Market based inflation forecast

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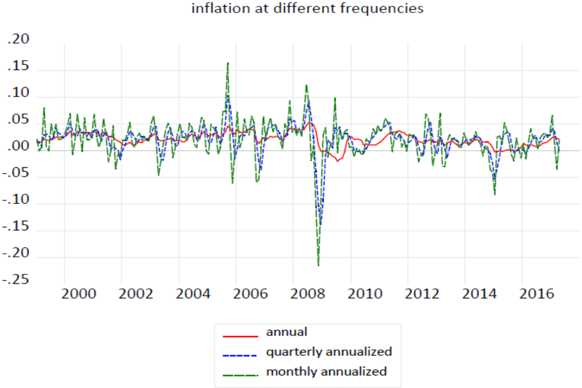
The object to our interest

The object of our interest is forecasting inflation.

- As an illustrative case consider the case of US Consumer Price Index for All Urban Consumers: All Items, seasonally adjusted observed at monthly frequency over the sample 1999-2015.
- Graphical analysis of the data shows clearly that inflation at different frequency (monthly, quarterly and annual) has different time-series properties in terms of persistence and volatility.

The object to our interest



Model Based, Survey Based and Asset Prices based Forecast of Inflation

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2. Multivariate TS methods (VAR, and model based forecasts)
3. Survey based
4. Asset price based forecasts
5. Model Based Forecast (for example the Phillips Curve)

Time-series methods

- Univariate ARIMA methods and multivariate VAR based approaches derived by reduced form of economic models are **backward-looking**.
- In addition, in the case of theoretical model a choice must be made on the frequency of the data that is more consistent with the generic "period" in the economic model, time-series methods have to make choices on the seasonal adjustment and on the level of aggregation at which the data are used in the forecasting model.
- Is it better to forecast directly the aggregate or to forecast inflation at higher frequencies and then aggregate them to obtain an annual forecast ?

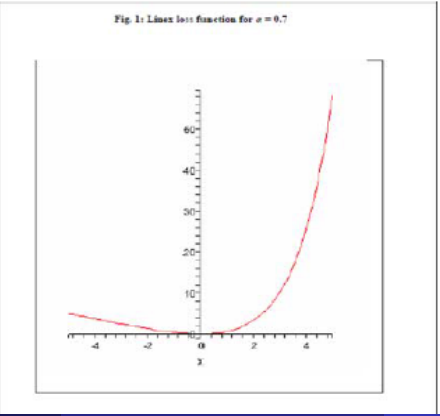
Survey based methods

- Survey based forecast reflect preferences of the forecasters.
- If the loss function is not symmetric, then inflation based forecast are different from expected inflation.
- Consider for example forecasters who have a LINEX loss function:

$$L = \frac{e^{\alpha x_t} - \alpha x_t - 1}{\alpha^2}$$

Survey based methods

If $\alpha > 0$, the Linex function is exponential to the right of the origin, and linear to the left. In this case, pessimistic forecasts of inflation (positive forecast errors) are more costly than optimistic forecasts (negative forecast errors).



Survey based methods

The optimal forecast is

$$f_{t,t+h} = E_t(\mu_{t+h}) + \frac{\alpha}{2} E_t(\sigma^2_{t+h})_t$$

Asset Pricing with Time-Varying Expected Returns

Consider a situation in which in each period k state of nature can occur and each state has a probability $\pi(k)$, in the absence of arbitrage opportunities the price of an asset i at time t can be written as follows:

$$P_{i,t} = \sum_{s=1}^{k} \pi_{t+1}(s) m_{t+1}(s) X_{i,t+1}(s)$$

where $m_{t+1}(s)$ is the discounting weight attributed to future pay-offs, which (as the probability π) is independent from the asset i, $X_{i,t+1}(s)$ are the payoffs of the assets (in case of stocks we have $X_{i,t+1} = P_{t+1} + D_{t+1}$, in case of zero coupon bonds, $X_{i,t+1} = P_{t+1}$), and therefore returns on assets are defined as $1 + R_{s,t+1} = \frac{X_{i,t+1}}{P_{i,t}}$.

