Marked Based Inflation Forecast

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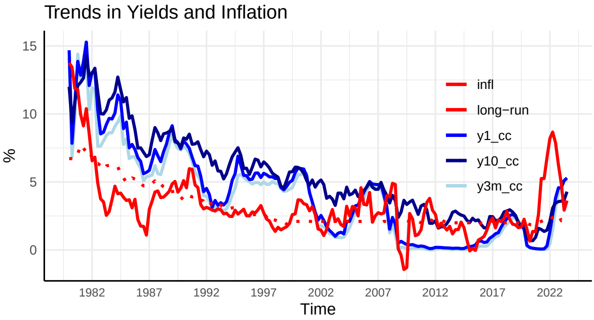
The object to our interest

The object of our interest is forecasting inflation using information from asset prices.

- As an illustrative case consider the case of US Consumer Price Index for All Urban Consumers: All Items, seasonally adjusted observed at monthly frequency over the sample 1980-2023.
- Annual inflation is mean reverting
- Asset Prices have (common) trends

Inflation and Bond Yields



Alternative Forecasting Methods

- Univariate time-series methods (ARMA and ARIMA)
- Multivariate TS methods (VAR, and model based forecasts)
- Model Based Forecast (for example the Phillips Curve, Blanchard-Bernanke(2023))
- Survey based
- Using big data (collecting online prices by webscraping, using NPL techniques) http://www.thebillionpricesproject.com/ http://truflation.com
- Asset price based forecasts

Time-series methods

- Univariate ARIMA methods and multivariate VAR based approaches are **backward-looking**.
- Using time-series on asset prices to forecast inflation requires dealing with the presence of unit-roots

Deterministic Trends

Trend stationary processes feature only a deterministic trend:

$$z_t = \alpha + \beta t + \epsilon_t.$$

The z_t process is non-stationary, but the non-stationarity is removed simply by regressing z_t on the deterministic trend.

Stochastic Trends

Consider, for example, the random walk process with a drift:

$$\begin{aligned} x_t &= a_0 + x_{t-1} + \epsilon_t, \\ \epsilon_t &\sim n.i.d.\left(0, \sigma_\epsilon^2\right). \end{aligned}$$

Recursive substitution yields

$$x_t = x_0 + a_0 t + \sum_{i=0}^{t-1} \epsilon_{t-i}, \tag{1}$$

which shows that the non-stationary series contains both a deterministic $(a_0 t)$ and a stochastic $\left(\sum_{i=0}^{t-1} \epsilon_{t-i}\right)$ trend.

Forecasting stochastic trends

An easy way to make a non-stationary series stationary is differencing:

$$\Delta x_t = x_t - x_{t-1} = (1 - L)\, x_t = a_0 + \epsilon_t.$$

However, note that when the equation above will be used to project x_{t+j} out-of-sample, only the deterministic trend will be forecasted and the variance of the forecasting error will explode as the forecasting horizon gets large.

Forecasting with a co-integrated system

Cointegration (i.e. the presence of common stochastic trends)) is a potential opportunity to deal with trends. Think of the case of the 1-year and the 10-year yields plotted before. A possible cointegrated VAR for these two variables can be written as :

$$\begin{aligned} Y_{t,t+10} - Y_{t,t+1} &= a_{1,0} + a_{1,1}(Y_{t-1,t+9} - Y_{t-1,t}) + v_{1,t+1} \\ \Delta Y_{t,t+1} &= a_{2,0} + a_{2,2}\Delta Y_{t-1,t} + v_{2,t+1} \end{aligned}$$

- this system works very well for 1-step ahead predictions, but for n-step ahead prediction it has the problem that 1-Year rate are predicted only with a deterministic trend and the 10-Year rate is forced to mantain its long run relations with the short-term rate.
- the long-term forecasting performance of a cointegrated system depends crucially on the predictions of the stochastic trend

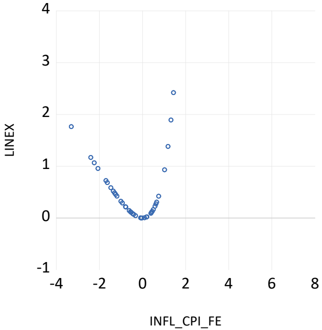
Survey based methods

- Survey based forecast reflect preferences of the forecasters.
- If the loss function is not symmetric, then inflation based forecast are different from expected inflation.
- Consider for example forecasters who have a LINEX loss function:

$$L = \frac{e^{\alpha x_t} - \alpha x_t - 1}{\alpha^2}$$

Survey based methods

If $\alpha > 0$, the Linex function is exponential to the right of the origin, and linear to the left. In this case, positive forecast errors for inflation are more costly than negative forecast errors.



