We consider three nested classes of preferences: Anscombe-Aumann expected utility preferences, Gilboa-Schmeidler preferences, and variational preferences a la MMR. Before formally defining them, we provide the axioms that characterize these preferences. For a thorough discussion of these assumptions, we refer the interested reader to [1], [12], and [17].

Axiom A. 1 (Weak Order) *The binary relation $\succsim$ is nontrivial, complete, and transitive.*

Axiom A. 2 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Axiom A. 3 (Continuity) *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$ are closed.*

Axiom A. 4 (Independence) *If $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \succsim g \Rightarrow \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h.$$

Definition 1 *A binary relation $\succsim$ on $\mathcal{F}$ is an Anscombe-Aumann expected utility preference if and only if it satisfies Weak Order, Monotonicity, Continuity, and Independence.*

By Anscombe and Aumann [1] (see also [17, Corollary 20]), $\succsim$ is an Anscombe-Aumann expected utility preference if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$ and a unique $p \in \Delta$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \int u(f)\, dp \qquad \forall f \in \mathcal{F},$$

represents $\succsim$.[2]

Gilboa-Schmeidler preferences differ from expected utility ones in terms of the Independence assumption. In fact, Gilboa and Schmeidler [12] weaken the Independence assumption and replace it with the following two postulates (see also [17, Lemma 1]):

Axiom A. 5 (C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha, \beta \in (0,1]$,*

$$\alpha f + (1-\alpha) x \succsim \alpha g + (1-\alpha) x \Rightarrow \beta f + (1-\beta) y \succsim \beta g + (1-\beta) y.$$

Axiom A. 6 (Uncertainty Aversion) *If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f + (1-\alpha) g \succsim f$.*

Definition 2 *A binary relation $\succsim$ on $\mathcal{F}$ is a Gilboa-Schmeidler preference if and only if it satisfies Weak Order, Monotonicity, Continuity, C-Independence, and Uncertainty Aversion.*

By Gilboa and Schmeidler [12] (see also [17, Proposition 19]), a binary relation $\succsim$ is a Gilboa-Schmeidler preference if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a unique closed and convex set $C \subseteq \Delta$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in C} \int u(f)\, dp \qquad \forall f \in \mathcal{F},$$

represents $\succsim$.

Finally, Maccheroni, Marinacci, and Rustichini [17] consider binary relations $\succsim$ on $\mathcal{F}$ that satisfy an even weaker assumption of Independence.

[2] That is, $f \succsim g$ if and only if $V(f) \geq V(g)$.

Axiom A. 7 (Weak C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Rightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

Definition 3 *A binary relation $\succsim$ on $\mathcal{F}$ is a variational preference if and only if it satisfies Weak Order, Monotonicity, Continuity, Weak C-Independence, and Uncertainty Aversion.*

By MMR [17, Theorem 3], a binary relation $\succsim$ is a variational preference if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a grounded, lower semicontinuous, and convex function $c : \Delta \to [0, \infty]$ such that $V : \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in \Delta} \left\{ \int u(f)\, dp + c(p) \right\} \qquad \forall f \in \mathcal{F}, \tag{2}$$

represents $\succsim$.

Given a binary relation $\succsim$ on $\mathcal{F}$, we define $\succsim^*$ as the *revealed unambiguous preference* of Ghirardato, Maccheroni, and Marinacci [10]:

$$f \succsim^* g \quad \Longleftrightarrow \quad \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h \qquad \forall \alpha \in (0,1], \forall h \in \mathcal{F}.$$

By Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [5], if $\succsim$ is a variational preference, then there exists a unique closed and convex set C^* such that

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in C^*.$$

The binary relation $\succsim^*$ is a Bewley preference (see Bewley [3]). Typically, $\succsim^*$ is interpreted as including the rankings for which the decision maker is sure, and the set C^* is interpreted as the uncertainty perceived by the decision maker.

3 Results

In this section we consider a variational preference $\succsim$ represented by a function V defined as in (2). Given V and u, the object of our study is the set $\mathcal{C} = \mathcal{C}(V, u)$ defined as the set of all functions $c : \Delta \to [0, \infty]$ which are grounded, lower semicontinuous, convex, and further satisfy (2). A notion that will play an important role in what follows is the correspondence $\pi_u : \mathcal{F}_{\text{int}} \rightrightarrows \Delta$, given by

$$\pi_u(f) = \left\{ p \in \Delta : \int u(f)\, dp \geq \int u(g)\, dp \implies f \succsim g \right\} \qquad \forall f \in \mathcal{F}_{\text{int}}.$$

The set $\pi_u(f)$ consists of the beliefs that rationalize the decision maker's preferences at f. These sets of local beliefs have been studied by Rigotti, Shannon, and Strzalecki [18] and by Hanany and Klibanoff [15].

In the next result, we show that $\mathcal{C}$ is a convex set and a complete lattice. In particular, $\mathcal{C}$ admits a minimum and a maximum element, $c^\star$ and $d^\star$. From a decision theoretic point of view, the function $c^\star$ is the function identified by MMR (see Remark 2 and Appendix B) and shown to capture the decision maker's uncertainty attitudes (see [17, Proposition 8]). On the other hand, the function $d^\star$ is a novel object that characterizes $\succsim^*$. Finally, point (v) shows that all functions $c \in \mathcal{C}$ coincide on the collection of local beliefs.

We conclude by computing $d^\star$ when $\succsim$ is a Gilboa-Schmeidler preference and when $\succsim$ is an Anscombe-Aumann expected utility preference.

Theorem 1 *Let $\succsim$ be a variational preference on $\mathcal{F}$. The following statements are true:*

(i) There exist $c^\star, d^\star \in \mathcal{C}$ *such that*

$$c^\star \leq c \leq d^\star \qquad \forall c \in \mathcal{C}.$$

(ii) $\mathcal{C}$ *is a convex set and a complete lattice.*

(iii) If $u(X)$ *is unbounded, then* $\mathcal{C}$ *is a singleton and* $c^\star = d^\star$*.*

(iv) $\mathrm{cl}(\mathrm{dom}\, d^\star) = C^*$*.*

(v) For each $f \in \mathcal{F}_{\mathrm{int}}$*, for each* $p \in \pi_u(f)$*, and for each* $c \in \mathcal{C}$*,*

$$c^\star(p) = c(p) = d^\star(p).$$

In particular,

$$C^* = \bar{\mathrm{co}}\left(\bigcup_{f \in \mathcal{F}_{\mathrm{int}}} \pi_u(f)\right). \tag{3}$$