Asset Pricing with Time-Varying Expected Returns

For the safe asset, whose payoffs do not depend on the state of nature, we have:

$$
\begin{aligned}
P_{s,t} &= X_{i,t+1} \sum_{s=1}^{k} \pi_{t+1}(s) m_{t+1}(s) \\
1 + R_{s,t+1} &= \frac{1}{\sum\limits_{j=1}^{m} \pi_{t+1}(s) m_{t+1}(s)} \\
1 + R_{s,t+1} &= \frac{1}{E_t(m_{t+1})}
\end{aligned}
$$

Asset Pricing with Time-Varying Expected Returns

consider now a risky asset :

$$\begin{aligned} E_t\left(m_{t+1}\left(1+R_{i,t+1}\right)\right) &= 1 \\ Cov\left(m_{t+1}R_{i,t+1}\right) &= 1-E_t\left(m_{t+1}\right)E_t\left(1+R_{i,t+1}\right) \\ E_t\left(1+R_{i,t+1}\right) &= -\frac{Cov\left(m_{t+1}R_{i,t+1}\right)}{E_t\left(m_{t+1}\right)}+\left(1+R_{s,t+1}\right) \end{aligned}$$

Turning now to excess returns we can write:

$$E_t\left(R_{i,t+1}-R_{s,t+1}\right)=-\left(1+R_{s,t+1}\right)cov\left(m_{t+1}R_{i,t+1}\right)$$

Assets whose returns are low when the stochastic discount factor is high (i.e. when agents values payoffs more) require an higher risk premium, i.e. an higher excess return on the risk-free rate.

Inflation indexed bonds

- The most common form of government bonds are nominal bonds that pay fixed coupons and principal.
- Inflation-indexed bonds, which in the U.S. are known as Treasury Inflation Protected Securities (TIPS), are bonds whose coupons and principal adjust automatically with the evolution of a consumer price index.
- They aim to pay investors a fixed inflation-adjusted coupon and principal, in other words they are real bonds and their yields are typically considered the best proxy for the term structure of real interest rates in the economy.

Inflation indexed bonds

- Investors holding either inflation-indexed or nominal government bonds are exposed to the risk of changing real interest rates.
- In addition to real interest rate risk, nominal government bonds expose investors to inflation risk while real bonds do not. When future inflation is uncertain, the coupons and principal of nominal bonds can suffer from the eroding effects of inflationary surprises.
- Finally, both the nominal and real bond are theoretically affected by a premium for liquidity risk. Liquidity risk, is, the risk of having to sell (or buy) a bond in a thin market and, thus, at an unfair price and with higher transaction costs.

Inflation indexed bonds

At time t the yields to maturity of nominal and real bonds maturing at T can be wriiten as follows:

$$\begin{aligned} Y^n_{t,T} &= rr_{t,T} + E_t\pi_{t,T} + RP^{rr}_t + RP^{\pi}_t \\ Y^r_{t,T} &= rr_{t,T} + RP^{rr}_t + RP^{liq}_t \end{aligned}$$

the difference in the yield to maturity, usually referred to as the breakeven inflation rate $B_{t,T}$, can be written as:

$$B_{t,T} = E_t\pi_{t,T} + RP^{\pi}_t - RP^{liq}_t$$

Further technical details

Further details complicate the pricing of real bonds.

- Inflation figures are published with a lag and therefore the principal value of inflation indexed bonds is adjusted with a lag (three-month in the US and the UK).
- Moreover the mechanism of indexation could be different for coupons and principal and it might have different provisions for positive and negative inflation.
- Finally the tax treatment of the bond might also differ. In the UK principal adjustments of inflation-linked gilts are not taxed. In the US, on the other hand, inflation-adjustments of principal are considered ordinary income for tax purposes.