Survey based methods

The optimal forecast under LINEX preferences is:

$$INFL^{f}_{t,t+h} = \mu_{t,t+h} + \frac{\alpha}{2}\sigma^{2}_{t,t+h}$$

Market Prices and Inflation Forecast

To extract expectations of future inflation from asset prices it is important to acknowledge that asset prices are affected by

- Expectations of future outcomes
- Compensation for risk (risk premia)

Asset Pricing with Time-Varying Expected Returns

Consider a situation in which in each period k state of nature can occur and each state has a probability $\pi(k)$, in the absence of arbitrage opportunities the price of an asset i at time t can be written as follows:

$$P_{i,t} = \sum_{s=1}^{k} \pi_{t+1}(s) m_{t+1}(s) X_{i,t+1}(s)$$

where $m_{t+1}(s)$ is the discounting weight attributed to future pay-offs, which (as the probability π) is independent from the asset i, $X_{i,t+1}(s)$ are the payoffs of the assets (in case of stocks we have $X_{i,t+1} = P_{t+1} + D_{t+1}$, in case of zero coupon bonds, $X_{i,t+1} = P_{t+1}$), and therefore returns on assets are defined as $1 + R_{s,t+1} = \frac{X_{i,t+1}}{P_{i,t}}$.

Asset Pricing with Time-Varying Expected Returns

For the safe asset, whose payoffs do not depend on the state of nature, we have:

$$
\begin{aligned}
P_{s,t} &= X_{i,t+1}\sum_{s=1}^{k}\pi_{t+1}(s)m_{t+1}(s) \\
1+R_{s,t+1} &= \frac{1}{\sum_{s=1}^{k}\pi_{t+1}(s)m_{t+1}(s)} \\
1+R_{s,t+1} &= \frac{1}{E_t(m_{t+1})}
\end{aligned}
$$

Asset Pricing with Time-Varying Expected Returns

consider now a risky asset :

$$
\begin{aligned}
E_t\left(m_{t+1}\left(1+R_{i,t+1}\right)\right) &= 1 \\
Cov\left(m_{t+1}R_{i,t+1}\right) &= 1-E_t\left(m_{t+1}\right)E_t\left(1+R_{i,t+1}\right) \\
E_t\left(1+R_{i,t+1}\right) &= -\frac{Cov\left(m_{t+1}R_{i,t+1}\right)}{E_t\left(m_{t+1}\right)}+\left(1+R_{s,t+1}\right)
\end{aligned}
$$

Turning now to excess returns we can write:

$$
E_t\left(R_{i,t+1}-R_{s,t+1}\right)=-\left(1+R_{s,t+1}\right)cov\left(m_{t+1}R_{i,t+1}\right)
$$

Assets whose returns are low when the stochastic discount factor is high (i.e. when agents values payoffs more) require an higher risk premium, i.e. an higher excess return on the risk-free rate.

Physical and risk-neutral probabilities

Consider an optimal portfolio equilibrium when an investor can choose between investing at period 0 in a risky asset and a in risk-free asset:

$$
\begin{aligned}
E\left[u'\left(Y_1\right)\left(1+r_r\right)\right] &= E\left[u'\left(Y_1\right)\left(1+r_{rf}\right)\right] \\
&= \left(1+r_{rf}\right)E\left[u'\left(Y_1\right)\right] \\
\frac{E\left[u'\left(Y_1\right)\left(1+r_r\right)\right]}{E\left[u'\left(Y_1\right)\right]} &= \left(1+r_{rf}\right) \\
E\left[\bar{m}\left(1+r_r\right)\right] &= \left(1+r_{rf}\right) \\
\bar{m} &= \frac{u'\left(Y_1\right)}{E\left[u'\left(Y_1\right)\right]}
\end{aligned}
$$

$\bar{m}$ is known as the "pricing kernel" .

