Market based inflation forecast

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Factor models of the term structure

Represent the dynamics of the term structure in a state-space framework:

$$y_{t,t+n} = \frac{-1}{n}\left(A_n + B_n' X_t\right) + \varepsilon_{t,t+n} \qquad \varepsilon_t \sim i.i.d.N(0,\sigma^2 I) \tag{1}$$

$$X_t = \mu + \Phi X_{t-1} + v_t \qquad v_t \sim i.i.d.N(0,\Omega) \tag{2}$$

- parsimonious representation
- it assumes stationarity
- no arbitrage generates cross-equation restrictions

Factor Models of the Term Structure

Gurkanyak et al. estimate the following interpolant at each point in time, by non-linear least squares, on the cross-section of yields:

$$\begin{aligned} y_{t,t+k} &= L_t + SL_t \frac{1-\exp\left(-\frac{k}{\tau_1}\right)}{\frac{k}{\tau_1}} + C_t^1 \left(\frac{1-\exp\left(-\frac{k}{\tau_1}\right)}{\frac{k}{\tau_1}} - \exp\left(-\frac{k}{\tau_1}\right) \right) \quad (3) \\ &\quad + C_t^2 \left(\frac{1-\exp\left(-\frac{k}{\tau_2}\right)}{\frac{k}{\tau_2}} - \exp\left(-\frac{k}{\tau_2}\right) \right) \end{aligned}$$

which is an extension originally proposed by Svensson(1994) on the original parameterization adopted by Nelson and Siegel (1987) that sets $C_t^2 = 0$.

Nelson-Siegel, Diebold-Li models of the Term Structure

Forward rates are easily derived as

$$f_{tk} = L_t + SL_t \exp\left(-\frac{k}{\tau_1}\right) + C_t^1 \frac{k}{\tau_1} \exp\left(-\frac{k}{\tau_1}\right) + C_t^2 \frac{k}{\tau_2} \exp\left(-\frac{k}{\tau_2}\right) \quad (4)$$

When maturity k goes to zero forward and spot rates coincide at $L_t + SL_t$, and when maturity goes to infinite forward and spot coincide at L_t. Terms in C_t^1 and C_t^2 describes two humps starting at zero at different starting points and ending at zero.

Nelson-Siegel, Diebold-Li models of the Term Structure

- L_t, SL_t, C_t^1, C_t^2, which are estimated as parameters in a cross-section of yields, can be interpreted as latent factors.
- L_t has a loading that does not decay to zero in the limit, while the loading on all the other parameters do so, therefore this parameter can be interpreted as the long-term factor, the level of the term-structure.
- The loading on SL_t is a function that starts at 1 and decays monotonically towards zero; it may be viewed a short-term factor, the slope of the term structure. In fact, $r_t^{rf} = L_t + SL_t$ is the limit when k goes to zero of the spot and the forward interpolant. We naturally interpret r_t^{rf} as the risk-free rate.
- C_t are medium term factor, in the sense that their loading start at zero, increase and then decay to zero (at different speed). Such factors capture the curvature of the yield curve.

A general state-space representation

To generalize the NS approach we can put the dynamics of the term structure in a state-space framework:

$$y_{t,t+n} = \frac{-1}{n}\left(A_n + B_n' X_t\right) + \varepsilon_{t,t+n} \qquad \varepsilon_t \sim i.i.d.N(0, \sigma^2 I) \tag{5}$$

$$X_t = \mu + \Phi X_{t-1} + v_t \qquad v_t \sim i.i.d.N(0, \Omega) \tag{6}$$

In the case of original NS we have

$$B_n' = \left[-n \;\; , -\left(\frac{1-e^{-\lambda n}}{\lambda}\right) \;\; , -\left(\frac{1-e^{-\lambda n}}{\lambda} - ne^{-\lambda n}\right)\right] \text{ and } A_n = 0$$

ATS Models

We present the Adrian et al(2013) approach, in which factors are first obtained by extracting the first K (K=5) principal components for the entire term structure.
A VAR will then be appropriately specified for the stationary state variables, X_t

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}, \tag{7}$$

$$v_{t+1} | \{X_s\}_{s=0}^{t} \sim \mathcal{N}(0, \Sigma), \tag{8}$$

The variables in X_t determine the market price of risk, λ_t in the following affine form:

$$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda_1 X_t), \tag{9}$$

The assumption of no-arbitrage implies that there exists a pricing kernel M_t such that:

$$P_t^{(n)} = E_t\left(M_{t+1} P_{t+1}^{(n-1)}\right), \tag{10}$$

for every $n > 0$ and $t \geq 0$. Where $P_t^{(n)} = exp\left[-nR_t^{(n)}\right]$ is the price of a zero coupon bond with maturity n.