Inflation Swaps

- An inflation swap is a bilateral contractual agreement. It requires one party (the ‘inflation payer’) to make periodic floating-rate payments linked to inflation, in exchange for predetermined fixed-rate payments from a second party (the ‘inflation receiver’).
- The most common structure is the zero-coupon inflation swap. The inflation receiver pays on a contract maturing in T years pays the Fixed leg = (1 + fixed rate)T x Nominal value to receive from the the inflation payer the variable Inflation leg = (Final price index/Starting price index) x Nominal value.

$$SW_{t,T} = E_t \pi_{t,T} + RP_t^{\pi}$$

Further Technical Details

In practice, inflation swap contracts have indexation lags.

$$SW_{t,T} = E_t \pi_{t-m,T-m} + RP_t^{\pi}$$

As the object of interest is inflation from time t onward, the following decomposition becomes relevant:

$$SW_{t,T} = \frac{m}{T-t} \pi_{t-m,t} + \frac{T-m-t}{T-t} E_t \pi_{t,T-m} + RP_t^{\pi}$$

which, in absence of publication lags on inflation data allows to derive the relevant inflation compensation from the fixed swap rate.

The distribution of market-based inflation expectations

Inflation options are instruments with non-linear payoffs:

- inflation caps pay-out if inflation exceeds a certain threshold
- inflations floors pay-out if inflation falls short of a certain threshold

By comparing the price of options that insure against different outcomes it is possible to infer the probability that investors assign to those different outcomes.

The distribution of market-based inflation expectations

More formally:

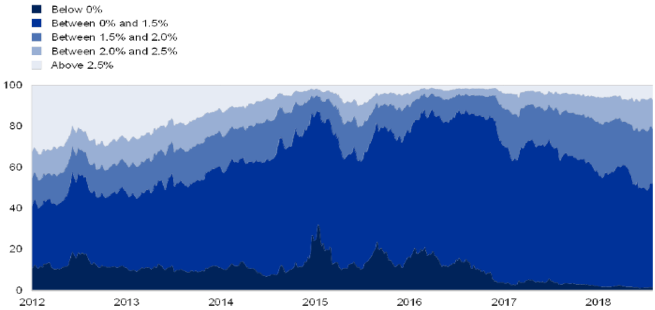
$$p_t = \sum \frac{1}{1 + r_f} P^Q(\pi) E_t \left(x_{t+1} \left(\pi \right) \right)$$

The price of an inflation linked-asset today depends on the risk neutral.probabilities $P^Q(\pi)$ and expected one-period ahead payoffs $E_t \left(x_{t+1} \left(\pi \right) \right)$ associated with different inflation events.
Given the knowledge of prices, payoffs and the risk-free rate it is then possible to recover risk-neutral probabilities of inflation.

The distribution of market-based inflation expectations

Option-implied risk-neutral distribution of euro area average inflation over the next five years

(percentages)



Physical and risk-neutral probabilities

Consider an optimal portfolio equilibrium when an investor can choose between investing at period 0 in a risky asset and a in risk-free asset:

$$
\begin{aligned}
E\left[u'\left(Y_1\right)\left(1+r_r\right)\right] &= E\left[u'\left(Y_1\right)\left(1+r_{rf}\right)\right] \\
&= \left(1+r_{rf}\right)E\left[u'\left(Y_1\right)\right] \\
\frac{E\left[u'\left(Y_1\right)\left(1+r_r\right)\right]}{E\left[u'\left(Y_1\right)\right]} &= \left(1+r_{rf}\right) \\
E\left[\bar{m}\left(1+r_r\right)\right] &= \left(1+r_{rf}\right) \\
\bar{m} &= \frac{u'\left(Y_1\right)}{E\left[u'\left(Y_1\right)\right]} = m\left(1+r_{rf}\right)
\end{aligned}
$$

$\bar{m}$ is known as the "pricing kernel" .