Physical and risk-neutral probabilities

Consider a simplified scenario in which there are only two states of the world (g,b)

$$
\begin{aligned}
E\left[\bar{m}\left(1+r_r\right)\right] &= \left(1+r_{rf}\right) \\
\pi^g \bar{m}^g\left(1+r_r^g\right)+\pi^b \bar{m}^b\left(1+r_r^b\right) &= \left(1+r_{rf}\right) \\
\bar{m}^g &= \frac{u'\left(Y_1^g\right)}{E\left[u'\left(Y_1\right)\right]},\ \bar{m}^b=\frac{u'\left(Y_1^b\right)}{E\left[u'\left(Y_1\right)\right]} \\
E\left[u'\left(Y_1\right)\right] &= \pi^g u'\left(Y_1^g\right)+\pi^b u'\left(Y_1^b\right)
\end{aligned}
$$

Physical and risk-neutral probabilities

We have

$$
\begin{aligned}
E\left[\bar{m}\right] &= 1 \\
\pi^g \bar{m}^g + \pi^b \bar{m}^b &= 1 \\
\pi^g \frac{u'\left(Y_1^g\right)}{E\left[u'\left(Y_1\right)\right]} + \pi^b \frac{u'\left(Y_1^b\right)}{E\left[u'\left(Y_1\right)\right]} &= 1 \\
E\left[u'\left(Y_1\right)\right] &= \pi^g u'\left(Y_1^g\right) + \pi^b u'\left(Y_1^b\right)
\end{aligned}
$$

Physical and risk-neutral probabilities

The quantity $\pi^i \bar{m}^i = q^i$ is naturally intepreted as a new probability measure of the state of the world i. The expected return for every risky asset under this new probability measure is the risk free rate:

$$
\begin{aligned}
\pi^g \bar{m}^g \left(1 + r_r^g\right) + \pi^b \bar{m}^b \left(1 + r_r^b\right) &= \left(1 + r_{rf}\right) \\
q^g \left(1 + r_r^g\right) + q^b \left(1 + r_r^b\right) &= \left(1 + r_{rf}\right)
\end{aligned}
$$

Note that $\frac{\pi^i \bar{m}^i}{\left(1 + r_{rf}\right)}$ is the price that an investor is willing to pay for a security that generates a marginal increase of wealth in the state of the world i and no change of wealth in other states of the world.

Bond Returns:YTM, Duration and HPR

Cash-flows from different type of bonds:

	$t+1$	$t+2$	$t+3$	...	T
general	CF_{t+1}	CF_{t+2}	CF_{t+3}	...	CF_T
coupon bond	C	C	C	...	$1+C$
1-period zero	1	0	0	...	0
2-period zero	0	1	0	...	0
...				...	
(T–t) -period zero	0	0	0	...	1

ZC Bonds

Define the relationship between price and yield to maturity of a zero-coupon bond as follows:

$$P_{t,T} = \frac{1}{(1 + Y_{t,T})^{T-t}}, \tag{2}$$

where $P_{t,T}$ is the price at time t of a bond maturing at time T, and $Y_{t,T}$ is yield to maturity. Taking logs we have the following relationship:

$$p_{t,T} = -(T-t)\, y_{t,T}, \tag{3}$$

which clearly illustrates that the elasticity of the yield to maturity to the price of a zero-coupon bond is the maturity of the security.

ZC Bonds

Price and YTM of zero-coupon bonds							
Mat	1	2	3	5	7	10	20
$P_{t,T}$	0.9524	0.9070	0.8638	0.7835	0.7106	0.6139	0.3769
$Y_{t,T}$	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
$p_{t,T}$	-0.0487	-0.0976	-0.1464	-0.2439	-0.3416	-0.4879	-0.9757
$y_{t,T}$	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488	0.0488

ZC Bonds

The one-period uncertain holding-period return on a bond maturing at time T, $r^T_{t,t+1}$, is then defined as follows:

$$r^T_{t,t+1} \equiv p_{t+1,T} - p_{t,T} = -(T-t-1)\, y_{t+1,T} + (T-t)\, y_{t,T} \quad (4)$$
$$= y_{t,T} - (T-t-1)\left(y_{t+1,T} - y_{t,T}\right),$$
$$= (T-t)\, y_{t,T} - (T-t-1)\, y_{t+1,T}, \quad (5)$$

which means that yields and returns differ by the a scaled measure of the change between the yield at time $t+1$, $y_{t+1,T}$, and the yield at time t, $y_{t,T}$. Think of a situation in which the one-year YTM stands at 4.1 per cent while the 30-year YTM stands at 7 per cent. If the YTM of the thirty year bonds goes up to 7.1 per cent in the following period, then the period returns from the two bonds is the same.