ATS Models

assume that the pricing kernel is exponentially affine, i.e.,

$$m_{t+1} = -R_t^1 - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}, \tag{11}$$

where $R_t^1 = log(P_t^{(1)}) = p_t^{(1)}$ is the continuously compounded risk-free rate, and $m_t = \log M_t$. The excess log returns are given by:

$$rx_{t+1}^{(n-1)} = p_{t+1}^{(n-1)} - p_t^{(n)} - R_t^1, , \tag{12}$$

where $p_t^{(n)} = \log P_t^{(n)}$. From (11) and (10) it follows that

$$1 = E_t\left[\exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}\right)\right]. \tag{13}$$

ATS Models

Assuming that $\left(rx_{t+1}^{(n-1)}, v'_{t+1}\right)$ are jointly normally distributed, we obtain:

$$\begin{aligned} 0 &= \log\left(E_t\left[exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}v_{t+1}\right)\right]\right) \\ &= E_t\left(rx_{t+1}^{(n-1)}\right) + \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right) - \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}\lambda_t'\lambda_t \\ &- Cov_t\left[rx_{t+1}^{(n-1)}, v'_{t+1}\Sigma^{-1/2}\lambda_t\right]. \end{aligned} \tag{14}$$

We can then write

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left[rx_{t+1}^{(n-1)}, v'_{t+1}\Sigma^{-1/2}\lambda_t\right] - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right), \tag{15}$$

and denote $\beta_t^{(n-1)'}$ as:

$$\beta_t^{(n-1)'} = \mathrm{Cov}_t\left[rx_{t+1}^{(n-1)}, v'_{t+1}\right]\Sigma^{-1}, \tag{16}$$

ATS Models

By substituting from (32) into (15), using (9), we have:

$$E_t\left(rx_{t+1}^{(n-1)}\right) = \beta_t^{(n-1)'}(\lambda_0 + \lambda_1 X_t) - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right), \tag{17}$$

The unexpected excess return can be decomposed in a component that is correlated with v_{t+1} and another component which is orthogonal:

$$rx_{t+1}^{(n-1)} - E_t\left(rx_{t+1}^{(n-1)}\right) = \beta_t^{(n-1)'}v_{t+1} + e_{t+1}^{(n-1)}, \tag{18}$$

Under the assumption that the return pricing error are i.i.d. with variance σ^2 and that β_t is constant, the generating process for log excess returns becomes:

$$p_{t+1}^{(n-1)} - p_t^{(n)} - R_t^1 = \underbrace{\beta^{(n-1)'}(\lambda_0 + \lambda_1 X_t)}_{\text{Expected Return}} - \underbrace{\frac{1}{2}\left(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2\right)}_{\text{Convexity Correction}} + \underbrace{\beta^{(n-1)'}v_{t+1}}_{\text{Return Innovation}} + e_{t+1}^{(n-1)}, \tag{19}$$

ATS Models

Stacking (19) across N maturities and T time-periods we have the following matrix-form representation:

$$rx = \beta'(\lambda_0 \iota_T' + \lambda_1 X_-) - \frac{1}{2}(B^* \text{vec}(\Sigma) + \sigma^2 \iota_N)\iota_T' + \beta' V + E, \tag{20}$$

rx is a NxT matrix of excess returns.
$\beta = [\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(N)}]$ is a KxN matrix of factor loadings.
ι_T and ι_N are Tx1 and Nx1 vectors of ones.
$X_- = [X_0, X_1, \ldots, X_{T-1}]$ is a KxT matrix of lagged pricing factors.
$B^* = [\text{vec}(\beta^{(1)}\beta^{(1)'}) \ldots \text{vec}(\beta^{(N)}\beta^{(N)'})]$ is an $N?K^2$ matrix.

