

# Market based inflation forecast

Carlo A Favero

February 2020

- 1 Assessing the importance of the risk premium
  - Single Equation based evidence
  
- 2 The Inflation Risk Premium: Pflueger-Viceira
  - The Construction of the Short-term Real Rate
  
- 3 The VAR based evidence
  
- 4 No-Arbitrage Affine Factor Models

# The importance of the risk premium

We consider two approaches to assess the importance of the risk premium:

- single equation based-evidence
- and VAR based evidence.

# Single Equation based evidence

To assess the importance of RP, three different implications of the Expectations Theory can be brought to the data:

$$y_{t,T} - (T - t - 1) E_t (y_{t+1,T} - y_{t,T}) = y_{t,t+1} + \phi_{t,T}.$$

$$f_{t,t+i,t+i+1} = E_t \left( y_{t+i,t+i+1} + \phi_{t+i,t+i+1}^T \right)$$

$$\begin{aligned} y_{t,t+n} - y_{t,t+1} &= \sum_{i=1}^{n-1} \left( 1 - \frac{1}{n} \right) E_t \Delta y_{t+i,t+i+1} \\ &\quad + \frac{1}{n} \sum_{i=1}^{n-1} \phi_{t+i,t+i+1}^T \end{aligned}$$

# Single Equation based evidence

(a) Estimate the following model :

$$y_{t+1,T} - y_{t,T} = \beta_0 + \beta_1 \frac{1}{T - t - 1} (y_{t,T} - y_{t,t+1}) + u_{t+1}$$

to test  $\beta_0 = 0, \beta_1 = 1$ .

(b) Estimate the following model :

$$S_{t,n}^* = \beta_0 + \beta_1 S_{t,n} + u_t$$

$$S_{t,n} = y_{t,t+n} - y_{t,t+1}$$

$$S_{t,n}^* = \sum_{i=1}^{n-1} \left(1 - \frac{1}{n}\right) \Delta y_{t+i,t+i+1}$$

to test  $\beta_0 = 0, \beta_1 = 1$ .

(c) Estimate the following model :

$$(y_{t+i,t+i+1} - y_{t,t+1}) = \beta_0 + \beta_1 (f_{t,t+i,t+i+1} - y_{t,t+1}) + u_{t+i+1}$$

to test  $\beta_0 = 0, \beta_1 = 1$ .

(d) Estimate the following model :

$$y_{t,T} - (T - t - 1) (y_{t+1,T} - y_{t,T}) - y_{t,t+1} = \beta_0 + \beta_1 (f_{t,t+1,t+1} - y_{t,t+1}) +$$

to test  $\beta_1 = 0$ .



# Single Equation based evidence

The empirical evidence shows that:

- i) high yields spreads fare poorly in predicting increases in long rates (see Campbell, 1995)
- ii) the change in yields does not move one-to-one with the forward spot spread (see Fama and Bliss, 1986)
- iii) period excess returns on long-term bond are predictable using the information in the forward-spot spread (see Cochrane, 1999, Cochrane-Piazzesi 2005)

# The Nominal Expectations Hypothesis

PV test in their data whether the term spread contains a time-varying risk premium by running the regression test :

$$(T - t) y_{t,T} - (T - t - 1) y_{t+1,T} - y_{t,t+1} = \beta_0 + \beta_1 (y_{t,T} - y_{t,t+1}) + u_{t+1}$$

# The Real Expectations Hypothesis

PV test in their data whether the real term spread contains a time-varying risk premium by running the regression test :

$$(T - t) y_{t,T}^{TIPS} - (T - t - 1) y_{t+1,T}^{TIPS} - y_{t,t+1}^{TIPS} = \beta_0 + \beta_1 \left( y_{t,T}^{TIPS} - y_{t,t+1}^{TIPS} \right) + u_t$$