Remark 1 In the unbounded case and for the more general class of uncertainty averse preferences, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [6, Proposition 11] show that the collection of all sets $\pi_u(f)$ characterizes $\succsim^*$. Equation (3) in point (v) extends this finding for the class of variational preferences to the bounded case. This fact could be proven in two ways: (a) by direct methods, as we do in Appendix B, (b) by showing (loosely speaking) that the set of Greenberg-Pierskalla differentials $\pi_u(f)$ coincides with the set of normalized Clarke's differentials (see our Theorem 2), and then invoking Ghirardato and Siniscalchi [11, Theorem 2], who show that C^* coincides with the closed convex hull of all normalized Clarke's differentials. Compared to [6, Proposition 11] the proof of (3) is rather different. In the proof of [6, Proposition 11], first C^* is shown to be equal to $\mathrm{cl}(\mathrm{dom}\, c^\star)$ and then $\mathrm{cl}(\mathrm{dom}\, c^\star)$ is shown to be equal to the closed convex hull of Greenberg-Pierskalla differentials $\pi_u(f)$. If $u(X) \neq \mathbb{R}$, this strategy of proof does not work. In fact, $c^\star(p) \leq \mathrm{length}\, u(X)$ for all $p \in \Delta$, see (4) below. Therefore, if $u(X)$ is bounded, then $\mathrm{cl}(\mathrm{dom}\, c^\star) = \Delta$, while, typically, both C^* and the closed convex hull of local beliefs are different from Δ. For such a reason, we need to prove the equality in (3) by direct methods (see also Proposition 3).[3]

Remark 2 Recall that $c^\star : \Delta \to [0, \infty]$ is such that

$$c^\star(p) = \sup_{f \in \mathcal{F}} \left\{ u(x_f) - \int u(f)\, dp \right\} \qquad \forall p \in \Delta, \tag{4}$$

where $x_f \sim f$ for all $f \in \mathcal{F}$ (see [17, Theorem 3]). Fix u. By [17, Proposition 6], if $u(X) = \mathbb{R}$, then $c^\star$ is the unique function in $\mathcal{C}$ representing $\succsim$. Point (v) of Theorem 1 can be seen as an extension of this uniqueness result. It shows that, even in the case $u(X)$ is bounded, each function c in $\mathcal{C}$ representing $\succsim$ coincides to $c^\star$ on the subdomain of local beliefs.

The next result confirms the intuition that the structure of a variational preference is summarized by the functions $c^\star$ and $d^\star$ associated to its representation. In fact, given a function $c : \Delta \to [0, \infty]$, it is enough to check whether it is lower semicontinuous, convex, and such that $c^\star \leq c \leq d^\star$ to conclude that $c \in \mathcal{C}$, that is, that c is also grounded and represents $\succsim$ as in (2).

[3] On the other hand, the proof contained in this paper relies on a property of strong monotonicity that variational preferences satisfy, but generic uncertainty averse preferences might not satisfy, namely,

$$f(s) \succ g(s) \quad \forall s \in S \implies f \succ g.$$

Corollary 1 *Let $\succsim$ be a variational preference on $\mathcal{F}$ and $c : \Delta \to [0, \infty]$ a lower semicontinuous and convex function. The following statements are equivalent:*

(i) $c^\star \leq c \leq d^\star$;

(ii) $c \in \mathcal{C}$.

The next results confirm that $d^\star$ captures an important behavioral trait of the decision maker. Its effective domain, aside from coinciding (up to closure) with the set of probabilities characterizing the revealed unambiguous preference $\succsim^*$, is the smallest closed convex set over which any function $c \in \mathcal{C}$ can be restricted to in (2).

Corollary 2 *Let $\succsim$ be a variational preference on $\mathcal{F}$. If $c \in \mathcal{C}$, then C^* is the smallest closed and convex subset of Δ such that the* min *in (2) can be restricted to.*

In particular, for each $c \in \mathcal{C}$ and for each $f \in \mathcal{F}$

$$V(f) = \min_{p \in C^*} \left\{ \int u(f)\, dp + c(p) \right\}.$$

Corollary 3 *If $\succsim$ is a Gilboa-Schmeidler preference on $\mathcal{F}$, then $d^\star = \delta_C$.*[4]

Recall that if the interval $u(X)$ is bounded, by (4), then $c^\star(p) \leq \operatorname{length} u(X) < \infty$ for all $p \in \Delta$. Therefore, unless $C = C^* = \Delta$, we have that $c^\star \neq d^\star$ and so $c^\star \neq \delta_C$.

Corollary 4 *If $\succsim$ is an Anscombe-Aumann expected utility preference on $\mathcal{F}$, then $d^\star = \delta_{\{p\}}$.*

Summing up, the triple $(u, c^\star, d^\star)$ characterizes the representation of variational preferences. Under unboundedness, the triple reduces to the pair $(u, c^\star)$. This latter case was the center of the analysis of [17].

A Nonsmooth Differentials

Recall that we denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions and by $ba(\Sigma)$ the set of all bounded and finitely additive set functions. We endow the former set with the supnorm. Thus, the latter set can be identified with the norm dual of $B_0(\Sigma)$. Recall also that we endow $ba(\Sigma)$ and any of its subsets with the weak* topology. We denote by $\langle \cdot, \cdot \rangle : B_0(\Sigma) \times ba(\Sigma) \to \mathbb{R}$ the dual pairing. The function $\langle \cdot, \cdot \rangle$ is defined by

$$\langle \varphi, p \rangle = \int \varphi dp \qquad \forall (\varphi, p) \in B_0(\Sigma) \times ba(\Sigma).$$

In the sequel, with a small abuse of notation, given $k \in \mathbb{R}$, we denote by k both the real number and the constant function on S that takes value k. Finally, we denote by K an interval of $\mathbb{R}$ such that $0 \in \operatorname{int} K$.

Consider a functional $I : B_0(\Sigma, K) \to \mathbb{R}$ and $\psi \in B_0(\Sigma, \operatorname{int} K)$. Define the

(i) *Clarke upper (directional) derivative* $I^\circ(\psi; \cdot) : B_0(\Sigma) \to [-\infty, \infty]$ at ψ by:

$$I^\circ(\psi; \xi) = \limsup_{\substack{\varphi \to \psi \\ t \downarrow 0}} \frac{I(\varphi + t\xi) - I(\varphi)}{t} \qquad \forall \xi \in B_0(\Sigma),$$

[4] Given $D \subseteq \Delta$, recall that $\delta_D : \Delta \to [0, \infty]$ is such that $\delta_D(p) = 0$ if $p \in D$ and $\delta_D(p) = \infty$ otherwise.

(ii) *Clarke lower (directional) derivative* $I_\circ(\psi;\cdot) : B_0(\Sigma) \to [-\infty,\infty]$ at ψ by:

$$I_\circ(\psi;\xi) = \liminf_{\substack{\varphi\to\psi \\ t\downarrow 0}} \frac{I(\varphi+t\xi) - I(\varphi)}{t} \qquad \forall \xi \in B_0(\Sigma).$$

It is easy to check that

$$I_\circ(\psi;\xi) \le I^\circ(\psi;\xi) \qquad \forall \psi \in B_0(\Sigma, \operatorname{int} K) \text{ and } \forall \xi \in B_0(\Sigma). \tag{5}$$

The *Clarke differential* $\partial^\circ I(\psi)$ at ψ is defined by

$$\partial^\circ I(\psi) = \{p \in ba(\Sigma) : \forall \varphi \in B_0(\Sigma) \quad \langle \varphi, p\rangle \le I^\circ(\psi;\varphi)\}.$$

At this level of generality, we can also define another notion of superdifferential:

$$\partial_{GP} I(\psi) = \{p \in \Delta : \forall \varphi \in B_0(\Sigma, K) \quad \langle \varphi, p\rangle \le \langle \psi, p\rangle \quad \Longrightarrow \quad I(\varphi) \le I(\psi)\}.$$

This notion of differential is common in Quasiconvex Analysis and it is due to Greenberg and Pierskalla [13]. If I is a normalized and concave niveloid,[5] then we can also define a third notion of superdifferential

$$\partial I(\psi) = \{p \in \Delta : \forall \varphi \in B_0(\Sigma, K) \quad I(\varphi) - I(\psi) \le \langle \varphi, p\rangle - \langle \psi, p\rangle\}.$$

This is the standard notion of superdifferential which is common in Convex Analysis. The following lemma can be found, for example, in Ghirardato, Maccheroni, and Marinacci [10] (see also Jahn [16, Chapter 3, Section 5] and Clarke [8, Proposition 2.1.2]).