V is a KxT matrix, and E is an NxT matrix

Estimation of the lambdas in ATS Models

1. Construct the pricing factors (X) from principal component analysis (PCA) of the cyclical components of yields derived in the first step. Estimate VAR (7) using OLS, decomposing the pricing factors into predictable components and factor innovations $\hat{V}$.

2. Regress excess returns on a constant, lagged pricing factors and contemporaneous pricing factor innovations according to

$$rX = a\iota_T' + \beta' \hat{V} + cX_- + E, \tag{21}$$

B^* can be computed from β and σ^2 from the residuals E.

3. From Eq (20), we can see $a = \beta' \lambda_0 - \frac{1}{2}(B^* \text{vec}(\Sigma) + \sigma^2 \iota_N)$ and $c = \beta' \lambda_1$. From these,the estimators are obtained

$$\hat{\lambda}_0 = \left(\hat{\beta}\hat{\beta}'\right)^{-1} \left(\hat{a} + \frac{1}{2}\left((\hat{B}^* \text{vec}(\hat{\Sigma}) + \hat{\sigma}^2 \iota_N\right)\right), \tag{22}$$

$$\hat{\lambda}_1 = \left(\hat{\beta}\hat{\beta}'\right)^{-1} \hat{\beta}\hat{c}. \tag{23}$$

ATSM: from returns to yields

Given the general expression for excess returns:

$$p_{t+1}^{(n-1)} - p_t^{(n)} - R_t^1 = \underbrace{\beta^{(n-1)'}(\lambda_0 + \lambda_1 X_t)}_{\text{Expected Return}} - \underbrace{\frac{1}{2}\left(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2\right)}_{\text{Convexity Correction}} + \underbrace{\beta^{(n-1)'} v_{t+1}}_{\text{Return Innovation}} + e_{t+1}^{(n-1)},$$

The price of bonds at all maturities can expressed as:

$$p_t^n = A_n + B_n' X_t + u_t^n, \tag{24}$$

where, as a consequence of no-arbitrage, recursive restrictions apply to A_n and B_n. The risk-free rate R_t^1 is expressed as a linear function of the underlying factors -

$$R_t^1 = \delta_0 + \delta_1' X_t + \epsilon_t, \tag{25}$$

so $A_1 = -\delta_0$ and $B_1 = -\delta_1$ We then have:

$$A_n = A_{n-1} + B_{n-1}'(-\lambda_0) + \frac{1}{2}(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2) - \delta_0, \tag{26}$$

$$B_n' = B_{n-1}'(\Phi - \lambda_1) - \delta_1', \tag{27}$$

$$\beta^{(n)} = B_n', \tag{28}$$

ATSM: understanding the restrictions

For the one-period bond R_t^1 we have

$$R_t^1 = \delta_0 + \delta_1' X_t + e_t^{(1)}, \tag{29}$$

Where parameters $\hat{\delta}_0$ and $\hat{\delta}_1$ can be estimated by projecting R_t^1 on the factors X_t.
For the two-period bond, we have

$$\begin{aligned} p_{t+1}^{(1)} - p_t^{(2)} - R_t^1 &= \beta^{(1)'}(\lambda_0 + \lambda_1 X_t) - \frac{1}{2}(\beta^{(1)'}\Sigma\beta^{(1)} + \sigma^2) + \beta^{(1)'}v_{t+1} + e_{t+1}^{(1)}, \\ p_{t+1}^{(1)} &= -\delta_0 - \delta_1'(\mu + \Phi X_t + v_{t+1}) \end{aligned}$$