Modelling the Term Structure

Apply the no arbitrage condition to a one-period bond (the safe asset) and a T-period bond:

$$\begin{aligned} E_t\left(r^T_{t,t+1} - r^1_{t,t+1}\right) &= E_t\left(r^T_{t,t+1} - y_{t,t+1}\right) = \phi^T_{t,t+1} \\ E_t\left(r^T_{t,t+1}\right) &= y_{t,t+1} + \phi^T_{t,t+1} \end{aligned}$$

Solving forward the difference equation $p_{t,T} = p_{t+1,T} - r^T_{t,t+1}$, we have :

$$\begin{aligned} y_{t,T} &= \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t\left(r^T_{t+i,t+i+1}\right) \\ &= \frac{1}{(T-t)} \sum_{i=0}^{n-1} E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right) \end{aligned}$$

Modelling the Term Structure

The model clearly shows that Bond yields are driven by two unobservable factors

- Expectations of future monetary policy (risk free) rates over the residual life of the bonds
- Compensation for risk (risk premia)

Modelling the Term Structure

By subtracting from the left-hand side and the right-hand side of the equation above the one-period rate, we have the following equation for the term spread:

$$y_{t,T} - y_{t,t+1} = \sum_{i=1}^{T-t-1} \left(1 - \frac{i}{T-t}\right) E_t \Delta y_{t+i,t+i+1} + \frac{1}{T-t} \sum_{i=1}^{T-t-1} \phi_{t+i,t+i+1}^{T} \quad (6)$$

The model clearly shows that if yields at long maturities are cointegrated with short-term rates

- there is a common stochastic trend affecting the entire term structure which is the trend in short-term rates
- Compensation for risk (risk premia) are stationary and therefore cyclical

Expectations Theory in Cointegrated VAR

Campbel-Shiller(1987)CS consider the case of the risk free rate $y_t^{(1)} = r_t$ and a very long term bond.

$$
\begin{aligned}
S_{t,T} = y_{t,T} - r_t &= \sum_{j=1}^{T-1} \gamma^j E[\Delta r_{t+j} \mid I_t] + E[\Phi_T \mid I_t] \\
&= S^*_{t,T} + E[\Phi_T \mid I_t]
\end{aligned}
$$

Expectations Theory in Cointegrated VAR

Assuming that $y_{t,T}$ and r_t are cointegrated with a cointegrating vector (1,-1), CS construct a bivariate stationary VAR in the first difference of the short-term rate and the spread :

$$\begin{aligned} \Delta r_t &= a(L)\Delta r_{t-1} + b(L)S_{t-1} + u_{1t} \\ S_t &= c(L)\Delta r_{t-1} + d(L)S_{t-1} + u_{2t} \end{aligned} \tag{7}$$

Expectations Theory in Cointegrated VAR

Stack the VAR as:

$$\begin{bmatrix} \Delta r_t \\ . \\ . \\ \Delta r_{t-p+1} \\ S_t \\ . \\ . \\ S_{t-p+1} \end{bmatrix} = \begin{bmatrix} a_1 & . & . & a_p & b_1 & . & . & b_p \\ 1 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & 1 & 0 & 0 & . & . & 0 \\ c_1 & . & . & c_p & d_1 & . & . & d_p \\ 0 & . & . & 0 & 1 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta r_{t-1} \\ . \\ . \\ \Delta r_{t-p} \\ S_{t-1} \\ . \\ . \\ S_{t-p} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ . \\ . \\ 0 \\ u_{2t} \\ . \\ . \\ 0 \end{bmatrix} \tag{8}$$

Expectations Theory in Cointegrated VAR

This can be written more succinctly as:

$$z_t = Az_{t-1} + v_t \tag{9}$$

The ET null puts a set of restrictions which can be written as :

$$g' z_t = \sum_{j=1}^{T-1} \gamma^j h' A^{j\prime} z_t \tag{10}$$

where $g\prime$ and $h\prime$ are selector vectors for S and Δr correspondingly (i.e. row vectors with 2p elements, all of which are zero except for the p+1st element of $g\prime$ and the first element of $h\prime$ which are unity).