Lemma 1 *Let K be an open interval. If $I : B_0(\Sigma, K) \to \mathbb{R}$ is locally Lipschitz, then $\partial^\circ I(\psi)$ is a nonempty, convex, and compact subset of $ba(\Sigma)$ at each $\psi \in B_0(\Sigma, K)$ with*

$$\partial^\circ I(\psi) = \{p \in ba(\Sigma) : \forall \varphi \in B_0(\Sigma) \quad \langle \varphi, p\rangle \ge I_\circ(\psi;\varphi)\}. \tag{6}$$

In particular, if I is monotone, then $\partial^\circ I(\psi) \subseteq ba_+(\Sigma)$ for all $\psi \in B_0(\Sigma, K)$.

Theorem 2 *If $I : B_0(\Sigma, K) \to \mathbb{R}$ is monotone, continuous, locally Lipschitz on $B_0(\Sigma, \operatorname{int} K)$, and quasi-concave, then for each $\psi \in B_0(\Sigma, \operatorname{int} K)$*

$$\left\{\frac{p}{\|p\|} : 0 \ne p \in \partial^\circ I(\psi)\right\} \subseteq \partial_{GP} I(\psi). \tag{7}$$

Moreover, if $0 \notin \partial^\circ I(\psi)$, then equality holds in (7).

Proof. Fix $\psi \in B_0(\Sigma, \operatorname{int} K)$. Consider $p \in ba(\Sigma)$ such that $0 \ne p \in \partial^\circ I(\psi)$. By Lemma 1 and since I is monotone, $\bar{p} = p/\|p\| \in \Delta$. Consider now $\varphi \in B_0(\Sigma, K)$. We prove three facts:

1. $\langle \varphi, \bar{p}\rangle < \langle \psi, \bar{p}\rangle \implies I(\varphi) \le I(\psi)$. Define $\varepsilon = \langle \psi - \varphi, p\rangle$. By assumption, we have that $\varepsilon > 0$. Since $p \in \partial^\circ I(\psi)$, $I^\circ(\psi;\psi-\varphi) \ge \langle \psi - \varphi, p\rangle = \varepsilon > 0$. By definition of $I^\circ(\psi;\psi-\varphi)$, we have that there exist $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ and $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq B_0(\Sigma)$ such that

$$\frac{I(\varphi_n + t_n(\psi-\varphi)) - I(\varphi_n)}{t_n} \to I^\circ(\psi;\psi-\varphi)$$

[5] See [7] for a definition of niveloid. Recall that a normalized niveloid is such that

1. $I(k) = k$ for all $k \in K$;
2. I is monotone;
3. $I(\varphi + k) = I(\varphi) + k$ for all $\varphi \in B_0(\Sigma, K)$ and $k \in \mathbb{R}$ such that $\varphi + k \in B_0(\Sigma, K)$.

where $0 < t_n \to 0$ and $\varphi_n \to \psi$. It follows that for n large enough

$$\frac{I(\varphi_n + t_n(\psi - \varphi)) - I(\varphi_n)}{t_n} \geq \frac{\varepsilon}{2}.$$

Since I is locally Lipschitz, we have that for n large enough

$$\begin{aligned}
\frac{\varepsilon}{2} &\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + \frac{I(\varphi_n + t_n(\psi - \varphi)) - I(\varphi_n + t_n(\varphi_n - \varphi))}{t_n} \\
&\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + \frac{K\|\varphi_n + t_n(\psi - \varphi) - \varphi_n - t_n(\varphi_n - \varphi)\|}{t_n} \\
&\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + \frac{K\|t_n(\psi - \varphi_n)\|}{t_n} \\
&\leq \frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} + K\|\psi - \varphi_n\|.
\end{aligned}$$

This implies that for n large enough

$$\frac{I(\varphi_n + t_n(\varphi_n - \varphi)) - I(\varphi_n)}{t_n} \geq \frac{\varepsilon}{4} > 0.$$

We can finally conclude that for n large enough

$$I(\varphi_n + t_n(\varphi_n - \varphi)) > I(\varphi_n). \tag{8}$$

Define $\alpha_n = (1 + t_n)^{-1} \in (0,1)$ for all $n \in \mathbb{N}$. Note that $\varphi_n = \alpha_n(\varphi_n + t_n(\varphi_n - \varphi)) + (1 - \alpha_n)\varphi$ for all $n \in \mathbb{N}$. Since I is quasiconcave and by (8), we have that for n large enough

$$I(\varphi_n) \geq \min\{I(\varphi_n + t_n(\varphi_n - \varphi)), I(\varphi)\} = I(\varphi).$$

Since I is continuous, it follows that $I(\varphi) \leq \lim_n I(\varphi_n) = I(\psi)$.

2. $\langle \varphi, \bar{p} \rangle = \langle \psi, \bar{p} \rangle$ and $\varphi \in B_0(\Sigma, \operatorname{int} K) \implies I(\varphi) \leq I(\psi)$. Since $p \neq 0$, there exists $\phi \in B_0(\Sigma)$ such that $\langle \phi, \bar{p} \rangle < 0$. Define
$$\varphi_n = \varphi + \frac{1}{n}\phi \qquad \forall n \in \mathbb{N}.$$
Since $\varphi \in B_0(\Sigma, \operatorname{int} K)$, note that $\{\varphi_n\}_{n \in \mathbb{N}}$ eventually belongs to $B_0(\Sigma, \operatorname{int} K)$. It is also immediate to see that $\langle \varphi_n, \bar{p} \rangle < \langle \psi, \bar{p} \rangle$ for all $n \in \mathbb{N}$. By point 1 and since I is continuous, we have that $I(\varphi) = \lim_n I(\varphi_n) \leq I(\psi)$.

3. $\langle \varphi, \bar{p} \rangle = \langle \psi, \bar{p} \rangle$ and $\varphi \notin B_0(\Sigma, \operatorname{int} K) \implies I(\varphi) \leq I(\psi)$. Define
$$\varphi_n = \left(1 - \frac{1}{n}\right)\varphi + \frac{1}{n}\psi \qquad \forall n \in \mathbb{N}.$$
Since $\varphi \in B_0(\Sigma, K)$ and $\psi \in B_0(\Sigma, \operatorname{int} K)$, note that $\{\varphi_n\}_{n \in \mathbb{N}}$ belongs to $B_0(\Sigma, \operatorname{int} K)$. Moreover, we also have that $\langle \varphi_n, \bar{p} \rangle = \langle \psi, \bar{p} \rangle$ for all $n \in \mathbb{N}$. By point 2 and since I is continuous, we have that $I(\varphi) = \lim_n I(\varphi_n) \leq I(\psi)$.