# Breakeven Inflation and the Inflation Expectations Hypothesis

given that breakeven inflation is defined as follows:

$$b_{t,T} = y_{t,T} - y_{t,T}^{TIPS}$$

then the excess returns on breakeven inflation is given by

$$(T - t) b_{t,T} - (T - t - 1) b_{t+1,T} - b_{t,t+1}$$

remembering that

$$b_{t,T} = E_t \pi_{t,T} + RP_t^\pi - RP_t^{liq}$$

When inflation and liquidity risk premia are constant the excess returns on break-even inflation are equal to a constant plus the expression

$$(T - t) \pi_{t,T}^e - (T - t - 1) \pi_{t+1,T}^e - \pi_{t,t+1}$$

which fluctuates zero when expectations are rational

# Breakeven Inflation and the Inflation Expectations Hypothesis

So the importance of inflation (and liquidity) risk premia can be captured by the following regression :

$$(T - t) b_{t,T} - (T - t - 1) b_{t+1,T} - b_{t,t+1} = \beta_0 + \beta_1 (b_{t,T} - b_{t,t+1}) + u_{t+1}$$

# The Construction of the short-term real rate

TIPS trade only at maturity of one year or longer so to construct a short term real rate assume zero inflation and risk premia over the quarter

$$y_{t,t+1}^{TIPS} = y_{t,t+1} - \pi_{t,t+1}^e$$

Next we assume that inflation expectations over the next quarter are rational and proxy for the ex-ante real short rate as the fitted value from the regression of this

quarter's realized real rate onto last quarter's realized real rate, the nominal short rate, and annual inflation up to time  $t$ .

# The Construction of the short-term real rate

Table 1  
Forecasted Real Short Rate

$y_{1,t}^s - \pi_{t+1}$	US	UK
$y_{1,t}^s$	0.57** (0.22)	0.46 (0.29)
$y_{1,t-1}^s - \pi_t$	0.08 (0.08)	-0.11 (0.07)
$(\pi_{t-2} + \pi_{t-1} + \pi_t + \pi_{t+1})/4$	0.08 (0.09)	0.03 (0.09)
p-value	0.00	0.00
R <sup>2</sup>	0.44	0.18

Overlapping quarterly real short rate returns onto the nominal short rate, last quarter's real short rate return and inflation over the past year.

Monthly data 1982.1-2009.12.

Newey-West standard errors with 4 lags in brackets.

\* and \*\* denote significance at the 5% and 1% level respectively.

p-value of the F-test for no predictability.

# The VAR based evidence

The VAR based evidence is due to Campbell-Shiller(1987) and considers the following version of the ET

$$S_{t,T} = S_{t,T}^* = \sum_{j=1}^{T-t-1} \gamma^j E[\Delta r_{t+j} \mid I_t] \quad (1)$$

(1) shows that a necessary condition for the ET to hold puts constraints on the long-run dynamics of the spread. In fact, the spread should be stationary being a weighted sum of stationary variables. Obviously, stationarity of the spread implies that, if yields are non-stationary, they should be cointegrated with a cointegrating vector (1,-1). However, the necessary and sufficient conditions for the validity of the ET impose restrictions both on the long-run and the short run dynamics.



# The VAR based evidence

Assuming that  $R_{t,T}$  and  $r_t$  are cointegrated with a cointegrating vector  $(1,-1)$ , CS construct a bivariate stationary VAR in the first difference of the short-term rate and the spread :

$$\begin{aligned}\Delta r_t &= a(L)\Delta r_{t-1} + b(L)S_{t-1} + u_{1t} \\ S_t &= c(L)\Delta r_{t-1} + d(L)S_{t-1} + u_{2t}\end{aligned}$$

# The VAR based evidence

Stack the VAR as:

$$\begin{bmatrix} \Delta r_t \\ . \\ . \\ \Delta r_{t-p+1} \\ S_t \\ . \\ . \\ S_{t-p+1} \end{bmatrix} = \begin{bmatrix} a_1 & . & . & a_p & b_1 & . & . & b_p \\ 1 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & 1 & 0 & 0 & . & . & 0 \\ c_1 & . & . & c_p & d_1 & . & . & d_p \\ 0 & . & . & 0 & 1 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & . & 0 \\ 0 & . & . & 0 & 0 & . & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta r_{t-1} \\ . \\ . \\ \Delta r_{t-p} \\ S_{t-1} \\ . \\ . \\ S_{t-p} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ . \\ . \\ 0 \\ u_{2t} \\ . \\ . \\ 0 \end{bmatrix} \quad (2)$$