Physical and risk-neutral probabilities

Consider a simplified scenario in which there are only two states of the world (g,b)

$$
\begin{aligned}
E\left[\bar{m}\left(1+r_r\right)\right] &= \left(1+r_{rf}\right) \\
\pi^g \bar{m}^g \left(1+r_r^g\right) + \pi^b \bar{m}^b \left(1+r_r^b\right) &= \left(1+r_{rf}\right) \\
\bar{m}^g &= \frac{u'\left(Y_1^g\right)}{E\left[u'\left(Y_1\right)\right]},\ \bar{m}^b = \frac{u'\left(Y_1^b\right)}{E\left[u'\left(Y_1\right)\right]} \\
E\left[u'\left(Y_1\right)\right] &= \pi^g u'\left(Y_1^g\right) + \pi^b u'\left(Y_1^b\right)
\end{aligned}
$$

Physical and risk-neutral probabilities

We have

$$
\begin{aligned}
E\left[\bar{m}\right] &= 1 \\
\pi^{g}\bar{m}^{g}+\pi^{b}\bar{m}^{b} &= 1 \\
\pi^{g}\frac{u'\left(Y_{1}^{g}\right)}{E\left[u'\left(Y_{1}\right)\right]}+\pi^{b}\frac{u'\left(Y_{1}^{b}\right)}{E\left[u'\left(Y_{1}\right)\right]} &= 1 \\
E\left[u'\left(Y_{1}\right)\right] &= \pi^{g}u'\left(Y_{1}^{g}\right)+\pi^{b}u'\left(Y_{1}^{b}\right)
\end{aligned}
$$

Physical and risk-neutral probabilities

The quantity $\pi^i \bar{m}^i = q^i$ is naturally intepreted as a new probability measure of the state of the world i. The expected return for every risky asset under this new probability measure is the risk free rate:

$$\begin{aligned}
\pi^g \bar{m}^g \left(1 + r_r^g\right) + \pi^b \bar{m}^b \left(1 + r_r^b\right) &= \left(1 + r_{rf}\right) \\
q^g \left(1 + r_r^g\right) + q^b \left(1 + r_r^b\right) &= \left(1 + r_{rf}\right)
\end{aligned}$$

Note that $\frac{\pi^i \bar{m}^i}{(1+r_{rf})}$ is the price that an investor is willing to pay for a security that generates a marginal increase of wealth in the state of the world i and no change of wealth in other states of the world.

Bond Returns

Cash-flows from different type of bonds:

	$t+1$	$t+2$	$t+3$	...	T
general	CF_{t+1}	CF_{t+2}	CF_{t+3}	...	CF_T
coupon bond	C	C	C	...	$1+C$
1-period zero	1	0	0	...	0
2-period zero	0	1	0	...	0
...				...	
(T–t) -period zero	0	0	0	...	1

ZC Bonds

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{\left(1 + Y_{t,T}\right)^{T-t}}, \tag{1}$$

where $P_{t,T}$ is the price at time t of a bond maturing at time T, and $Y_{t,T}$ is yield to maturity. Taking logs we have the following relationship:

$$p_{t,T} = -\left(T - t\right) y_{t,T}, \tag{2}$$

which clearly illustrates that the elasticity of the yield to maturity to the price of a zero-coupon bond is the maturity of the security.

ZC Bonds

Price and YTM of zero-coupon bonds						
Maturity	1	2	3	5	7	10
$P_{t,T}$	0.9524	0.9070	0.8638	0.7835	0.7106	0.6139
$Y_{t,T}$	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
$p_{t,T}$	-0.0487	-0.0976	-0.1464	-0.2439	-0.3416	-0.4879
$y_{t,T}$	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488

ZC Bonds

The one-period uncertain holding-period return on a bond maturing at time T, $r_{t,t+1}^{T}$, is then defined as follows:

$$r_{t,t+1}^{T} \equiv p_{t+1,T} - p_{t,T} = -\left(T-t-1\right) y_{t+1,T} + \left(T-t\right) y_{t,T} \qquad (3)$$
$$= y_{t,T} - \left(T-t-1\right)\left(y_{t+1,T} - y_{t,T}\right),$$
$$= \left(T-t\right) y_{t,T} - \left(T-t-1\right) y_{t+1,T}, \qquad (4)$$

which means that yields and returns differ by the a scaled measure of the change between the yield at time $t+1$, $y_{t+1,T}$, and the yield at time t, $y_{t,T}$. Think of a situation in which the one-year YTM stands at 4.1 per cent while the 30-year YTM stands at 7 per cent. If the YTM of the thirty year bonds goes up to 7.1 per cent in the following period, then the period returns from the two bonds is the same.