By points 1, 2, and 3, we proved that, if $\varphi \in B_0(\Sigma, K)$ is such that $\langle \varphi, \bar{p} \rangle \leq \langle \psi, \bar{p} \rangle$, then $I(\varphi) \leq I(\psi)$. Thus, $p/\|p\| = \bar{p} \in \partial_{GP} I(\psi)$, proving (7).

Suppose $0 \notin \partial^\circ I(\psi)$. Let $\bar{p} \in \partial_{GP} I(\psi)$. If $\xi \in \ker \bar{p} = \{\varphi \in B_0(\Sigma) : \langle \varphi, \bar{p} \rangle = 0\}$, then we have that $\langle \psi + t\xi, \bar{p} \rangle \leq \langle \psi, \bar{p} \rangle$ for all $t \geq 0$. Since $\bar{p} \in \partial_{GP} I(\psi)$, this implies that $I(\psi + t\xi) \leq I(\psi)$ for all $t \geq 0$. By definition of $I_\circ(\psi; \xi)$, it follows that $I_\circ(\psi; \xi) \leq 0$. By the Hahn-Banach Theorem, there exists a continuous linear functional $p : B_0(\Sigma) \to \mathbb{R}$ such that $\langle \xi, p\rangle \geq I_\circ(\psi;\xi)$ for all $\xi \in B_0(\Sigma)$ and $\langle \xi, p\rangle = 0$ for all $\xi \in \ker \bar{p}$. In particular, we have that $\ker \bar{p} \subseteq \ker p$. By Lemma 1, we can conclude that $p \in \partial^\circ I(\psi)$. By Lemma 1 and since $0 \notin \partial^\circ I(\psi)$ and I is monotone, it follows that $0 \neq p \geq 0$. By the Fundamental Theorem of Duality (see [2, Theorem 5.91]) and since $\ker \bar{p} \subseteq \ker p$ and $0 \neq p \geq 0$, there exists $\alpha > 0$ such that $p = \alpha \bar{p}$. We can conclude that $\alpha = \|p\|$ and $\bar{p} = p/\|p\|$, proving the equality in (7) when $0 \notin \partial^\circ I(\psi)$. ■

Lemma 2 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized and concave niveloid. If $J : B_0(\Sigma) \to \mathbb{R}$ is a normalized and concave niveloid which extends I, then for each $\psi \in B_0(\Sigma, \operatorname{int} K)$*

$$\partial_{GP} I(\psi) = \partial_{GP} J(\psi).$$

Proof. Fix $\psi \in B_0(\Sigma, \operatorname{int} K)$. Consider $p \in \partial_{GP} J(\psi)$. It follows that for each $\varphi \in B_0(\Sigma)$

$$\langle \varphi, p\rangle \leq \langle \psi, p\rangle \quad \Longrightarrow \quad J(\varphi) \leq J(\psi).$$

Since J extends I, we have that for each $\varphi \in B_0(\Sigma, K)$

$$\langle \varphi, p\rangle \leq \langle \psi, p\rangle \quad \Longrightarrow \quad I(\varphi) = J(\varphi) \leq J(\psi) = I(\psi),$$

proving that $p \in \partial_{GP} I(\psi)$. On the other hand, consider $p \in \partial_{GP} I(\psi)$. By contradiction, assume that $p \notin \partial_{GP} J(\psi)$. It follows that there exists $\bar{\varphi} \in B_0(\Sigma)$ such that

$$\langle \bar{\varphi}, p\rangle \leq \langle \psi, p\rangle \quad \text{and} \quad J(\bar{\varphi}) > J(\psi).$$

Since $\psi \in B_0(\Sigma, \operatorname{int} K)$, there exists $\lambda \in (0,1)$ such that $\lambda\bar{\varphi} + (1-\lambda)\psi \in B_0(\Sigma, \operatorname{int} K)$. It is immediate to check that $\langle \lambda\bar{\varphi} + (1-\lambda)\psi, p\rangle \leq \langle \psi, p\rangle$. Since J is concave and extends I, we also have that

$$I(\lambda\bar{\varphi} + (1-\lambda)\psi) = J(\lambda\bar{\varphi} + (1-\lambda)\psi) \geq \lambda J(\bar{\varphi}) + (1-\lambda) J(\psi) > J(\psi) = I(\psi),$$

a contradiction with $p \in \partial_{GP} I(\psi)$. ■

Corollary 5 *If $I : B_0(\Sigma, K) \to \mathbb{R}$ is a normalized and concave niveloid, then*

$$\partial^\circ I(\psi) = \partial I(\psi) = \partial_{GP} I(\psi) \qquad \forall \psi \in B_0(\Sigma, \operatorname{int} K).$$

Proof. Since I is a niveloid, I is monotone and Lipschitz continuous. Fix $\psi \in B_0(\Sigma, \operatorname{int} K)$. It is immediate to see that $\partial I(\psi) \subseteq \partial^\circ I(\psi)$. Next, we show that $\partial_{GP} I(\psi) \subseteq \partial I(\psi)$. Consider $p \in \partial_{GP} I(\psi)$. By [7, Theorem 5], there exists $J : B_0(\Sigma) \to \mathbb{R}$ which extends I and is a normalized and concave niveloid. By Lemma 2, it follows that $p \in \partial_{GP} J(\psi)$. Consider $\varphi \in B_0(\Sigma, K)$. Define $k = \langle \psi, p\rangle - \langle \varphi, p\rangle$. It follows that $\langle \varphi + k, p\rangle \leq \langle \psi, p\rangle$. Since $p \in \partial_{GP} J(\psi)$, this implies that $J(\varphi + k) \leq J(\psi)$. Since J is a niveloid that extends I, we can conclude that

$$I(\varphi) - I(\psi) + k = J(\varphi) + k - J(\psi) = J(\varphi + k) - J(\psi) \leq 0,$$

that is, $I(\varphi) - I(\psi) \leq -k = \langle \varphi, p\rangle - \langle \psi, p\rangle$. Since φ was arbitrarily chosen in $B_0(\Sigma, K)$, it follows that $p \in \partial I(\psi)$.

We can conclude that

$$\partial_{GP} I(\psi) \subseteq \partial I(\psi) \subseteq \partial^\circ I(\psi). \tag{9}$$

Next, consider $p \in \partial^\circ I(\psi)$. By Lemma 1, $p \geq 0$ and since I is a normalized and concave niveloid, then $1 = I^\circ(\psi; 1) \geq \langle 1, p\rangle \geq I_\circ(\psi; 1) = 1$, proving that $\langle 1, p\rangle = 1$. It follows that $0 \neq p \in \Delta$ and $\|p\| = 1$. This implies that $\left\{\frac{p}{\|p\|} : 0 \neq p \in \partial^\circ I(\psi)\right\} = \partial^\circ I(\psi)$ and $0 \notin \partial^\circ I(\psi)$. By Theorem 2, we can also conclude that

$$\partial^\circ I(\psi) = \left\{\frac{p}{\|p\|} : 0 \neq p \in \partial^\circ I(\psi)\right\} = \partial_{GP} I(\psi). \tag{10}$$

Since ψ was arbitrarily chosen and by (9), the statement follows. ■

B Proofs

Next we study a normalized and concave niveloid $I : B_0(\Sigma, K) \to \mathbb{R}$. We denote by β an element in $\operatorname{int} K$. Note that the constant function β belongs to $B_0(\Sigma, \operatorname{int} K)$.