This can be written more succinctly as:

$$z_t = Az_{t-1} + v_t$$

# The VAR based evidence

The ET null puts a set of restrictions which can be written as :

$$g'z_t = \sum_{j=1}^{T-1} \gamma^j h' A^{j'} z_t \quad (3)$$

where  $g'$  and  $h'$  are selector vectors for  $S$  and  $\Delta r$  correspondingly ( i.e. row vectors with  $2p$  elements, all of which are zero except for the  $p+1$ st element of  $g'$  and the first element of  $h'$  which are unity). Since the above expression has to hold for general  $z_t$ , and, for large  $T$ , the sum converges under the null of the validity of the ET, it must be the case that:

$$g' = h' \gamma A (I - \gamma A)^{-1} \quad (4)$$

# The VAR based evidence

which implies:

$$g'(I - \gamma A) = h' \gamma A \quad (5)$$

and we have the following constraints on the individual coefficients of VAR(??):

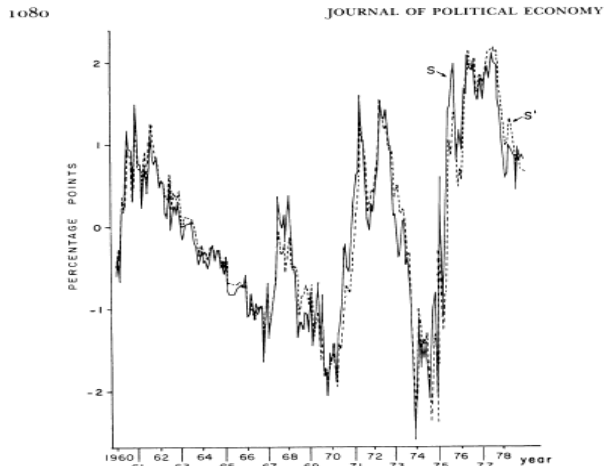
$$\{c_i = -a_i, \forall i\}, \{d_1 = -b_1 + 1/\gamma\}, \{d_i = -b_i, \forall i \neq 1\} \quad (6)$$

The above restrictions are testable with a Wald test.

# The VAR based evidence

By doing so using US data between the fifties and the eighties Campbell and Shiller (1987) rejected the null of the ET. However, when CS construct a theoretical spread  $S_{t,T}^*$ , by imposing the (rejected) ET restrictions on the VAR they find that, despite the statistical rejection of the ET,  $S_{t,T}^*$  and  $S_{t,T}$  are strongly correlated.

# The VAR based evidence



# Affine TS Models

Consider an n-period zero coupon bond  $R_{t,t+1}^T \equiv \frac{P_{t+1,T}}{P_{t,T}} - 1$ , using the standard pricing equation we have:

$$\begin{aligned} 1 &= E_t \left[ \left( 1 + R_{t,t+1}^T \right) M_{t+1} \right] \\ P_{t,T} &= E_t [P_{t+1,T} M_{t+1}] \end{aligned}$$

solving forward and using the terminal condition  $P_{T,T} = 1$ , we have

$$P_{t,T} = E_t [M_{t+1} M_{t+2} \dots M_T] = E_t [M_{t,T}]$$

The no-arb condition shows that a term structure model is nothing but a time-series model for the SDF

Assume that the SDF are jointly lognormally distributed and take logs:

$$p_{t,T} = E_t [m_{t+1} + p_{t+1,T}] + \frac{1}{2} \text{Var}_t [m_{t+1} + p_{t+1,T}]$$

$$p_{t,T} = p_{t,t+1} + E_t [p_{t+1,T}] + \frac{1}{2} \text{Var}_t [p_{t+1,T}] + \text{Cov}_t [m_{t+1} + p_{t+1,T}]$$

as

$$p_{t,t+1} = -y_{t,t+1} = E_t [m_{t+1}] + \frac{1}{2} \text{Var}_t [m_{t+1}]$$



- Affine TS models make assumptions to ensure that all log bond prices are linear in state variables
  - *Completely affine* models assume that both log bond prices and the log SDF are linear
  - *Essentially affine* models allow non linearities in the log SDF specification

# No-Arbitrage Affine Factor Models

First building block is the dynamics of the factors determining risk premium:

$$X_t = \mu + \Phi X_{t-1} + \Sigma \epsilon_t$$

interpret this as a companion form representation.