A simple model of the Term Structure

Apply the no arbitrage condition to a one-period bond (the safe asset) and a T-period bond:

$$
\begin{aligned}
E_t\left(r^T_{t,t+1} - r^1_{t,t+1}\right) &= E_t\left(r^T_{t,t+1} - y_{t,t+1}\right) = \phi^T_{t,t+1} \\
E_t\left(r^T_{t,t+1}\right) &= y_{t,t+1} + \phi^T_{t,t+1}
\end{aligned}
$$

A simple model of the Term Structure

Solving forward the difference equation $p_{t,T} = p_{t+1,T} - r^T_{t,t+1}$, we have :

$$
\begin{aligned}
p_{t,T} &= -\sum_{i=0}^{n-1} E_t \left(r^T_{t+i,t+i+1} \right) \\
p_{t,T} &= -\left(T - t \right) y_{t,T}, \\
y_{t,T} &= \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t \left(r^T_{t+i,t+i+1} \right) \\
&= \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t \left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1} \right)
\end{aligned}
$$

Forward Rates

Forward rates are returns on an investment at time t, made in the future at time t' with maturity at time T. The return on this strategy is equivalent to the return on a strategy that buys at time t zero coupon with maturity T and sells at time t the same amount of bonds with maturity t'.
The price of the investment strategy is $\left(-\left(T-t\right)y_{t,T}+\left(t'-t\right)y_{t,t'}\right)$and using the usual formula that links prices to returns we have :

$$f_{t,t',T}=\frac{\left(T-t\right)y_{t,T}-\left(t'-t\right)y_{t,t'}}{T-t'} \tag{5}$$

Forward Rates

Applying the general formula to specific maturities we have :

$$f_{t,t+1,t+2} = 2y_{t,t+2} - y_{t,t+1} \tag{6}$$

$$f_{t,t+2,t+3} = 3y_{t,t+3} - 2y_{t,t+2} \tag{7}$$

$$f_{t,t+3,t+4} = 4y_{t,t+4} - 3y_{t,t+3} \tag{8}$$

$$f_{t,t+n-1,t+n} = ny_{t,t+n} - (n-1)y_{t,t+n-1} \tag{9}$$

Forward Rates

Using all these equations we have:

$$y_{t,t+n} = \frac{1}{n}\left(y_{t,t+1} + f_{t,t+1,t+2} + f_{t,t+2,t+3} + \ldots + f_{t,t+n-1,t+n}\right) \quad (10)$$

$$y_{t,t+n} = \frac{1}{n}\sum_{i=0}^{n-1} E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right)$$

$$f_{t,t+i,t+i+1} = E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right)$$

Forward Rates

Think of using forward rates to assess the impact of monetary policy. Let us analyze a potential movement of spot and forward rates around a shift in the central bank target rate.

Before CB intervention

1-year spot and forward rates					
maturity	i=1	i=2	i=3	i=4	i=5
$y_{t,t+i}$	0.05	0.05	0.05	0.05	0.05
$f_{t,t+i,t+i+1}$	0.05	0.05	0.05	0.05	

Forward Rates

After CB intervention :

1-year spot and forward rates					
maturità	i=1	i=2	i=3	i=4	i=5
$y_{t,t+i}$	0.06	0.06	0.05	0.045	0.04
$f_{t,t+i,t+i+1}$	0.06	0.03	0.03	0.02	

Please remember that the interpretion of future forward as expected rates requires some assumption on the risk premium.