Expectations Theory in Cointegrated VAR

Since the above expression has to hold for general z_t, and, for large T, the sum converges under the null of the validity of the ET, it must be the case that:

$$g' = h'\gamma A(I - \gamma A)^{-1} \tag{11}$$

which implies:

$$g'(I - \gamma A) = h'\gamma A \tag{12}$$

and we have the following constraints on the individual coefficients of VAR :

$$\{c_i = -a_i, \forall i\}, \{d_1 = -b_1 + 1/\gamma\}, \{d_i = -b_i, \forall i \neq 1\} \tag{13}$$

The above restrictions are testable with a Wald test. By doing so using US data between the fifties and the eighties Campbell and Shiller (1987) rejected the null of the ET.

Cointegration and Present Value Models

However, when CS construct a theoretical spread $S^*_{t,T}$, by imposing the (rejected) ET restrictions on the VAR they find that, despite the statistical rejection of the ET, $S^*_{t,T}$ and $S_{t,T}$ are strongly correlated.

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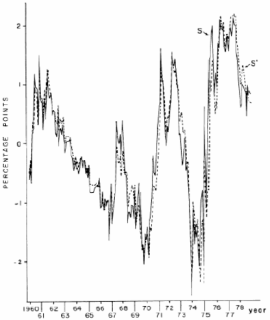

FIG. 1.—Term structure: deviations from means of long-short spread S_t and theoretical spread S'_t.

Forward Rates

Forward rates are returns on an investment at time t, made in the future at time t' with maturity at time T. The return on this strategy is equivalent to the return on a strategy that buys at time t zero coupon with maturity T and sells at time t the same amount of bonds with maturity t'.
The price of the investment strategy is $\left(-(T-t)\,y_{t,T}+(t'-t)\,y_{t,t'}\right)$and using the usual formula that links prices to returns we have :

$$f_{t,t',T}=\frac{(T-t)\,y_{t,T}-(t'-t)\,y_{t,t'}}{T-t'} \tag{14}$$

Forward Rates

Applying the general formula to specific maturities we have :

$$f_{t,t+1,t+2} = 2y_{t,t+2} - y_{t,t+1} \tag{15}$$

$$f_{t,t+2,t+3} = 3y_{t,t+3} - 2y_{t,t+2} \tag{16}$$

$$f_{t,t+3,t+4} = 4y_{t,t+4} - 3y_{t,t+3} \tag{17}$$

$$f_{t,t+n-1,t+n} = ny_{t,t+n} - (n-1)y_{t,t+n-1} \tag{18}$$

Forward Rates

Using all these equations we have:

$$y_{t,t+n} = \frac{1}{n}\left(y_{t,t+1} + f_{t,t+1,t+2} + f_{t,t+2,t+3} + ... + f_{t,t+n-1,t+n}\right) \quad (19)$$

$$y_{t,t+n} = \frac{1}{n}\sum_{i=0}^{n-1} E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right)$$

$$f_{t,t+i,t+i+1} = E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right)$$

Instantaneous Forward Rates

Define the instanteous froward as the forwad rate on the contract with infinitesimal maturity:

$$f_{t,t'} = \lim_{T \to t'} f_{t,t',T} \tag{20}$$

given the sequence of forward rates you can define forward rate at any settlement date as follows :

$$f_{t,t',T} = \frac{\int_{\tau=t'}^{T} f_{\tau t} d\tau}{(T - t')}$$

Instantaneous Forward Rates

As a consequence the relationship between spot and forward rate is written as:

$$y_{t,T} = \frac{\int_{\tau=t}^{T} f_{\tau t} d\tau}{(T-t)}$$

and therefore

$$f_{t,T} = y_{t,T} + (T-t)\,\frac{\partial y_{t,T}}{\partial T} \tag{21}$$

so instantaneous forward rates and spot rates coincide at the very short and very long-end of the term structure, forward rates are above spot rates when the yield curve slopes positevely and forward rates are below spot rates when the yield curve slopes negatively.