Second is a specification for the one-period rate  $r_t$  which is assumed to be a linear function of the factors:

$$r_t = \delta_0 + \delta_1' X_t$$

# No-Arbitrage Affine Factor Models

The third is a pricing kernel. The assumption of no-arbitrage guarantees the existence of a risk-neutral measure  $Q$  such that the price of any asset  $V_t$  that does not pay any dividends at time  $t+1$  satisfies the following relation:

$$V_t = E_t^Q (\exp(-r_t) V_{t+1})$$

the Radon-Nikodym derivative (which converts the risk neutral measure to the data-generating measure) is denoted by  $\tilde{\zeta}_{t+1}$ .

# No-Arbitrage Affine Factor Models

So for any random variable  $Z_{t+1}$  we have

$$E_t^Q(Z_{t+1}) = E_t(\zeta_{t+1}Z_{t+1}) / \zeta_t$$

The assumption of no-arbitrage allows us to price any nominal bond in the economy.

Assume that  $\zeta_{t+1}$  follows the log-normal process:

$$\zeta_{t+1} = \zeta_t \exp\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\epsilon_{t+1}\right)$$

where  $\lambda_t$  are the time-varying market prices of risk associated with the sources of uncertainty  $\epsilon_t$ .

# No-Arbitrage Affine Factor Models

Parameterize  $\lambda_t$  as an affine process:

$$\lambda_t = \lambda_0 + \lambda_1 X_t$$

define the pricing kernel  $m_{t+1}$  as

$$m_{t+1} = \exp(-r_t) \tilde{\zeta}_{t+1} / \tilde{\zeta}_t$$

substituting from the processes for the short-rate and  $\tilde{\zeta}_{t+1}$  we have:

$$m_{t+1} = \exp \left( -\delta_0 - \delta'_1 X_t - \frac{1}{2} \lambda'_t \lambda_t - \lambda'_t \varepsilon_{t+1} \right)$$

# No-Arbitrage Affine Factor Models

Now the total one-period gross return of any nominal asset satisfies:

$$E_t(m_{t+1}R_{t+1}) = 1$$

If  $p_t^n$  represents the price of an n-period zero coupon bond, then we can use this equation to compute recursively bond prices as:

$$p_t^{n+1} = E_t(m_{t+1}p_{t+1}^n)$$

# No-Arbitrage Affine Factor Models

Guess that the log of bond prices are linear functions of the state variable we have:

$$p_t^n = \exp(A_n + B_n' X_t)$$

This guess is easily verified for the one-period bond, in which case we have:

$$\begin{aligned} p_t^1 &= E_t(m_{t+1}) = \exp(-r_t) \\ &= \exp(\delta_0 + \delta_1' X_t) \end{aligned}$$

# No-Arbitrage Affine Factor Models

But it also applies to n-period bonds, in which case we have:

$$\begin{aligned} p_t^{n+1} &= E_t (m_{t+1} p_{t+1}^n) \\ &= E_t \exp \left( -\delta_0 - \delta'_1 X_t - \frac{1}{2} \lambda'_t \lambda_t - \lambda'_t \varepsilon_{t+1} + A_n + B'_n X_{t+1} \right) \\ &= \exp \left( -\delta_0 - \delta'_1 X_t - \frac{1}{2} \lambda'_t \lambda_t + A_n \right) E_t \left[ \exp \left( -\lambda'_t \varepsilon_{t+1} + B'_n X_{t+1} \right) \right] \\ &= \exp \left( -\delta_0 - \delta'_1 X_t - \frac{1}{2} \lambda'_t \lambda_t + A_n \right) \\ &\quad E_t \left[ \exp \left( -\lambda'_t \varepsilon_{t+1} + B'_n (\mu + \Phi X_t + \Sigma \varepsilon_{t+1}) \right) \right] \\ &= \exp \left( -\delta_0 + (B'_n \Phi - \delta'_1) X_t - \frac{1}{2} \lambda'_t \lambda_t + A_n + B'_n \mu \right) \\ &\quad E_t \left[ \exp \left( -\lambda'_t \varepsilon_{t+1} + B'_n \Sigma \varepsilon_{t+1} \right) \right] \end{aligned}$$