By using the second equation in the first one, we have:

$$\begin{aligned} p_t^{(2)} &= -\delta_0 - \delta_1' X_t - e_t^{(1)} - \delta_0 - \delta_1'(\mu + \Phi X_t + v_{t+1}) \\ &- \beta^{(1)'}(\lambda_0 + \lambda_1 X_t) + \frac{1}{2}\left(\beta^{(1)'}\Sigma\beta^{(1)} - \sigma^2\right) - \beta^{(1)'}v_{t+1} - e_{t+1}^{(1)} \end{aligned}$$

which illustrates that

$$A_2 = A_1 + B_1'(-\lambda_0) + \frac{1}{2}(\beta^{(n-1)'}\Sigma\beta^{(n-1)} + \sigma^2) - \delta_0, \tag{30}$$

$$B_2' = B_1'(\Phi - \lambda_1) - \delta_1', \tag{31}$$

$$\beta^{(1)} = B_1', \beta^{(2)} = B_2' \tag{32}$$

ATSM: Measuring the Term Premia

Model simulation in two scenarios, a baseline with all parameters set are their fitted values and an alternative one in which the market price of risk is set to zero, i.e. $\lambda_0 = \lambda_1 = 0$, allows to compute term premia as the differences between the model implied yields and the risk neutral yields.

The problem with standard ATSM models

- The data tell us that yields are drifting, but excess returns are cyclical.
- In standard ATSM models a few factors, modelled by a Vector Autoregressive Process, are the common drivers of the dynamics of both the expected path of future one-period rates and the term premia. As yields drift, factors exhibit a high level of persistence. When these factors are modeled using a VAR (Vector Autoregression), the forecast of future one-period rates gradually converges to the mean of the sample used for estimation.
- Standard ATSM models tend to generate term premia that are a-cyclical and parallel to the secular trend in yields.
- These features of the term-premia are a by-product of the specification strategy for the dynamics of yields and excess-returns that adopts a common autoregressive factor structure for them.

A potential solution

Specify a model that takes into account trends and cycles:

$$\begin{aligned}
r_t^{(1)} &= r_t^{*,(1)} + u_t^{(1)} && (33)\\
r_t^{*,(1)} &= \gamma_1 MY_t + \gamma_2 \Delta y_t^{pot} + \gamma_3 \pi_t^{LR} && (34)\\
r_t^{(n)} &= r_t^{*,(1)} + u_t^{(n)} && (35)\\
p_t^{(n)} &= p_t^{*,(n)} + A_n + B_n' X_t + \varepsilon_t^{(n)}, && (36)\\
X_t &= \mu + \Phi X_{t-1} + v_{t+1} && (37)
\end{aligned}$$

in which the factors X_t are extracted from the cyclical components of yields, u_t^n, after the completion of the first stage of estimation. The standard recursive restrictions apply to A_n and B_n

Market Prices and Inflation Forecast

There are several market instruments from which inflation forecasts can be derived

- Inflation indexed bonds
- Inflation Swaps
- Inflation Options

Inflation indexed bonds

- The most common form of government bonds are nominal bonds that pay fixed coupons and principal.
- Inflation-indexed bonds, which in the U.S. are known as Treasury Inflation Protected Securities (TIPS), are bonds whose coupons and principal adjust automatically with the evolution of a consumer price index.
- They aim to pay investors a fixed inflation-adjusted coupon and principal, in other words they are real bonds and their yields are typically considered the best proxy for the term structure of real interest rates in the economy.

Inflation indexed bonds

- Investors holding either inflation-indexed or nominal government bonds are exposed to the risk of changing real interest rates.
- In addition to real interest rate risk, nominal government bonds expose investors to inflation risk while real bonds do not. When future inflation is uncertain, the coupons and principal of nominal bonds can suffer from the eroding effects of inflationary surprises.
- Finally, both the nominal and real bond are theoretically affected by a premium for liquidity risk. Liquidity risk, is, the risk of having to sell (or buy) a bond in a thin market and, thus, at an unfair price and with higher transaction costs.

Inflation indexed bonds

At time t the yields to maturity of nominal and real bonds maturing at T can be wriiten as follows:

$$\begin{aligned} Y^n_{t,T} &= rr_{t,T} + E_t\pi_{t,T} + RP^{rr}_t + RP^{\pi}_t \\ Y^r_{t,T} &= rr_{t,T} + RP^{rr}_t + RP^{liq}_t \end{aligned}$$

the difference in the yield to maturity, usually referred to as the breakeven inflation rate $B_{t,T}$, can be written as:

$$B_{t,T} = E_t\pi_{t,T} + RP^{\pi}_t - RP^{liq}_t$$

Further technical details

Further details complicate the pricing of real bonds.