Proposition 1 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized niveloid. I is concave if and only if there exists a grounded, lower semicontinuous, and convex function $c : \Delta \to [0, \infty]$ such that*

$$I(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + c(p)\} \qquad \forall \varphi \in B_0(\Sigma, K). \tag{11}$$

Moreover, there exists a minimal function $c^\star : \Delta \to [0, \infty]$, defined by

$$c^\star(p) = \sup_{\varphi \in B_0(\Sigma, K)} \{I(\varphi) - \langle \varphi, p \rangle\} \qquad \forall p \in \Delta,$$

which is grounded, lower semicontinuous, convex, and satisfies (11).

Proof. For a proof see Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini [7]. ■

Given I, define as $\mathcal{C} = \mathcal{C}(I)$ the class of functions $c : \Delta \to [0, \infty]$ such that c is grounded, lower semicontinuous, convex, and represents I as in (11). By Proposition 1, if I is a normalized and concave niveloid, then $\mathcal{C}$ is nonempty. Given $c \in [0, \infty]^\Delta$ and $\psi \in B_0(\Sigma, K)$, we define $M_c(\psi)$ by

$$M_c(\psi) = \{p \in \Delta : \langle \psi, p \rangle + c(p) = I(\psi)\}.$$

If $c \in \mathcal{C}$, then $M_c(\psi) \neq \emptyset$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$.

Proposition 2 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized and concave niveloid. If $c \in \mathcal{C}$, then $M_c(\psi) = \partial_{GP} I(\psi)$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$. In particular, $\arg\min c = M_c(\beta) = \partial_{GP} I(\beta)$, that is,*

$$c(p) = 0 \textit{ if and only if } p \in \partial_{GP} I(\beta).$$

Proof. Pick c in $\mathcal{C}$. Define $J_c : B_0(\Sigma) \to \mathbb{R}$ by $J_c(\varphi) = \min_{p \in \Delta} \{\langle \varphi, p \rangle + c(p)\}$ for all $\varphi \in B_0(\Sigma)$. It is immediate to verify that the concave conjugate of J_c, $J_c^* : ba(\Sigma) \to [-\infty, \infty)$, is such that $J_c^*(p) = -c(p)$ if $p \in \Delta$ and $J_c^*(p) = -\infty$ otherwise. Pick ψ in $B_0(\Sigma, \operatorname{int} K)$. By [4, Proposition 4.4.1], Corollary 5, and Lemma 2, it follows that

$$M_c(\psi) = \partial J_c(\psi) = \partial_{GP} J_c(\psi) = \partial_{GP} I(\psi), \tag{12}$$

proving the main statement. Finally, consider the case $\psi = \beta$. By (12), we have that $M_c(\beta) = \partial_{GP} I(\beta)$. Next, consider $p \in \arg\min c$, that is, since c is grounded, consider $p \in \Delta$ such that $c(p) = 0$. It follows that $I(\beta) = \beta = \langle \beta, p \rangle = \langle \beta, p \rangle + c(p)$, proving that $p \in M_c(\beta)$. On the other hand, if $p \in M_c(\beta)$, then $\beta = I(\beta) = \langle \beta, p \rangle + c(p) = \beta + c(p)$, proving that $c(p) = 0$ and $p \in \arg\min c$. ■

Remark 3 *By Proposition 2, we can conclude that if $c \in \mathcal{C}$, then for each $\psi \in B_0(\Sigma, \operatorname{int} K)$ the set $\partial_{GP} I(\psi)$ is nonempty since $\emptyset \neq M_c(\psi) = \partial_{GP} I(\psi)$.*

Theorem 3 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be a normalized and concave niveloid. The following statements are true:*

1. *$\mathcal{C}$ is a complete lattice.*
2. *There exist $c^\star, d^\star \in \mathcal{C}$ such that $c^\star \leq c \leq d^\star$ for all $c \in \mathcal{C}$.*

3. *If c is lower semicontinuous, convex, and such that $c^\star \leq c \leq d^\star$, then $c \in \mathcal{C}$.*

4. *$\mathcal{C}$ is a convex set.*

5. $\mathrm{cl}\left(\mathrm{dom}\, d^\star\right) = \bar{\mathrm{co}}\left(\bigcup_{\psi \in B_0(\Sigma, \mathrm{int}\, K)} \partial_{GP} I\left(\psi\right)\right)$.

6. *For each $\psi \in B_0\left(\Sigma, \mathrm{int}\, K\right)$, for each $p \in \partial_{GP} I\left(\psi\right)$, and for each $c \in \mathcal{C}$*

$$c^\star\left(p\right) = c\left(p\right) = d^\star\left(p\right).$$

Proof. Given I, consider $\mathcal{C}$. By Proposition 1, it follows that $\mathcal{C} \neq \emptyset$.

1. Consider a nonempty subset $\{c_\gamma\}_{\gamma \in \Gamma} \subseteq \mathcal{C}$. Define $d : \Delta \to [0, \infty]$ by

$$d\left(p\right) = \sup_{\gamma \in \Gamma} c_\gamma\left(p\right) \qquad \forall p \in \Delta. \tag{13}$$

Step 1: d is lower semicontinuous and convex.

Proof of the Step. By (13) and since each element c in $\mathcal{C}$ is lower semicontinuous and convex, the statement follows (see [2, Lemma 2.41] and [14, Lemma 4.26]). □

Step 2: For each $\psi \in B_0\left(\Sigma, \mathrm{int}\, K\right)$ we have that $M_d\left(\psi\right) = \partial_{GP} I\left(\psi\right) \neq \emptyset$. In particular, d is grounded.

Proof of the Step. Consider $\bar{p} \in \partial_{GP} I\left(\psi\right)$. By Proposition 2, we have that $\langle \psi, \bar{p} \rangle + c\left(\bar{p}\right) = I\left(\psi\right)$ for all $c \in \mathcal{C}$. In particular, $c_\gamma\left(\bar{p}\right) = I\left(\psi\right) - \langle \psi, \bar{p} \rangle$ for all $\gamma \in \Gamma$. Since d is defined as the pointwise supremum over $\{c_\gamma\}_{\gamma \in \Gamma}$, we can conclude that $d\left(\bar{p}\right) = I\left(\psi\right) - \langle \psi, \bar{p} \rangle$, that is, $\langle \psi, \bar{p} \rangle + d\left(\bar{p}\right) = I\left(\psi\right)$, proving that $\bar{p} \in M_d\left(\psi\right)$. Since $\{c_\gamma\}_{\gamma \in \Gamma}$ is nonempty, consider $c_{\bar{\gamma}} \in \{c_\gamma\}_{\gamma \in \Gamma}$. Consider $\bar{p} \in M_d\left(\psi\right)$. Since $c_{\bar{\gamma}}$ represents I as in (11) and by construction of d, it follows that