# No-Arbitrage Affine Factor Models

$$= \exp \left( \begin{array}{c} -\delta_0 + (B'_n \Phi - \delta'_1) X_t + A_n + B'_n (\mu - \Sigma \lambda_0) \\ + \frac{1}{2} B'_n \Sigma \Sigma' B_n - B'_n \Sigma \lambda_1 X_t \end{array} \right)$$

here the last step uses log-normality and the fact that

$$\lambda'_t \lambda_t = \lambda'_t \text{var}(\varepsilon_{t+1}) \lambda_t.$$

# No-Arbitrage Affine Factor Models

By matching coefficients we now have:

$$\begin{aligned}A_{n+1} &= -\delta_0 + A_n + B'_n (\mu - \Sigma \lambda_0) + \frac{1}{2} B'_n \Sigma \Sigma' B_n \\ B'_{n+1} &= B'_n (\Phi - \Sigma \lambda_1) - \delta'_1\end{aligned}$$

# No-Arbitrage Affine Factor Models

To sum up, we can characterize a traditional Affine TS model as follows:

$$y_{t,t+n} = \frac{-1}{n} (A_n + B'_n X_t) + \varepsilon_{t,t+n} \quad \varepsilon_t \sim i.i.d.N(0, \sigma^2 I) \quad (7)$$

$$X_t = \mu + \Phi X_{t-1} + v_t \quad v_t \sim i.i.d.N(0, \Omega) \quad (8)$$

$$b_{n+1} = \frac{1}{(n+1)} \left[ \sum_{i=0}^n (\Phi' - \lambda'_1 \Omega)^i \right] b_1$$
$$a_{n+1} = a_1 - \frac{1}{(n+1)} \sum_{i=0}^n B^{(i)}, \text{ where } B^{(i)} = B'_i (\mu - \Omega \lambda_0) + \frac{1}{2} B'_i \Omega B_i.$$

# Risk Premium

Given the knowledge of the model parameters the risk premium can be derived naturally:

$$RP_t^n = y_{t,t+n} - \frac{1}{n} E_t \sum_{j=0}^{n-1} r_{t+j}$$

$$\begin{aligned} E_t r_{t+j} &= \delta_0 + \delta_1' E_t X_{t+j} \\ &= \delta_0 + \delta_1' (\bar{\mu} + \Phi^j (X_t - \bar{\mu})) \end{aligned}$$

where

$$\begin{aligned} X_t &= \mu + \Phi X_{t-1} + \Sigma \epsilon_t \\ \bar{\mu} &= (I - \Phi)^{-1} \mu \end{aligned}$$

# Risk Premium

and, in absence of measurement error, we have:

$$\begin{aligned}y_{t,t+n} &= \frac{-1}{n} (A_n + B'_n X_t) \\ &= \bar{A}_n + \bar{B}'_n X_t\end{aligned}$$

$$\begin{aligned}RP_t^n &= \bar{A}_n - \sum \frac{\delta_0}{n} - \delta'_1 \left( I - \frac{1}{n} \sum_{j=0}^{n-1} \Phi^j \right) \bar{\mu} + \\ &\quad + \left( \bar{B}'_n - \delta'_1 \frac{1}{n} \sum_{j=0}^{n-1} \Phi^j \right) X_t\end{aligned}$$

but  $B'_{n+1} = B'_n (\Phi - \Sigma \lambda_1) - \delta'_1$ , so when  $\Sigma = 0$  or  $\lambda_1 = 0$  the term multiplying  $X_t$  in the last expression vanishes and the risk premium becomes constant.

Recently the no-arbitrage approach has been extended to include some observable macroeconomic factors in the state vector and to explicit allow for a Taylor-rule type of specification for the risk-free one period rate. This extension of the small information set using macroeconomic variable can be further expanded by moving to large-information set. In this case rather than including in the state vector some specific macroeconomic variables, common factors can be extracted from a large panel of macroeconomic variables using static principal components, as suggested by Stock and Watson (2002).