- Inflation figures are published with a lag and therefore the principal value of inflation indexed bonds is adjusted with a lag (three-month in the US and the UK).
- Moreover the mechanism of indexation could be different for coupons and principal and it might have different provisions for positive and negative inflation.
- Finally the tax treatment of the bond might also differ. In the UK principal adjustments of inflation-linked gilts are not taxed. In the US, on the other hand, inflation-adjustments of principal are considered ordinary income for tax purposes.

Inflation Swaps

- An inflation swap is a bilateral contractual agreement. It requires one party (the 'inflation payer') to make periodic floating-rate payments linked to inflation, in exchange for predetermined fixed-rate payments from a second party (the 'inflation receiver').
- The most common structure is the zero-coupon inflation swap. The inflation receiver pays on a contract maturing in T years pays the Fixed leg = (1 + fixed rate)T x Nominal value to receive from the the inflation payer the variable Inflation leg = (Final price index/Starting price index) x Nominal value.

$$SW_{t,T} = E_t \pi_{t,T} + RP_t^{\pi}$$

Further Technical Details

In practice, inflation swap contracts have indexation lags.

$$SW_{t,T} = E_t \pi_{t-m,T-m} + RP_t^{\pi}$$

As the object of interest is inflation from time t onward, the following decomposition becomes relevant:

$$SW_{t,T} = \frac{m}{T-t}\pi_{t-m,t} + \frac{T-m-t}{T-t}E_t \pi_{t,T-m} + RP_t^{\pi}$$

which, in absence of publication lags on inflation data allows to derive the relevant inflation compensation from the fixed swap rate.

The distribution of market-based inflation expectations

Inflation options are instruments with non-linear payoffs:

- inflation caps pay-out if inflation exceeds a certain threshold
- inflations floors pay-out if inflation falls short of a certain threshold

By comparing the price of options that insure against different outcomes it is possible to infer the probability that investors assign to those different outcomes.

The distribution of market-based inflation expectations

More formally:

$$p_t = \sum \frac{1}{1 + r_f} P^Q(\pi) X_{t+1}(\pi)$$

The price of an inflation linked-asset today depends on the risk neutral probabilities $P^Q(\pi)$ and one-period ahead payoffs $(X_{t+1}(\pi))$ associated with different inflation events.
Given the knowledge of prices, payoffs and the risk-free rate it is then possible to recover risk-neutral probabilities of inflation.

Inflation Caps and Floors

- A (zero-coupon) inflation cap is a contract entered into at time t. The seller of the cap promises to pay a fraction $\max((1+\pi(n))^n - (1+k)^n; 0)$ of a notional underlying principal as a single payment in n years' time, where $\pi(n)$ denotes the average annual CPI inflation rate from t to t+n, and k denotes the strike of the cap. The notional underlying principal is normalized to 1. In exchange for this, the buyer makes an up-front payment of $p_t(k, n)$.
- A zero coupon inflation floor is identical except that the payment is A zero-coupon inflation floor is identical, except that the payment is $\max((1+k)^n - (1+\pi(n))^n; 0)$

Building the distribution of inflation expectations from caps and floor

- Suppose you obtain daily quotes on zero-coupon inflation caps and floors at different strike prices (say -2,-1,0,1,2,3,4,5,6) and that inflation over the next n years can be approximated as having only integer support.

- A butterfly portfolio that involves buying one cap with strike of $(k-1)$ percent and one with strike $(k+1)$ percent while shorting two caps with a strike of k percent, is a pure Arrow-Debreu security with a payoff of 1 if inflation is k percent and zero otherwise, for any integer k. So the price of this security will be:

$$p_t = \frac{1}{1+r_f} P^Q(\pi = k)$$

- As for the tails, if an investor buys an inflation cap at 5 percent and shorts one at 6 percent, then this investor receives 1 if inflation is 6 percent or more and zero otherwise. This gives the probability of inflation being 6 percent or higher. The same works for the left tail

A correction

- A butterfly portfolio that involves buying one cap with strike of $(k-1)$ percent and one with strike (k) percent while shorting two caps with a strike of k percent, is a pure Arrow-Debreu security with a payoff of 1 if inflation is k percent and zero otherwise, for any integer k. So the price of this security will be:

$$p_t = \frac{1}{1+r_f} P^Q(\pi = k)$$

- The usual proxy adopted for r_f is the nominal risk free at the appropriate maturity, which we can write as follows:

$$r_f^{obs} = rr_f + \pi^*$$

  where π^* is the inflation target of the monetary policy maker and rr_f is the real risk-free rate.