$$I\left(\psi\right) = \langle \psi, \bar{p} \rangle + d\left(\bar{p}\right) \geq \langle \psi, \bar{p} \rangle + c_{\bar{\gamma}}\left(\bar{p}\right) \geq \min_{p \in \Delta}\left\{\langle \psi, p \rangle + c_{\bar{\gamma}}\left(p\right)\right\} = I\left(\psi\right),$$

proving that $\bar{p} \in M_{c_{\bar{\gamma}}}\left(\psi\right)$. By Proposition 2, this implies that $\bar{p} \in \partial_{GP} I\left(\psi\right)$. From the previous part of the proof, it follows that $M_d\left(\beta\right) = \partial_{GP} I\left(\beta\right) \neq \emptyset$. Since I is normalized, if $\bar{p} \in M_d\left(\beta\right)$, then $\beta = I\left(\beta\right) = \langle \beta, \bar{p} \rangle + d\left(\bar{p}\right) = \beta + d\left(\bar{p}\right)$, that is, $d\left(\bar{p}\right) = 0$, proving that d is grounded. □

Step 3: $I\left(\varphi\right) = \min_{p \in \Delta}\left\{\langle \varphi, p \rangle + d\left(p\right)\right\}$ for all $\varphi \in B_0\left(\Sigma, K\right)$, that is, d satisfies (11).

Proof of the Step. Consider $\varphi \in B_0\left(\Sigma, \mathrm{int}\, K\right)$. Since $\{c_\gamma\}_{\gamma \in \Gamma}$ is nonempty, consider $c_{\bar{\gamma}} \in \{c_\gamma\}_{\gamma \in \Gamma}$. By definition of d, it follows that $\langle \varphi, p \rangle + c_{\bar{\gamma}}\left(p\right) \leq \langle \varphi, p \rangle + d\left(p\right)$ for all $p \in \Delta$. By Step 2, we have that there exists $\bar{p} \in M_d\left(\varphi\right)$, that is, $I\left(\varphi\right) = \langle \varphi, \bar{p} \rangle + d\left(\bar{p}\right)$. We can conclude that

$$\min_{p \in \Delta}\left\{\langle \varphi, p \rangle + d\left(p\right)\right\} \leq \langle \varphi, \bar{p} \rangle + d\left(\bar{p}\right) = I\left(\varphi\right) = \min_{p \in \Delta}\left\{\langle \varphi, p \rangle + c_{\bar{\gamma}}\left(p\right)\right\} \leq \min_{p \in \Delta}\left\{\langle \varphi, p \rangle + d\left(p\right)\right\},$$

proving that I coincides to J_d on $B_0\left(\Sigma, \mathrm{int}\, K\right)$. Since both functionals are Lipschitz continuous on $B_0\left(\Sigma, K\right)$, the statement follows. □

Step 4: $\mathcal{C}$ is a complete lattice, that is, given a nonempty subset $\{c_\alpha\}_{\alpha \in A} \subseteq \mathcal{C}$ there exists $\hat{c}$ and $\hat{d}$ such that

$$\hat{c} \leq c_\alpha \leq \hat{d} \qquad \forall \alpha \in A$$

where $\hat{c}$ is the greatest lower bound for $\{c_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$ and $\hat{d}$ is the least upper bound for $\{c_\alpha\}_{\alpha \in A}$ in $\mathcal{C}$.

Proof of the Step. Define $\hat{c}, \hat{d} : \Delta \to [0,\infty]$ by

$$\hat{c}(p) = \sup_{c' \in \{c \in \mathcal{C} : \forall \alpha \in A \; c \leq c_\alpha\}} c'(p) \text{ and } \hat{d}(p) = \sup_{\alpha \in A} c_\alpha(p) \qquad \forall p \in \Delta.$$

In the first case, we have that $\{c_\gamma\}_{\gamma\in\Gamma} = \{c \in \mathcal{C} : \forall \alpha \in A \; c \leq c_\alpha\} \ni c^\star$ and $\hat{c} = d$ where d is defined as in (13). In the second case, we have that $\{c_\gamma\}_{\gamma\in\Gamma} = \{c_\alpha\}_{\alpha\in A}$ and $\hat{d} = d$ where d is again defined as in (13). In light of these observations and Steps 1-3, we have that both $\hat{c}$ and $\hat{d}$ are elements of $\mathcal{C}$. By construction, it is immediate to see that $\hat{c} \leq c_\alpha \leq \hat{d}$ for all $\alpha \in A$ and $\hat{c}$ is the greatest lower bound for $\{c_\alpha\}_{\alpha\in A}$ in $\mathcal{C}$ and $\hat{d}$ is the least upper bound for $\{c_\alpha\}_{\alpha\in A}$ in $\mathcal{C}$. □

Step 4 proves point 1 of the statement.

2. Since $\mathcal{C}$ is a complete lattice, it follows that there exist a minimum and a maximum element. By Proposition 1, the minimum element is $c^\star$. By the proof of point 1, the maximum element $d^\star : \Delta \to [0,\infty]$ is defined by

$$d^\star(p) = \sup_{c\in\mathcal{C}} c(p) \qquad \forall p \in \Delta.$$[6]

3. Consider a lower semicontinuous and convex function $c : \Delta \to [0,\infty]$ such that $c^\star \leq c \leq d^\star$. By Proposition 2 and since $c^\star, d^\star \in \mathcal{C}$, we have that $c^\star(p) = 0 = d^\star(p)$ if and only if $p \in \partial_{GP} I(\beta)$. Since $\partial_{GP} I(\beta) \neq \emptyset$, consider $\bar{p} \in \partial_{GP} I(\beta)$. It follows that $0 = c^\star(\bar{p}) \leq c(\bar{p}) \leq d^\star(\bar{p}) = 0$, that is, c is grounded. Finally, observe that for each $\varphi \in B_0(\Sigma, K)$ we have that

$$I(\varphi) = \min_{p\in\Delta} \{\langle\varphi, p\rangle + c^\star(p)\} \leq \min_{p\in\Delta} \{\langle\varphi, p\rangle + c(p)\} \leq \min_{p\in\Delta} \{\langle\varphi, p\rangle + d^\star(p)\} = I(\varphi),$$

proving that c satisfies (11), that is, $c \in \mathcal{C}$.

4. Consider $c_1, c_2 \in \mathcal{C}$ and fix $\lambda \in (0,1)$. Define $c_\lambda = \lambda c_1 + (1-\lambda) c_2$. The convex linear combination of lower semicontinuous and convex functions is lower semicontinuous and convex (see Clarke [9, Propositions 2.13 and 2.20]). Finally, by point 2, we have that $c^\star \leq c_1, c_2 \leq d^\star$. This implies that $c^\star \leq c_\lambda \leq d^\star$. By point 3, we can conclude that $c_\lambda \in \mathcal{C}$.