Instantaneous Forward Rates

Define the instantaneous forward as the forward rate on the contract with infinitesimal maturity:

$$f_{t,t'} = \lim_{T \to t'} f_{t,t',T} \tag{11}$$

given the sequence of forward rates you can define forward rate at any settlement date as follows :

$$f_{t,t',T} = \frac{\int_{\tau=t'}^{T} f_{\tau t} d\tau}{(T - t')}$$

Instantaneous Forward Rates

As a consequence the relationship between spot and forward rate is written as:

$$y_{t,T} = \frac{\int_{\tau=t}^{T} f_{\tau t} d\tau}{(T-t)}$$

and therefore

$$f_{t,T} = y_{t,T} + (T-t)\frac{\partial y_{t,T}}{\partial T} \tag{12}$$

so instantaneous forward rates and spot rates coincide at the very short and very long-end of the term structure, forward rates are above spot rates when the yield curve slopes positevely and forward rates are below spot rates when the yield curve slopes negatively.

Factor Models of the Term Structure

Gurkanyak et al. estimate the following interpolant at each point in time, by non-linear least squares, on the cross-section of yields:

$$\begin{aligned} y_{t,t+k} &= L_t + SL_t \frac{1-\exp\left(-\frac{k}{\tau_1}\right)}{\frac{k}{\tau_1}} + C_t^1 \left(\frac{1-\exp\left(-\frac{k}{\tau_1}\right)}{\frac{k}{\tau_1}} - \exp\left(-\frac{k}{\tau_1}\right) \right) \\ &\quad + C_t^2 \left(\frac{1-\exp\left(-\frac{k}{\tau_2}\right)}{\frac{k}{\tau_2}} - \exp\left(-\frac{k}{\tau_2}\right) \right) \end{aligned}$$

which is an extension originally proposed by Svensson(1994) on the original parameterization adopted by Nelson and Siegel (1987) that sets $C_t^2 = 0$.

Forward rates are easily derived as

$$f_{tk} = L_t + SL_t \exp\left(-\frac{k}{\tau_1}\right) + C_t^1 \frac{k}{\tau_1} \exp\left(-\frac{k}{\tau_1}\right) + C_t^2 \frac{k}{\tau_2} \exp\left(-\frac{k}{\tau_2}\right) \tag{14}$$

When maturity k goes to zero forward and spot rates coincide at $L_t + SL_t$, and when maturity goes to infinite forward and spot coincide at L_t. Terms in C_t^1 and C_t^2 describes two humps starting at zero at different starting points and ending at zero.

Factor Models of the Term Structure

- L_t, SL_t, C_t^1, C_t^2, which are estimated as parameters in a cross-section of yields, can be interpreted as latent factors.
- L_t has a loading that does not decay to zero in the limit, while the loading on all the other parameters do so, therefore this parameter can be interpreted as the long-term factor, the level of the term-structure.
- The loading on SL_t is a function that starts at 1 and decays monotonically towards zero; it may be viewed a short-term factor, the slope of the term structure. In fact, $r_t^{rf} = L_t + SL_t$ is the limit when k goes to zero of the spot and the forward interpolant. We naturally interpret r_t^{rf} as the risk-free rate.
- C_t are medium term factor, in the sense that their loading start at zero, increase and then decay to zero (at different speed). Such factors capture the curvature of the yield curve.

A general state-space representation

To generalize the NS approach we can put the dynamics of the term structure in a state-space framework:

$$y_{t,t+n} = \frac{-1}{n}\left(A_n + B_n' X_t\right) + \varepsilon_{t,t+n} \qquad \varepsilon_t \sim i.i.d.N(0,\sigma^2 I) \tag{15}$$

$$X_t = \mu + \Phi X_{t-1} + v_t \qquad v_t \sim i.i.d.N(0,\Omega) \tag{16}$$

In the case of original NS we have

$$B_n' = \left[-n \ , - \left(\frac{1-e^{-\lambda n}}{\lambda}\right) \ , - \left(\frac{1-e^{-\lambda n}}{\lambda} - ne^{-\lambda n}\right)\right] \text{ and } A_n = 0$$