- However, the nominal risk free rate when $\pi = k$ is

$$r_f = rr_f + k$$

- so if r_f^{obs} so to extract risk neutral probabilities from the otion price the following formula should be used

$$p_t = \frac{1}{1+r_f^{obs}} \frac{1+rr+\pi^*}{1+rr+k} P^Q(\pi = k)$$

- the correction is neglible for k close to the inflation target but it becomes more relevant when k is far away from the target (deflation or inflation disasters)

YTM and Risk Premia

Apply the no arbitrage condition to a one-period bond (the safe asset) and a T-period bond:

$$\begin{aligned} E_t\left(r^T_{t,t+1} - r^1_{t,t+1}\right) &= E_t\left(r^T_{t,t+1} - y_{t,t+1}\right) = \phi^T_{t,t+1} \\ E_t\left(r^T_{t,t+1}\right) &= y_{t,t+1} + \phi^T_{t,t+1} \end{aligned}$$

Solving forward the difference equation $p_{t,T} = p_{t+1,T} - r^T_{t,t+1}$, we have :

$$\begin{aligned} y_{t,T} &= \frac{1}{(T-t)} \sum_{i=0}^{T-1} E_t\left(r^T_{t+i,t+i+1}\right) \\ &= \frac{1}{(T-t)} \sum_{i=0}^{T-1} E_t\left(y_{t+i,t+i+1} + \phi^T_{t+i,t+i+1}\right) \end{aligned}$$

$$y_{t,T} - y_{t,t+1} = \sum_{i=1}^{T-t-1} \left(1 - \frac{i}{T-t}\right) E_t \Delta y_{t+i,t+i+1} + \frac{1}{T-t} \sum_{i=1}^{T-t-1} \phi^T_{t+i,t+i+1}$$

HPR and risk-premia

The one-period uncertain holding-period return on a bond maturing at time T, $r^T_{t,t+1}$, is then defined as follows:

$$\begin{aligned} r^T_{t,t+1} &\equiv p_{t+1,T} - p_{t,T} = -\left(T-t-1\right) y_{t+1,T} + \left(T-t\right) y_{t,T} \qquad (38) \\ &= y_{t,T} - \left(T-t-1\right)\left(y_{t+1,T} - y_{t,T}\right), \\ &= \left(T-t\right) y_{t,T} - \left(T-t-1\right) y_{t+1,T}, \qquad (39) \end{aligned}$$

No arbitrage + RE implies:

$$\left(r^T_{t,t+1} - y^1_{t,t+1}\right) = \phi^T_{t,t+1} + u_{t,t+1}$$

HPR and risk-premia: a regression approach

Think of the following regression

$$(T-t)\,y_{t,T}-(T-t-1)\,y_{t+1,T}-y_{t,t+1}=\beta_0+\beta_1\left(y_{t,T}-y_{t,t+1}\right)+u_{t+1}$$

It is a predictive regression that will deliver predictability of excess returns under the two following hypothesis

- the one period risk premium is persistent
- the term spread it is not dominated by the expected monetary policy component

If the regression reveals predictability we have:

$$\begin{aligned}\hat{\phi}^{T}_{t,t+1} &= \hat{\beta}_0+\hat{\beta}_1\left(y_{t,T}-y_{t,t+1}\right)\\ \hat{\phi}^{T}_{t,t+T} &= \frac{1}{T}\sum_{i=1}^{T}E_t\hat{\phi}^{T}_{t+i-1,t+i}\end{aligned}$$

so the period term premium can be proxied by pairing the predictive regression with a model for the persistent one period risk premium