5. Fix $\varphi \in B_0(\Sigma, \operatorname{int} K)$. By Proposition 2, recall that $M_{d^\star}(\psi) = \partial_{GP} I(\psi)$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$. Define D as

$$D = \bar{\text{co}}\left(\bigcup_{\psi\in B_0(\Sigma,\operatorname{int} K)} \partial_{GP} I(\psi)\right).$$

Define also $d : \Delta \to [0,\infty]$ by $d(p) = d^\star(p)$ if $p \in D$ and $d(p) = \infty$ if $p \notin D$. In other words, $d(p) = \sup\{d^\star(p), \delta_D(p)\}$ for all $p \in \Delta$. It is immediate to check that d is grounded, lower semicontinuous, convex, and such that $d \geq d^\star$. On the other hand, by definition, we have that $M_d(\varphi) \supseteq \partial_{GP} I(\varphi) \neq \emptyset$. It follows that $I(\varphi) = \min_{p\in\Delta}\{\langle\varphi, p\rangle + d(p)\}$. Since φ was chosen to be generic, we have that $I = J_d$ on $B_0(\Sigma, \operatorname{int} K)$, that is, d satisfies (11) on $B_0(\Sigma, \operatorname{int} K)$.[7] Since both I and J_d are Lipschitz continuous on $B_0(\Sigma, K)$, we have that $I = J_d$ on $B_0(\Sigma, K)$, that is, d satisfies (11) on $B_0(\Sigma, K)$, thus $d \in \mathcal{C}$. By construction of $d^\star$, we can conclude that $d \leq d^\star$, thus, $d = d^\star$. In turn, this yields $\operatorname{dom} d^\star \subseteq D$. Since D is closed, we have that $\operatorname{cl}(\operatorname{dom} d^\star) \subseteq D$. In order to derive the opposite inclusion, observe that $M_{d^\star}(\psi) \subseteq \operatorname{dom} d^\star$ for all $\psi \in B_0(\Sigma, \operatorname{int} K)$. By Proposition 2, we can conclude that

$$\bigcup_{\psi\in B_0(\Sigma,\operatorname{int} K)} \partial_{GP} I(\psi) = \bigcup_{\psi\in B_0(\Sigma,\operatorname{int} K)} M_{d^\star}(\psi) \subseteq \operatorname{dom} d^\star.$$

[6] $d^\star$ can also be obtained as the Fenchel-Moreau biconjugate of the function $c^\star + \delta_R$ where $R = \bigcup_{\psi\in B_0(\Sigma,\operatorname{int} K)} \partial_{GP} I(\psi)$.

[7] Recall that J_d is such that $J_d(\varphi) = \min_{p\in\Delta}\{\langle\varphi, p\rangle + d(p)\}$ for all $\varphi \in B_0(\Sigma)$.

Since $d^\star$ is convex, $\operatorname{dom} d^\star$ is convex. This implies that

$$\operatorname{co}\left(\bigcup_{\psi\in B_0(\Sigma,\operatorname{int}K)}\partial_{GP}I(\psi)\right)\subseteq\operatorname{dom} d^\star.$$

By taking the closure, we obtain that $D\subseteq\operatorname{cl}(\operatorname{dom} d^\star)$, proving the statement.

6. Consider $\psi\in B_0(\Sigma,\operatorname{int}K)$, $p\in\partial_{GP}I(\psi)$, and $c\in\mathcal{C}$. By point 2, we have that $c^\star,d^\star\in\mathcal{C}$ and $c^\star\le c\le d^\star$. By Proposition 2, we also have that $M_{c^\star}(\psi)=\partial_{GP}I(\psi)=M_{d^\star}(\psi)$. It follows that

$$\langle\psi,p\rangle+c^\star(p)=I(\psi)=\langle\psi,p\rangle+d^\star(p).$$

Since $c^\star\le c\le d^\star$, this implies that

$$c^\star(p)=c(p)=d^\star(p).$$

Since ψ, p, and c were arbitrarily chosen, the statement follows. ∎

Before proving the results of Section 3, we need some extra notation and an extra ancillary fact. Given a normalized niveloid $I:B_0(\Sigma,K)\to\mathbb{R}$, we define $\succcurlyeq^\circ$ to be the binary relation on $B_0(\Sigma,K)$ such that

$$\varphi\succcurlyeq^\circ\psi\Longleftrightarrow I(\lambda\varphi+(1-\lambda)\phi)\ge I(\lambda\psi+(1-\lambda)\phi)\ \ \forall\lambda\in(0,1],\forall\phi\in B_0(\Sigma,K). \tag{14}$$

By [5], it follows that $\succcurlyeq^\circ$ is an affine (conic) preorder (see also [10, Appendix A]). Given I, define

$$D=\overline{\operatorname{co}}\left(\bigcup_{\psi\in B_0(\Sigma,\operatorname{int}K)}\partial_{GP}I(\psi)\right).$$

Proposition 3 *If $I:B_0(\Sigma,K)\to\mathbb{R}$ is a normalized and concave niveloid, then*

$$\varphi\succcurlyeq^\circ\psi\quad\Longleftrightarrow\quad\int\varphi dp\ge\int\psi dp\qquad\forall p\in D.$$

Proof. We proceed by steps.

Step 1: Let $\varphi_1,\varphi_2\in B_0(\Sigma,\operatorname{int}K)$. (i) implies (ii), (ii) implies (iii), and (iii) implies (iv) where
(i) $\varphi_1\succcurlyeq^\circ\varphi_2$;
(ii) For each $\phi\in B_0(\Sigma,\operatorname{int}K)$ and for each $\lambda\in(0,1]$

$$I(\lambda\varphi_1+(1-\lambda)\phi)\ge I(\lambda\varphi_2+(1-\lambda)\phi);$$

(iii) For each $\phi\in B_0(\Sigma,\operatorname{int}K)$ and for each $\lambda\in(0,1]$ there exists $\delta_{\lambda,\phi}>0$ such that for each $\delta'\in(0,\delta_{\lambda,\phi})$

$$I\left(\lambda\varphi_1+(1-\lambda)\phi+\delta'\right)>I(\lambda\varphi_2+(1-\lambda)\phi);$$

(iv) For each $\psi\in B_0(\Sigma,\operatorname{int}K)$ and for each $p\in\partial_{GP}I(\psi)$

$$\langle\varphi_1,p\rangle\ge\langle\varphi_2,p\rangle.$$

Proof of the Step. (i) implies (ii). It follows by the definition of $\succcurlyeq^\circ$.

(ii) implies (iii). Consider $\phi\in B_0(\Sigma,\operatorname{int}K)$ and $\lambda\in(0,1]$. Since $\varphi_1,\varphi_2,\phi\in B_0(\Sigma,\operatorname{int}K)$, it follows that $\lambda\varphi_1+(1-\lambda)\phi,\lambda\varphi_2+(1-\lambda)\phi\in B_0(\Sigma,\operatorname{int}K)$. This implies that there exists $\delta_{\lambda,\phi}>0$ such that $\lambda\varphi_1+(1-\lambda)\phi+\delta'\in B_0(\Sigma,\operatorname{int}K)$ for all $\delta'\in(0,\delta_{\lambda,\phi})$. Since (ii) holds and I is a normalized niveloid, we have that

$$I\left(\lambda\varphi_1+(1-\lambda)\phi+\delta'\right)=I(\lambda\varphi_1+(1-\lambda)\phi)+\delta'>I(\lambda\varphi_2+(1-\lambda)\phi),$$

proving the statement.

(iii) implies (iv). Consider now $\psi \in B_0(\Sigma, \operatorname{int} K)$ and $p \in \partial_{GP} I(\psi)$. Since $\psi \in B_0(\Sigma, \operatorname{int} K)$, there exists $\phi \in B_0(\Sigma, \operatorname{int} K)$ and $\lambda \in (0,1)$ such that $\psi = \lambda \varphi_2 + (1-\lambda)\phi$. By definition of $\partial_{GP} I(\psi)$, we have that for each $\psi' \in B_0(\Sigma, K)$

$$I(\psi') > I(\psi) \quad \Longrightarrow \quad \langle \psi', p \rangle > \langle \psi, p \rangle .$$

Since $I(\lambda\varphi_1 + (1-\lambda)\phi + \delta') > I(\lambda\varphi_2 + (1-\lambda)\phi) = I(\psi)$ for all $\delta' \in (0, \delta_{\lambda,\phi})$, it follows that

$$\langle \lambda\varphi_1 + (1-\lambda)\phi + \delta', p \rangle > \langle \psi, p \rangle = \langle \lambda\varphi_2 + (1-\lambda)\phi, p \rangle \qquad \forall \delta' \in (0, \delta_{\lambda,\phi}) .$$

Since $\lambda \in (0,1)$, it follows that $\langle \varphi_1, p \rangle \geq \langle \varphi_2, p \rangle$. □

Step 2: Let $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$. $\varphi_1 \succcurlyeq^\circ \varphi_2$ only if $\langle \varphi_1, p \rangle \geq \langle \varphi_2, p \rangle$ for all $p \in D$.

Proof of the Step. Consider $\beta \in \operatorname{int} K$. It follows that $\frac{1}{2}\varphi_1 + \frac{1}{2}\beta, \frac{1}{2}\varphi_2 + \frac{1}{2}\beta \in B_0(\Sigma, \operatorname{int} K)$ and $\frac{1}{2}\varphi_1 + \frac{1}{2}\beta \succcurlyeq^\circ \frac{1}{2}\varphi_2 + \frac{1}{2}\beta$. By Step 1, we have that

$$\frac{1}{2}\langle \varphi_1, p \rangle + \frac{1}{2}\beta = \left\langle \frac{1}{2}\varphi_1 + \frac{1}{2}\beta, p \right\rangle \geq \left\langle \frac{1}{2}\varphi_2 + \frac{1}{2}\beta, p \right\rangle = \frac{1}{2}\langle \varphi_2, p \rangle + \frac{1}{2}\beta \quad \forall p \in \bigcup_{\psi \in B_0(\Sigma, \operatorname{int} K)} \partial_{GP} I(\psi) .$$

It follows that $\langle \varphi_1, p \rangle \geq \langle \varphi_2, p \rangle$ for all $p \in D$, proving the statement. □

Step 3: Let $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$. If $\langle \varphi_1, p \rangle \geq \langle \varphi_2, p \rangle$ for all $p \in D$, then $\varphi_1 \succcurlyeq^\circ \varphi_2$.

Proof of the Step. By point 5 of Theorem 3, we have that $D = \operatorname{cl}(\operatorname{dom} d^\star)$. First, observe that

$$I(\psi) = \min_{p \in \Delta} \{\langle \psi, p \rangle + d^\star(p)\} = \min_{p \in D} \{\langle \psi, p \rangle + d^\star(p)\} \qquad \forall \psi \in B_0(\Sigma, K) . \tag{15}$$

Consider $\phi \in B_0(\Sigma, K)$ and $\lambda \in (0,1]$. If $\langle \varphi_1, p \rangle \geq \langle \varphi_2, p \rangle$ for all $p \in D$, then $\langle \lambda\varphi_1 + (1-\lambda)\phi, p \rangle \geq \langle \lambda\varphi_2 + (1-\lambda)\phi, p \rangle$ for all $p \in D$. It follows that

$$\langle \lambda\varphi_1 + (1-\lambda)\phi, p \rangle + d^\star(p) \geq \langle \lambda\varphi_2 + (1-\lambda)\phi, p \rangle + d^\star(p) \qquad \forall p \in D.$$

By (15), it follows that $I(\lambda\varphi_1 + (1-\lambda)\phi) \geq I(\lambda\varphi_2 + (1-\lambda)\phi)$. Since ϕ and λ were arbitrarily chosen, it follows that $\varphi_1 \succcurlyeq^\circ \varphi_2$. □

By Steps 2 and 3, it follows that $\varphi_1 \succcurlyeq^\circ \varphi_2$ if and only if $\langle \varphi_1, p \rangle \geq \langle \varphi_2, p \rangle$ for all $p \in D$, proving the statement. ■

Proof of Theorem 1 and Corollary 1. By [17, Lemma 28 and Theorem 3] and since $\succsim$ is a variational preference, there exist a nonconstant affine function $u : X \to \mathbb{R}$ and a normalized and concave niveloid $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that

$$f \succsim g \iff I(u(f)) \geq I(u(g)) .$$

Note that the function $V : \mathcal{F} \to \mathbb{R}$, in (2), is $V = I \circ u$. Without loss of generality, we can assume that $0 \in \operatorname{int} u(X)$. It is immediate to verify that $\mathcal{C}(V, u) = \mathcal{C}(I)$, $u(\mathcal{F}_{\text{int}}) = B_0(\Sigma, \operatorname{int} u(X))$, $\pi_u(f) = \partial_{GP} I(u(f))$ for all $f \in \mathcal{F}_{\text{int}}$. Thus, points (i), (ii), and the first part of (v) follow from points 2, 4, 1, and 6 of Theorem 3. Point (iii) follows from [17, Proposition 6]. Next, observe that $f \succsim^* g$ if and only if $u(f) \succcurlyeq^\circ u(g)$. By point 5 of Theorem 3 and Proposition 3, it follows that

$$f \succsim^* g \iff \int u(f)\, dp \geq \int u(g)\, dp \qquad \forall p \in \operatorname{cl}(\operatorname{dom} d^\star) ,$$

proving that $C^* = \operatorname{cl}(\operatorname{dom} d^\star)$ and point (iv) of Theorem 1. Thus, the second part of point (v) follows from point 5 of Theorem 3. Corollary 1 follows from points 2 and 3 of Theorem 3. ■

Proof of Corollary 3 and Corollary 4. We retain the notation of the proof of Theorem 1. By [12], if $\succsim$ is a Gilboa-Schmeidler preference, then there exists a closed and convex set $C \subseteq \Delta$ such that I can be chosen to be

$$I(\varphi) = \min_{p \in C} \int \varphi dp \qquad \forall \varphi \in B_0(\Sigma, u(X)).$$

This implies that $\delta_C \in \mathcal{C}$. By Theorem 1, we have that $\delta_C \leq d^\star$. At the same time, $\partial_{GP} I(\beta) = C$. By Proposition 2, this implies that $d^\star(p) = 0$ for all $p \in C$. Thus, we can conclude that $d^\star \leq \delta_C$, that is, $d^\star = \delta_C$. Since Anscombe-Aumann expected utility preferences are a particular case of Gilboa-Schmeidler preferences with $C = \{p\}$, Corollary 4 also follows. ■

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