The Nominal Expectations Hypothesis

PV test in their data whether the term spread contains a time-varying risk premium by running the regression test :

$$(T-t)\, y_{t,T} - (T-t-1)\, y_{t+1,T} - y_{t,t+1} = \beta_0 + \beta_1 \left(y_{t,T} - y_{t,t+1} \right) + u_{t+1}$$

The Real Expectations Hypothesis

PV test in their data whether the real term spread contains a time-varying risk premium by running the regression test :

$$(T-t)\,y_{t,T}^{TIPS} - (T-t-1)\,y_{t+1,T}^{TIPS} - y_{t,t+1}^{TIPS} = \beta_0 + \beta_1\left(y_{t,T}^{TIPS} - y_{t,t+1}^{TIPS}\right) + u_{t+1}$$

Breakeven Inflation and the Inflation Expectations Hypothesis

given that breakeven inflation is defined as follows:

$$b_{t,T} = y_{t,T} - y_{t,T}^{TIPS}$$

then the excess returns on breakeven inflation is given by

$$(T-t)\, b_{t,T} - (T-t-1)\, b_{t+1,T} - b_{t,t+1}$$

remebering that

$$b_{t,T} = E_t \pi_{t,T} + RP_t^{\pi} - RP_t^{liq}$$

When inflation and liquidity risk premia are constant the excess returns on break-even inflation are equal to a constant plus the expression

$$(T-t)\, \pi_{t,T}^{e} - (T-t-1)\, \pi_{t+1,T}^{e} - \pi_{t,t+1}$$

which fluctuates around zero when expectations are rational

Breakeven Inflation and the Inflation Expectations Hypothesis

So the inflation (and liquidity) risk premia can be measured via the following regression strategy :

$$
\begin{aligned}
(T-t)\, b_{t,T} - (T-t-1)\, b_{t+1,T} - b_{t,t+1} &= \beta_0 + \beta_1 \left(b_{t,T} - b_{t,t+1} \right) + u_{t+1} \\
\hat{\phi}^T_{t,t+1} &= \hat{\beta}_0 + \hat{\beta}_1 \left(b_{t,T} - b_{t,t+1} \right) \\
\hat{\phi}^T_{t,t+T} &= \frac{1}{T} \sum_{i=1}^{T} E_t \hat{\phi}^T_{t+i-1,t+i}
\end{aligned}
$$

Implementation requires the following steps

- TIPS trade only at maturity of one year or longer so a short-term (one-quarter) real rate needs to be constructed
- run the predictive regression and extract the one period inflation risk premia
- given predictive model for the one period inflation risk premia construct the T period inflation risk premia
- subtract the T period inflation risk premia form the corrispondent measure of breakeven inflation to derive inflation expectations

The Construction of the short-term real rate

TIPS trade only at maturity of one year or longer so to construct a short term real rate assume zero inflation and risk premia over the quarter

$$y_{t,t+1}^{TIPS} = y_{t,t+1} - \pi_{t,t+1}^{e}$$

Next we assume that inflation expectations over the next quarter are rational and proxy for the ex-ante real short rate as the fitted value from the regression of this quarter's realized real rate onto last quarter's realized real rate, the nominal short rate,and annual inflation up to time t.

The Construction of the short-term real rate

Table 1
Forecasted Real Short Rate

$y^{\$}_{1,t} - \pi_{t+1}$	US	UK
$y^{\$}_{1,t}$	0.57**	0.46
	(0.22)	(0.29)
$y^{\$}_{1,t-1} - \pi_t$	0.08	−0.11
	(0.08)	(0.07)
$(\pi_{t-3} + \pi_{t-2} + \pi_{t-1} + \pi_t)/4$	0.08	0.03
	(0.09)	(0.09)
$p - value$	0.00	0.00
R^2	0.44	0.18

Overlapping quarterly real short rate returns onto the nominal short rate, last quarter's real short rate return and inflation over the past year.
Monthly data 1982.1-2009.12.
Newey-West standard errors with 4 lags in brackets.
* and ** denote significance at the 5% and 1% level respectively.
p-value of the F-test for no predictability.

Break-even Inflation and Inflation Expectations